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On Sensitivity of Characteristic Transfer Functions of Multivariable Control Systems

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Abstract

In the paper, a systematic treatment of sensitivity analysis of multivariable control systems from a perspective of the characteristic transfer functions (CTFs) method is given. The CTFs method (also called Characteristic Gain Loci method) allows one to associate with an *N*-dimensional multi-input multi-output (MIMO) system a set of *N* independent single-input single-output (SISO) characteristic systems and thereby to reduce the analysis and design of a MIMO system to analysis and design of *N* SISO systems. The formulas are derived determining the sensitivity functions of the CTFs and sensitivity vectors of the canonical basis axes to small variations of parameters of general type MIMO systems. The relations between the sensitivity functions of the open-loop and closed-loop MIMO systems are established. Two illustrative examples are considered. The first of them concerns the sensitivity of a two-dimensional not robust system with large degree of skewness of the canonical basis axes. In the second example, the sensitivity of the control system of a hexacopter (multirotor UAV with six rotors) to small degradations of the motors' efficiency is analyzed.

Keywords: multivariable control system; characteristic transfer functions; canonical basis; sensitivity; parameter perturbation

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1. Introduction

Multivariable or multi-input multi-output (MIMO) control systems are ubiquitous in modern industry and technology, including such fields as mechatronics and robotics, chemical and power industries, electrical and aerospace engineering, and many others [1–3]. The article addresses an important issue in multivariable feedback control which always attracted the interest of scientists and researchers. In a broader context, the matter concerns the analysis of sensitivity of feedback control systems to small variation of parameters, an issue which is closely related to conventional robustness analysis.

Various aspects of sensitivity analysis of control systems are presented in numerous articles, textbooks and monographs [4–12]. Thus, the bibliography in the published in 1994 monograph "Theory of Sensitivity in Dynamic Systems" by Prof. M. Eslami [6] contained more than 2500 titles on sensitivity theory and related topics. One of the first publications devoted to sensitivity analysis of multivariable feedback systems was the article by J. Cruz and W. Perkins [8], in which some frequency-domain criteria involving the sensitivity matrix were derived.

At present, the theory and methods of multivariable feedback control are mostly inherited from general feedback control [1–4,13], where optimal, adaptive, robust and some other methods are predominant [14–24]. For the most part, these methods are based on state-space representation of control systems, which, being quite effective from the computational perspective and having many other substantial features, often lacks in physical clarity that is inherent in common frequency-domain and root-domain methods of classical feedback control [13]. The point is that many important structural features of multivariable control systems that can be easily seen from the input-output (matrix) description of the MIMO systems are often significantly "disguised" in the corresponding state-space equations.

In this respect, in the 1970s, a prominent British theorist A.G. Macfarlane proposed with his colleagues the Characteristic Transfer Functions (CTFs) method (also called Characteristic Locus or Characteristic Gain Loci method), which was based on the transfer matrix representation of MIMO control systems [25–28].

Formally, the CTFs method allows one to associate with an N-dimensional (that is having N inputs and N outputs) MIMO system a set of N independent single-input single-output (SISO) characteristic systems, and, as a consequence, to reduce the stability analysis and design of an interconnected MIMO system to stability analysis and design of N SISO systems [1–4].

In fact, the ideas of the CTFs method created exceptional opportunities to extend the classical feedback control to multivariable systems. However, no effective design procedures for MIMO systems were proposed in [25–28]. Moreover, the CTFs method was claimed in [23,24] as unreliable and not robust.

In monograph [3], the main classical methods of feedback control were extended to MIMO case based on the CTFs method. The central feature of the presented in [3] methods is that design of any N-dimensional MIMO system can be reduced to design of a certain SISO system using standard classical methods and techniques.

The concept of characteristic gain loci sensitivity was first introduced by I. Postlethwaite [29]. The introduced in [29] *sensitivity indices* characterize the sensitivity of the gain loci to small perturbations of the return-ratio matrix of the MIMO system. The sensitivity indices are calculated by the left and right eigenvectors of the return-ratio matrix and give information about stability with respect to small perturbation of that matrix. The issue of robustness of MIMO systems, where the plant is described using the CTFs method is addressed in [30]. In that paper, a sophisticated two-stage design procedure to improve system robustness is proposed based on the notion of *commutative controller* [28]. At the first stage, a dynamic pre-compensator is designed which transforms, at least, at some predefined frequencies, the plant transfer matrix into a normal matrix. Then, at the second stage, a commutative controller is designed for the pre-compensated (normalized) plant. If the design is correct, then the corresponding CTFs of the pre-compensated plant and the commutative controller are just multiplied together. However, generally, it is impossible to find a realizable matrix compensator that transforms the transfer matrix of the plant into a normal matrix. For that reason, there are no effective design procedures based on the notion of commutative controllers.

At the same time, many important aspects of sensitivity of MIMO systems, such as sensitivity of canonical basis axes, relation of the sensitivity of the CTFs of open-loop and closed-loop systems, etc., have not been discussed in [29,30] or in any other publications.

In this paper, the issue of sensitivity of general type MIMO systems to small variations of parameters is systematically addressed from the perspective of the CTFs method.

The paper is organized as follows. A short presentation of the main ideas of the CTFs method is given in Section 2. Section 3 is the central and is devoted to derivation of formulas determining the sensitivity of the CTFs and the canonical basis axes to small variations (perturbations) of parameters. Section 4 presents two examples of applying the results of the previous section to specific MIMO control systems. The first of them describes the notorious two-dimensional system which was used in [23,24] as an example illustrating that the CTFs method is not robust. In the section, a detailed explanation is given why the conclusions made in [23,24] are erroneous. The second example concerns the sensitivity analysis of the hexacopter's control system to small degradations of motors' efficiency.

2. Canonical Representation of MIMO Control Systems

Consider an N-dimensional multi-input multi-output (MIMO) feedback control system in Figure 1. Here $\varphi(s)$, $f(s)$, $\varepsilon(s)$ denote the Laplace transforms of the N-dimensional input, output, and error vector signals $\varphi(t), f(t), \varepsilon(t)$, respectively (we shall regard them as elements of some N-dimensional complex space $\mathbb{C}^N$); $W(s)=\{w_{kr}(s)\}$ is a square transfer function matrix of the open-loop system of order $N \times N$ with entries $w_{kr}(s)$ ($k,r=1,2,...,N$) which are scalar proper rational functions in complex variable s [3].

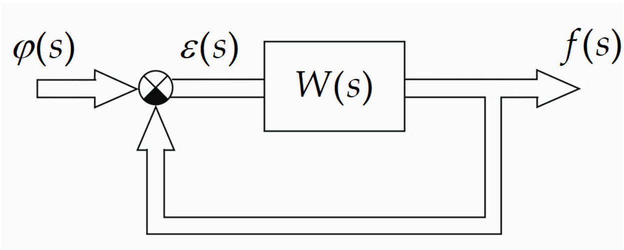

Figure 1. Block-diagram of a MIMO control system.

The output $f(s)$ and error $\varepsilon(s)$ vectors, where

$$\varepsilon(s) = \varphi(s) - f(s), \tag{1}$$

are related to the input vector $\varphi(s)$ by the following operator equations:

$$f(s) = T(s)\varphi(s), \qquad \varepsilon(s) = S(s)\varphi(s) \tag{2}$$

where

$$T(s) = \left[I + W(s)\right]^{-1} W(s) \ , \qquad S(s) = \left[I + W(s)\right]^{-1} \tag{3}$$

are the transfer function matrices of the closed-loop MIMO system with respect to output and error signals, and I is the unit matrix. The transfer matrices $S(s)$ and $T(s)$ are usually called the sensitivity function matrix and complementary sensitivity function matrix [3].

Based on the method of Characteristic Transfer Functions (CTF), the transfer matrix of the open-loop MIMO system $W(s)$ can be represented, using dyadic notation and similarity transformation, in the following canonical forms [3]:

$$W(s) = \sum_{i=1}^{N} c_i(s) > q_i(s) < c_i^+(s) = C(s)\,diag\{q_i(s)\}C^{-1}(s)\ , \tag{4}$$

where complex scalar function $q_i(s)$ ($i=1,2,...,N$) ("eigenvalues" of $W(s)$) are called the CTFs of the open-loop system (further, for simplicity, all $q_i(s)$ are assumed distinct); $c_i(s)$ are linearly-independent normalized eigenvectors of $W(s)$ which constitute the canonical basis of the open-loop MIMO system; $c_1^+(s)$ are vectors dual to $c_i(s)$ (vectors of the dual basis), and the modal matrix $C(s)$ is composed of vector-columns $c_i(s)$.

Substitution the canonical representations of $W(s)$ (4) into the complementary sensitivity function matrix $T(s)$ and the sensitivity function matrix $S(s)$ (3) yields

$$T(s) = \sum_{i=1}^{N} c_i(s) > \frac{q_i(s)}{1 + q_i(s)} < c_i^+(s) = C(s)\,diag\left\{\frac{q_i(s)}{1 + q_i(s)}\right\}C^{-1}(s) \tag{5}$$

$$S(s) = \sum_{i=1}^{N} c_i(s) > \frac{1}{1 + q_i(s)} < c_i^+(s) = C(s)\,diag\left\{\frac{1}{1 + q_i(s)}\right\}C^{-1}(s) \tag{6}$$

Examination of (4)-(6) shows that the canonical basis of the closed-loop MIMO system coincides with the canonical basis of the open-loop system [3]. Moreover, the CTFs of the closed-loop MIMO system with respect to the output and error are related to $q_i(s)$ by the very same relationships as common transfer functions of open-loop and closed-loop single-input single-output (SISO) feedback systems. Geometrically, all this is illustrated in Figures 2 and 3.

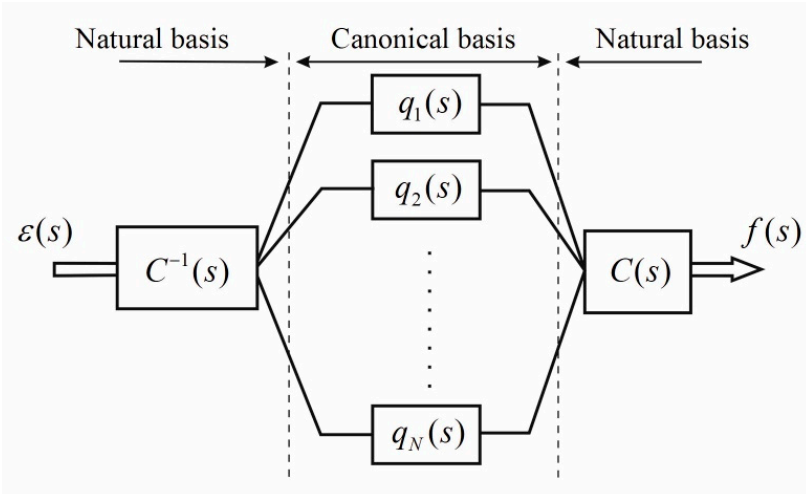

Figure 2. Canonical representation of the open-loop MIMO system.

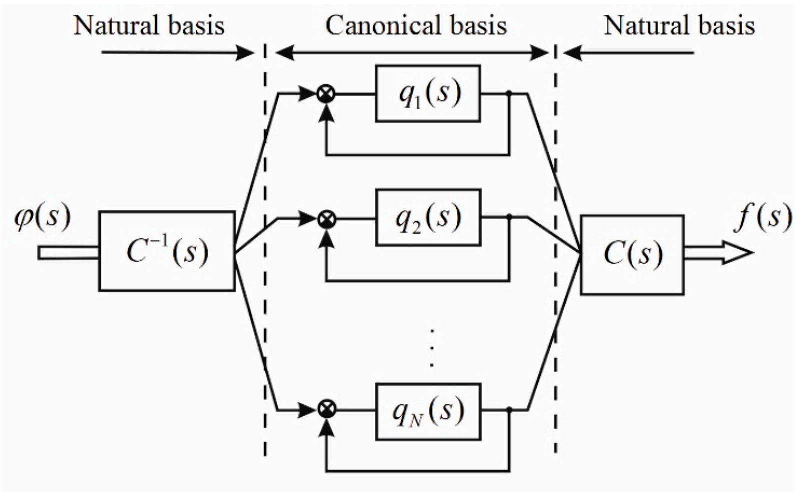

Figure 3. Canonical representation of the closed-loop MIMO system.

The stability of the multivariable system in Figure 1 is determined by roots of the characteristic equation [1–3]

$$\det[I + W(s)] = \prod_{i=1}^{N}[1 + q_i(s)] = 0 \tag{7}$$

As can be seen from (7), the multivariable system in Figure 1 is stable if all N SISO characteristic systems are stable. It should be noted that the stability of characteristic systems can be analyzed using conventional methods of classical feedback control, in particular, using the Nyquist criterion [1,3,13].

3. Sensitivity of Characteristic Transfer Functions and Canonical Basis Axes

In this section, we discuss, from the positions of the CTFs method, the influence of small perturbations (variations) of parameters on structural and geometrical features of linear MIMO systems. The main attention is paid to the sensitivity functions of SISO characteristic systems and canonical basis axes, in the case of small variations of the MIMO system parameters.

Consider the linear MIMO system in Figure 1 and assume that the open-loop transfer matrix $W(s)$ depends continuously on m varying parameters α_r $(r=1,2,\dots,m)$, forming the vector α. Further, to emphasize the dependence of $W(s)$ on α, we shall write $W(s,\alpha)$. Denote by α_o the vector of nominal values of α, and by $\Delta\alpha$ - the vector of variations, i.e. assume $\alpha=\alpha_o+\Delta\alpha$. Let us know the CTFs and canonical basis of the MIMO system with the nominal values α_o. A question arises, how the CTFs and canonical basis axes are changed in the case of small variations of the parameters α, i.e. for $(\Delta\alpha_r)^2\approx 0$? We solve this task as a first approximation, based on the perturbation theory of linear operators in finite-dimension Hilbert spaces [31,32].

Let, because of parameters perturbations, the transfer matrix $W(s,\alpha_o)$ can be expressed as

$$W(s,\alpha)=W(s,\alpha_o)+\Delta W(s,\alpha) \tag{8}$$

where $\Delta W(s,\alpha)$ is the variation of $W(s,\alpha_o)$ caused by the perturbations $\Delta\alpha$. As we assume continuous dependence of $W(s,\alpha)$ on α_r, the variation $\Delta W(s,\alpha)$ in (8) can be represented, for small $\Delta\alpha_r$, with the help of the Taylor series

$$\Delta W(s,\alpha)=\sum_{r=1}^{m}U_r^W(s)\Delta\alpha_r + \dots, \tag{9}$$

where

$$U_r^W(s)=\left.\frac{\delta W(s,\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o} \qquad r=1,2,\dots,m \tag{10}$$

are the first-order sensitivity matrices of the transfer matrix $W(s,\alpha)$ with respect to variations $\Delta\alpha_r$, evaluated for the nominal vector $\alpha=\alpha_o$. The dots in (9) and in what follows mean that the terms of the higher order of smallness, as compared with $\Delta\alpha_r$, are omitted.

As $W(s,\alpha)$ is assumed differentiable with respect to α_r, the following expansions for the CTFs $q_i(s,\alpha)$ and canonical basis axes $c_i(s,\alpha)$ are valid

$$q_i(s,\alpha)=q_i(s,\alpha_o)+\sum_{r=1}^{m}S_{ir}^q(s)\Delta\alpha_r + \dots \tag{11}$$

and

$$c_i(s,\alpha)=c_i(s,\alpha_o)+\sum_{r=1}^{m}\Upsilon_{ir}^q\Delta\alpha_r + \dots, \qquad i=1,2,\dots,N \tag{12}$$

where

$$S_{ir}^q(s)=\left.\frac{\delta q_i(s,\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o}, \qquad \Upsilon_{ir}^q(s)=\left.\frac{\delta c_i(s,\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o} \qquad r=1,2,\dots,m \tag{13}$$

are the unknown first-order sensitivity functions of $q_i(s,\alpha)$ and sensitivity vectors of $c_i(s,\alpha)$.

Since (11) and (12) are the eigenvalues and eigenvectors of the varied matrix $W(s,\alpha)$ (8), and allowing for (9), we can write down

$$\left[W(s,\alpha_o) + \sum_{r=1}^{m} U_r^W(s)\Delta\alpha_r + \dots\right]\left[c_i(s,\alpha_o) + \sum_{r=1}^{m} \Upsilon_{ir}^q \Delta\alpha_r + \dots\right] =$$

$$= \left[q_i(s,\alpha_o) + \sum_{r=1}^{m} S_{ir}^q(s)\Delta\alpha_r + \dots\right]\left[c_i(s,\alpha_o) + \sum_{r=1}^{m} \Upsilon_{ir}^q \Delta\alpha_r + \dots\right] \quad (14)$$

$$r = 1, 2, \dots, m$$

Carrying out the multiplication and comparing the first-order terms with respect to the corresponding perturbations $\Delta\alpha_r$ in both sides of (14), we obtain the following system of mN equations:

$$U_r^W(s)c_i(s,\alpha_o) + W(s,\alpha_o)\Upsilon_{ir}^q(s) = q_i(s,\alpha_o)\Upsilon_{ir}^q(s) + S_{ir}^q(s)c_i(s,\alpha_o) \quad (15)$$

$$i = 1, 2, \dots, N; \quad r = 1, 2, \dots, m$$

Multiply, in the form of scalar product, both sides of (15) by the vector $c_i^+(s,\alpha_o)$, which is dual to the eigenvector $c_i(s,\alpha_o)$ of the initial (unperturbed) MIMO system. Recalling that the first vector in the scalar product should be taken as complex conjugate, we obtain

$$< U_r^W(s)c_i(s,\alpha_o), c_i^+(s,\alpha_o) > + < W(s,\alpha_o)\Upsilon_{ir}^q(s), c_i^+(s,\alpha_o) > =$$

$$= \tilde{q}_i(s,\alpha_o) < \Upsilon_{ir}^q(s), c_i^+(s,\alpha_o) > + \tilde{S}_{ir}^q(s) \quad (16)$$

$$i = 1, 2, \dots, N; \quad r = 1, 2, \dots, m$$

Note that if the matrix $W(s,\alpha_o)$ has the canonical representation

$$W(s,\alpha_o) = C(s,\alpha_o)\, diag\{q_i(s,\alpha_o)\}\, C^{-1}(s,\alpha_o) \quad (17)$$

then the conjugate matrix $W^*(s,\alpha_o)$ is

$$W^*(s,\alpha_o) = (C^{-1}(s,\alpha_o))^*\, diag\{\tilde{q}_i(s,\alpha_o)\}\, C^*(s,\alpha_o). \quad (18)$$

From here, it is clear that the eigenvector $c_i^+(s,\alpha_o)$, which is dual to the eigenvector $c_i(s,\alpha_o)$ of $W(s,\alpha_o)$, corresponds to the CTF $\tilde{q}_i(s,\alpha_o)$ of $W^*(s,\alpha_o)$. Therefore

$$< W(s,\alpha_o)\Upsilon_{ir}^q(s), c_i^+(s,\alpha_o) > = < \Upsilon_{ir}^q(s), W^*(s,\alpha_o)c_i^+(s,\alpha_o) >$$

$$= \tilde{q}_i(s,\alpha_o) < \Upsilon_{ir}^q(s), c_i^+(s,\alpha_o) > \quad (19)$$

Substituting (19) into (16) and conjugating both sides in (16), yields the final expression for the sensitivity function $S_{ir}^q(s)$:

$$S_{ir}^q(s) = < c_i^+(s,\alpha_o), U_r^W c_i(s,\alpha_o) > \quad (20)$$

$$i = 1, 2, \dots, N; \quad r = 1, 2, \dots, m.$$

It is important to note that, for any fixed r, the right sides in (20) represent (for $i = 1, 2, \dots, N$) the diagonal elements of the matrix $U_r^W(s)$ (10), where the latter is evaluated in the basis composed of the vectors $c_i(s,\alpha_o)$. Hence, the open-loop MIMO system sensitivity functions $S^q_{ir}(s)$ with respect to small perturbations of the r th parameter α_r are equal to the diagonal elements of the sensitivity matrix $U^W_r(s)$ evaluated in the canonical basis of the unperturbed MIMO system. Multiplying both sides in (16) by $\Delta\alpha_r$, we come to a conclusion that a similar relationship takes place, as a first approximation, between the finite variations $\Delta q_{ir} = S^W_{ir}\Delta\alpha_r$ and the diagonal elements of the matrix of finite variations $\Delta W_r(s) = U^W_r(s)\Delta\alpha_r$.

Proceed now to the sensitivity vectors $\Upsilon^q_{ir}(s)$ (13) of the open-loop MIMO system's canonical basis axes. Toward this end, multiply both sides of (15) by the dual vectors $c^+_k(s,\alpha_o)$ $(k=1,2,\dots,N;\ \ k\neq i)$. After some simple transformations this yields:

$$< c^+_k(s,\alpha_o), \Upsilon^q_{ir}(s) > = \frac{n^r_{ki}(s)}{q_i(s,\alpha_o) - q_k(s,\alpha_o)} \tag{21}$$

$$k=1,2,\dots,N;\ \ k\neq i, \qquad r=1,2,\dots,m,$$

where

$$n^r_{ki}(s) = <c^+_k(s,\alpha_o), U^W_r(s)\, c_i(s,\alpha_o)>$$

are the elements of the i th column (except for $n^r_{ii}(s)$) of the matrix $U^W_r(s)$ (10), represented in the canonical basis of the unperturbed MIMO system. Notice that the scalar products on the left side of (21) are the coordinates of the vector $\Upsilon^q_{ir}(s)$ along the k th axis $c_k(s,\alpha_o)$ of the initial canonical basis. It is easy to show, using the normalizing condition for the varied eigenvector $c_i(s,\alpha_o)$ (12) (Gelfand 1966), that the coordinate of $\Upsilon^q_{ir}(s)$ along the i th axis $c_i(s,\alpha_o)$ is equal to zero. Therefore, we can write

$$\Upsilon^q_{ir}(s) = \sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{n^r_{ki}(s)}{q_i(s,\alpha_o) - q_k(s,\alpha_o)}\, c_k(s,\alpha_o), \tag{22}$$

$$i=1,2,\dots,N; \qquad r=1,2,\dots,m$$

Thus, expressions (20) – (22) determine the sensitivity functions $S^q_{ir}(s)$ of the CTFs $q_i(s,\alpha)$ of the open-loop MIMO system and the sensitivity vectors $\Upsilon^q_{ir}(s)$ of the canonical basis axes $c_i(s,\alpha)$ with respect to small perturbations of the parameters α_r. These expressions show that for evaluating $S^q_{ir}(s)$ and $\Upsilon^q_{ir}(s)$ it is enough to find the representation

$$N_r(s) = C^{-1}(s,\alpha_o)\, U^W_r(s)\, C(s,\alpha_o), \tag{23}$$

$$r=1,2,\dots,m$$

of the matrices $U^W_r(s)$ (10) in the canonical basis of the unperturbed MIMO system. The diagonal elements $n^r_{ii}(s)$ of the matrix $N_r(s)$ (23) are equal to $S^q_{ir}(s)$, and the non-diagonal elements $n^r_{ik}(s)$

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, after dividing by the differences $[q_i(s,\alpha_o) - q_k(s,\alpha_o)]$, give the coordinates of $\Upsilon_{ir}^q(s)$ in the mentioned basis.

Consider now the sensitivity of the closed-loop MIMO system, restricting ourselves, for brevity, to the sensitivity transfer matrix $S(s)$ (3). Assume that because of the parameter's perturbations $\Delta\alpha$, the initial transfer matrix $S(s,\alpha_o)$ becomes

$$S(s,\alpha) = S(s,\alpha_o) + \Delta S(s,\alpha) \tag{24}$$

where $\Delta S(s,\alpha)$ is the variation of $S(s,\alpha_o)$ caused by the perturbations $\Delta\alpha$. Let us find the expression for $S(s,\alpha)$ assuming that the corresponding variation $\Delta W(s,\alpha)$ of the open-loop MIMO system is known. In that case, the matrix $S(s,\alpha)$ (24) can be represented in the form

$$S(s,\alpha) = [I + W(s,\alpha_o) + \Delta W(s,\alpha)]^{-1} \tag{25}$$

From (3) and (12), it is easy to see that the variation $\Delta W(s,\alpha)$ is equal to the difference of the inverse transfer matrices $S^{-1}(s,\alpha)$ and $S^{-1}(s,\alpha_o)$, i.e.

$$S^{-1}(s,\alpha) - S^{-1}(s,\alpha_o) = \Delta W(s,\alpha) \text{ ю}$$

Multiplying this relation from the left by $S(s,\alpha_o)$ and from the right by $S(s,\alpha)$ yields

$$S(s,\alpha_o) - S(s,\alpha) = -\Delta S(s,\alpha) = S(s,\alpha_o)\Delta W(s,\alpha)\, S(s,\alpha) \tag{26}$$

from which, taking into account (24), after simple algebraic transformations we obtain the following expression for the variation $\Delta S(s,\alpha)$ of the transfer matrix $S(s,\alpha_o)$ caused by the variation $\Delta W(s,\alpha)$

$$\Delta S(s,\alpha) = -[I + S(s,\alpha_o)\Delta W(s,\alpha)]^{-1} S(s,\alpha_o)\Delta W(s,\alpha) S(s,\alpha_o) \tag{27}$$

In principle, this expression is valid for any, not necessarily small, variations $\Delta W(s,\alpha)$. Recalling that the task above was solved as a first approximation, preserving only the terms linear with respect to $\Delta\alpha_r$ determine this approximation for $\Delta S(s,\alpha)$. To this end, expand the inverse matrix on the right side of (27) into the infinite Neumann series [32]:

$$[I + S(s,\alpha_o)\Delta W(s,\alpha)]^{-1} = \sum_{i=0}^{\infty} [S(s,\alpha_o)\Delta W(s,\alpha)]^i \tag{28}$$

This series converges for $||S(s,\alpha_o)\Delta W(s,\alpha)|| < 1$, which always takes place in practice for small $\Delta\alpha_r$. Substituting (28) in (27), and neglecting the terms of the higher order of smallness, gives a first approximation for the variation $\Delta S(s,\alpha)$

$$\Delta S(s,\alpha) = -S(s,\alpha_o)\Delta W(s,\alpha) S(s,\alpha_o)\,. \tag{29}$$

Owing to the continuous dependence of the transfer matrix $S(s,\alpha)$ on parameters α_r, the variation $\Delta S(s,\alpha)$ can be written in the form

$$\Delta S(s,\alpha) = \sum_{r=1}^{m} U_r^S(s)\Delta\alpha_r + ... \tag{30}$$

where

$$U_r^S(s) = \left.\frac{\delta S(s,\alpha)}{\delta \alpha_r}\right|_{\alpha=\alpha_o} \quad r = 1, 2, \ldots, m \tag{31}$$

are the first-order sensitivity matrices of the transfer matrix $S(s,\alpha)$ with respect to the variations of the r th parameter α_r. Then, based on (29), (30), and (9), we find the following formulae establishing relationships between the sensitivity matrices (10) and (31) of the open-loop and closed-loop MIMO system transfer matrices

$$U_r^S(s) = -S(s,\alpha_o)U_r^W(s)S(s,\alpha_o) \quad r = 1, 2, \ldots, m \tag{32}$$

The CTFs $S_i(s)$ of the closed-loop MIMO system have the following form.

$$S_i(s,\alpha) = \frac{1}{1 + q_i(s,\alpha)} \quad i = 1, 2, \ldots, N$$

Expand these functions into the Taylor series:

$$S_i(s,\alpha) = S_i(s,\alpha_o) + \sum_{r=1}^{m} S_{ir}^S(s)\Delta\alpha_r + \ldots, \quad i = 1, 2, \ldots, N, \tag{33}$$

where

$$S_{ir}^S(s) = \left.\frac{\delta S_i(s,\alpha)}{\delta \alpha_r}\right|_{\alpha=\alpha_o} \quad r = 1, 2, \ldots, m \tag{34}$$

are the first-order sensitivity functions of the CTFs $S_i(s,\alpha)$ with respect to the variations of the r th parameter α_r.

Since the canonical bases of the open-loop and closed-loop MIMO system coincide, the expansion (12) is also valid for the axes $c_i(s,\alpha)$ of the closed-loop system, where instead of $\Upsilon_{ir}^q(s)$ should be introduced the designations $\Upsilon_{ir}^S(s)$. Proceeding in the same manner, as in analyzing the open-loop MIMO system, it is easy to show that for evaluating $S_{ir}^S(s)$ and $\Upsilon_{ir}^S(s)$ it is necessary to find the representation of the matrices $U_r^S(s)$ (31) in the canonical basis of the unperturbed MIMO system. That representation has the form

$$M_r(s) = C^{-1}(s,\alpha_o)U_r^S(s)C(s,\alpha_o) \quad r = 1, 2, \ldots, m \tag{35}$$

The diagonal elements $m_{ii}^r(s)$ of the matrix $M_r(s)$ are equal to the sensitivity functions $S_{ir}^S(s)$ (34), and the non-diagonal elements $m_{ki}^r(s)$, divided by the differences $[S_i(s,\alpha_o) - S_k(s,\alpha_o)]$

$(k = 1, 2, \ldots, N,\ k \neq i)$, yield the coordinates of the sensitivity vectors $\Upsilon_{ir}^{S}(s)$ of the i th canonical axis of the closed-loop MIMO system.

Expressions (32) and (35) enable us to determine the relationship between sensitivity functions of the open-loop and closed-loop MIMO system, i.e. to determine the effect caused on the MIMO system sensitivity by the introduction of feedback. Substituting (32) into (35), using the canonical representation of the transfer matrix $S(s,\alpha_o)$ (47) and accounting for (23) gives:

$$M_r(s) = -diag\left\{\frac{1}{1 + q_i(s,\alpha_o)}\right\} N_r(s)\, diag\left\{\frac{1}{1 + q_i(s,\alpha_o)}\right\} \quad r = 1, 2, \ldots, m \tag{36}$$

from which, finding the diagonal elements on the right side, yields

$$S_{ir}^{S}(s) = -\frac{1}{[1 + q_i(s,\alpha_o)]^2} S_{ir}^{q}(s) \quad i = 1, 2, \ldots, N; \quad r = 1, 2, \ldots, m. \tag{37}$$

Expressions (37) relate the sensitivity functions of the closed-loop and open-loop SISO characteristic systems and are completely analogous to those well-known in the classical control theory [4,13]. If we let $s = j\omega$, then from (37) it is clear that for those frequencies ω for which $|q_i(j\omega,\alpha_o)| >> 1$ (usually it is the low-frequency region), the introduction of feedback decreases the sensitivity of the CTFs to parameters variations. In the high-frequency region, where $|q_i(j\omega,\alpha_o)| \approx 0$, the sensitivity of the closed- and open-loop CTFs is approximately the same. Finally, in the region of the resonance frequencies, where $|S_i(j\omega,\alpha_o)| > 1$, the feedback deteriorates the sensitivity of the CTFs.

The non-diagonal elements $m_{ki}^{r}(s)$ of the matrix $M_r(s)$ (36) have the form

$$m_{ki}^{r}(s) = -\frac{n_{ik}^{r}(s)}{[1 + q_i(s,\alpha_o)][1 + q_k(s,\alpha_o)]} \quad k, i = 1, 2, \ldots, N \quad (k \neq i) \tag{38}$$

Dividing both sides of (38) by the differences $[S_i(s,\alpha_o) - S_k(s,\alpha_o)]$, we obtain the following equalities

$$\frac{m_{ki}^{r}(s)}{S_i(s,\alpha_o) - S_k(s,\alpha_o)} = \frac{n_{ki}^{r}(s)}{q_i(s,\alpha_o) - q_k(s,\alpha_o)} \quad k, i = 1, 2, \ldots, N \quad (k \neq i) \tag{39}$$

But, according to the stated above, the left-hand terms in (39) are equal (for some i) to the coordinates of the se1nsitivity vectors $\Upsilon_{ir}^{S}(s)$ of the i th canonical axis of the closed-loop MIMO system, and the right-hand terms are equal to the coordinates of the sensitivity vectors $\Upsilon_{ir}^{q}(s)$ of the corresponding (in fact, the same) axis of the open-loop system. Besides, all these coordinates are determined with respect to the same canonical basis of the unperturbed MIMO system.

Consequently, the vectors $\Upsilon_{ir}^{S}(s)$ and $\Upsilon_{ir}^{q}(s)$ are equal to each other, i.e. the introduction of feedback does not affect the sensitivity of the MIMO system canonical basis to small variations of parameters.

4. Examples

4.1. Two-Dimensional Not Robust System

Below, we consider the celebrated example from [23,24], owing to which an erroneous conclusion about inefficiency and lack of robustness of the CTFs method has become quite common in technical literature.

Given the two-dimensional open-loop system with the following transfer function matrix

$$W(s) = \frac{1}{(s+1)(s+2)}\begin{pmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{pmatrix} . \tag{40}$$

which can be represented in the form

$$W(s) = \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix}\begin{pmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{2}{s+2} \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix} , \tag{41}$$

The expression (41) means that for this system the CTFs $q_1(s)$ and $q_2(s)$ have the form of the first order transfer functions:

$$q_1(s) = \frac{1}{s+1}, \qquad q_2(s) = \frac{2}{s+2} , \tag{42}$$

which imply an infinite gain margin and phase margin equal to 180°. The gain loci $q_1(j\omega)$ and $q_2(j\omega)$ of the system are shown in Figure 4a and confirm that conclusion.

However, as shown in [23], introducing into the system a diagonal regulator

$$K = \begin{pmatrix} 1.13 & 0 \\ 0 & 0.88 \end{pmatrix} , \tag{43}$$

i.e. increasing the gain of the first channel by 0.13 and decreasing the of the second channel by 0.12, makes the closed-loop system unstable.

This, at first sight, indisputable evidence that the stability margins calculated with the help of the CTFs may lead to a wrong conclusion, in fact demands more careful treatment. As is shown in [3], the discussed MIMO system indeed remains stable for arbitrary large but the *same* gains in the separate channels, since in that case the canonical basis of the system is unchanged. However, if the matrices $W(s)$ (41) and K (43) are multiplied, then the resulting modal decomposition of the system changes drastically. The characteristic gain loci $q_1(j\omega)$ and $q_2(j\omega)$ of the system with the regulator K (43) are shown in Figure 4b, from which it is clear that the MIMO system is unstable, and its new CTFs are very far from those in Figure 4a.

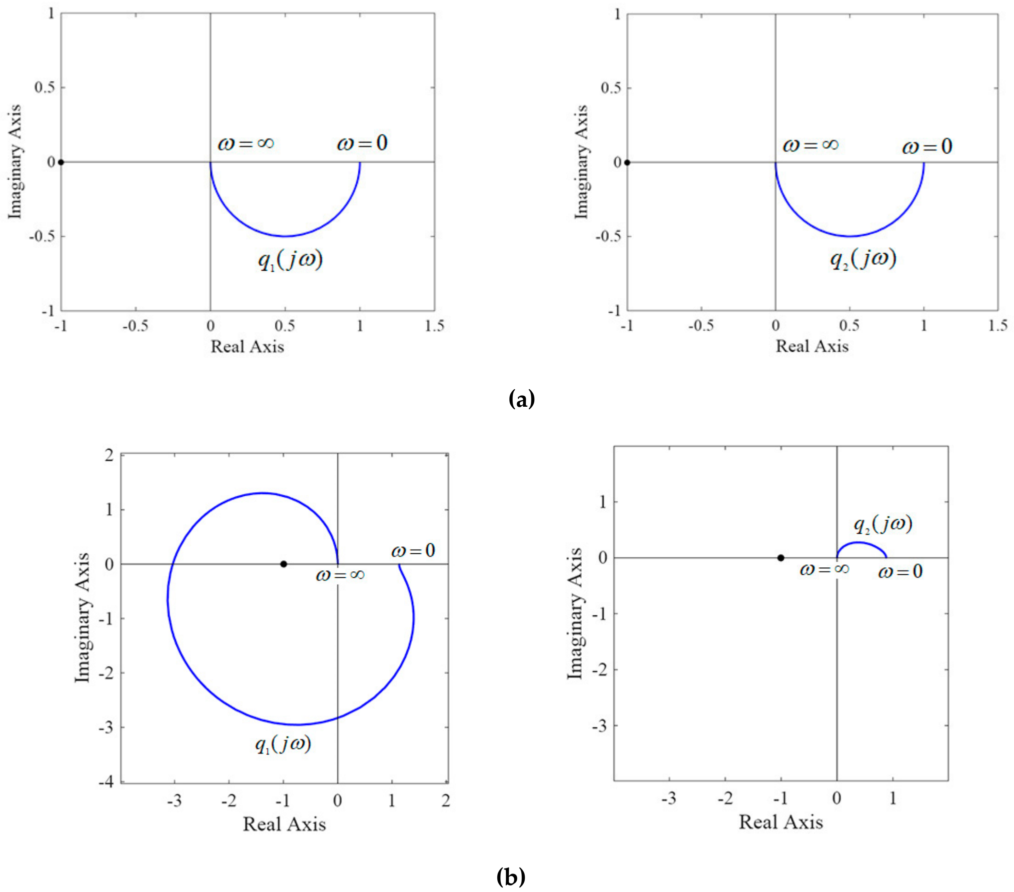

Figure 4. Stability analysis of a two-dimensional system (41): **(a)** CTFs in case of the unit regulator $K=I$; **(b)** CTFs in case of the perturbed regulator K (43).

Let us discuss now the sensitivity of that system with respect to additive perturbations of gains of the diagonal controller. Let the nominal values of the gains be unity and designate their small perturbations by ΔK_1 and ΔK_2. Then the open-loop system can be represented as

$$W(s)=\underbrace{\begin{pmatrix}64.5368 & -73.7564\\ -63.7809 & 74.4111\end{pmatrix}}_{C^{-1}}\begin{pmatrix}\dfrac{1}{s+1} & 0\\ 0 & \dfrac{2}{s+2}\end{pmatrix}\underbrace{\begin{pmatrix}0.7593 & 0.7526\\ 0.6508 & 0.6885\end{pmatrix}}_{C}\begin{pmatrix}1+\Delta K_1 & 0\\ 0 & 1+\Delta K_2\end{pmatrix} \quad (44)$$

where, unlike (41), the columns of the modal matrix C are normalized. It is easy to compute that the matrices $N_1(s)$ and $N_2(s)$ (23) in this case are

$$N_1(s)=\begin{pmatrix}\dfrac{49}{s+1} & \dfrac{56}{s+1}\\ -\dfrac{84}{s+2} & -\dfrac{96}{s+2}\end{pmatrix},\qquad N_2(s)=\begin{pmatrix}-\dfrac{48}{s+1} & -\dfrac{56}{s+1}\\ \dfrac{84}{s+2} & \dfrac{98}{s+2}\end{pmatrix}. \quad (45)$$

Since the diagonal elements of $N_1(s)$, $N_2(s)$ (45) are equal to the sensitivity functions $S_{i1}^{q}(s)$ and $S_{i2}^{q}(s)$ $(i=1,2)$ of the open-loop CTFs with respect to ΔK_1 and ΔK_2, the CTFs of the open-loop varied system can be represented as a first approximation, based on (11)-(13), in the form

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$$q_1(s) = \frac{1}{s+1} + \underbrace{\frac{49}{s+1}}_{S_{11}^q(s)}\Delta K_1 - \underbrace{\frac{48}{s+1}}_{S_{21}^q(s)}\Delta K_2\,,$$
$$q_2(s) = \frac{2}{s+2} - \underbrace{\frac{96}{s+2}}_{S_{12}^q(s)}\Delta K_1 + \underbrace{\frac{98}{s+2}}_{S_{22}^q(s)}\Delta K_2 \qquad (46)$$

The magnitude plots $|S_{11}^q(j\omega)|$ and $|S_{12}^q(j\omega)|$ of the first open-loop characteristic system are shown in Figure 6. Similar plots for the second characteristic system practically coincide with those in Figure 5.

Inspection of these plots shows that the CTFs of the open-loop system are very sensitive in the low-frequency region to the variations ΔK_1 and ΔK_2. Thus, at the zero frequency $\omega = 0$, the sensitivity functions $S_{i1}^q(s)$ and $S_{i2}^q(s)$ $(i=1,2)$ are equal to

$$S_{11}^q(j0) = 49\,,\quad S_{21}^q(j0) = -48\,,\quad S_{12}^q(j0) = -48\,,\quad S_{22}^q(j0) = 49\,. \qquad (47)$$

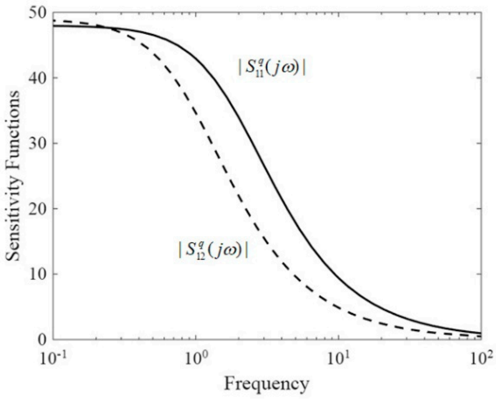

Figure 5. Sensitivity functions $|S_{11}^q(j\omega)|$ and $|S_{12}^q(j\omega)|$.

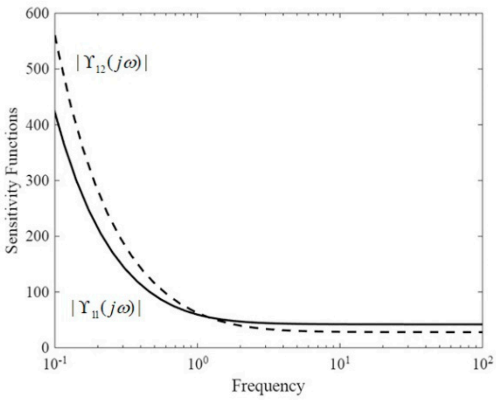

Figure 6. Sensitivity functions $|\Upsilon_{11}(j\omega)|$ and $|\Upsilon_{12}(j\omega)|$.

Further, inspection of (46) allows drawing interesting conclusions concerning the character of influence of variations ΔK_1 and ΔK_2. The most dangerous are variations of different signs. Thus, if we assume that ΔK_1 and ΔK_2 are equal by their absolute values and have opposite signs, i.e. $\Delta K_2 = -\Delta K_1$ (assuming for definiteness $\Delta K_1 > 0$), then instead of (46) we obtain

$$q_1(s) = \frac{1}{s+1} + \frac{97}{s+1}|\Delta K_1|, \qquad q_2(s) = \frac{2}{s+2} - \frac{194}{s+2}|\Delta K_2| \tag{48}$$

where the additive variations $\Delta q_i(s)$ of the CTFs have opposite signs. At the same time, in case of equal perturbations $\Delta K_1 = \Delta K_2 = \Delta K$, we have

$$q_1(s) = \frac{1+\Delta K}{s+1}, \qquad q_2(s) = \frac{2(1+\Delta K)}{s+2}, \tag{49}$$

i.e. the variations of the CTFs are proportional to ΔK.

Let us proceed now to the sensitivity of the canonical basis axes. For the normalized canonical basis axes of the varied system, based on (13), (23), (44), (45) and carrying out some simple transformations we have

$$c_1(s) = \begin{bmatrix} 0.7593 \\ 0.6508 \end{bmatrix} + \underbrace{\frac{42(s+1)}{s}\begin{bmatrix} 0.7593 \\ 0.6508 \end{bmatrix}}_{\Upsilon_{11}^q(s)}\Delta K_1 - \underbrace{\frac{42(s+1)}{s}\begin{bmatrix} 0.7593 \\ 0.6508 \end{bmatrix}}_{\Upsilon_{21}^q(s)}\Delta K_2 \tag{50}$$

$$c_2(s) = \begin{bmatrix} 0.7526 \\ 0.6885 \end{bmatrix} + \underbrace{\frac{28(s+2)}{s}\begin{bmatrix} 0.7526 \\ .6885 \end{bmatrix}}_{\Upsilon_{21}^q(s)}\Delta K_1 - \underbrace{\frac{28(s+2)}{s}\begin{bmatrix} 0.7526 \\ .6885 \end{bmatrix}}_{\Upsilon_{22}^q(s)}\Delta K_2 \tag{51}$$

It is clear from (50), (51) that the canonical basis axes of the system are rather sensitive to the variations ΔK_1 and ΔK_2. The plots $|\Upsilon_{11}(j\omega)|$ and $|\Upsilon_{12}(j\omega)|$ are shown in Figure 6. From these plots and (50), (51) we see that the magnitudes of the sensitivity functions tend to infinity at zero frequency, i.e. the canonical basis axes are extremely sensitive to variations of the gains. At the same time, for the equal values and signs of ΔK_1 and ΔK_2 (i.e. for $\Delta K_1 = \Delta K_2$), the corresponding terms in (50), (51) are mutually compensated and in such situations the canonical basis does not depend at all on the variations ΔK_1 and ΔK_2.

Finaly, consider the sensitivity of the CTFs of the closed-loop system. With the help of formulae (37), it is easy to find

$$\begin{aligned} S_{11}^S(s) &= -\frac{49(s+1)}{(s+2)^2}, \quad & S_{21}^S(s) &= \frac{96(s+2)}{(s+4)^2}, \\ S_{12}^S(s) &= \frac{48(s+1)}{(s+2)^2}, \quad & S_{22}^S(s) &= -\frac{98(s+2)}{(s+4)^2}. \end{aligned} \tag{52}$$

Comparison of (52) and (47) shows that the introduction of feedback decreases the sensitivity of the CTFs to the variations ΔK_1 and ΔK_2. Thus, at the zero frequency $\omega = 0$, these functions are equal to

$$S_{11}^S(j0) = -12.25, \quad S_{21}^S(j0) = 12, \quad S_{12}^S(j0) = 12, \quad S_{22}^S(j0) = -12.25$$

i.e. the feedback decreases the sensitivity to variations ΔK_1 and ΔK_2 approximately by a factor of four. The magnitude plots $|S_{11}^S(j\omega)|$ and $|S_{12}^S(j\omega)|$ are shown in Figure 7 (two other plots practically coincide with the given in Figure 7). Note that the sensitivity functions of the CTFs of the closed-loop system have insignificant resonance peaks.

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Hence, such an extraordinary behavior of the discussed two-dimensional system in the case of small ΔK_1 and ΔK_2 is due to extremely high sensitivity to non-identical variations of these parameters. Especially this becomes apparent if the variations have opposite signs.

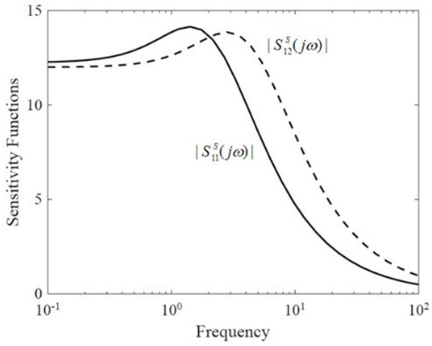

Figure 7. Sensitivity functions $|S_{11}^{S}(j\omega)|$ and $|S_{12}^{S}(j\omega)|$.

The reason for all this is quite simple. As is known [14–19], the robustness of feedback systems depends on H_∞ norm of the transfer matrices $S(s)$ or $T(s)$ (3). Using, for example, the canonical representation (6) we have

$$\begin{aligned} \|S(j\omega)\|_\infty &= \left\| C(j\omega)\, diag\left\{ \frac{1}{1+q_i(j\omega)} \right\} C^{-1}(j\omega) \right\|_\infty \le \\ &\le \nu\left[C(j\omega)\right] \max_i \left\| \frac{1}{1+q_i(j\omega)} \right\|_\infty \end{aligned} \tag{53}$$

where

$$\nu\left[C(j\omega)\right] = \|C(j\omega)\| \left\|C^{-1}(j\omega)\right\| \tag{54}$$

is the condition number of the modal matrix $C(j\omega)$ [4]. In general, the condition number is unity for orthogonal bases and tends to infinity if the axes of the basis approach to a linearly dependent reference frame. This means that the robustness of MIMO control systems deteriorates with an increase of the condition number $\nu\left[C(j\omega)\right]$ (54).

In this respect, the angle between the canonical bases axes of the system (41) is about 179.45°, that is, practically, the axes belong to the same line, and the condition number $\nu(C)$ (54) is equal to 196. The generalized frequency-response characteristics of the closed-loop system are shown in Figure 8 by vinous color. The exact value of H_∞ norm is 16.47. This indicates that the system is not robust and the admissable norm of the additive perturbation is 0.061 (note that the norm of the regulator K (43) is 1.13).

In other words, the discussed example just shows that the system (40), but not the CTFs method, is not robust.

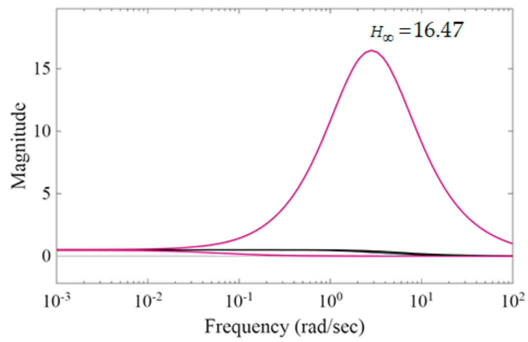

Figure 8. Generalized frequency responses of the closed-loop system.

4.2. Control System of a Hexacopter

Consider control system of the hexacopter shown in Figure 9. Generally, irrespective of the number of motors, the flight altitude z and the vector of rotation angles (roll ϕ, pitch θ, and yaw ψ) are usually chosen as four control variables in the UAVs' control systems [33].

(a) **(b)**

Figure 9. Hexacopter, **(a)** general view ; **(b)** motors' allocation.

Below, we shall assume that the angles and angular velocities of the UAV are so small that nonlinear terms in the dynamics equations of rotational motions can be neglected, the cosines of all angles are approximately equal to one and sines of the angles are zero. On these conditions, the dynamics equations of multirotor UAVs have the following linearized form [33,34]:

$$\frac{d^2 z(t)}{dt^2} = \frac{1}{m} T_\Sigma(t) - g \quad , \tag{53}$$

$$J \frac{d\omega(t)}{dt} = \tau(t) \, , \tag{54}$$

where m is the mass of the UAV; g – the gravitational constant; J – diagonal tensor of inertia with the components I_x, I_y, I_z on the principal diagonal; $\omega(t)$ - vector of angular velocities in the body-fixed frame; vector $\tau(t) = [\tau_x, \tau_y, \tau_z]^T$ combines the principal non-conservative forces and moments applied to the UAV airframe by the aerodynamics of the six rotors (assuming no external disturbances);

$$T_\Sigma = \sum_{i=1}^{6} T_i \tag{55}$$

- the total thrust at hover, where T_i is the thrust generated by the i th rotor ($i = 1, 2,\ldots, 6$). Denoting by $\bar{T}$ the 6-dimensional vector of thrusts T_i ($\bar{T} = [T_1, T_2, ..., T_6]^T$), the mapping of $\bar{T}$ to the vector $[T_\Sigma, \tau]^T$ can be written in matrix form

$$\begin{bmatrix} T_\Sigma \\ \tau \end{bmatrix} = D_M \Lambda_M \bar{T}, \quad \Lambda_M = diag\left\{\lambda_i^M\right\}. \tag{56}$$

In (54), the 4×6 full-rank numerical matrix D_M for the motors' allocation in Figure 9b is [34]

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.866L & 0.866L & 0 & -0.866L & -0.866L \\ -L & -L/2 & L/2 & L & L/2 & -L/2 \\ -k_\psi & k_\psi & -k_\psi & k_\psi & -k_\psi & k_\psi \end{bmatrix}, \tag{57}$$

where L is the distance of the motors from the center of mass, k_ψ is a design parameter of the motors [33], and λ_i^M $(0 < \lambda_i^M \le 1)$ $(i = 1, 2, \ldots, 6)$ are the motors' efficiency degradation parameters. For properly functioning motors, the matrix Λ_M is equal to 6×6 identity matrix I (or $I_{6\times6}$, to indicate the order of the matrix I)..

The matrix block diagram of the UAV's control system is shown in Figure 10 [34].

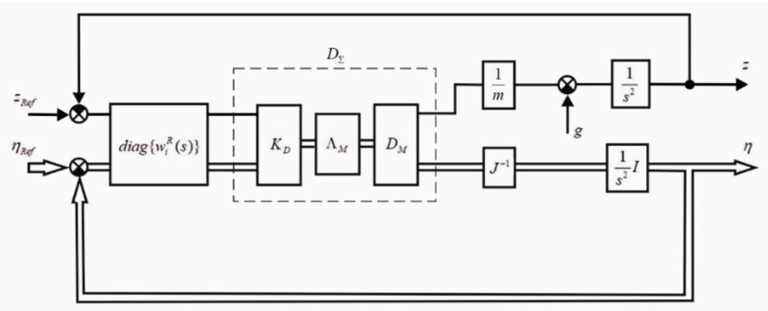

Figure 10. Matrix block diagram of the UAV's control system.

Commonly, the matrix regulator $K_{\text{Reg}}(s)$ in such systems is taken in the form

$$K_{\text{Reg}}(s) = K_R diag\{w_i^R(s)\}. \tag{58}$$

where K_R is a constant matrix, and $w_i^R(s)$ are scalar transfer functions of the regulators in separate channels. Usually, standard Proportional-Integral-Derivative (PID) regulators are used as $w_i^R(s)$ in (58).

The transfer matrix of the open-loop control system of the hexacopter in Figure 10 has the form:

$$W(s) = \frac{1}{s^2} M_\Sigma^{-1} D_M \Lambda_M K_R diag\left\{w_i^R(s)\right\}, \tag{59}$$

where

$$M_\Sigma = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_x & 0 & 0 \\ 0 & 0 & I_y & 0 \\ 0 & 0 & 0 & I_z \end{bmatrix} \tag{60}$$

The matrix K_R in (58) and (59) is usually chosen as

$$K_R = D_M^{+} M_{\Sigma}, \tag{61}$$

$$D_M^{+} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ -\sqrt{3}/6L & \sqrt{3}/6L & \sqrt{3}/6L & 0 & -\sqrt{3}/6L & -\sqrt{3}/6L \\ -1/3L & 1/6L & 1/6L & \sqrt{3}/6L & 1/6L & -1/6L \\ -1/6k_{\psi} & 1/6k_{\psi} & -1/6k_{\psi} & 1/6k_{\psi} & -1/6k_{\psi} & 1/6k_{\psi} \end{pmatrix}^{T} \tag{62}$$

where T is the sign of matrix transposition and D_M^{+} denotes the 6×4 Moore-Penrose pseudoinverse of the matrix D_M (57) [33,34].

In what follows, we shall assume for simplicity that all regulators $w_i^R(s)$ in (58) are identical, i.e. $w_i^R(s) = w^R(s)$.

In case of no motors' efficiency degradations (i.e. if $\Lambda_M = I_{6x6}$), we have $M_{\Sigma}^{-1} D_M K_R = I_{4\times4}$, i.e. the kinematic cross-connections between four separate channels of the systems in Figure 10 are compensated and the transfer matrix of the open-loop system reduces to the form

$$W(s) = \frac{w^R(s)}{s^2} I_{4\times4} \tag{62}$$

Correspondingly, the matrix block diagram in Figure 10 splits into 4 isolated SISO channels in Figure 11.

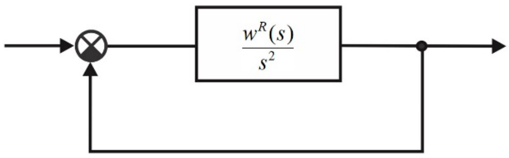

Figure 11. Block diagram of isolated separate channels in case of ideal motors ($\Lambda_M = I_{6x6}$).

On the other hand, if there are some degradations of motors' efficiency, i.e. if, then instead of (59) we have

$$W(s) = \frac{w^R(s)}{s^2} B_{\Sigma}, \tag{63}$$

where the matrix

$$B_{\Sigma} = M_{\Sigma}^{-1} D_M \Lambda_M K_R = M_{\Sigma}^{-1} D_M \Lambda_M D_M^{+} M, \tag{64}$$

is not an identity matrix. Consequently, if $\Lambda_M \neq I_{6x6}$, then the control system of the hexacopter belongs to the class of cross-connected MIMO control systems.

Let the parameters of the hexacopter in equations (53), (54) are : $m = 2.5\ kg$, $I_x = I_y = 0.5\ kg\cdot m^2$, $I_z = 1.5\ kg\cdot m^2$. The matrix M_{Σ} (60) in this case has the form

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$$M_{\Sigma} = \begin{bmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1.5 \end{bmatrix}. \tag{65}$$

The identical proportional-integral-derivative (PID) regulators with the first order filter $w^R(s)$ in the separate channels are taken in the form:

$$w^R(s) = K_P + \frac{K_I}{s} + K_D \frac{s}{T_f s + 1} \tag{66}$$

where $K_P = 0.1857,\ K_I = 0.008693,\ K_D = 0.9739,\ T_f 0.1834$. The parameters of the PID regulator are obtained using the function *pidtune* in MATLAB [35].

The frequency response and step response characteristics of separate channels in case of ideal motors (when $\Lambda_M = I_{6\times 6}$) are shown in Figures 11–13. In Figure 11, the black line is the graph of the plant (a double integrator), the blue line represents the PID regulator (66), and the transfer function of the resulting open-loop channel is shown by the brown line.

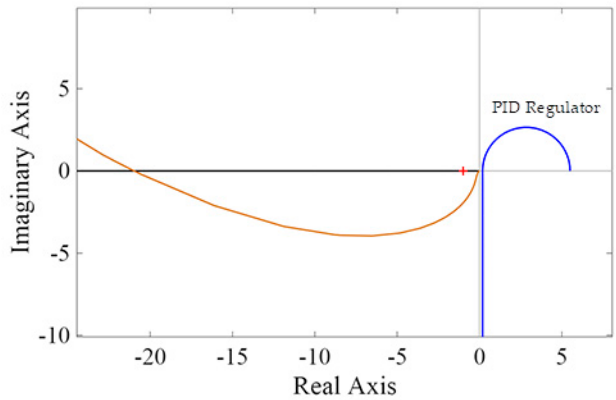

Figure 11. Nyquist plots of the isolated channels ($\Lambda_M = I_{6\times 6}$).

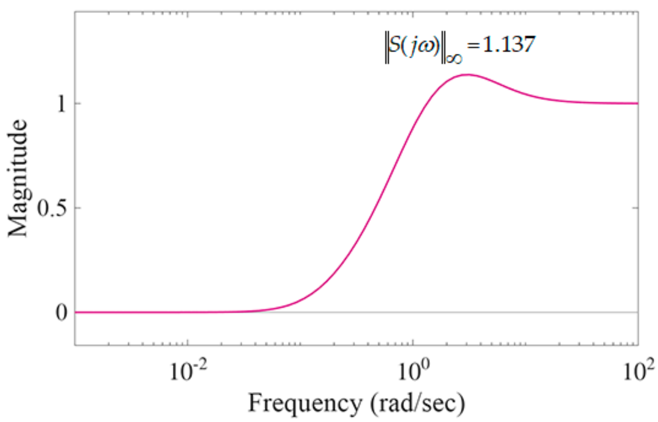

Figure 12. Frequency-response characteristic of the sensitivity function of the isolated channels ($\Lambda_M = I_{6\times 6}$).

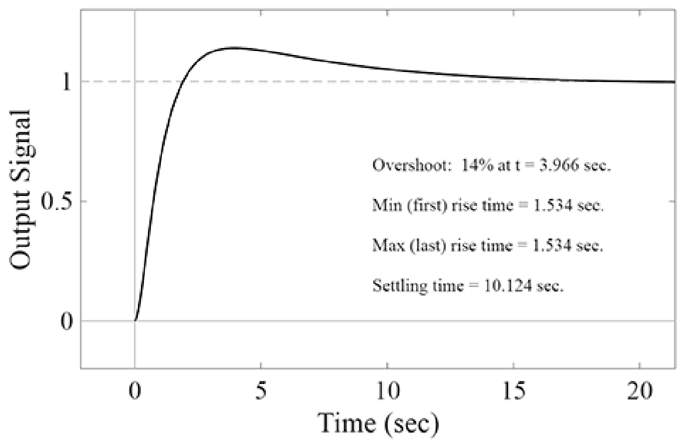

Figure 13. Step response of the isolated channels ($\Lambda_M = I_{6\times6}$).

Let us analyze now the sensitivity of the hexacopter's control system to the motors' efficiency small degradations supposing that the matrix Λ_M in (59) has the following form:

$$\Lambda_M = \begin{bmatrix} 0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.95 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.77 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.85 \end{bmatrix}. \quad (67)$$

The matrix of cross-connections B_Σ (64) in this case is

$$B_\Sigma = \begin{bmatrix} 0.853 & 0.010 & 0.004 & 0.160 \\ 0.130 & 0.855 & -0.017 & 0.173 \\ 0.050 & -0.017 & 0.852 & -0.017 \\ 0.040 & 0.004 & -0.00 & 0.853 \end{bmatrix} \quad (68)$$

and the CTFs of the open-loop system are:

$$\begin{aligned} q_1(s) &= \frac{5.2355(s+0.096)(s+0.0902)}{s^3(s+5.453)} \\ q_2(s) &= \frac{4.2319(s+0.096)(s+0.0902)}{s^3(s+5.453)} \\ q_3(s) &= \frac{4.754(s+0.096)(s+0.0902)}{s^3(s+5.453)} \\ q_4(s) &= \frac{4.5397(s+0.096)(s+0.0902)}{s^3(s+5.453)}. \end{aligned} \quad (69).$$

Due to the structural features of the control system, the modal matrix $C(s)$ in (4) is constant, i.e. $C(s) = C$, and equal to

$$C = \begin{bmatrix} -0.410 & -0.623 & 0.056 & 0.023 \\ -0.891 & 0.490 & -0.508 & -0.824 \\ 0.018 & 0.541 & 0.860 & -0.563 \\ 0.196 & 0.280 & 0.016 & 0.064 \end{bmatrix}. \quad (70)$$

The Nyquist plot, generalized frequency responses and step response of the hexacopter's control system with the motors' degradation parameters given by the matrix Λ_M (67) are shown in Figures 14–16. In Figure 14, the four coinciding black lines are the graphs of the plant, the blue line represents the identical PID regulator (66) in separate channels, and the gain loci of the CTFs (69) are shown by brown lines. The vinous lines in Figure 15 represent the majorant and minorant of the generalized frequency response characteristic and the black lines depict the sensitivity functions of the closed-loop characteristic systems.

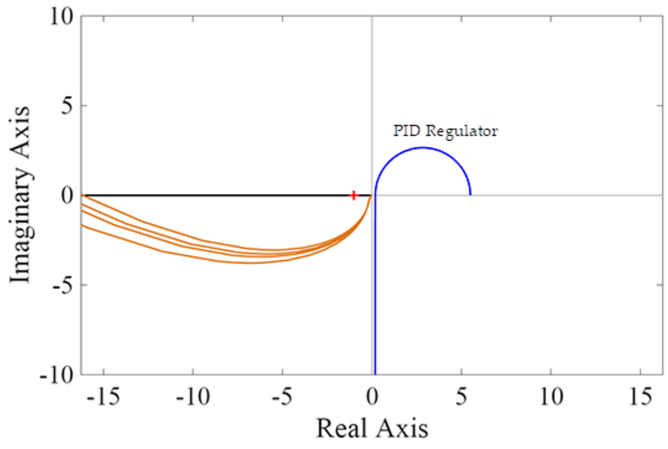

Figure 14. Nyquist plots of the CTFs ($\Lambda_M \neq I_{6\times6}$):.

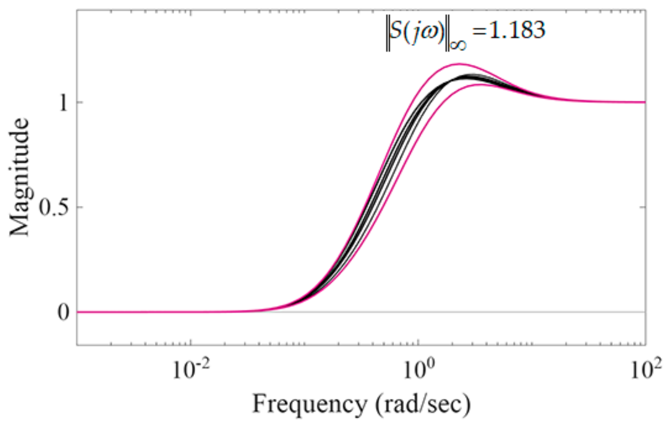

Figure 15. Generalized frequency-response characteristics of the sensitivity function matrix.

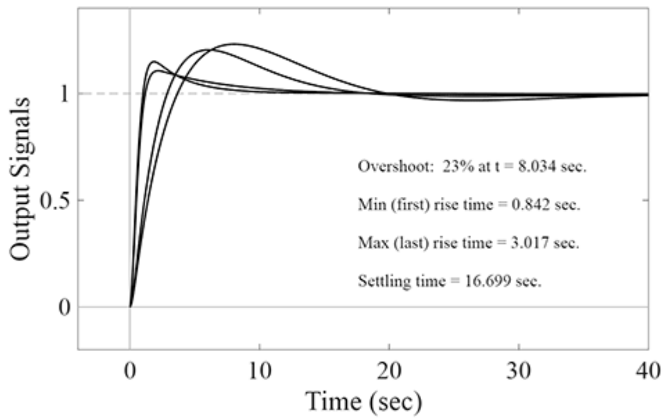

Figure 16. Step responses of the control system ($\Lambda_M \neq I_{6\times6}$).

Using the expressions (23), we get the following three matrices that determine the sensitivity of the CTFs and canonical basis axes to small efficiency degradations λ_1^M, λ_2^M, and λ_3^M of the first three motors:

$$N_1(s) = \frac{5.496(s+0.096)(s+0.0902)}{s^3(s+5.453)} \begin{bmatrix} 0.17 & 0 & -0.07 & -0.33 \\ 0 & 0 & 0 & 0 \\ -0.83 & 0 & 0.33 & 1.67 \\ -0.08 & 0 & 0.03 & 0.17 \end{bmatrix} \quad (71)$$

$$N_2(s) = \frac{5.496(s+0.096)(s+0.0902)}{s^3(s+5.453)} \begin{bmatrix} 0.17 & 0.06 & -0.03 & 0.33 \\ 0.72 & 0.25 & -0.14 & 1.44 \\ -0.42 & -0.14 & 0.08 & -0.83 \\ 0.08 & 0.03 & -0.02 & 0.17 \end{bmatrix} \quad (72)$$

$$N_3(s) = \frac{5.496(s+0.096)(s+0.0902)}{s^3(s+5.453)} \begin{bmatrix} 0.17 & 0.06 & 0.03 & -0.33 \\ 0.72 & 0.25 & 0.14 & -1.44 \\ 0.42 & 0.14 & 0.08 & -0.83 \\ -0.08 & -0.03 & -0.02 & 0.17 \end{bmatrix} \quad (73)$$

Due to the symmetry of the motors' allocations, the matrices $N_4(s)$, $N_5(s)$, and $N_6(s)$ for other three motors are actually equal in pairs (up to the signs of non-diagonal terms) to matrices (71)-(73).

These expressions, together with (35), (37), (39), can be used for analyzing sensitivity of the hexacopters' control systems with respect to variation of parameters of motors, sensors, regulators, etc.

5. Discussion and Conclusion

In the paper, the issue of sensitivity of multivariable control systems is discussed from the perspective of the CTFs method. The formulas are derived determining the sensitivity functions of the CTFs and sensitivity vectors of the canonical basis axes to small variations of parameters of general type MIMO control systems. The relations between the sensitivity functions of the open-loop and closed-loop MIMO systems are established. It is shown that the sensitivity functions of the CTFs of the open-loop and the closed-loop MIMO systems are related to each other by the same expressions as standard sensitivity functions of the open-loop and closed- loop SISO systems. Besides, the sensitivity vectors of the canonical basis axes are not affected by introduction of feedback loop.

In future research it is intended to extend the results of the paper to special structural classes of MIMO control systems, such as uniform system, symmetric and anti-symmetric systems, etc.

It should be noted that the CTFs method was successfully used in aerospace engineering, particularly, in the design of high-precision guidance systems of two astronomical telescopes mounted on the space station "MIR". Nevertheless, in the technical literature on multivariable feedback control that method is often claimed as not robust and, therefore, unreliable in practical design. That erroneous conclusion has become quite common because of two main reasons.

First, it is due to the notorious example attributed usually to J. Doyle and G. Stein [23], where it is shown that a two-dimensional system having, at first sight, infinite gain margins calculated through the CTFs becomes unstable in case of small additive perturbations of the controller's gains. In the paper, a detailed analysis of the sensitivity and robustness of the system in [23] is given and it is shown that incorrect conclusion about robustness was made because of unproper application of the CTFs method (unproper treatment of the results). In short, the example in [23] can be considered just as a good illustrative example of a not robust control system, but not as evidence of unreliability of the CTFs method. The point is that robustness of MIMO control systems essentially depends not only on stability margins of the SISO characteristic systems but also on the skewness of the canonical basis axes (or condition number of the modal matrix) which in this case is quite close to critical.

The second reason is more significant. Design of MIMO control systems is computationally intensive and cannot be performed without specialized computer-aided control system design (CACSD) tools [35–40]. Unfortunately, at present, there are no acknowledged CACSD tools based on the CTFs method. It is worth mentioning that in the 1990s, the *Multivariable Frequency-Domain Toolbox (MFD Toolbox)* was developed by J.M. Maciejowski, M.P. Ford, and J.B. Boyle at Cambridge Control Ltd [41]. The MATLAB-based MFD toolbox was designed for the analysis and design of linear MIMO systems. Among some other design techniques, the MFD Toolbox supported also the CTFs method both for continuous-time and discrete-time systems. However, no information can be found about current situation with that toolbox.

In this respect, a new software package *MIMO Control Toolbox* is being developed at the Aerial Robotics Center of the National Polytechnic University of Armenia [42]. The toolbox works in the MATLAB environment and is designed for analysis and design of control systems in robotics and mechatronics, as well as in many other fields. The principal feature of the *MIMO Control Toolbox* is that the design of any *N*-dimensional MIMO control system is reduced to the design of a certain fictitious SISO control system. Another key feature of the toolbox is that it includes about 250 new MATLAB classes (objects) describing all the main structural types of MIMO control systems known from scientific and technical literature.

The frequency and time response characteristics of the hexacopter's control system in Figures 11–16 were obtained with the help of a special interactive graphical user interface (GUI) which is a part of the package *MIMO Control Toolbox* and which can be viewed as an extension to multivariable case of the well-known GUI *Control Systems Designer* in MATLAB [35]. The *MIMO Control Toolbox* can be used both as a computer-aided control systems design tool in various areas of industry and technics, and for teaching the fundamentals of classical and modern feedback control at educational institutions.

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Abbreviations

The following abbreviations are used in this manuscript:

MIMO	Multi-Input Multi-Output
SISO	Single-Input Single-Output
UAV	Unmanned Aerial Vehicle
PID	Proportional-Integral-Derivative
CACSD	Computer-Aided Control System Design
GUI	Graphical User Interface

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