Formal Calculation

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Abstract: Formal Calculation, using auxiliary form to calculate various nested sums, based on binary or Gaussian coefficients and providing three different expressions. In addition to computation, it introduces a large number of new concepts and methods for analysis. With just a little skill, it can easily obtain dozens of theorems. It has undergone three years of development. This article summarizes it and provides a large number of latest achievements, showcasing many analytical methods. This article includes various relationships between two types of Stirling numbers, associated Stirling numbers, generalized Eulerian numbers and polynomials, and provides a theorem of symmetry and extended Wolstenholme theorems.

Keywords: Formal Calculation; nested sums; Gaussian coefficient; Stirling number; associated Stirling numbers; Eulerian number and polynomial; Wolstenholme theorem

1. Introduction

To calculate all products of M distinct integers in [1,N-1], items needs to be divided into 2^{M-1} categories.

For example, calculate ∑3-products in [1,6]

①$1\times2\times3+2\times3\times4+3\times4\times5+4\times5\times6=1\times2\times3\binom{7}{4}$

②$1\times2\times4+(1\times2+2\times3)\times5+(1\times2+2\times3+3\times4)\times6=1\times2\times4\binom{7}{5}$

③$1\times3\times4+(1\times4+2\times4)\times5+(1\times5+2\times5+3\times5)\times6=1\times3\times4\binom{7}{5}$

④$1\times3\times5+(1\times3+1\times4+2\times4)\times6=1\times3\times5\binom{7}{6}$

An item$(\mathrm{I}_1,\mathrm{I}_2,\mathrm{I}_3)$, $\mathrm{I}_i+1=\mathrm{I}_{i+1}$ means continuity(A), $\mathrm{I}_i+1<\mathrm{I}_{i+1}$ means discontinuity(B)

So ①=AA,②=AB,③=BA,④=BB.Items of a category have the same discontinuities at the same positions.

Continuity means the distance between factors remains constant. Discontinuity assumes the distance between factors can be arbitrarily expanded. As N increases, the number of items will also increase.In ②=AB:

N-1=6,products={1×2×6,2×3×6,3×4×6},1×2,2×3,3×4 keep continuity. 2×6,3×6,4×6 keep discontinuity.

N-1=7,products={1×2×7,2×3×7,3×4×7,4×5×7}

Start here and going through several promotions, [1][2][3] introduced Formal Calculation:

M-series:Serie_i={K_i,K_i+D_i,K_i+ 2D_i…K_i+(N-1)D_i}, i∈[1,M], K_i,D_i∈ring with identity elements.

Use PS=[K_1:D_1, K_2:D_2...K_M:D_M] to represent the M-series.

[K_1:D, K_2:D...K_M:D] is abbreviated as [K_1, K_2…K_M]:D. [K_1,K_2…K_M]:1 is abbreviated as [K_1,K_2…K_M]

Recursive definie operator $\nabla^p, p\in\mathbb{Z}$:

$\nabla^0 f(n)=f(n), \sum_{n=0}^{N-1}\nabla^1 f(n+1)=f(N), \sum_{n=0}^{N-1} f(n+1)=\nabla^{-1} f(N)$

$\nabla^1 f(n)=\nabla f(n)=f(n)-f(n-1)$, *it's a little different from* Differential calculation.

$If\ f(n)=\sum A_i\binom{N_i}{M_i}$ *and* M_i *is not changed with* N, *then* $\nabla^p f(N)=\sum A_i\binom{N_i-p}{M_i-p}$

Use PT=[T_1,T_2...T_M] to indicate some products in M-series.By default,the following uses:

PS=[K_1:D_1,K_2:D_2…K_M:D_M],PT=[T_1,T_2...T_M]

PS1=[K_1:D_1,K_2:D_2…K_M:D_M,K_{M+1}:D_{M+1}]=[PS, K_{M+1}:D_{M+1}],PT1=[PT, T_{M+1}= T_M+2-p]

Recursive define SUN(N,PS,PT), abbreviated as SUM(N):

$SUM(N,[K_1:D_1],[T_1=1])=\sum_{n=0}^{N-1}(K_1+n\times D_1)$

$SUM(N, PS1, PT1) = \sum_{n=0}^{N-1}(K_{M+1} + n \times D_{M+1}) \times \nabla^p \; SUM(n+1, PS, PT)$

This is actually nested summation. For example:

$$SUM(N, PS, [1,2 \dots M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$$

$$SUM(N, PS, [1,3 \dots 2M-1]) = \sum_{n_M=0}^{N-1}(K_M + n_M D_M) \dots \sum_{n_2=0}^{n_3}(K_2 + n_2 D_2) \sum_{n_1=0}^{n_2}(K_1 + n_1 D_1)$$

$$SUM(N, PS, [1,2,4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_2 + nD_2)(K_1 + nD_1)$$

$$SUM(N, PS, [1,3,4]) = \sum_{n=0}^{N-1} (K_3 + nD_3)\,(K_2 + nD_2) \sum_{n_1=0}^{n} (K_1 + n_1 D_1)$$

$$SUM(N, PS, [1,4]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \sum_{n_2=0}^{n} \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$$

Formal Computation was once named the shape of numbers because PS and PT specify the shape of the product factors and extensively use the concepts of continuity, discontinuity, and position. It has little to do with trigonometric numbers, squares numbers, etc. To avoid confusion, it was renamed.

If $T_i< T_{i+1}, T_i \in N$, define PB(PT)=count of discontinuities of PT. If compared to [1,2...M], PB(PT)=T_M-M.

Also define PB(PS)=count of discontinuities of PS,it's compared to a certain benchmark.

If PB (PT)=PB (PS) and their discontinuities are at the same positions, then say them have the same shape.

The following use K to represent the set$\{K_1,K_2...K_M\}$, T to represent the set$\{T_1,T_2...T_M\}$

$Use\ the\ auxiliary\ Form: (T_1 + K_1)(T_2 + K_2) \dots (T_M + K_M) = \sum \prod_{i=1}^{M} X_i\,, X_i = T_i\ or\ K_i$

X(T)=Count of $\{X_1, X_2...X_M\}\in T$, X_{T-1}=Count of $\{X_1, X_2...X_{i-1}\}\in T$, X_{K-1}=Count of $\{X_1, X_2...X_{i-1}\}\in K$, $X_{T-1}+X_{K-1}$=i-1

Don't swap X_i,then each $\prod X_i$ corresponds to one expression in the SUM(N).

1.1)$g = X(T), SUM(N, PS, PT) =$

$$\xrightarrow{Form_1} \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{N-1-g} = \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{T_M-M+1+g}, B_i = \begin{cases}(T_i - X_{K-1})D_i; X_i = T_i \\ K_i + X_{T-1}D_i; X_i = K_i\end{cases}$$

$$\xrightarrow{Form_2} \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{N-1} = \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{T_M-M+1+g}, B_i$$

$$= \begin{cases}(T_i - X_{K-1})D_i; X_i = T_i \\ K_i + (X_{K-1} - T_i)D_i; X_i = K_i\end{cases}$$

$$\xrightarrow{Form_3} \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{T_M+1}, B_i = \begin{cases}-K_i + (T_i - X_{T-1})D_i; X_i = T_i \\ K_i + X_{T-1}D_i; X_i = K_i\end{cases}$$

$$H_i(g), short\ for\ H_i(g, PS, PT), is\ also\ defined\ above = \sum_{\prod X_i\ with\ X(T)=g} \prod_{i=1}^{M} B_i$$

$Use\ H(g)\ to\ represent\ the\ three\ parameters\ H_{1,2,3}(g).$

$Define\ F_M^{K=\{K_1,K_2 \dots K_M\}} = \sum(M\ distinct\ products, factors\ \in K), F_M^N\ is\ short\ for\ F_M^{\{1,2\dots N\}}, F_0^K = 0$

$Define\ E_M^K = \sum(M\ products, factors \in K), E_M^N\ is\ short\ for\ E_M^{\{1,2\dots N\}}, E_0^K = 0$

$Easy\ to\ obtain: F_M^{\{1,1\dots 1\}, count\ of\ 1=N} = \binom{N}{M}, E_M^N = \sum_{\lambda_1+\lambda_2+\dots+\lambda_N=M, \lambda_i \ge 0} 1^{\lambda_1} 2^{\lambda_2} \dots N^{\lambda_N}$

1.2) $\nabla\ SUM(N, PS, [1,2 \dots M]) = \prod_{i=1}^{M}(K_i + (N-1)D_i) = \prod_{i=1}^{M}(K_i + nD_i)$

1.3)$SUM(N, [\dots PS \dots], [\dots T+1, T+2 \dots T+M \dots]), K_i\ can\ exchange\ order.$

1.4)$SUM(N, [L_1, L_2 \dots L_P, PS], [L_1, L_2 \dots L_P, PT]) = \prod_{i=1}^{P} L_i\ SUM(N, PS, PT) \rightarrow$

T_1 can be greater than $1. T \in \mathbb{N}$

1.5)$SUM(N, [1,1 \dots 1], [1,2 \dots M]) = SUM(N, [1,1 \dots 1], [2,3 \dots M]) = 1^M + 2^M + \cdots + N^M$

1.6)$SUM(N, [1,1 \dots 1], [1,3 \dots 2M-1]) = SUM(N, [1,1 \dots 1], [3,5 \dots 2M-1]) = S_2(N+M, N) = E_M^N$

1.7)$SUM(N, [1,2 \dots M], [1,3 \dots 2M-1]) = SUM(N, [2,3 \dots M], [3,5 \dots 2M-1]) = S_1(N+M, N) = F_M^{N+M-1}$

S_1(N,M) is unsigned stirling number of the first kind. S_2(N,M) is stirling number of the second kind.

1.8)$SUM(N,PT,PT) = \prod_{i=1}^{M} T_i \binom{N+T_M}{T_M+1}$.

This is an important conclusion. It is the origin of Formal Computation and why ①②③④ came into being.

It generalizes the formula $\sum_{n=0}^{N-1}\binom{n}{M} = \binom{N}{M+1} = \frac{1}{M!}SUM(N-M,[1,2\ldots M],[1,2\ldots M])$ *in nested summation.*

$\mathrm{SUM}(4,[1,2,3],[1,2,3]) =$ ①

$\mathrm{SUM}(4-1,[1,2,4],[1,2,4]) =$ ②, $\mathrm{SUM}(4-1,[1,3,4],[1,3,4]) =$ ③

$\mathrm{SUM}(4-2,[1,3,5],[1,3,5]) =$ ④

$\mathrm{SUM}(\mathrm{N}-PB(PT),\mathrm{PT},\mathrm{PT}) =$ ①②③④ $=$ 1.8), $\mathrm{PB}(\mathrm{PS})$ *compares with* $[1,2\ldots M]$

$\mathrm{SUM}(\mathrm{N},[1,2,3],[1,3,5]) = \mathrm{SUM}(\mathrm{N},[2,3],[3,5]) =$ ① + ② + ③ + ④, $\sum_{n=0}^{N-1}\mathrm{M}-\mathrm{Products} =$ 1.7)

1.9)$If\ PT = [T_1 = 1, T_2\ldots T_M], T_1 < T_2 <\ldots< T_M, then\ (Count\ of\ products\ in\ SUM(N)) = \binom{N+PB(PT)}{PB(PT)+1}$

$\mathrm{SUM}(\mathrm{N},[2,3],[3,5]) \xrightarrow{Form} (2+T_1)(3+T_2)\ , T_M - M = 5-2 = 3 \xrightarrow{Form_1}$

$= T_1 \times T_2\binom{N+3}{6} + \{K_1\times(T_2-X_{K-1}) + T_1\times(K_2+X_{T-1})\}\binom{N+3}{5} + K_1\times K_2\binom{N+3}{4} \xrightarrow{X_{K-1}=X_{T-1}=1}$

$= 3\times 5\binom{N+3}{6} + \{2\times 4 + 3\times 4\}\binom{N+3}{5} + 2\times 3\binom{N+3}{4}$

$\mathrm{SUM}(\mathrm{N},[1,2,3],[1,3,5]) \xrightarrow{Form} (1+T_1)(2+T_2)(3+T_3), T_M - M = 5-3 = 2 \xrightarrow{Form_1}$

$= 1\times 3\times 5\binom{N+2}{6} + 35\binom{N+2}{5} + 26\binom{N+2}{4} + 1\times 2\times 3\binom{N+2}{3}$

$Here: H_1(1) = 1\times 2\times(T_3 - X_{K-1}) + 1\times(T_2 - X_{K-1})\times(3+X_{T-1}) + T_1\times(2+X_{K-1})\times(3+X_{K-1})$

$= 1\times 2\times(5-2) + 1\times(3-1)\times(3+1) + 1\times(2+1)\times(3+1) = 26$

Formal Calculation can be calculated in the *ring with identity elements.*

$\mathrm{PS}=\left[\begin{pmatrix}7&3\\1&2\end{pmatrix}:\begin{pmatrix}1&3\\4&2\end{pmatrix},\begin{pmatrix}2&1\\1&2\end{pmatrix}:\begin{pmatrix}2&3\\1&2\end{pmatrix}\right], \mathrm{PT} = [1,3] \xrightarrow{Form} (1+\begin{pmatrix}7&3\\1&2\end{pmatrix})(3+\begin{pmatrix}2&1\\1&2\end{pmatrix})$

$H_1(0) = \begin{pmatrix}7&3\\1&2\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix} = \begin{pmatrix}17&13\\4&5\end{pmatrix}$

$H_1(1) = 1\times\left[\begin{pmatrix}2&1\\1&2\end{pmatrix}+\begin{pmatrix}2&3\\1&2\end{pmatrix}\right]\begin{pmatrix}1&3\\4&2\end{pmatrix} + \begin{pmatrix}7&3\\1&2\end{pmatrix}\times(3-1)\begin{pmatrix}2&3\\1&2\end{pmatrix} = \begin{pmatrix}44&70\\28&38\end{pmatrix}$

$H_1(2) = 1\times\begin{pmatrix}1&3\\4&2\end{pmatrix}\times 3\times\begin{pmatrix}2&3\\1&2\end{pmatrix} = \begin{pmatrix}15&27\\30&48\end{pmatrix}$

$\mathrm{SUM}(\mathrm{N},\mathrm{PS},\mathrm{PT}) = \begin{pmatrix}15&27\\30&48\end{pmatrix}\binom{N+1}{4} + \begin{pmatrix}44&70\\28&38\end{pmatrix}\binom{N+1}{3} + \begin{pmatrix}17&13\\4&5\end{pmatrix}\binom{N+1}{2}$

$Form_1$ *is obtained by guess and proved by induction. There have three forms because:* $\sum_{n=0}^{N-1} n\binom{n+K}{M}$

$= (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1} = (M+1)\binom{N+K+1}{M+2} - (1+K)\binom{N+K}{M+1} = (M-K)\binom{N+K+1}{M+2} + (1+K)\binom{N+K}{M+2}$

Property of H(g)

2.1)$H_1(g,PT,PT) = \prod_{i=1}^{M} T_i\binom{M}{g}, H_2(g<M,PT,PT) = H_3(g>0,PT,PT) = 0 \rightarrow$ 1.8)

The recursive relationship of H(g) can be obtained by definition.

$H_1(g,PS1,PT1) = H_1(g-1)(\mathrm{T}_{M+1} - [\mathrm{M}-(\mathrm{g}-1)])\mathrm{D}_{M+1} + H_1(g)(\mathrm{K}_{M+1} + \mathrm{gD}_{M+1})$

$H_2(g,PS1,PT1) = H_2(g-1)(\mathrm{T}_{M+1} - [\mathrm{M}-(\mathrm{g}-1)])\mathrm{D}_{M+1} + H_2(g)(\mathrm{K}_{M+1} + [-\mathrm{T}_{M+1}+\mathrm{M}-\mathrm{g}]\mathrm{D}_{M+1})$

$H_3(g,PS1,PT1) = H_3(g-1)(-\mathrm{K}_{M+1} + (\mathrm{T}_{M+1} - [\mathrm{g}-1])\mathrm{D}_{M+1}) + H_3(g)(\mathrm{K}_{M+1} + \mathrm{gD}_{M+1})$

Using the recursive relationships and induction to prove→

2.2)$H_1(g) = \sum_{k=g}^{M} H_2(k)\binom{k}{g} = \sum_{k=0}^{g} H_3(k)\binom{M-k}{M-g}, Inversion \rightarrow$

2.3)$H_2(g) = \sum_{k=g}^{M}(-1)^{k+g}H_1(k)\binom{k}{g}, H_3(g) = \sum_{k=0}^{g}(-1)^{k+g}H_1(k)\binom{M-k}{M-g}$

Calculation with 2.2)→

2.4) $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g)2^g = \sum_{g=0}^{M} H_3(g)2^{M-g}$

2.5) $\sum_{g=0}^{M} H_1(g)\binom{A}{B-g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{B} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{B-g}$

It indicates Form1=Form2=Form3. If regardless of the actual meaning, PT's domain can be extended to ℂ. A=B→

2.6) $\sum_{g=0}^{M} H_1(g)\binom{A}{g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{g} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{M}$

2.7) $Induction \rightarrow \sum_{g=0}^{M} H_1(g)\, g\binom{A+1}{B-g} = \sum_{g=0}^{M} H_2(g)\, g\binom{A+g}{B-1} = \sum_{g=0}^{M}\{H_3(g) g\binom{A+M-g}{B-g} + MH_3(g)\binom{A+M-g}{B-1-g}\}$

2.8) $H_1(g,[AD:D,PS],[A,PT]) = AD\{H_1(g)+H_1(g-1)\} \rightarrow H_1(g,[1,PS],[1,PT]) = H_1(g)+H_1(g-1)$

$In\ 1.1), H(g) = \prod B_i = \prod_{X_i \in T} B_i \prod_{X_i \in K} B_i$

$Define\ H(g,T) = H(g,T,PS,PT) = \prod_{X_i \in T} B_i\, ; H(g,\sum T) = \sum H(g,T), also\ define\ H(g,K), H(g,\sum K)$

2.9) $If\ D_i = 1\ and\ T_{i+1} - T_i = 1\ then\ H_1(g,T) = \prod_{i=1}^{g} T_i\, ; H_1(g,\sum K,[1,1\ldots1],PT) = E_{M-g}^{g+1}$

Using induction to prove→

2.10) $If\ D_i = 1, H_1(g,\sum K) = F_{M-g}^{K}E_0^{g} + F_{M-g-1}^{K}E_1^{g} + \ldots + F_0^{K}E_{M-g}^{g}$

2.11) $If\ D_i = 1, H_1(g,\sum T) = F_g^{T}E_0^{M-g} - F_{g-1}^{T}E_1^{M-g} + \ldots + (-1)^g F_0^{T}E_g^{M-g}$

2.12) $If\ D_i = 1\ and\ T_{i+1} - T_i = K_{i+1} - K_i = 1\ then\ H_1(g) = \binom{M}{N} T_1 \ldots T_g \times K_{g+1} \ldots K_M$

$\lambda\ is\ a\ primitive\ unit\ root, \lambda^M = 1, PS = [\lambda^1,\lambda^2\ldots\lambda^M] \rightarrow F_M^K = (-1)^{M+1}, F_{0<x<M}^{K} = 0$

$SUM(N,PS,[1,2\ldots M]) \xrightarrow{2.10)\rightarrow H_1(g>0,\sum K)=E_{M-g}^{g}} M!\binom{N}{M+1}E_0^M + \cdots + 1!\binom{N}{2}E_{M-1}^1 + 0!\binom{N}{1}(-1)^{M+1}$

$$\begin{cases} \nabla\ SUM(N) = \prod_{i=1}^{M}(\lambda^i+n) = \sum_{g=1}^{M} g!\, S_2(M,g)\binom{n}{g} + (-1)^{M+1} \xrightarrow{Definition\ of\ S_2(M,g)} n^M + (-1)^{M+1} \\ \prod_{i=1}^{M}(\lambda^i+n) \xrightarrow{\text{Vieta theorem}} n^M + (-1)^{M+1} \end{cases}$$

$Define\ E_p^q \odot ([T_1,T_2\ldots],C)$
$= \sum_{\lambda_1+\lambda_2+\ldots+\lambda_q=p,\lambda_i\geq 0} 1^{\lambda_1}2^{\lambda_2}\ldots q^{\lambda_q} \times (T_1+\lambda_1 C)(T_2+\lambda_1 C+\lambda_2 C)\ldots(T_{q-1}+\lambda_1 C+\lambda_2 C+\cdots+\lambda_{q-1}C)$

2.12) $H_1(g,[1,1\ldots1],PT = [T_i = T_1 + (i-1)(C+1)]) = E_{M-g}^{g+1} \odot (PT,C)$

2.13) $H_3(g,[1,1\ldots1],PT = [T_i = T_1 + (i-1)(C+1)]) = E_{M-g}^{g+1} \odot ([T_i = (T_1 - 1) + (i-1)C], C+1)$

$PB(PS) compares\ with\ [K_1,K_2\ldots K_M]$

$Define\ MIN_g(PS) = K_1 \times \sum_{PB(PS)=g} \prod_{i=2}^{M}(K_i+\lambda_i D_i), \lambda_1 = 0, \begin{cases} \lambda_i = \lambda_{i+1}, continuity \\ \lambda_i + 1 = \lambda_{i+1}, discontinuity \end{cases}$

$MIN_g = MIN_g(M), 0 \leq g \leq M-1, is\ short\ for\ MIN_g([1,2\ldots M])$

It's easy to understand MIN$_g$.Imagine that discontinuity is a hole. Factors from small to large with a distance of 2 as an hole.MIN$_g$ =∑products with g holes.

For example M=3,MIN$_0$=1×2×3,MIN$_1$=1×2×4+1×3×4,MIN$_2$=1×3×5,they are used to calcute ∑3-proudcts.

Due to historical reasons, they are called MIN, which is actually the first item of SUM().D$_i$ can be less than 0, at this point, they are not the minimum terms.

2.14) $K_1 \times H_1\left(g,[K_2:D_2,K_3:D_3\ldots K_M:D_M],\left[\frac{K_2}{D_2}+1,\frac{K_3}{D_3}+2\ldots\frac{K_M}{D_M}+(M-1)\right]\right) = MIN_g([K_1:D_1\ldots K_M:D_M])$

$PS = [1,1\ldots1], PT = [2,3\ldots M] \rightarrow (g+1)!\,E_{M-g-1}^{g+1} = (g+1)!\,S_2(M,g+1) = MIN_g([1,1\ldots1])$

Table 2. 1:H(g) of special functions.

PS	PT	H1(g)	H2(g)	H3(g)
[1,1…1]	[1,2…M]	$g!\,E_{M-g}^{g+1} =$ $g!\,S_2(M+1,g+1)$	$(-1)^{M-g}g!\,E_{M-g}^{g} =$ $(-1)^{M-g}g!\,S_2(M,g)$	$E_{M-g}^{g+1} \odot ([0,0\ldots],1)$ $= \left\langle {M \atop M-1-g} \right\rangle = \left\langle {M \atop g} \right\rangle$
[1,1…1]	[2,3…M]	$(g+1)!\,S_2(M,g+1)$	$(-1)^{M-1-g}(g+1)!\,S_2(M,g+1)$	$E_{M-1-g}^{g+1} \odot ([1,1\ldots],1)$ $= \left\langle {M \atop M-1-g} \right\rangle = \left\langle {M \atop g} \right\rangle (*)$
[1,1…1]	[1,3…2M-1]	$E_{M-g}^{g+1} \odot (PT,1)$	$(-1)^{M-g}MIN_{g-1}$	$E_{M-g}^{g+1} \odot ([0,1,2\ldots],2)$
[1,1…1]	[3,5…2M-1]	$E_{M-1-g}^{g+1} \odot (PT,1) =$ $S_{2,2}(M+1+g,g+1)$	$(-1)^{M-1-g}MIN_g =$ $(-1)^{M-1-g}S_{1,2}(M+1+g,g+1)$	$E_{M-1-g}^{g+1} \odot ([2,3,4\ldots],2)$
[1,2…M]	[1,3…2M-1]	$MIN_g + MIN_{g-1}$	$1 \times (-1)^{M-g}E_{M-g}^{g} \odot ([3,5\ldots],1)$	$1 \times E_g^{M-g} \odot ([2,3,4\ldots],2)$
[2,3…M]	[3,5…2M-1]	$MIN_g =$ $S_{1,2}(M+1+g,g+1)$	$(-1)^{M-1-g}E_{M-1-g}^{g+1} \odot ([3\ldots],1) =$ $(-1)^{M-1-g}S_{2,2}(M+1+g,g+1)$	$E_g^{M-g} \odot ([2,3,4\ldots],2)$

$\left\langle {M \atop g} \right\rangle$ *is Eulerian number, defined as* $N^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$; $\left\langle {M \atop M-1-g} \right\rangle$ $= \left\langle {M \atop g} \right\rangle$, $(*)$ *is its another formula*[5].

$S_{1,2}(M+1+g,g+1)$ and $S_{2,2}(M+1+g,g+1)$ are described in the following section.

2. H(g) and Associated Stirling Numbers

3.1) By definition: $\mathrm{MIN_g}(M) = \sum \frac{(M+g)!}{i_1 i_2 \ldots i_g}$, $2 \le i_1 < i_2 < \ldots < i_g \le M+g-1, i_{j+1}-i_j \ge 2$

Associated Stirling Numbers of the first kind $S_{1,r}(n,k)$ is defined as the number of permutations of a set of *n* elements having exactly *k* cycles, all length >= *r*.

$(1) S_{1,r}(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+\ldots+i_k=n, i_j \ge r} \frac{1}{i_1 i_2 \ldots i_k}$

$(2) S_{1,r}(n+1,k) = n S_{1,r}(n,k) + (n)_{r-1} S_{1,r}(n-r+1,k-1), n \ge kr$

$(3) S_{1,r}(n,k) = \sum \frac{(n-1)!}{i_1 i_2 \ldots i_{k-1}}$, $r \le i_1 < i_2 < \ldots < i_{K-1} \le n-r, i_{j+1}-i_j \ge r$, [4]

Derived from (2) or (3)→

$3.2) \mathrm{MIN_g}(M) = S_{1,2}(M+1+g,g+1) = \frac{(M+1+g)!}{(g+1)!} \sum_{i_1+i_2+\ldots+i_{g+1}=M+1+g} \frac{1}{i_1 i_2 \ldots i_{g+1}}, i_j \ge 2$

Table 3. 1 :$MIN_g(M)=S_{1,2}(M+1+g,g+1)$.

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	2	3					
M=3	6	20	15				
M=4	24	130	210	105			
M=5	120	924	2380	2520	945		
M=6	720	7308	26432	44100	34650	10395	
M=7	5040	64224	303660	705320	866250	540540	135135

$MIN_0 = M!, MIN_1 = (M+1)!\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{M}\right), MIN_{M-1} = (2M-1)!!$

$MIN_{M-2} = \frac{2(M-1)}{3}(2M-1)!!, M \ge 2; MIN_{M-3} = (M-1)(M-2)(2M-3)!!\frac{4M-3}{9}, M \ge 3$

Associated Stirling Numbers of the second kind $S_{2,r}(n,k)$ is defined as the number of permutations of a set of *n* elements having exactly *k* blocks, all length >= *r*.

$(4)\mathrm{S}_{2,r}(\mathrm{n},\mathrm{k}) = \frac{\mathrm{n}!}{\mathrm{k}!}\sum_{i_1+i_2+\ldots+i_k=n, i_j\geq r}\frac{1}{i_1!i_2!\ldots i_k!}$

$(5)\mathrm{S}_{2,r}(\mathrm{n}+1,\mathrm{k}) = k\mathrm{S}_{2,r}(\mathrm{n},\mathrm{k}) + \binom{n}{r-1}\mathrm{S}_{2,r}(\mathrm{n}-\mathrm{r}+1,\mathrm{k}-1), n \geq kr$

Derived from (5)→

$3.3)\mathrm{H}_1(\mathrm{g},[1,1\ldots1],[3,5\ldots2\mathrm{M}-1]) = E_{M-1-g}^{g+1} \odot (PT,1) = \mathrm{S}_{2,2}(\mathrm{M}+1+\mathrm{g},\mathrm{g}+1)$

Table 3. 2 :H1(g,[1,1…1],[3,5…2M-1])=S2,2(M+1+g,g+1).

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	1	3					
M=3	1	10	15				
M=4	1	25	105	105			
M=5	1	56	490	1260	945		
M=6	1	119	1918	9450	17325	10395	
M=7	1	246	6825	56980	190575	270270	135135

$SUM(N,[2,3\ldots M],[3,5\ldots 2M-1])$

$= \sum_{g=0}^{M-1} S_{1,2}(M+1+g,g+1)\binom{N+M}{M+1+g} = \sum_{g=1}^{M} S_{1,2}(M+g,g)\binom{N+M}{M+g} = S_1(N+M,N) = S_{1,1}(N+M,N)$

$= \frac{(N+M)!}{N!}\sum_{i_1+i_2+\ldots+i_N=N+M, i_j\geq1}\frac{1}{i_1i_2\ldots i_N} = \frac{(N+M)!}{N!}\sum_{g=1}^{M}\sum_{i_1+i_2+\cdots+i_N=N+M, i_j\geq1, (count\ of\ i_j>1)=g}\frac{1}{i_1i_2\ldots i_N} \rightarrow$

$\frac{(N+M)!}{N!}\sum_{i_1+i_2+\ldots+i_N=N+M, i_j\geq1, (count\ of\ i_j>1)=g}\frac{1}{i_1i_2\ldots i_N} \xrightarrow{\binom{N}{N-g}=slecct\ N-g\ quantities\ of\ 1\ from\ N}$

$= \frac{(N+M)!}{N!}\binom{N}{N-g}\sum_{i_1+i_2+\ldots+i_g=g+M, i_j>1,}\frac{1}{i_1i_2\ldots i_g} = \binom{N+M}{M+g}S_{1,2}(M+g,g).$

Use the same way:

$S_{1,r}(rN+M,N) = \frac{(rN+M)!}{N!}\sum_{i_1+i_2+\cdots+i_N=rN+M, i_j\geq r}\frac{1}{i_1i_2\ldots i_N}$

$= \frac{(rN+M)!}{N!}\sum_{g=1}^{M}\sum_{i_1+i_2+\ldots+i_N=rN+M, i_j\geq r, (g=count\ of\ i_j)>r}\frac{1}{i_1i_2\ldots i_N}$

$= \sum_{g=1}^{M}\frac{(rN+M)!}{N!}\binom{N}{g}\frac{1}{r^{N-g}}\sum_{i_1+i_2+\ldots+i_g=rg+M, i_j\geq r+1}\frac{1}{i_1i_2\ldots i_g} = \sum_{g=1}^{M}\frac{1}{r^{N-g}}\frac{(rN+M)!}{N!}\binom{N}{g}\frac{g!}{(rg+M)!}S_{1,r+1}(rg+M,g)$

$3.4)S_{1,r}(rN+M,N) = \sum_{g=1}^{M}\frac{1}{r^{N-g}}\frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M}S_{1,r+1}(rg+M,g)$

$3.5)S_{2,r}(rN+M,N) = \sum_{g=1}^{M}\frac{1}{r!^{N-g}}\frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M}S_{2,r+1}(rg+M,g)$

P is a prime number.(1)(4)→

$\mathrm{Max}(\mathrm{g}+1,\mathrm{M}-\mathrm{g}+3) \leq \mathrm{P} \leq \mathrm{M}+\mathrm{g}, S_{1,2}(M+g,g) \equiv S_{2,2}(M+g,g) \equiv 0\mathrm{MOD\ P}, 2 \leq g \leq M \rightarrow$

$3.6)S_1(P,N) \equiv S_2(P,N) \equiv 0\mathrm{MOD\ P}, 2 \leq \mathrm{N} \leq \mathrm{P}-1$

$3.7)S_1(N+M,N) \equiv S_2(N+M,N) \equiv \begin{cases}1\ \mathrm{MOD\ P}, \mathrm{N} \equiv 1\ \mathrm{MOD\ P}\\ 0\ \mathrm{MOD\ P}, \mathrm{N} \not\equiv 1\ \mathrm{MOD\ P}\end{cases}, \mathrm{P} = \mathrm{M}+2$

Proof by induction and recurrence formula of $S_1(N,M)$ and $S_2(N,M)$

$3.8)\mathrm{S}_1(\mathrm{P}-1,\mathrm{g}) \equiv 1\ \mathrm{MOD\ P}, \mathrm{g}!\,\mathrm{S}_2(\mathrm{P}-1,\mathrm{g}) \equiv (-1)^{g+1}\mathrm{MOD\ P}, 1 \leq \mathrm{g} \leq \mathrm{P}-1$

3. Application of SUM(N) and H(g)

$4.1)\ \prod_{i=1}^{M}(K_i+nD_i)\ \ can\ be\ decomposed\ into\ three\ forms\ by\ 1.2)$

$4.2)SUM(N,PS,PT) =$

$SUM(N,[1,1\ldots1,PS],[1,1\ldots1,PT])\ , expand\ \sum_{g=0}^{M}(\ldots)\ \ to\ \sum_{g=0}^{M+count\ of\ 1\ added}(\ldots)$

$PS = [1,1\ldots1], PT = [1,2\ldots M] \xrightarrow{2.2)} H_1(M) = \sum_{g=0}^{M}H_3(g)\ , H_1(1) = \sum_{g=1}^{M}H_2(g)\ g \rightarrow$

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4.3) $\sum_{g=0}^{M}\left\langle {M \atop g} \right\rangle = M!; \sum_{g=1}^{M}(-1)^{M-g} g \times g!\, S_2(M,g) = 2^M - 1$

$PS = [1,1 \ldots 1], PT = [2,3 \ldots M] \xrightarrow{2.2)}$

4.4) $g!\, S_2(M,g) = \sum_{k=g}^{M}(-1)^{M-K} k!\, S_2(M,k)\binom{k-1}{g-1} = \sum_{k=0}^{g-1}\left\langle {M \atop k} \right\rangle \binom{M-1-k}{M-g}, 1 \le g \le M$

$\sum_{k=0}^{M}(-1)^{M-K} k!\, S_2(M,k) = 1!\, S_2(M,1) = 1$

$SUM(g,[2,3 \ldots M],[3,5 \ldots 2M-1]), SUM(g,[1,1 \ldots 1],[3,5 \ldots 2M-1]) \xrightarrow{2.2)}$

4.5) $\mathrm{S}_{1,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = \sum_{k=g}^{M}(-1)^{M-k}\mathrm{S}_{2,2}(\mathrm{M}+\mathrm{k},\mathrm{k})\binom{k-1}{g-1}, \mathrm{S}_{2,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = \sum_{k=g}^{M}(-1)^{M-k}\mathrm{S}_{1,2}(\mathrm{M}+\mathrm{k},\mathrm{k})\binom{k-1}{g-1}$

$SUM(N,[1,1 \ldots 1],[2,3 \ldots M]) \xrightarrow{2.4)}$

4.6) $\sum_{g=1}^{M} g!\, S_2(M,g) = \sum_{g=1}^{M}(-1)^{M-g} g!\, S_2(M,g) 2^{g-1} = \sum_{g=1}^{M}\left\langle {M \atop M-g} \right\rangle 2^{M-g}$

$SUM(N,[1,1 \ldots 1],[3,5 \ldots 2M-1]), SUM(N,[2,3 \ldots M],[3,5 \ldots 2M-1]) \xrightarrow{2.4)}$

4.7) $\sum_{g=1}^{M}\mathrm{S}_{2,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = \sum_{g=1}^{M}(-1)^{M-g}\mathrm{S}_{1,2}(\mathrm{M}+\mathrm{g},\mathrm{g})2^{g-1}; \sum_{g=1}^{M}\mathrm{S}_{1,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = \sum_{g=1}^{M}(-1)^{M-g}\mathrm{S}_{2,2}(\mathrm{M}+\mathrm{g},\mathrm{g})2^{g-1}$

$SUM(N,[1,1 \ldots 1],[2,3 \ldots M]) \xrightarrow{2.7)}$

4.8) $\sum_{g=1}^{M} g!\, S_2(M,g)(g-1)\binom{A+1}{g} = \sum_{g=1}^{M}(-1)^{M-g} g!\, S_2(M,g)(g-1)\binom{A+g-1}{g}$

$H_1(g,[1,2 \ldots M],[1,2 \ldots M]) = H_1(M-g) = M!\binom{M}{g} \xrightarrow{2.10)} g!\left(F_{M-g}^{M}E_0^{g}+\ldots+F_0^{M}E_{M-g}^{g}\right) \rightarrow$

4.9) $M!\binom{M}{g} = g!\sum_{i=0}^{M-g} S_1(M+1,g+1+i)S_2(g+i,g) = (M-g)!\sum_{i=0}^{g} S_1(M+1,M+1-i)S_2(M-i,M-g)$

$H_1(g,[1,1 \ldots 1],[1,2 \ldots M]) = g!\, S_2(M+1,g+1) \xrightarrow{2.10)} g!\left(F_{M-g}^{\{1,1\ldots1\}}E_0^{g}+\ldots+F_0^{\{1,1\ldots1\}}E_{M-g}^{g}\right) \rightarrow$

4.10) $S_2(M+1,g+1) = \sum_{i=0}^{M-g}\binom{M}{g+i} S_2(g+i,g) = \sum_{i=0}^{M-g}\binom{M}{i} S_2(M-i,g)$

$H_1(g,[1,2 \ldots M],[1,2 \ldots M]) = H_1(g,[M \ldots 2,1],[1,2 \ldots M]) = M!\binom{M}{g} \xrightarrow{2.11)} \frac{M!}{g!}\left(F_g^{M}E_0^{M-g}+\ldots+(-1)^g F_0^{M}E_g^{M-g}\right) \rightarrow$

4.11) $g!\binom{M}{g} = \sum_{i=0}^{g} S_1(M+1,M+1-g+i)S_2(M-g+i,M-g)(-1)^g$

$H_1(g,[K+i],[T+i]) \xrightarrow{2.12)} \binom{\mathrm{M}}{g} T_1 \ldots T_g \times K_{g+1} \ldots K_M \xrightarrow{2.10)} T_1 \ldots T_g(\ldots) \xrightarrow{2.11)} K_{g+1} \ldots K_M(\ldots)$

4.12) $\prod_{i=g+1}^{M}(\mathrm{K}+\mathrm{i})\binom{M}{g} = F_{M-g}^{\{K+i\}}E_0^{g} + \cdots + F_0^{\{K+i\}}E_{M-g}^{g}; \prod_{i=1}^{g}(\mathrm{T}+\mathrm{i})\binom{M}{g} = F_g^{\{\mathrm{T}+i\}}E_0^{M-g}+\ldots(-1)^g F_0^{\{\mathrm{T}+i\}}E_g^{M-g}$

$SUM(1,[1,1 \ldots 1],[3,5 \ldots 2M-1]) = 1, SUM(1,[2,3 \ldots M],[3,5 \ldots 2M-1]) = M! \xrightarrow{Form_2} \sum_{g=0}^{M-1} H_2(g)\binom{1+M+g}{M+1+g} \rightarrow$

4.13) $\sum_{g=0}^{M-1}(-1)^{M-1-g}MIN_g = \sum_{g=1}^{M}(-1)^{M-g}\mathrm{S}_{1,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = 1; \sum_{g=1}^{M}(-1)^{M-g}\mathrm{S}_{2,2}(\mathrm{M}+\mathrm{g},\mathrm{g}) = M!$

$\mathrm{PS1} = [\mathrm{D}_2:\mathrm{D}_2\ldots\mathrm{D}_\mathrm{M}:\mathrm{D}_\mathrm{M}], \mathrm{PT1} = \left[\frac{\mathrm{K}_2}{\mathrm{D}_2}+1 \ldots \frac{\mathrm{K}_\mathrm{M}}{\mathrm{D}_\mathrm{M}}+\mathrm{M}-1\right] \rightarrow \mathrm{K}_1 \times H_2(g) = (-1)^{M-1-g}MIN_g[\mathrm{K}_1:\mathrm{D}_1,\mathrm{K}_2:\mathrm{D}_2 \ldots] \rightarrow$

4.14) $\sum_{g=0}^{M-1}(-1)^{M-1-g}MIN_g(PS) = K_1\prod_{i=2}^{M} D_i$

$SUM(2,[1,1 \ldots 1],[3,5 \ldots 2M-1]) = S_2(M+2,2) = 2^{M+1}-1 = \sum_{g=0}^{M-1} H_2(g)\binom{2+M+g}{1+M+g}, combination\ 4.13) \rightarrow$

4.15) $\sum_{g=0}^{M-1}(-1)^{M-1-g} g\, MIN_g = 2^{M+1} - M - 3$

$SUM(N,[T,T \ldots T],[T,T \ldots T]) \xrightarrow{1.8)} T^M\binom{N+T}{T+1} \xrightarrow{2.1)} \sum_{g=0}^{M} T^M\binom{M}{g}\binom{N+T-M}{T-M+1+g} \rightarrow$

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4.16) $\sum_{g=0}^{M}\binom{M}{g}\binom{N+T-M}{T-M+1+g}=\binom{N+T}{T+1}$

4.17) $P\ is\ prime, SUM(P,[\ldots]:D,[1,2\ldots M])=\sum_{g=0}^{M}H_1(g)\binom{P}{1+g}\to$ $\begin{cases}\equiv 0MODP, M<P-1\\ \equiv -1MODP, M=P-1\end{cases},(P,D)=1$

Exclude products≡0 MOD P , then a series of congruence can be obtained.For Example:

$1^M+2^M+\ldots+(P-1)^M\equiv\begin{cases}0MODP, 0<M<P-1\\ -1MODP, M=P-1\end{cases}$

$1^A2^B+2^A3^B+\ldots+(P-2)^A(P-1)^B\equiv\begin{cases}0MODP, 0<A+B<P-1, 0\le A,B\in\mathbb{N}\\ -1MODP, A+B=P-1, 0\le A,B\in\mathbb{N}\end{cases}$

$1^A3^B+2^A4^B+\ldots+(P-3)^A(P-1)^B+(P-1)^A1^B\equiv\begin{cases}0MODP, 0<A+B<P-1, 0\le A,B\in\mathbb{N}\\ -1MODP, A+B=P-1, 0\le A,B\in\mathbb{N}\end{cases}$

$1^A3^B5^C+2^A4^B6^C+\cdots+(P-5)^A(P-3)^B(P-1)^C+(P-3)^A(P-1)^B1^C+(P-1)^A1^B3^C$ $\equiv\begin{cases}0MODP, 0<A+B+C<P-1, 0\le A,B,C\in\mathbb{N}\\ -1MODP, A+B+C=P-1, 0\le A,B,C\in\mathbb{N}\end{cases}$

Derived from 4.17) →

4.18) $P\ is\ prime, \sum_{,\lambda_1+\lambda_2+\ldots+\lambda_k=q,1\le q\le P-2,\lambda_j\ge 0}1^{\lambda_1}2^{\lambda_2}\ldots(P-1)^{\lambda_k}\equiv 0\text{MOD P}$

$PS1=[K_1:D_1\ldots K_i:D_i,K_{i+1}+D_{i+1}:D_{i+1},K_{i+2}+D_{i+2}:D_{i+2}\ldots],PT1=[T_1\ldots T_i,T_{i+1}-1,T_{i+2}-1\ldots T_M-1]$

4.19) $Classification\ principle: SUM(N)=SUM(N,PS,PT1)+SUM(N-1,PS1,PT)$

Simply put, PS and PS have the same shape.

$SUM(N,[1,2,3],[1,3,5])=SUM(N,[1,2,3],[1,2,4])+SUM(N-1,[1,3,4],[1,3,5])$
$=\{SUM(N,PT,[1,2,3])+SUM(N-1,PT,[1,2,4])\}+\{SUM(N-1,PT,[1,3,4])+SUM(N-2,PT,[1,3,5])\}$

If $T_i=T_{i+1}$, it is also established.

$SUM(N,[1,2],[1,2])=2\binom{N+2}{3}=SUM(N,[1,2],[1,1])+SUM(N-1,[1,3],[1,2])$
$=SUM(N,[2],[1])+SUM(N-1,[3],[2])=\binom{N}{2}+2\binom{N}{1}+2\binom{N}{3}+3\binom{N}{2}$

The classification principle also applies to the following Formal Calculation.

PT's domain can be extended to $\mathbb{C}$. The Recursive Formula of H (g) can be used for.

4.20) $Solving\ the\ Equation: F(M+1,g)=F(M,g)(aM+bg+c)+F(M,g-1)(xM+bg+z), b\neq 0$

$H_1(g,PS1,PT1)=H_1(g-1)(\mathrm{T}_{M+1}-[\mathrm{M}-(\mathrm{g}-1)])\mathrm{D}_{M+1}+H_1(g)(\mathrm{K}_{M+1}+\mathrm{gD}_{M+1})$

$\begin{cases}aM+bg+c=\mathrm{K}_{M+1}+\mathrm{gD}_{M+1}\\ xM+bg+z=(\mathrm{T}_{M+1}-[\mathrm{M}-(\mathrm{g}-1)])\mathrm{D}_{M+1}\end{cases}$

$\xrightarrow{\text{eliminate } g}\begin{cases}\mathrm{D}_{M+1}=b, \mathrm{K}_{M+1}=aM+c, PS=[a(i-1)+c:b]\\ \mathrm{T}_{M+1}=M+1+\frac{xM+z}{b}, PT=[\frac{x(i-1)+z}{b}+i]\end{cases}$

$H_2(g), H_3(g)also\ has\ similar\ properties.$ This way, we can use the attributes of H(g) for analysis.

$eg: F(M+1,g)=(g+1)F(M,g)+(g+1)F(M,g-1)\to PS=[1,1\ldots 1], PT=[2,3\ldots M+1]$

$It's\ (g+1)!S_2(M+1,g+1)=(g+1)(g+1)!S_2(M,g+1)+(g+1)g!S_2(M,g)$

$SUM(N,PS1,PT1)=SUM(N-1,PS1,PT1)+[\mathrm{K}_{M+1}+(N-1)\mathrm{D}_{M+1}]\ \nabla^P\ SUM(N)\ can\ do\ the\ same.$

$SUM(N,[1,2\ldots M],[1,3\ldots 2M-1])\to S_1(N+M,N)=S_1(N+M-1,N-1)+(N+M-1)S_1(N+M-1,N)$

4. Combinatorial identity of SUM(N)

$R-FOLD\ SUM: \sum_{(r)}^{N}f(k)=\sum_{k_r=1}^{N}\cdots\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2}f(k_1)=\sum_{k_r=0}^{N-1}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}f(k_1+1)$

5.1) $\sum_{(r)}^{N}\nabla^P\ SUM(k,PS,PT)=\nabla^{P-r}\ SUM(N,PS,PT)$

5.2) $\sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla\, SUM(k_r, PS, [1,2 \dots M]) = SUM(N, PS, [1+(r-1), 2+(r-1) \dots M+(r-1)])$

[Proof]

$PS1 = [1{:}0, 1{:}0 \dots 1{:}0, PS], PT1 = [1,3 \dots 2r-3, 1+2(r-1), 2+2(r-1) \dots M+2(r-1)]$

$PT = [1,2 \dots M], PTx = [1+(r-1), 2+(r-1) \dots M+(r-1)]$

$f(N) = \sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla\, SUM(k_r, PS, PT) = \sum_{k_r=1}^{N} \nabla\, SUM(k_r, PS, PT) \dots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} 1$

$PT = [1,2 \dots M]. According\ to\ definition\ of\ SUM(N) \rightarrow$

$f(N) = SUM(N, PS1, PT1) \xrightarrow{M+2(r-1)-(r-1+M)=r-1} \sum_{g=0}^{M+r-1} H_1(g, PS1, PT1) \binom{N+r-1}{r+g}$

$In\ H_1(g > M, PS1, PT1), i < r, X_i = \begin{cases} (T_i - X_{K-1})D_i = 0; X_i = T_i \\ K_i + X_{T-1}D_i = 1; X_i = K_i \end{cases} = 0$

$In\ H_1(g \le M, PS1, PT1), \prod_{i<r} X_i = 1; i \ge r, X_i = \begin{cases} (T_i - X_{K-1})D_i; X_i = T_i \\ K_i + X_{T-1}D_i; X_i = K_i \end{cases} = H_1(g, PS, PTx)$

$\rightarrow f(N) = \sum_{g=0}^{M} H_1(g, PS1, PT1) \binom{N+r-1}{r+g} = SUM(N, PS, PTx)$

q.e.d.

5.3) $\sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla\, SUM(k_i, PS, [1,2 \dots M]) = \nabla^{i-r}\, SUM(N, PS, [1+(i-1), 2+(i-1) \dots M+(i-1)])$

$\sum_{k_r=0}^{N} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \binom{k_i}{j} = \frac{1}{j!} \nabla^{i-r}\, SUM(N+1, [0,-1,-2 \dots -j+1], [1+(i-1), 2+(i-1) \dots j+(i-1)])$

$\xrightarrow{H_1(g<j)=0, T_M - M = i-1} \frac{1}{j!} \nabla^{i-r} \frac{(j+i-1)!}{(i-1)!} \binom{N+1+i-1}{i+j} = \binom{j+i-1}{i-1} \binom{N+i-(i-r)}{i+j-(i-r)} = \binom{j+i-1}{i-1} \binom{N+r}{r+j} \rightarrow$

$record\ at\ [6]{:}\ Lemma\ 1$

$PS = \left[1,2 \dots T_1 - P_1, \frac{K_1(T_1+1-P_1)}{T_1 D_1}, 1,2 \dots T_2 - P_2, \frac{K_2(T_2+1-P_2)}{T_2 D_2}\right], PT = [1,2,3 \dots T_1 + T_2 + 2 - P_1 - P_2]$

5.4) $\sum_{n=0}^{N-1} \nabla^{P_1}\, SUM(n+1, [K_1{:}D_1], [T_1])\, \nabla^{P_2}\, SUM(n+1, [K_2{:}D_2], [T_2]) = \frac{T_1 T_2 D_1 D_2}{(T_1+1-P_1)!(T_2+1-P_2)!} SUM(N, PS, PT)$

5.5) $\sum_{0 \le n_1 \le n_2 \le \dots \le n_M \le N-1} (K + n_1 + n_2 + \dots + n_M) = \nabla^{\binom{M}{2}}\, SUM\left(N, [K], \left[\binom{M+1}{2}\right]\right) = \binom{M+1}{2}\binom{N+M-1}{M+1} + K\binom{N+M-1}{M}$

$\sum_{0 \le n_1 \le \dots \le n_p \le n} (n_1 + \dots + n_p) = \sum_{n_p=0}^{n} \dots \sum_{n_1=0}^{n_2} (n_1 + \dots + n_p) = \binom{p+1}{2} \binom{n+p}{p+1} = \frac{pn}{2} \binom{n+p}{p} record\ at\ [6]{:}\ Theorem\ 2$

5.6) $\sum_{0 \le n_1 \le \dots \le n_M \le N-1} (K + n_1 D_1 + \dots + n_M D_M) = \nabla^{\binom{M}{2}}\, SUM\left(N, \left[K{:}(D_1 + 2D_2 + \dots + MD_M)\binom{M+1}{2}^{-1}\right], \left[\binom{M+1}{2}\right]\right)$

$= \frac{D_1 + 2D_2 + \dots + MD_M}{\binom{M+1}{2}} \binom{M+1}{2}\binom{N+M-1}{M+1} + K\binom{N+M-1}{M}$

Use 1.2)-1.4) and properties of H(g) can be used to derive combinatorial identities.For Example:

$\binom{n+A}{A}\binom{n+M+B}{M} = \frac{1}{A!M!} \nabla\, SUM(N, [1,2 \dots A, B+1, B+2 \dots B+M], [1,2 \dots A, A+1, A+2 \dots A+M])$

$= \frac{1}{M!} \nabla\, SUM(N, [B+1 \dots B+2, B+M], [A+1, A+2 \dots A+M]) \xrightarrow{2.12)}$

$H_1(g) = \binom{M}{g} [B+M]_{M-g} [A+1]^g = \binom{M}{g} (M-g)! \binom{M+B}{M-g} g! \binom{A+g}{g} = M! \binom{A+g}{g} \binom{M+B}{M-g}$

5.7) $\binom{n+A}{A}\binom{n+M+B}{M} = \sum_{g=0}^{M} \binom{A+g}{g} \binom{M+B}{M-g} \binom{n+A}{A+g} = \sum_{g=0}^{M} (-1)^{M-g} \binom{A-B}{M-g} \binom{A+g}{g} \binom{n+A+g}{A+g} = \sum_{g=0}^{M} \binom{A-B}{g} \binom{M+B}{M-g} \binom{n+A+M-g}{A+M}$

5.8) $\binom{n+X}{A}\binom{n+Y}{M} = \sum_{q=0}^{A} \binom{M+q}{q} \binom{M+X-Y}{A-q} \binom{n+Y}{M+q}, 0 \le Y \le M$

[Proof]

$= \frac{1}{A!M!} \nabla\, SUM(N, [X, X-1 \dots X-A+1, Y, Y-1 \dots Y-M+1], [1,2 \dots A+M])$

$= \frac{1}{\text{A!M!}} \nabla\, SUM(N, [1,2..Y, 0, -1 \dots -(M-Y)+1, X \dots X-A+1], [1,2 \dots \text{A}+\text{M}])$
$= \frac{\text{Y!}}{\text{A!M!}} \nabla\, SUM(N, [0, -1..-(M-Y)+1, X \dots X-A+1], [Y+1, Y+2 \dots A+M])$
$If\ \text{H}_1(\text{g}) \neq 0, X_1, X_2 \dots X_{\text{M}-\text{Y}}\ must\ belong\ to\ T. let\ C = A+M-Y$
$\text{H}_1(\text{g} \geq \text{M}-\text{Y}, \sum\text{K}) \neq 0 \xrightarrow{\text{count of K}=\binom{C-(M-Y)}{\text{C}-\text{g}}} \binom{C-M+Y}{\text{C}-\text{g}} [\text{X}+(\text{M}-\text{Y})]_{\text{C}-\text{g}} = \binom{A}{\text{C}-\text{g}} [\text{X}+(\text{M}-\text{Y})]_{\text{C}-\text{g}}$
$\frac{\text{Y!}}{\text{A!M!}} H_1(g \geq \text{M}-\text{Y}) = \frac{\text{Y!}}{\text{A!M!}} \binom{\text{A}}{M+A-Y-\text{g}} [\text{X}+\text{M}-\text{Y}]_{A+M-Y-\text{g}} [Y+1]^g \xrightarrow{\text{q}:=-(\text{M}-\text{Y}-\text{g})}$
$\frac{\text{Y!}}{\text{A!M!}} H_1(\text{M}-\text{Y}+\text{q}) = \frac{\text{Y!}}{\text{A!M!}} \binom{\text{A}}{A-q} \binom{\text{X+M}-\text{Y}}{A-q} (A-q)! \binom{\text{M+q}}{M-y+q} (M-y+q)! = \binom{\text{M+q}}{\text{q}} \binom{\text{M+X}-\text{Y}}{\text{A}-\text{q}}$
$\binom{\text{n+X}}{\text{A}}\binom{\text{n+Y}}{\text{M}} = \frac{\text{Y!}}{\text{A!M!}} \nabla \sum_{g=M-Y}^{C} H_1(g) \binom{N+(A+M)-C}{(A+M)-C+1+q} = \frac{\text{Y!}}{\text{A!M!}} \sum_{g=M-Y}^{A+M-Y} H_1(g) \binom{n+Y}{\text{Y}+g}$
q.e.d.
$Compare\ with\ 5.7), there\ has\ no\ formula\ for\ \binom{\text{n+X}}{\text{A}}\binom{\text{n+Y}}{\text{M}}, Y > M, X \neq A$

$\text{PS} = [0, -1 \dots -(B-1), A-(M-B-1): -1 \dots A: -1], \text{PT} = [1,2 \dots \text{M}] \rightarrow$
$5.9)\binom{\text{n}}{\text{B}}\binom{\text{A}-\text{n}}{\text{M}-\text{B}} = \sum_{\text{g}=0}^{\text{M}-\text{B}} (-1)^{M-B-g} \binom{\text{A}-\text{M}+\text{g}}{\text{g}} \binom{M-g}{\text{B}} \binom{n}{\text{M}-\text{g}}, n \geq 0$

$\text{PS} = [0, -1 \dots -A+1, 0, -1 \dots -\text{B}+1], \text{PT} = [1,2 \dots \text{A}+\text{B}] \rightarrow$
$5.10)\binom{\text{n}}{\text{A}}\binom{\text{n}}{\text{B}} = \sum_{\text{g}=0}^{\text{B}} \binom{\text{B}}{\text{g}} \binom{\text{A+B}-\text{g}}{\text{B}-\text{g}} \binom{\text{n}}{\text{A+B}-\text{g}}$, record at [7]: (6.44)

$5.11) \prod_{i=1}^{M} (A+2i+n) = \binom{n+A}{A}^{-1} \sum_{g=0}^{M} (2[M-g]-1)!! \binom{2M-g}{g} [A+1]^g \binom{n+A+g}{A+g}, A \in \mathbb{N}\ or\ A = 0$
[Proof]
PS=[A+2,A+4…A+2M],PT=[A+1,A+2…A+M]
$H_2(g, \sum K) = \text{SUM}(\text{g}+1, [1,3 \dots 2(\text{M}-\text{g})-1], [1,3 \dots 2(\text{M}-\text{g})-1]) \xrightarrow{1.8)} (2(\text{M}-\text{g})-1)!! \binom{\text{2M}-\text{g}}{\text{g}}$
$H_2(g, T) = [A+1]^g$
$\binom{\text{n+A}}{\text{A}} \prod_{i=1}^{M} (A+2i+n) = \frac{1}{\text{A!}} \nabla\, \text{SUM}(\text{N}, [1,2 \dots \text{A}, \text{PS}], [1,2 \dots \text{A}, \text{PT}]) = \nabla\, \text{SUM}(\text{N}) =$
$\sum_{\text{g}=0}^{\text{M}} H_2(g) \binom{\text{n+A+g}}{\text{A+g}}$
q.e.d.
$5.12) \prod_{i=1}^{M} (2i+n) = \sum_{g=0}^{M} (2[M-g]-1)!! \binom{2M-g}{g} g! \binom{n+A+g}{A+g}$

SUM(N,[1,3…2M-1],[1,2…M])= SUM(N,[3,5…2M-1],[2,3…M])→
$5.13) \prod_{i=1}^{M} (2i-1+n) = \sum_{g=0}^{M-1} (2[\text{M}-1-\text{g}]-1)!! \binom{2(M-1)-g}{g} (g+1)! \binom{\text{n}+1+\text{g}}{\text{g}+1}$

$5.14) SUM(N, [\text{A}+1, \text{A}+3 \dots \text{A}+2\text{M}-1], [1,3 \dots 2\text{M}-1]) = \sum_{g=0}^{M} [\text{A}]^{M-g} (2g-$
$1)!! \binom{M+g}{2g} \binom{\text{N+M}-1+\text{g}}{\text{M+g}}$
[Proof]
$H_2(g, K) = [A]^{M-g}, H_2(g, T) = \text{SUM}(\text{M}-\text{g}+1, [1,3 \dots 2g-1], [1,3 \dots 2\text{g}-1]) \xrightarrow{1.8)} (2g-1)!! \binom{\text{M+g}}{2\text{g}}$
q.e.d.
$5.15) \sum_{n_M=0}^{N-1} (2M+n_M) \dots \sum_{n_2=0}^{n_3} (4+n_2) \sum_{n_1=0}^{n_2} (2+n_1) = \sum_{g=0}^{M} \frac{(M+g)!}{\text{g!}2^g} \binom{\text{N+M}-1+\text{g}}{\text{M+g}} =$
$\sum_{q=0}^{M} \binom{\text{N}-1+\text{g}}{\text{g}} \frac{[N+g]^M}{2^g}$
$5.16) SUM(\text{N}, [\text{A}, \text{A}+1 \dots \text{A}+\text{M}-1]: 2, [1,3 \dots 2\text{M}-1]) = \binom{M+N-1}{M} [A+M+N-2]_M$
[Proof]
$\text{H}_1(\text{g}, \sum\text{K}, [A, A+1 \dots \text{A}+(\text{M}-1)], [1,2 \dots \text{M}])$
$= SUM(g+1, [A, A+1 \dots \text{A}+(\text{M}-1-\text{g})]: 2, [1,3 \dots 2(M-\text{g})-1])$
$= \text{H}_1(\text{g}, \sum\text{K}, [\text{A}+(\text{M}-1) \dots A+1, A], [1,2 \dots \text{M}]) = \binom{M}{g} [\text{A}+(\text{M}-1)]_{\text{M}-\text{g}}$

$SUM(g+1, [A, A+1 \dots \text{A}+(\text{M}-1-\text{g})]: 2, [1,3 \dots 2(M-\text{g})-1]) = \binom{M}{g} [\text{A}+(\text{M}-1)]_{\text{M}-\text{g}} \xrightarrow{M:=M-g}$
$SUM(g+1) = \binom{M+g}{g} [\text{A}+(\text{M}+\text{g}-1)]_{\text{M}} \xrightarrow{N:=g+1} SUM(N) = \binom{M+N-1}{M} [A+M+N-2]_M$
q.e.d.

$5.17)\mathrm{SUM}(\mathrm{N},[1,2\ldots\mathrm{M}]:2,[1,3\ldots2\mathrm{M}-1])=\mathrm{M}!\binom{\mathrm{N}+\mathrm{M}-1}{\mathrm{M}}^2\xrightarrow{M=1}1+3+\cdots+(2\mathrm{N}-1)=\mathrm{N}^2$

$\frac{1}{M!}SUM(\mathrm{N},[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}]:2,[1,3\ldots2\mathrm{M}-1])=\sum_{g=0}^{\mathrm{M}}\mathrm{H}_1(\mathrm{g})\binom{\mathrm{N}+\mathrm{M}-1}{\mathrm{M}+\mathrm{g}}=\binom{M+N-1}{M}\binom{A+M+N-1}{M}$
$\xrightarrow{5.7),A=M,B=A}$

$=\nabla\, SUM(\mathrm{N},\mathrm{PS1}=[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}],\mathrm{PT1}=[\mathrm{M}+1,\mathrm{M}+2\ldots2\mathrm{M}])=$
$\sum_{g=0}^{\mathrm{M}}\mathrm{H}_1(\mathrm{g},\mathrm{PS1},\mathrm{PT1})\binom{\mathrm{N}+\mathrm{M}-1}{\mathrm{M}+\mathrm{g}}\rightarrow$

$5.18)\frac{1}{M!}\mathrm{H}(\mathrm{g},[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}]:2,[1,3\ldots2\mathrm{M}-1])=\mathrm{H}(\mathrm{g},[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}],[\mathrm{M}+1,\mathrm{M}+2\ldots2\mathrm{M}])$

$\mathrm{H}_1(\mathrm{g},[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}]:2,[1,3\ldots2\mathrm{M}-1])=$
$M!\binom{\mathrm{M}+\mathrm{g}}{\mathrm{g}}\binom{\mathrm{M}+\mathrm{A}}{\mathrm{M}-\mathrm{g}}.The\ definition\ of\ H(g) is\ complex.eg:$

$\mathrm{H}_1(1)=2\{(\mathrm{A}+1)\ldots(\mathrm{A}+\mathrm{M}-1)\mathrm{xM}+(\mathrm{A}+1)\ldots(\mathrm{A}+\mathrm{M}-2)(\mathrm{A}+\mathrm{M}+2)\mathrm{x}(M-1)+\cdots+(A+4)(A+5)\ldots(A+M+2)\mathrm{x}1\}\rightarrow$

$5.19)\sum_{\mathrm{n}=0}^{\mathrm{M}-1}(\mathrm{M}-\mathrm{n})\prod_{k=1}^{M-1}(A+k+3\left[\frac{\mathrm{k}+\mathrm{n}}{M}\right])=\frac{(M+1)!}{2}\binom{\mathrm{M}+\mathrm{A}}{\mathrm{M}-1},[\mathrm{n}]\ is\ truncates\ integer.$

$5.20)SUM(\mathrm{N},[\mathrm{A}+2,\mathrm{A}+4\ldots\mathrm{A}+2\mathrm{M}]:3,[1,3\ldots2\mathrm{M}-1])=\sum_{y=0}^{\mathrm{M}}\binom{A+N-1+y}{y}\binom{N+M-1-y}{M-y}\frac{[M+N-y]^M}{2^{M-y}}$

[Proof]

$PS1=[\mathrm{A}+2,\mathrm{A}+4\ldots\mathrm{A}+2\mathrm{M}],PT1=[\mathrm{A}+1,\mathrm{A}+2\ldots\mathrm{A}+\mathrm{M}]$

$\mathrm{H}_1(\mathrm{g},\mathrm{T},PS1,\mathrm{PT1})=[\mathrm{A}+1]^g$

$\mathrm{H}_1(\mathrm{g},\sum\mathrm{K},PS1,\mathrm{PT1})=SUM(g+1,[\mathrm{A}+2,\mathrm{A}+4\ldots\mathrm{A}+2(\mathrm{M}-\mathrm{g})]:3,[1,3\ldots2(M-g)-1])$

$\mathrm{H}_1(\mathrm{g},\mathrm{PS1},\mathrm{PT1})\xrightarrow{2.2)}\sum_{\mathrm{x}=\mathrm{g}}^{\mathrm{M}}\mathrm{H}_2(\mathrm{x},\mathrm{PS1},\mathrm{PT1})\binom{\mathrm{x}}{\mathrm{g}}\xrightarrow{5.11)}\sum_{\mathrm{x}=\mathrm{g}}^{\mathrm{M}}(2[\mathrm{M}-\mathrm{x}]-1)!!\binom{2M-x}{x}[\mathrm{A}+1]^x\binom{\mathrm{x}}{\mathrm{g}}$

$SUM(g+1,[\mathrm{A}+2,\mathrm{A}+4\ldots\mathrm{A}+2(\mathrm{M}-\mathrm{g})]:3,[1,3\ldots2(M-g)-1]$
$=\sum_{\mathrm{x}=\mathrm{g}}^{\mathrm{M}}(2[\mathrm{M}-\mathrm{x}]-1)!!\binom{2M-x}{x}[\mathrm{A}+1]^x\binom{\mathrm{x}}{\mathrm{g}}[\mathrm{A}+1]^{-g}$
$=\sum_{\mathrm{x}=\mathrm{g}}^{\mathrm{M}}(2[\mathrm{M}-\mathrm{x}]-1)!!\frac{(2M-x)!}{(2m-2x)!\mathrm{x}!}\frac{(\mathrm{A}+\mathrm{x})!}{A!}\frac{\mathrm{x}!}{g!(x-g)!}\frac{\mathrm{A}!}{(A+g)!}=\sum_{\mathrm{x}=\mathrm{g}}^{\mathrm{M}}\frac{(2M-x)!}{(m-x)!2^{M-x}}\frac{(A+x)!}{(A+\mathrm{g})!\mathrm{g}!(x-g)!}$
$\xrightarrow{\mathrm{y}:=\mathrm{x}-\mathrm{g}}\sum_{\mathrm{y}=0}^{\mathrm{M}-\mathrm{y}}\frac{(2M-g-y)!}{(M-g-y)!2^{(M-g-y)}}\frac{(A+y+g)!}{(A+g)!y!g!}\xrightarrow{M:=M-g}$

$\mathrm{SUM}(\mathrm{g}+1,[\mathrm{A}+2,\mathrm{A}+4\ldots\mathrm{A}+2\mathrm{M}]:3,[1,3\ldots2\mathrm{M}-1])=\sum_{\mathrm{y}=0}^{\mathrm{M}}\frac{(2M+g-y)!}{(M-y)!2^{M-y}}\frac{(A+y+g)!}{g!y!(\mathrm{A}+\mathrm{g})!}$
$\xrightarrow{N:=g+1} conclusion$

q.e.d.

The identity of PS=[K₁,K₂…K_M]:D similar to 5.7)-5.20) can also be obtained.

5. Matrix of SUM(N)

Consider H(g) as variables,list SUM(N),SUM(N+1)…SUM(N+M),we can obtain a (M+1) ×(M+1) matrix

Let P=N+T_M-M,Q=N-1,define A(P,Q,M), respectively corresponding to the three forms

$$\mathrm{A}_1(\mathrm{P},\mathrm{Q},\mathrm{M})=\begin{bmatrix}\binom{\mathrm{P}}{\mathrm{Q}} & \cdots & \binom{\mathrm{P}}{\mathrm{Q}-\mathrm{M}}\\ \vdots & \ddots & \vdots\\ \binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}+\mathrm{M}} & \cdots & \binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}}\end{bmatrix},\mathrm{A}_2(\mathrm{P},\mathrm{Q},\mathrm{M})=\begin{bmatrix}\binom{\mathrm{P}}{\mathrm{Q}} & \cdots & \binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}}\\ \vdots & \ddots & \vdots\\ \binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}+\mathrm{M}} & \cdots & \binom{\mathrm{P}+2\mathrm{M}}{\mathrm{Q}+\mathrm{M}}\end{bmatrix},\mathrm{A}_3(\mathrm{P},\mathrm{Q},\mathrm{M})=\begin{bmatrix}\binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}} & \cdots & \binom{\mathrm{P}}{\mathrm{Q}-\mathrm{M}}\\ \vdots & \ddots & \vdots\\ \binom{\mathrm{P}+2\mathrm{M}}{\mathrm{Q}+\mathrm{M}} & \cdots & \binom{\mathrm{P}+\mathrm{M}}{\mathrm{Q}}\end{bmatrix}$$

$6.1)||\mathrm{A}_1(\mathrm{P},\mathrm{Q},\mathrm{M})||=\ ||\mathrm{A}_2(\mathrm{P},\mathrm{Q},\mathrm{M})||=\ ||\mathrm{A}_3(\mathrm{P},\mathrm{Q},\mathrm{M})||$

$6.2)\ ||\mathrm{A}(\mathrm{P},0,\mathrm{M})||=1,||\mathrm{A}(\mathrm{P},1,\mathrm{M})||=\binom{\mathrm{P}+\mathrm{M}}{1+\mathrm{M}},||\mathrm{A}(\mathrm{P},\mathrm{Q}>0,\mathrm{M})||=\prod_{g=0}^{Q-1}\frac{\binom{\mathrm{P}+\mathrm{M}-\mathrm{g}}{1+\mathrm{M}}}{\binom{1+\mathrm{M}+\mathrm{g}}{1+\mathrm{M}}}$

$6.3)||\mathrm{A}(\mathrm{P},\mathrm{Q},\mathrm{M})||=\ ||\mathrm{A}(\mathrm{P},\mathrm{P}-\mathrm{Q},\mathrm{M})||$

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$$A_1(P,0,M)=\begin{bmatrix}\binom{P}{0} & \cdots & 0\\ \vdots & \ddots & \vdots\\ \binom{P+M}{M} & \cdots & \binom{P+M}{0}\end{bmatrix}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ \binom{P+M}{M} & \cdots & 1\end{bmatrix}$$

$$A_3(P,0,M)=\begin{bmatrix}\binom{P+M}{0} & \cdots & 0\\ \vdots & \ddots & \vdots\\ \binom{P+2M}{M} & \cdots & \binom{P+M}{0}\end{bmatrix}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ \binom{P+2M}{M} & \cdots & 1\end{bmatrix}$$

$A_2(1,0,M)$ can be changed to $\begin{bmatrix}\binom{0}{0} & \cdots & \binom{M}{0}\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \binom{M}{M}\end{bmatrix}$

If SUM(N) or ∇SUM(N) is easy to obtained,then H(g) can be calculated with the Cramer's law.

Table 6. 1: Calculate H(g) with matrix,T_M>=M

$\mathbf{H_1(g)=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{1+T_M-M+g}{T_M-M+k}SUM(k)}$.	$\mathbf{SUM(N,[1,2\ldots M],[1,2\ldots M])=M!\binom{M+N}{M+1}\rightarrow \binom{M}{g}=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{1+g}{k}\binom{M+k}{M+1}}$ $\mathbf{SUM(N,[2,3\ldots M],[3,5\ldots 2M-1])=S_1(M+N,N)\rightarrow}$ $\mathbf{MIN_g=\sum_{k=1}^{g+1}(-1)^{g+k+1}\binom{M+g+1}{M+k}S_1(M+k,k)}$
$H_1(g)=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{T_M-M+g}{T_M-M+k-1}\nabla SUM(k)$	$\nabla SUM(N,[1,2\ldots M],[1,2\ldots M])=M!\binom{M+N-1}{M}\rightarrow \binom{M}{g}=\sum_{k=0}^{g}(-1)^{g+k}\binom{g}{k}\binom{M+k}{M}$ $\nabla SUM(N,[1,1\ldots 1],[2,3\ldots M])=N^M\rightarrow$ $S_2(M,g)=\frac{1}{g!}\sum_{k=0}^{g}(-1)^{g+k}\binom{g}{k}k^M=\frac{1}{g!}\sum_{k=0}^{g}(-1)^{k}\binom{g}{k}(g-k)^M$ It's a known formula
$z(k)=\sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i}\nabla SUM(k)$ $H_2(g)=\sum_{k=g}^{M}(-1)^{M+k}\binom{k-1}{g-1}z(k)$	$\nabla SUM(N,[1,1\ldots 1],[2,3\ldots M])=N^M$ $z(k)=\sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i}i^M=k!S_2(M,k)\rightarrow$ $g!S_2(M,g)=\sum_{k=g}^{M}(-1)^{M+k}\binom{k-1}{g-1}k!S_2(M,k)\rightarrow$ 4.4)
$H_3(g)=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{2+T_M}{g+1-k}SUM(k)$	
$H_3(g)=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{1+T_M}{g+1-k}\nabla SUM(k)$	$\nabla SUM(N,[1,1\ldots 1],[2,3\ldots M])=N^M\rightarrow$ $\left\langle {M \atop g} \right\rangle=\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{M+1}{g+1-k}k^M$ $=\sum_{k=0}^{g}(-1)^{k}\binom{M+1}{k}(g+1-k)^M$, *it's* a known formula

Many conclusions of [8] can be obtained through the matrix. Use 5.16)-5.18) we can get some new *identities*.

6. Formal Calculation of Gaussian Coefficients

Gaussian Coefficients: $G_M^N=\begin{bmatrix}N\\M\end{bmatrix}_q=\frac{(q^N-1)(q^{N-1}-1)\ldots(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)\ldots(q-1)}, q\neq 0,1$

(1)$G_0^N=1, G_{M<0\ or\ M>N}^N=0, G_M^N=G_{N-M}^N$
(2)$G_M^N=q^M G_M^{N-1}+G_{M-1}^{N-1}=G_M^{N-1}+q^{N-M}G_{M-1}^{N-1}$
(3)$G_M^N=\sum_{n=0}^{N-1}q^n G_M^{n+K}=q^{M-K}G_{M+1}^{N+K}$

Formal Calculation of Gaussian Coefficients has been obtained by [3].

It uses $q^n(K_i+G_1^n)$ instead of K_i+q^n *and nested summation can also be performed.*

$\nabla_q^0 f(N)=f(N), \sum_{n=0}^{N-1}q^n\,\nabla_q^1 f(n+1)=f(N), \sum_{n=0}^{N-1}q^n f(n+1)$
$=\nabla_q^{-1}f(N)$. *Recursive define* ∇_q^P

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$\mathrm{SUM}_q(\mathrm{N},[\mathrm{K}_1:\mathrm{D}_1],[\mathrm{T}_1=1])=\sum_{n=0}^{N-1} q^n\,(\mathrm{K}_1+\mathrm{D}_1\mathrm{G}_1^n)$

$\mathrm{SUM}_q(\mathrm{N},[\mathrm{K}_1:\mathrm{D}_1,\mathrm{K}_2:\mathrm{D}_2],[\mathrm{T}_1=1,\mathrm{T}_2=\mathrm{T}_1+2-\mathrm{P}])=\sum_{n=0}^{N-1} q^n(\mathrm{K}_2+\mathrm{D}_2\mathrm{G}_1^n)\,\nabla_q^P\,\mathrm{SUM}_q(\mathrm{n}+1,[\mathrm{K}_1:\mathrm{D}_1],[1])$

$Recursive\ define\ \mathrm{SUM}_q(\mathrm{N},PS,PT)$

$The\ Form:(\mathrm{T}_1+\mathrm{K}_1)(\mathrm{T}_2+\mathrm{K}_2)\dots(\mathrm{T}_M+\mathrm{K}_M)=\sum\prod_{i=1}^M \mathrm{X}_i$

7.1) $\mathrm{H}^q(\mathrm{g})=\mathrm{H}^q(\mathrm{g},\mathrm{PS},\mathrm{PT})=\sum_{\mathrm{X(T)}=g}\prod_{i=1}^M B_i\,,\mathrm{SUM}_q(N)=$

$$\begin{cases} \xrightarrow{\mathrm{Form}_1}\sum_{g=0}^{M}\mathrm{H}_1^q(\mathrm{g})\mathrm{G}_{\mathrm{N-1-g}}^{\mathrm{N+T_M-M}}=\sum_{g=0}^{M}\mathrm{H}_1^q(\mathrm{g})\mathrm{G}_{\mathrm{T_M-M+1+g}}^{\mathrm{N+T_M-M}},\mathrm{B_i}=\begin{cases} \mathrm{q}^{\{\mathrm{T_i-T_{i-1}}\}\mathrm{X_{T-1}}+1}\mathrm{G}_1^{\mathrm{T_i-X_{K-1}}}\mathrm{D_i};\mathrm{X_i}=\mathrm{T_i} \\ \mathrm{q}^{(\mathrm{T_i-T_{i-1}}-1)\mathrm{X_{T-1}}}\left(\mathrm{K_i}+\mathrm{G}_1^{\mathrm{X_{T-1}}}\mathrm{D_i}\right);\mathrm{X_i}=\mathrm{K_i}\end{cases} \\ \xrightarrow{\mathrm{Form}_2}\sum_{g=0}^{M}\mathrm{H}_2^q(\mathrm{g})\mathrm{G}_{\mathrm{N-1}}^{\mathrm{N+T_M-M+g}}=\sum_{g=0}^{M}\mathrm{H}_2^q(\mathrm{g})\mathrm{G}_{\mathrm{T_M-M+1+g}}^{\mathrm{N+T_M-M+g}},\mathrm{B_i}=\begin{cases} \mathrm{q}^{-(\mathrm{T_i-X_{K-1}})}\mathrm{G}_1^{\mathrm{T_i-X_{K-1}}}\mathrm{D_i};\mathrm{X_i}=\mathrm{T_i} \\ \mathrm{K_i}-\mathrm{q}^{-(\mathrm{T_i-X_{K-1}})}\mathrm{G}_1^{\mathrm{T_i-X_{K-1}}}\mathrm{D_i};\mathrm{X_i}=\mathrm{K_i}\end{cases} \\ \xrightarrow{\mathrm{Form}_3}\sum_{g=0}^{M}\mathrm{H}_3^q(\mathrm{g})\mathrm{G}_{\mathrm{N-1-g}}^{\mathrm{N+T_M-g}}=\sum_{g=0}^{M}\mathrm{H}_3^q(\mathrm{g})\mathrm{G}_{\mathrm{T_M+1}}^{\mathrm{N+T_M-g}},\mathrm{B_i}=\begin{cases} \mathrm{q}^{(\mathrm{T_i-T_{i-1}}-1)\mathrm{X_{T-1}}+1}\{(\mathrm{q}^{\mathrm{X_{T-1}}}\mathrm{G}_1^{\mathrm{T_i}}-\mathrm{q}^{\mathrm{T_i}}\,\mathrm{G}_1^{\mathrm{X_{T-1}}})\mathrm{D_i}-\mathrm{K_i}\mathrm{q}^{\mathrm{T_i}}\};\mathrm{X_i}=\mathrm{T_i} \\ \mathrm{q}^{(\mathrm{T_i-T_{i-1}}-1)\mathrm{X_{T-1}}}\left(\mathrm{K_i}+\mathrm{G}_1^{\mathrm{X_{T-1}}}\mathrm{D_i}\right);\mathrm{X_i}=\mathrm{K_i}\end{cases}\end{cases}$$

$\lim_{q\to1}H^q(g)=H(g)\,,\lim_{q\to1}SUM_q(N)=SUM(N)$

7.2) $\nabla_q^1\,\mathrm{SUM}_q(\mathrm{N},PS,[1,2\dots\mathrm{M}])=\prod_{i=1}^M(\mathrm{K_i}+\mathrm{D_i}\mathrm{G}_1^n)$

7.3) $SUM_q(N,[\dots PS\dots],[\dots T+1,T+2\dots T+M\dots]),K_i\ can\ exchange\ order.$

Derived from (3), Form2 is the simplest,Let X=TM-M-p:

7.4) $\nabla_q^P\,SUM(N)\xrightarrow{\mathrm{Form}_1}\sum_{g=0}^{M}\mathrm{H}_1^q(\mathrm{g})\mathrm{G}_{\mathrm{X+1+g}}^{\mathrm{N+X}}\,\mathrm{q}^{-\mathrm{gp}}\xrightarrow{\mathrm{Form}_2}\sum_{g=0}^{M}\mathrm{H}_2^q(\mathrm{g})\mathrm{G}_{\mathrm{X+1+g}}^{\mathrm{N+X+g}}\xrightarrow{\mathrm{Form}_3}\sum_{g=0}^{M}\mathrm{H}_3^q(\mathrm{g})\mathrm{G}_{X+M+1}^{\mathrm{N+X+M-g}}\mathrm{q}^{-\mathrm{gp}}$

$\mathrm{PS}=[q^{-T_1}G_1^{T_1},q^{-T_2}G_1^{T_2}\dots q^{-T_M}G_1^{T_M}],\to\mathrm{H}_2^q(\mathrm{g}<\mathrm{M})=0\to SUM_q(N)=\prod_{i=1}^M q^{-T_i}\,G_1^{T_i}G_{T_M+1}^{N+T_M}$

define $\mathrm{n}_{q-}=q^{-\mathrm{n}}G_1^{\mathrm{n}},\mathrm{n}_{q+}=q^{\mathrm{n}}G_1^{\mathrm{n}},\mathrm{n}_{q-\ or\ q+}!=1_q2_q\dots\mathrm{n}_q,0_{q-}!=0_{q+}!=1,\mathrm{Form}_2\to$

7.5) $SUM_q\left(N,\left[\mathrm{L}_{1_{q-}},\mathrm{L}_{2_{q-}}\dots\mathrm{L}_{\mathrm{P}_{q-}},PS\right],[L_1,L_2\dots L_P,PT]\right)=\prod_{i=1}^P\mathrm{L}_{\mathrm{i}_{q-}}SUM_q(N,PS,PT)\,,similar\ to\ 1.4)$

This indicates T1 can be greater than 1, T is defined in ℕ.

7.6) $SUM_q\left(N,\left[\mathrm{T}_{1_{q-}},\mathrm{T}_{2_{q-}}\dots\mathrm{T}_{\mathrm{M}_{q-}}\right],\mathrm{PT}\right)=\prod_{i=1}^M\mathrm{T}_{\mathrm{i}_{q-}}\,G_{T_M+1}^{N+T_M},similar\ to\ 1.8)$

7.7) $G_{\mathrm{M+1}}^{N+\mathrm{M}}=q^{\binom{\mathrm{M+1}}{2}}\sum_{g=0}^{M}\{q^{\binom{\mathrm{g+1}}{2}-(\mathrm{M+1})(\mathrm{M-g})}\sum_{1\le i_1<i_2<\dots<i_{M-g}\le M}q^{i_1}q^{i_2}\dots q^{i_{M-g}}\}\mathrm{G}_{1+\mathrm{g}}^{\mathrm{N}}$

[Proof]

It is difficult to derive directly from (2).

$SUM(N,[1_{q-},2_{q-}\dots\mathrm{M}_{q-}],[1,2\dots\mathrm{M}])\xrightarrow{7.6)}M_{q-}!\,G_{\mathrm{M+1}}^{N+\mathrm{M}}=SUM(N,[\mathrm{M}_{q-}\dots1_{q-}],[1,2\dots\mathrm{M}])$
$=\sum_{g=0}^{M}\mathrm{H}_1^q(\mathrm{g})\mathrm{G}_{1+\mathrm{g}}^{\mathrm{N}}$

$$B_i\xrightarrow{Form_1}\begin{cases} q^{X_{T-1}+1}G_1^{T_i-X_{K-1}}=q^{X_{T-1}+1}G_1^{i-X_{K-1}}=q^{X_{T-1}+1}G_1^{X_{T-1}+1}=q^{X_T}G_1^{X_T};X_i=T_i \\ K_i+G_1^{X_{T-1}}=q^{-(M+1-i)}G_1^{M+1-i}+G_1^{X_{T-1}}=\frac{q^{\mathrm{M+1-i+X_{T-1}-1}}}{q^{\mathrm{M+1-i}}(q-1)}=\frac{q^{\mathrm{M-X_{K-1}-1}}}{q^{\mathrm{M+1-i}}(q-1)};X_i=K_i\end{cases}$$

$H_1^q(g,T)=g_{q+}!;H_1^q(g,\sum K)=G_1^MG_1^{M-1}\dots G_1^{g+1}q^{-(\mathrm{M+1})(\mathrm{M-g})}\sum_{1\le i_1<i_2<\dots<i_{M-g}\le M}q^{i_1}q^{i_2}\dots q^{i_{M-g}}$

$\frac{\mathrm{H}_1^q(g,[\mathrm{M}_{q-}\dots2_{q-},1_{q-}],[1,2\dots\mathrm{M}])}{M_{q-}!}=\frac{q^{1+2+\cdots g}q^{1+2+\cdots M}\sum_{1\le i_1<i_2<\dots<i_{M-g}\le M}q^{i_1}q^{i_2}\dots q^{i_{M-g}}}{q^{(\mathrm{M+1})(\mathrm{M-g})}}$

q.e.d.

$G_{\mathrm{M+1}}^{N+\mathrm{M}}=\sum_{g=0}^{M}f(g)\mathrm{G}_{1+\mathrm{g}}^{\mathrm{N}}\to f(0)=1,f(M)=q^{\mathrm{M(M+1)}}$

$G_3^{N+2}=q^6\mathrm{G}_3^{\mathrm{N}}+q(q^1+q^2)\mathrm{G}_2^{\mathrm{N}}+\mathrm{G}_1^{\mathrm{N}}$

$G_4^{N+3}=q^{12}\mathrm{G}_4^{\mathrm{N}}+q^5(q^1+q^2+q^3)\mathrm{G}_3^{\mathrm{N}}+q^{-1}(q^1q^2+q^1q^3+q^2q^3)\mathrm{G}_2^{\mathrm{N}}+\mathrm{G}_1^{\mathrm{N}}$
$=q^{12}\mathrm{G}_4^{\mathrm{N}}+(q^6+q^7+q^8)\mathrm{G}_3^{\mathrm{N}}+(q^2+q^3+q^4)\mathrm{G}_2^{\mathrm{N}}+\mathrm{G}_1^{\mathrm{N}}=Equation\ derived\ from\ (2)$

$E_M^N=\sum_{\lambda_1+\lambda_2+\dots+\lambda_N=M,\lambda_i\ge0}1^{\lambda_1}2^{\lambda_2}..N^{\lambda_N}=S_2(\mathrm{N+M,N})$

$define\ E_{q\,M}^{\ N}=\sum_{\lambda_1+\lambda_2+\dots+\lambda_N=M,\lambda_i\ge0}G_1^{1\lambda_1}G_1^{2\lambda_2}\dots G_1^{N\lambda_N}=S_2^q(\mathrm{N+M,N})$

$define\ E_{q-\,M}^{\ \ N}=\sum_{\lambda_1+\lambda_2+\dots+\lambda_N=M,\lambda_i\ge0}G_1^{1\lambda_1}G_1^{2\lambda_2}\dots G_1^{N\lambda_N}q^{-(\lambda_1+2\lambda_2+\dots+N\lambda_N)}$

$define\ E_{q_M}^{\ N} \odot ([T_1, T_2 \dots], K) =$
$\sum_{\lambda_1..+\lambda_N=M,\lambda_i\geq 0} G_1^{1\lambda_1} \dots G_1^{N\lambda_N}\, G_1^{T_1+\lambda_1 K} G_1^{T_2+\lambda_1 K+\lambda_2 K} \dots G_1^{T_{q-1}+\lambda_1 K+\lambda_2 K+..+\lambda_{N-1} K}$

Table 7.1 : $S_2^q(\mathrm{N},\mathrm{M})$

	M=0	M=1	M=2	M=3	M=4
N=0	1				
N=1	0	1			
N=2	0	1	1		
N=3	0	1	$G_1^1 + G_1^2$	1	
N=4	0	1	$G_1^1 G_1^1 + G_1^1 G_1^2 + G_1^2 G_1^2$	$G_1^1 + G_1^2 + G_1^3$	1

7.8) $(G_1^N)^M = \sum_{g=1}^M g_{q+}!\, S_2^q(M,g) G_g^N q^{-g}$, similar to $\mathrm{N^M} = \sum_{\mathrm{g}=1}^{\mathrm{M}} \mathrm{g}!\, \mathrm{S}_2(\mathrm{M},\mathrm{g}) \binom{\mathrm{N}}{\mathrm{g}}$

[Proof]

$PS = [1_{q-}, 1_{q-} \dots 1_{q-}], PT = [2{,}3 \dots M], D_i = 1, B_i \xrightarrow{Form_1} \begin{cases} q^{X_{T-1}+1} G_1^{T_i - X_{K-1}}; X_i = T_i \\ 1_{q-} + G_1^{X_{T-1}} = q^{-1} G_1^{X_{T-1}+1}; X_i = K_i \end{cases}$

$H_1^q(g,T) = \frac{(g+1)_{q+}!}{q^{g+1}}, H_1^q(g,\sum K) = q^{-(\mathrm{M}-1-\mathrm{g})} E_{q_{M-1-g}}^{\ g+1} \to H_1^q(g) = q^{-\mathrm{M}}(g+1)_{q+}!\, E_{q_{M-1-g}}^{\ g+1}$

$1_{q-} + G_1^n = \frac{1}{\mathrm{q}} + \frac{q^{\mathrm{n}}-1}{\mathrm{q}-1} = q^{-1} G_1^{n+1} \to$

$q^{-1} \nabla_q^1 SUM_q(N, [1_{q-}, 1_{q-} \dots 1_{q-}], [2{,}3 \dots M]) \xrightarrow{7.5)} \nabla_q^1 SUM_q(N, [1_{q-}, 1_{q-} \dots 1_{q-}], [1{,}2 \dots M]) =$
$(q^{-1} G_1^N)^M$

$= q^{-1} \sum_{g=0}^{M-1} H_1^q(g) G_{1+g}^N q^{-g} = \sum_{g=0}^{M-1} q^{-\mathrm{M}}(g+1)_{q+}!\, S_2^q(M,g+1) G_{1+g}^N q^{-g}$

$(G_1^N)^M = q^{\mathrm{M}-1} \sum_{g=0}^{M-1} q^{-\mathrm{M}}(g+1)_{q+}!\, S_2^q(M,g+1) G_{1+g}^N q^{-g}$

q.e.d.

Table 7. 2 H(g) of PS=[1,1…1],PT=[1,2…M].

	SUM(N)	SUMq(N)
	PS=[1,1...1] PT=[1,2…M]	PS=[1_q,1_{q-}…1_q] PT=[1,2…M]
H1(g)	$\mathrm{g}!\, \mathrm{S}_2(\mathrm{M}+1,\mathrm{g}+1)$	$\mathrm{q}^{-(\mathrm{M}-\mathrm{g})} \mathrm{g}_{\mathrm{q}+}!\, \mathrm{S}_2^{\mathrm{q}}(\mathrm{M}+1,\mathrm{g}+1)$
H2(g)	$(-1)^{\mathrm{M}-\mathrm{g}} \mathrm{g}!\, E_{M-g}^{g}$	$\mathrm{q}^{-(\mathrm{M}-\mathrm{g})}(-1)^{(\mathrm{M}-\mathrm{g})} \mathrm{g}_{\mathrm{q}-}!\, E_{q-_{M-g}}^{\ g}$
H3(g)	$\left\langle {\mathrm{M} \atop \mathrm{M}-1-\mathrm{g}} \right\rangle = \mathrm{E}_{\mathrm{M}-\mathrm{g}}^{\mathrm{g}+1} \odot ([0{,}0 \dots], 1)$	$\mathrm{q}^{\binom{\mathrm{g}+1}{2}-(\mathrm{M}-\mathrm{g})} \mathrm{E}_{\mathrm{q}_{\mathrm{M}-\mathrm{g}}}^{\ \mathrm{g}+1} \odot ([0{,}0 \dots], 1)\ (*)$
	PS=[1,1…1] PT=[2,3…M]	PS=[1_q,1_{q-}…1_q] PT=[2,3…M]
H1(g)	$(\mathrm{g}+1)!\, \mathrm{S}_2(\mathrm{M},\mathrm{g}+1)$	$\mathrm{q}^{-\mathrm{M}}(\mathrm{g}+1)_{\mathrm{q}+}!\, \mathrm{S}_2^{\mathrm{q}}(\mathrm{M},\mathrm{g}+1)$
H2(g)	$(-1)^{\mathrm{M}-1-\mathrm{g}}(\mathrm{g}+1)!\, E_{M-1-g}^{g+1}$	$\mathrm{q}^{-\mathrm{M}}(-1)^{(\mathrm{M}-1-\mathrm{g})}(\mathrm{g}+1)_{\mathrm{q}-}!\, E_{q-_{M-1-g}}^{\ g+1}$
H3(g)	$\left\langle {\mathrm{M} \atop \mathrm{M}-1-\mathrm{g}} \right\rangle = \mathrm{E}_{\mathrm{M}-1-\mathrm{g}}^{\mathrm{g}+1} \odot ([1{,}1 \dots], 1)$	$\mathrm{q}^{\binom{\mathrm{g}+2}{2}-1-(\mathrm{M}-1-\mathrm{g})} \mathrm{E}_{\mathrm{q}_{\mathrm{M}-1-\mathrm{g}}}^{\ \mathrm{g}+1} \odot ([1{,}1 \dots], 1)$

[Proof of (*)]

$H_3^q(g) \to B_i \xrightarrow{Form_3} \begin{cases} q\{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) - 1_{q-} q^{T_i}\}; X_i = T_i \\ 1_{q-} + G_1^{X_{T-1}} = q^{-1} G_1^{X_{T-1}+1}; X_i = K_i \end{cases}$

$H_3^q(g,\sum K) = \mathrm{q}^{-(\mathrm{M}-\mathrm{g})} E_{q_{M-g}}^{\ g+1}$

$H_3^q(g,T) = \prod q\{(q^{X_{T-1}} G_1^i - q^i G_1^{X_{T-1}}) - \mathrm{q}^{-1} q^i\} = \prod(\mathrm{q}-1)^{-1}\mathrm{q}\{\mathrm{q}^{X_{T-1}}(\mathrm{q}^{\mathrm{i}}-1) - \mathrm{q}^{\mathrm{i}}(\mathrm{q}^{X_{T-1}}-1) -$
$\mathrm{q}^{\mathrm{i}-1}(\mathrm{q}-1)\}$

$= \prod \frac{q^i - q^{1+X_{T-1}}}{\mathrm{q}-1} = \prod q^{1+X_{T-1}} \frac{q^{X_{K-1}}-1}{\mathrm{q}-1} = \prod q^{X_T} \frac{q^{X_{K-1}}-1}{\mathrm{q}-1} = q^{1+2+\dots g} \prod G_1^{X_{K-1}}$

$\prod G_1^{X_{K-1}} = \mathrm{G}_1^{\lambda_1} \mathrm{G}_1^{\lambda_1+\lambda_2} \dots \mathrm{G}_1^{\lambda_1+\lambda_2+\dots+\lambda_{g-1}}$

q.e.d.

7.9)Induction $\to \mathrm{PT} = [1{,}2 \dots \mathrm{M}], q^{-\binom{g+1}{2}} H_1^q(\mathrm{g}) = q^{\binom{g+1}{2}} \sum_{\mathrm{k}=0}^{\mathrm{M}} H_2^q(\mathrm{k}) \mathrm{G}_{\mathrm{g}}^{\mathrm{k}}$

7.10)Induction $\to H_1^q(0) = \sum_{\mathrm{k}=0}^{\mathrm{M}} H_2^q(\mathrm{k})$

Other situations are relatively complex and no conclusions similar to 2.2) have been drawn.

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$$SUM_q\left(N,\left[1_{q-},1_{q-}\dots 1_{q-}\right],[1,3\dots 2M-1]\right)$$
$$=\sum_{n_M=0}^{N-1} q^{n_M-1}G_1^{n_M+1}\dots\sum_{n_2=0}^{n_3} q^{n_2-1}G_1^{n_2+1}\sum_{n_1=0}^{n_2} q^{n_1-1}G_1^{n_1+1}$$
$$=q^{-2M}\sum_{1\le I_1\le I_2\le\dots\le I_M\le N} G_1^{I_1}G_1^{I_2}\dots G_1^{I_M}q^{I_1+I_2+\dots+I_M}=$$
$$q^{-2M}\sum_{\lambda_1+\lambda_2+\dots+\lambda_N=M,\lambda_i\ge 0} G_1^{1}{}^{\lambda_1}G_1^{2}{}^{\lambda_2}\dots G_1^{N}{}^{\lambda_N}q^{\lambda_1+2\lambda_2+\dots+N\lambda_N}$$

$$SUM_q\left(N,\left[1_{q-},2_{q-}\dots M_{q-}\right],[1,3\dots 2M-1]\right)$$
$$=\sum_{n_M=0}^{N-1} q^{n_M-M}G_1^{n_M+M}\dots\sum_{n_2=0}^{n_3} q^{n_2-2}G_1^{n_2+2}\sum_{n_1=0}^{n_2} q^{n_1-1}G_1^{n_1+1}$$
$$=q^{-(M+1)M}\sum_{1\le I_1<I_2<\dots<I_M\le N+M-1} G_1^{I_1}G_1^{I_2}\dots G_1^{I_M}q^{I_1+I_2+\dots+I_M}$$

7.11) $P_1<T_1, P_2<T_2, T_1>0, T_2>0, PT=[1,2,3\dots T_1+T_2+2-P_1-P_2]$

$$PS=\left[1_{q-},2_{q-}\dots(T_1-P_1)_{q-},\frac{K_1G_1^{T_1+1-P_1}}{q^{1-P_1}G_1^{T_1}D_1},1_{q-},2_{q-}\dots(T_2-P_2)_{q-},\frac{K_2G_1^{T_2+1-P_2}}{q^{1-P_2}G_1^{T_2}D_2}\right]$$
$$\sum_{n=0}^{N-1} q^n\,\nabla^{P_1}SUM_q(n+1,[K_1{:}D_1],[T_1])\,\nabla^{P_2}SUM_q(n+1,[K_2{:}D_2],[T_2])$$
$$=\frac{1}{(T_1-P_1)_{q-}!}\frac{q^{1-P_1}G_1^{T_1}D_1}{G_1^{T_1-P_1+1}}\frac{1}{(T_2-P_2)_{q-}!}\frac{q^{1-P_2}G_1^{T_2}D_2}{G_1^{T_2-P_2+1}}SUM_q(N,PS,PT)$$

7.12) $\sum_{n=0}^{N-1} q^n\prod_{i=1}^{M}(K_i+q^nD_i)=\sum_{g=0}^{M}H_1^q(g)G_{g+1}^N,\ B_i=\begin{cases}q^{X_T}(q^{X_T}-1)D_i; X\in T\\ K_i+q^{X_T}D_i; X\in K\end{cases}$

7.13) Induction $\rightarrow \sum_{n=0}^{N-1}\prod_{i=1}^{M}(K_i+q^nD_i)=\sum_{g=1}^{M}f(g)G_g^N+N\prod_{i=1}^{M}K_i$

$$f(g)=\sum_{X(T)=g}\prod_{i=1}^{M}\begin{cases}1; X\in T, X_{T-1}=0\\ q^{X_{T-1}}(q^{X_{T-1}}-1)D_i; X\in T, X_{T-1}>0\\ K_i; X\in K, X_{T-1}=0\\ K_i+q^{X_{T-1}-1}D_i; X\in K, X_{T-1}>0\end{cases}$$

Similar to Section 6, P=N+T_M-M,Q=N-1,define A^q(P,Q,M)

$$A_1^q(P,Q,M)=\begin{bmatrix}G_Q^P & \cdots & G_{Q-M}^P\\ \vdots & \ddots & \vdots\\ G_{Q+M}^{P+M} & \cdots & G_Q^{P+M}\end{bmatrix}, A_2^q(P,Q,M)=\begin{bmatrix}G_Q^P & \cdots & G_Q^{P+M}\\ \vdots & \ddots & \vdots\\ G_{Q+M}^{P+M} & \cdots & G_{Q+M}^{P+2M}\end{bmatrix}, A_3^q(P,Q,M)=$$
$$\begin{bmatrix}G_Q^{P+M} & \cdots & G_{Q-M}^P\\ \vdots & \ddots & \vdots\\ G_{Q+M}^{P+2M} & \cdots & G_Q^{P+M}\end{bmatrix}$$

Turn it into an upper triangular matrix:

7.14) $\left\|A_2^q(P,Q,M)\right\|=\frac{G_Q^P}{G_0^P}\frac{G_{Q+1}^{P+2}}{G_1^{P+2}}\frac{G_{Q+2}^{P+4}}{G_2^{P+4}}\dots\frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}}q^{(P+1)+2(P+2)+\dots+M(P+M)}=\left\|A_2^q(P,P-Q,M)\right\|$

7.15) $\left\|A_2^q(P,0,M)\right\|=q^{(P+1)+2(P+2)+\dots+M(P+M)};\ \left\|A_2^q(P,1,M)\right\|=G_{1+M}^{P+M}q^{(P+1)+2(P+2)+\dots+M(P+M)}$

$$A_2^q(P,Q,M)\xrightarrow{Row_{i+1}:=Row_{i+1}-Row_i\text{ and repeat}}\text{Change the first column to to }[G_Q^P,q^{Q+1}G_{Q+1}^P,q^{2(Q+2)}G_{Q+2}^P\dots]^T$$
$$q^{(Q+1)+2(Q+2)\dots+M(Q+M)}\begin{bmatrix}G_Q^P & \cdots & G_Q^{P+M}\\ \vdots & \ddots & \vdots\\ G_{Q+M}^P & \cdots & G_{Q+M}^{P+M}\end{bmatrix}\xrightarrow{\text{transpose}}q^{\{\dots\}}\begin{bmatrix}G_Q^P & \cdots & G_{Q+M}^P\\ \vdots & \ddots & \vdots\\ G_Q^{P+M} & \cdots & G_{Q+M}^{P+M}\end{bmatrix}=q^{\{\dots\}}A_1(P,P-Q,M)\rightarrow$$
$$\left\|A_1^q(P,P-Q,M)\right\|=\frac{G_Q^P}{G_0^P}\frac{G_{Q+1}^{P+2}}{G_1^{P+2}}\frac{G_{Q+2}^{P+4}}{G_2^{P+4}}\dots\frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}}q^{(P-Q)+2(P-Q)+\dots+M(P-Q)}$$

$$A_1^q(P,Q,M)=\begin{bmatrix}G_Q^P & \cdots & G_{Q-M}^P\\ \vdots & \ddots & \vdots\\ G_{Q+M}^{P+M} & \cdots & G_Q^{P+M}\end{bmatrix}\xrightarrow{use\ (2), col_i=col_i+q^x col_{i+1}\text{ and repeat}}\begin{bmatrix}G_Q^{P+M} & \cdots & G_{Q-M}^P\\ \vdots & \ddots & \vdots\\ G_{Q+M}^{P+2M} & \cdots & G_Q^{P+M}\end{bmatrix}=A_3^q(P,Q,M)$$

7.16) $\left\|A_{1,3}^q(P,Q,M)\right\|=\frac{G_Q^P}{G_0^P}\frac{G_{Q+1}^{P+2}}{G_1^{P+2}}\frac{G_{Q+2}^{P+4}}{G_2^{P+4}}\dots\frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}}q^{Q\binom{M+1}{2}};\ \left\|A_{1,3}^q(P,0,M)\right\|=1;\ \left\|A_{1,3}^q(P,1,M)\right\|=q^{\binom{M+1}{2}}G_{1+M}^{P+M}$

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So $H^q(\mathrm{g})$can be calculated by the matrix .

7. Application of SUMq(N) and Hq(g)

8.1)$q^{Mn}=\sum_{g=0}^{M} q^{-(M-g)}\begin{bmatrix}M\\g\end{bmatrix}_{q^{-1}}\prod_{i=1}^{g}(q^n-q^{-i})=\sum_{\mathrm{g}=0}^{\mathrm{M}} \mathrm{G}_{\mathrm{g}}^{\mathrm{M}}\prod_{i=0}^{g-1}(q^n-q^i)$

$=\sum_{\mathrm{g}=0}^{\mathrm{M}} \mathrm{G}_{\mathrm{g}}^{\mathrm{M}}(-1)^g q^{\binom{\mathrm{g}}{2}}G_{\mathrm{M}}^{\mathrm{n+M-g}}=\mathrm{q}^{-\binom{M+1}{2}}\sum_{\mathrm{g}=0}^{\mathrm{M}}\mathrm{q}^{\binom{M-g}{2}}\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}\prod_{i=0}^{g-1}(\mathrm{q}^{n+g-i}-1)$

[Proof]

$q^{Mn}=\nabla\ SUM_q(N,[1,1\ldots 1]:q-1,[1,2\ldots \mathrm{M}])$

$=\sum_{g=0}^{M} H_1^q(g)G_{\mathrm{g}}^{\mathrm{n}}q^{-\mathrm{g}}=\sum_{g=0}^{M} H_2^q(g)G_{\mathrm{g}}^{\mathrm{n+g}}=\sum_{g=0}^{M} H_3^q(g)G_{\mathrm{M}}^{\mathrm{n+M-g}}q^{-\mathrm{g}}$

$H_2^q(g,T)=(q-1)^{\mathrm{g}}\mathrm{g}_{q-}!$

$H_2^q(g,\sum K)=\sum\prod\left\{1-\frac{q^{i-\mathrm{X_{K-1}}-1}}{q^{i-\mathrm{X_{K-1}}}(q-1)}(q-1)\right\}=\sum\prod\frac{1}{q^{i-\mathrm{X_{K-1}}}}=\sum\prod\frac{1}{q^{1+\mathrm{X_{T-1}}}}=q^{-(M-g)}\sum\prod\frac{1}{q^{\mathrm{X_T}}}$

MacMahon [9] observes $\mathrm{G}_{\mathrm{K}}^{\mathrm{M}}=\sum_{w\,\in\,\Omega(0^{M-K},1^{K})} q^{inv(w)}$, that is, all words $\mathrm{w}=\mathrm{w}_1\mathrm{w}_2\ldots\mathrm{w}_M$ with $\mathrm{M}-$ K zeroes and K ones, and $\mathrm{inv}(\cdot)$ denotes the inversion statistic defined

$\rightarrow\ H_2^q(g,\sum K)=q^{-(M-g)}\begin{bmatrix}M\\M-g\end{bmatrix}_{q^{-1}}=q^{-(M-g)}\begin{bmatrix}M\\g\end{bmatrix}_{q^{-1}}$ → The first equation.

$H_1^q(g,T)=(q-1)^{\mathrm{g}}\mathrm{g}_{q+}!\,;H_1^q(g,\sum K)=\sum\left\{1+\frac{q^{\mathrm{X_{T-1}}-1}}{(q-1)}(q-1)\right\}=\sum\prod \mathrm{q}^{\mathrm{X_{T-1}}}=\sum\prod\mathrm{q}^{\mathrm{X_T}}=\mathrm{G}_{\mathrm{M-g}}^{\mathrm{M}}=\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}$

$q^{Mn}=\sum_{\mathrm{g}=0}^{\mathrm{M}}(q-1)^{\mathrm{g}}\mathrm{g}_{q+}!\,\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}q^{-\mathrm{g}}\mathrm{G}_{\mathrm{g}}^{\mathrm{n}}=\sum_{\mathrm{g}=0}^{\mathrm{M}} q^{-\mathrm{g}}\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}q^{\binom{\mathrm{g}+1}{2}}\prod_{i=0}^{g-1}(q^{n-i}-1)\rightarrow$ *The* second equation

$H_3^q(g,\sum K)=\mathrm{G}_{\mathrm{g}}^{\mathrm{M}},H_3^q(g,T)=(-1)^g q^{\binom{\mathrm{g}+1}{2}},q^{Mn}=\sum_{\mathrm{g}=0}^{\mathrm{M}}\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}(-1)^g q^{\binom{\mathrm{g}+1}{2}}q^{-\mathrm{g}}G_{\mathrm{M}}^{\mathrm{n+M-g}}\rightarrow$

The third equation

$\nabla\ SUM_q(N,[\mathrm{q}^1:(q-1)\mathrm{q}^1,\mathrm{q}^2:(q-1)\mathrm{q}^2\ldots\mathrm{q}^M:(q-1)\mathrm{q}^M],[1,2\ldots M])=q^{n+1}q^{n+2}\ldots q^{n+M}=q^{Mn+\binom{M+1}{2}}$

$B_i\xrightarrow{Form_2}\begin{cases}\mathrm{q}^{-(\mathrm{T_i}-\mathrm{X_{K-1}})}\{\mathrm{q}^{(\mathrm{T_i}-\mathrm{X_{K-1}})}-1\}\mathrm{q}^{T_i}=\mathrm{q}^{T_i}-\mathrm{q}^{\mathrm{X_{K-1}}}=\mathrm{q}^{\mathrm{X_{K-1}}}(\mathrm{q}^{T_i-\mathrm{X_{K-1}}}-1);X_i=T_i\\ \mathrm{K_i}-\mathrm{q}^{-(\mathrm{T_i}-\mathrm{X_{K-1}})}\mathrm{G}_1^{\mathrm{T_i}-\mathrm{X_{K-1}}}\mathrm{D_i}=\mathrm{q}^{\mathrm{X_{K-1}}};X_i=K_i\end{cases}$

$H_2^q(g,K)=\mathrm{q}^0\mathrm{q}^1\ldots\mathrm{q}^{\mathrm{M-g-1}}=\mathrm{q}^{\binom{M-g}{2}};H_2^q(g,\sum T)=\prod_{i=0}^{g}(\mathrm{q}^i-1)\,\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}\rightarrow$ *The fouth* equation

$B_i\xrightarrow{Form_3}\begin{cases}\mathrm{q}^1\{(\mathrm{q}^{\mathrm{X_{T-1}}}\mathrm{G}_1^{\mathrm{T_i}}-\mathrm{q}^{\mathrm{T_i}}\,\mathrm{G}_1^{\mathrm{X_{T-1}}})\mathrm{D_i}-\mathrm{K_i}\mathrm{q}^{\mathrm{T_i}}\}=-\mathrm{q}^{i+\mathrm{X_T}};X_i=T_i\\ \mathrm{K_i}+\mathrm{G}_1^{\mathrm{X_{T-1}}}\mathrm{D_i}=\mathrm{q}^{i+\mathrm{X_{T-1}}}=\mathrm{q}^{i+\mathrm{X_T}};X_i=K_i\end{cases}$ *, Every item of* $\mathrm{H}_3^{\mathrm{q}}(\mathrm{g})$ *has* $\mathrm{q}^{1+2+\cdots+M}$

$\prod(X\in T)$ *of* $\mathrm{H}_3^{\mathrm{q}}(\mathrm{g})q^{-\binom{M+1}{2}}=(-1)^g\mathrm{q}^{1+2+\ldots+g},\sum\prod(X\in K)$ *of* $\mathrm{H}_3^{\mathrm{q}}(\mathrm{g})q^{-\binom{M+1}{2}}=\sum\prod\mathrm{q}^{\mathrm{X_T}}=\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}$

$q^{Mn}=q^{-\binom{M+1}{2}}\sum_{\mathrm{g}=0}^{\mathrm{M}}(-1)^g q^{\binom{M+1}{2}}q^{\binom{\mathrm{g}+1}{2}}\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}\,G_{\mathrm{M}}^{\mathrm{n+M-g}}q^{-g}=\sum_{\mathrm{g}=0}^{\mathrm{M}}\mathrm{G}_{\mathrm{g}}^{\mathrm{M}}(-1)^g q^{\binom{\mathrm{g}}{2}}G_{\mathrm{M}}^{\mathrm{n+M-g}}=$

The third equation

q.e.d.

$SUM_q(N,[1,1\ldots 1]:0,[1,3\ldots 2\mathrm{M}-1])\xrightarrow{Form_2}\mathrm{G}_{\mathrm{M}}^{\mathrm{N+M-1}}\rightarrow\sum_{0\le\lambda_1\le\ldots\le\lambda_M\le N}\mathrm{q}^{\lambda_1+\ldots+\lambda_M}=\mathrm{G}_{\mathrm{M}}^{\mathrm{N+M}}\rightarrow$

record at [6]: 3.1

8.2) $\sum_{0\le\lambda_1\le\lambda_2\le\ldots\le\lambda_N\le M-N}\mathrm{q}^{\lambda_1+\lambda_2+\ldots+\lambda_N}=\mathrm{G}_{\mathrm{M-N}}^{\mathrm{M}}=\mathrm{G}_{\mathrm{N}}^{\mathrm{M}}=\sum_{w\,\in\,\Omega(0^{N},1^{M-N})}q^{inv(w)}$

8.3) $\begin{bmatrix}N+M-1\\M\end{bmatrix}_{q^2}=\sum_{\mathrm{g}=0}^{\mathrm{M}}(-1)^g q^{g^2}\begin{bmatrix}M\\g\end{bmatrix}_{q^2}G_{\mathrm{2M}}^{\mathrm{N+2M-1-g}}$

[Proof]

$SUM_q(N,[\mathrm{q}^1:(q-1)\mathrm{q}^1,\mathrm{q}^3:(q-1)\mathrm{q}^3\ldots\mathrm{q}^{T_M}:(q-1)\mathrm{q}^{T_M}],[1,3\ldots 2M-1])$

$=\sum_{n_M=0}^{N-1}q^{n_M}q^{n_M+2M-1}\ldots\sum_{n_1=0}^{n_2}q^{n_1}q^{n_1+1}=q^{M^2}\sum_{n_M=0}^{N-1}q^{2n_M}\ldots\sum_{n_1=0}^{n_2}q^{2n_1}$

$=q^{M^2}\sum_{0\le\lambda_1\le\lambda_2\le\ldots\le\lambda_M\le N-1}\mathrm{q}^{2(\lambda_1+\lambda_2+\ldots+\lambda_M)}=q^{M^2}\begin{bmatrix}N+M-1\\M\end{bmatrix}_{q^2}=\sum_{\mathrm{g}=0}^{\mathrm{M}}\mathrm{H}_3^{\mathrm{q}}(\mathrm{g})G_{\mathrm{2M}}^{\mathrm{N+2M-1-g}}$

$B_i\xrightarrow{Form_3}\begin{cases}\mathrm{q}^{1+\mathrm{X_{T-1}}}\{(\mathrm{q}^{\mathrm{X_{T-1}}}\mathrm{G}_1^{\mathrm{T_i}}-\mathrm{q}^{\mathrm{T_i}}\,\mathrm{G}_1^{\mathrm{X_{T-1}}})\mathrm{D_i}-\mathrm{K_i}\mathrm{q}^{\mathrm{T_i}}\}=-\mathrm{q}^{\mathrm{T_i}-1+2\mathrm{X_T}};\mathrm{X_i}=\mathrm{T_i}\\ \mathrm{q}^{\mathrm{X_{T-1}}}(\mathrm{K_i}+\mathrm{G}_1^{\mathrm{X_{T-1}}}\mathrm{D_i})=\mathrm{q}^{\mathrm{T_i}+2\mathrm{X_{T-1}}}=\mathrm{q}^{\mathrm{T_i}+2\mathrm{X_T}};\mathrm{X_i}=\mathrm{K_i}\end{cases}$ *, Every item of* $\mathrm{H}_3^{\mathrm{q}}(\mathrm{g})$ *has* $\mathrm{q}^{1+3+\cdots+2M-1}$

$H_3^q(g)=(-1)^g q^{(1+3+\cdots+(2M-1))-g+2\binom{\mathrm{g}+1}{2}}\sum\prod_{X\in K}\mathrm{q}^{2\mathrm{X_T}}=(-1)^g q^{M^2+g^2}\begin{bmatrix}M\\g\end{bmatrix}_{q^2}$

q.e.d.

8.4) $\sum_{1 \le i_1 < i_2 < \ldots < i_M \le N+M-1} q^{i_1} q^{i_2} \ldots q^{i_M} = q^{\frac{M(M+1)}{2}} \sum_{g=0}^{M} (-1)^g q^{\frac{g^2}{2}} \begin{bmatrix} M \\ g \end{bmatrix}_q \begin{bmatrix} N+2M-1-g \\ 2M \end{bmatrix}_{q^{\frac{1}{2}}}$

[Proof]

$SUM_q(N, [q^2 : (q-1)q^2, q^4 : (q-1)q^4 \ldots q^{2M} : (q-1)q^{2M}], [1,3 \ldots 2M-1])$

$= \sum_{n_M=0}^{N-1} q^{n_M} q^{n_M+2M} \ldots \sum_{n_2=0}^{n_3} q^{n_2} q^{n_2+4} \sum_{n_1=0}^{n_2} q^{n_1} q^{n_1+2} =$

$\sum_{n_M=0}^{N-1} q^{2(n_M+M)} \ldots \sum_{n_2=0}^{n_3} q^{2(n_2+2)} \sum_{n_1=0}^{n_2} q^{2(n_1+1)}$

$B_i \xrightarrow{Form_3} \begin{cases} -q^{2i-1+2X_T}; X_i = T_i \\ q^{2i+2X_T}; X_i = K_i \end{cases}, Every\ item\ of\ H_3^q(g)\ has\ q^{2+4+\cdots+2M}$

$H_3^q(g) = (-1)^g q^{2(1+2+\ldots+M)-g+2\binom{g+1}{2}} \sum \prod_{X \in K} q^{2X_T} = (-1)^g q^{M(M+1)+g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}$

$SUM_q(N) = \sum_{g=0}^{M} (-1)^g q^{M(M+1)+g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-1-g} = \sum_{1 \le i_1 < i_2 < \ldots < i_M \le N+M-1} q^{2i_1} q^{2i_2} \ldots q^{2i_M}$

q.e.d.

8.5) $\sum_{k_r=1}^{N} \ldots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+\ldots+k_r} \nabla SUM_q(k_i, PS, [1,2 \ldots M]) = \nabla^{i-r} SUM_q(N, PS, [T_j = j + (i - 1)])$

8.6) $(K+qD)^M = (q-1)D \sum_{g=0}^{M-1} (K+D)^g (K+qD)^{M-1-g} + (K+D)^M$

[Proof]

$PS = [K+D, K+D \ldots K+D] : (q-1)D, PT = [1,2 \ldots M], B_i \xrightarrow{Form_1} \begin{cases} q^{X_T}(q^{X_T}-1)D; X_i = T_i \\ K + q^{X_{T-1}} D; X_i = K_i \end{cases}$

$H_1^q(1) = q(q-1)D \sum_{a+b=M-1, a,b \ge 0} (K+D)^a (K+qD)^b$

$SUM_q(2) - SUM_q(1) = q \prod_{i=1}^{M} (K + D + G_1^1(q-1)D) = q \prod_{i=1}^{M} (K+qD) = H_1^q(1) + H_1^q(0) G_1^2 - H_1^q(0) G_1^1$

$= H_1^q(1) + q H_1^q(0) = H_1^q(1) + q(K+D)^M$

q.e.d.

8. Multiparameter Formal Calcution

$2-$ parameters Formal Calculation calculate *nested summation of* $K_i + \binom{n}{1} D_{1,i} + \binom{n}{2} D_{2,i}$

SUM(N, PS, PT) changed to

SUM(N, [$K_1, K_2 \ldots K_M$], [$T_{1,1}$:$D_{1,1} \ldots T_{1,M}$:$D_{1,M}$], [$T_{2,1}$: $D_{2,1} \ldots T_{2,M}$: $D_{2,M}$])=SUM(N,PS,PT_1,PT_2), always $T_i = T_{1,i} = T_{2,i}$

Use the Form $(K_1 + T_{1,1} + T_{2,1})(K_2 + T_{1,2} + T_{2,2}) \ldots (K_M + T_{1,M} + T_{2,M}) = \sum \prod_{i=1}^{M} X_i$

$X(PT_1) = count\ of\ X \in PT_1, X(PT_2) = count\ of\ X \in PT_2; X(PT) = X(PT_1) + 2X(PT_2)$

$X_{PT_1} = count\ of\ \{X_1, X_2 \ldots X_i\} \in PT_1, X_{PT_2} = count\ of\ \{X_1, X_2 \ldots X_i\} \in PT_2, X_{PT} = X_{PT_1} + 2X_{PT_2}$

9.1) $g = X(T), H(g) = \sum_{\prod X_i\ with\ X(T)=g} \prod_{i=1}^{M} B_i, SUM(N, PS, PT_1, PT_2) =$

$\xrightarrow{Form_1} \sum_{g=0}^{2M} H_1(g) \binom{N+T_M-M}{T_M-M+1+g}, B_i = \begin{cases} K_i + X_{PT} D_{1,i} + \binom{X_{PT}}{2} D_{2,i}; X_i = K_i \\ (T_i - i + X_{PT}) D_{1,i} + (T_i - i + X_{PT})(X_{PT} - 1) D_{2,i}; X_i = PT_{1,i} \\ \binom{T_i - i + X_{PT}}{2} D_{2,i}; X_i = PT_{2,i} \end{cases}$

$\xrightarrow{Form_2} \sum_{g=0}^{2M} H_2(g) \binom{N+T_M-M+g}{T_M-M+1+g}, B_i =$

$\begin{cases} K_i - (T_i - i + X_{PT} + 1) D_{1,i} + \binom{T_i - i + X_{PT} + 2}{2} D_{2,i}; X_i = K_i \\ (T_i - i + X_{PT}) D_{1,i} - (T_i - i + X_{PT})(T_i - i + X_{PT} + 1) D_{2,i}; X_i = PT_{1,i} \\ \binom{T_i - i + X_{PT}}{2} D_{2,i}; X_i = PT_{2,i} \end{cases}$

$\xrightarrow{Form_3} \sum_{g=0}^{2M} H_3(g) \binom{N+T_M+M-g}{T_M+M+1} B_i =$

$\begin{cases} K_i + X_{PT-1} D_{1,i} + \binom{X_{PT-1}}{2} D_{2,i}; X_i = K_i \\ -2K_i + (T_i + i - 1 - 2X_{PT-1}) D_{1,i} + (T_i + i - X_{PT-1}) X_{PT-1} D_{2,i}; X_i = PT_{1,i} \\ K_i - (T_i + i - 1 - X_{PT-1}) D_{1,i} + \binom{T_i + i - X_{PT-1}}{2} D_{2,i}; X_i = PT_{2,i} \end{cases}$

[Proof]

[3] has obtained Form1 and ignores Form2,3. All that remains is some technical work.Here proves Form2.

$(*)\ \sum_{n=0}^{N-1} n\binom{n+K}{M} = (M+1)\binom{N+K+1}{M+2} - (K+1)\binom{N+K}{M+1}$

$(**)\ \sum_{n=0}^{N-1}\binom{n}{2}\binom{n+K}{M} = \binom{M+2}{2}\binom{N+K+2}{M+3} - (M+1)(K+2)\binom{N+K+1}{M+2} + \binom{K+2}{2}\binom{N+K}{M+1},\quad M > K$

$PS1 = [PS, K_{M+1}], PT1_1 = [PT_1, T_{M+1} = T_M + 2 - p{:}\, D_{1,M+1}], PT1_2 = [PT_2, T_{M+1} = T_M + 2 - p{:}\, D_{2,M+1}]$

$Use\ induction, it\ can\ be\ verified\ when\ M = 1, suppose\ SUM(N) = \sum_{g=0}^{2M} H_2(g)\binom{N+T_M-M+g}{T_M-M+1+g}$

$Let\ A = T_M - M + 1 - p = T_{M+1} - (M+1)$

$SUM(N, PS1, PT1_1, PT1_2)$

$= \sum_{n=0}^{N-1}(K_{M+1} + nD_{1,M+1} + \binom{n}{2}D_{2,M+1})\ \nabla^p \sum_{g=0}^{2M} H_2(g)\binom{n+1+T_M-M+g}{T_M-M+1+g}$

$= \sum_{n=0}^{N-1}\sum_{g=0}^{2M}(K_{M+1} + nD_{1,M+1} + \binom{n}{2}D_{2,M+1})H_2(g)\binom{n+A+g}{A+g}$

$= \sum_{g=0}^{2M}(K_{M+1} - (A+g+1)D_{1,M+1} + \binom{A+g+2}{2}D_{M+1,2})H_2(g)\binom{N+A+g}{A+g+1}$ ①

$+ \sum_{g=0}^{2M}((A+g+1)D_{1,M+1} - (A+g+1)(A+g+2)D_{2,M+1})H_2(g)\binom{N+A+g+1}{A+g+2}$ ②

$+ \sum_{g=0}^{2M}\binom{A+g+2}{2}D_{2,M+1}H_2(g)\binom{N+A+g+2}{A+g+3}$ ③

① $\binom{N+A+g}{A+g+1} = \binom{N+T_{M+1}-(M+1)+g}{T_{M+1}-(M+1)+1+g}, X_i = K_i, X_{PT}\ unchanged, X_{PT} = g, T_{M+1} = T_i, M+1 = i$

② $\binom{N+A+g+1}{A+g+2} = \binom{N+T_{M+1}-(M+1)+(g+1)}{T_{M+1}-(M+1)+1+(g+1)}, X_i = T_{1,i}, X_{PT} = g+1, T_{M+1} = T_i, M+1 = i$

③ $\binom{N+A+g+2}{A+g+3} = \binom{N+T_{M+1}-(M+1)+(g+2)}{T_{M+1}-(M+1)+1+(g+2)}, X_i = T_{2,i}, X_{PT} = g+2, T_{M+1} = T_i, M+1 = i$

Form2 has been proved. Use the following two formulas to prove Form3.

$\sum_{n=0}^{N-1} n\binom{n+K}{M} = -(K-M)\binom{N+K+2}{M+3} + (2K-M+1)\binom{N+K+1}{M+3} - (K+1)\binom{N+K}{M+3}$

$\sum_{n=0}^{N-1}\binom{n}{2}\binom{n+K}{M} = \binom{M-K}{2}\binom{N+K+2}{M+3} + (K+2)(M-K)\binom{N+K+1}{M+3} + \binom{K+2}{2}\binom{N+K}{M+3}$

q.e.d.

According to this method, it can be extended to multiparameter SUM(N) and multiparameter SUMq(N).

$1. Serie_i{:}\ \mathrm{K_i} + \binom{n}{1}\mathrm{D_{1,i}} + \binom{n}{2}\mathrm{D_{2,i}} + \binom{n}{3}\mathrm{D_{3,i}} + \cdots or\ \mathrm{K_i} + \mathrm{G_1^n D_{1,i}} + \mathrm{G_2^n D_{2,i}} + \mathrm{G_3^n D_{3,i}} + \cdots; T_{1,i} = T_{2,i} = T_{3,i} \ldots$

$2. X(PT) = X(PT_1) + 2X(PT_2) + 3X(PT_3) \ldots; X_{PT} = X_{PT_1} + 2X_{PT_2} + 3X_{PT_3} \ldots$

$3. Use\ X_{PT}\ or\ X_{PT-1}\ Indicate\ the\ positions.$

9. A theorem of symmetry

$There\ have\ 2.1) \to H_1(g, PT, \mathrm{PT}) = \prod_{i=1}^{M} T_i\binom{M}{g} = \prod_{i=1}^{M} T_i\binom{M}{g, M-g}$

$Promote\ it, the\ Set\ \{T_1, T_2 \ldots T_M\}\ come\ form\ p\ source\ S_1, S_2 \ldots S_P, T_i \in S_j\ means\ T_i\ come\ from\ source\ j$

$define\ Diff(S_x, S_y) = Z_{x,y}, x \neq y; Diff(S_y, S_x) = -Diff(S_x, S_y); Diff(S_x, S_x) = 0$

$define\ Diff(T_i, T_j) = Diff(S_x, S_y), T_i \in S_x, T_j \in S_y$

$define\ W(g_1, g_2 \ldots g_p, [T_1, T_2 \ldots T_M]) = \sum_{g_1+g_2+\ldots+g_p=M, g_i\geq 0}\prod_{i=1}^{M}\{T_i + \sum_{j<i} Diff(T_j, T_i)\}$

$Obvious{:}\ H_1(g, PT, \mathrm{PT}) = W(g, M-g, [T_1, T_2 \ldots T_M]), Z_{x,y} = -1, x < y$

$[T_1, T_2 \ldots T_M]\ can\ be\ arranged\ arbitrarily, Z_{x,y(x\neq y)}\ can\ be\ arbitrarily\ too.$

$10.1) Induction \to W(g_1, g_2 \ldots g_p, [T_1, T_2 \ldots T_M]) = \prod_{i=1}^{M} T_i\binom{M}{g_1, g_2 \cdots g_p}. A\ conclusion\ of\ [10].$

$Promote\ to\ Gaussian\ coefficient{:}\ G_{g>0}^{0} = 0; G_g^{M<0}\ is\ still\ defined$

$define\ Diff_q(S_x, S_y) = -1, x < y; Diff_q(S_y, S_x) = -Diff_q(S_x, S_y); Diff_q(S_x, S_x) = 0$

$define\ W_q(g_1, g_2 \ldots g_p, [T_1, T_2 \ldots T_M]) = \sum_{g_1+g_2+\ldots+g_p=M, g_i\geq 0}\prod_{i=1}^{M} G_1^{T_i+\sum_{j<i} Diff_q(T_j, T_i)}\ q^{\sum_{j<i, Diff_q(T_j, T_i)=-1} 1}$

$10.2) W_q(g_1, g_2, [T_1, T_2 \ldots T_M]) = \prod_{i=1}^{M} G_1^{T_i}\, G_{g_1}^{M}$

[Proof]

$Obvious{:}\ It\ holds\ when\ M = 1\ or\ g_1 = 0\ or\ g_1 = M;$

$Verified\ by\ calculation{:}\ It\ holds\ when\ M = 2{:}\ G_1^{T_1}G_1^{T_2+1} + qG_1^{T_1}G_1^{T_2-1} = G_1^{T_1}G_1^{T_2}G_1^2$

$When\ M+1, prove\ by\ induction: PT=[T_1,T_2\dots T_M], PT1=[T_1,T_2\dots T_M,T_{M+1}]$
$W_q(g_1,g_2+1,PT1)=W_q(g_1,g_2,PT)G_1^{T_{M+1}+g_1}+W_q(g_1-1,g_2+1,PT)G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$
$=\prod_{i=1}^{M}G_1^{T_i}G_{g_1}^{M}G_1^{T_{M+1}+g_1}+\prod_{i=1}^{M}G_1^{T_i}G_{g_1-1}^{M}G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$
$=\prod_{i=1}^{M}G_1^{T_i}\{G_{g_1}^{M}G_1^{T_{M+1}+g_1}+G_{g_1-1}^{M}G_1^{T_{M+1}-(M+1-g_1)}q^{M+1-g_1}\}$
$\prod_{i=1}^{M+1}G_1^{T_i}G_{g_1}^{M+1}=\prod_{i=1}^{M}G_1^{T_i}G_1^{T_{M+1}}(q^{g_1}G_{g_1}^{M}+G_{g_1-1}^{M})$
$Just\ need\ to\ prove\ G_1^{T_{M+1}}(q^{g_1}G_{g_1}^{M}+G_{g_1-1}^{M})=G_{g_1}^{M}G_1^{T_{M+1}+g_1}+G_{g_1-1}^{M}G_1^{T_{M+1}-(M+1-g_1)}q^{M+1-g_1}$
$Both\ sides\ multiply\ (q^{g_1}-1)(q^{g_1-1}-1)\dots(q-1)\ and\ divide\ (q^M-1)(q^{M-1}-1)\dots(q^{M-g_1+2}-1)$
$Left=(q^{T_{M+1}+g-1}+\dots+q^{g})(q^{M-g_1+1}-1)+(q^{T_{M+1}-1}+\dots+q^{0})(q^{g}-1)$
$Right=(q^{T_{M+1}+g-1}+\dots+q^{0})(q^{M-g_1+1}-1)+(q^{T_{M+1}-1}+\dots+q^{M+1-g_1})(q^{g}-1)$
$Left-Right=(q^{g}-1)(q^{M-g}+\cdots+1)-(q^{M-g_1+1}-1)(q^{g-1}+\cdots+1)$
$=(q^{M}+\dots+q^{g})-(q^{M-g}+\dots+1)+(q^{g-1}+\dots+1)-(q^{M}+\dots+q^{M-g_1+1})=0$
q.e.d.

$define\ \begin{bmatrix} M \\ g_1\dots g_p\end{bmatrix}_q=\frac{(q^M-1)(q^{M-1}-1)\dots(q^1-1)}{\prod_{i=1}^{g_1}(q^i-1)\prod_{i=1}^{g_2}(q^i-1)\dots\prod_{i=1}^{g_p}(q^i-1)}, g_1+g_2+\dots+g_p=M$

$10.3) W_q(g_1,g_2\dots g_p,[T_1,T_2\dots T_M])=\prod_{i=1}^{M}G_1^{T_i}\begin{bmatrix} M \\ g_1,g_2\dots g_p\end{bmatrix}_q$

[Proof]
Only needs to prove that it holds when $p=3$
$W_q(g_1,g_2+g_3,[T_1,T_2\dots T_M])=\prod_{i=1}^{M}G_1^{T_i}\begin{bmatrix} g_1+g_2+g_3 \\ g_1,g_2+g_3\end{bmatrix}_q$
Every item has g_2+g_3 $factors\ come\ from\ Source_2, divide\ them\ to\ g_2xSource_2+g_3xSource_3$
$g_1-factors$ are invariant, $(g_2+g_3)-factors$ are variant.
$\sum\prod(\text{variant}\ factors)=W_q(g_2,g_3,[X_1,X_2\dots X_{g_2+g_3}])\ \prod_{i=1}^{g_2+g_3}G_1^{X_i}\begin{bmatrix} g_2+g_3 \\ g_2,g_3\end{bmatrix}_q$
$W_q(g_1,g_2,g_3)=\prod_{i=1}^{M}G_1^{T_i}\begin{bmatrix} g_1+g_2+g_3 \\ g_1,g_2+g_3\end{bmatrix}_q\begin{bmatrix} g_2+g_3 \\ g_2,g_3\end{bmatrix}_q=\prod_{i=1}^{M}G_1^{T_i}\begin{bmatrix} g_1+g_2+g_3 \\ g_1,g_2,g_3\end{bmatrix}_q$
The rest is to prove that the change of $Diff_q(T_j,T_i)\ meets\ the\ definition.$
Obvious： $g_1-(\text{invariant}\ factors)\ meet\ the\ definition.$
$When\ W_q(g_2,g_3)\ is\ embedded\ in\ W_q(g_1,g_2,g_3), all\ Diff_q(T_j\in Source_1,T_i\in Source_{2,3})\ will\ add\ 1.$
q.e.d.

10.Eulerian polynomials and Beyond

The main proof method of $\sum_{n=0}^{N-1}q^n n^M$ comes from [2]. Extensive promotion here.

$11.1)\sum_{n=0}^{N-1}q^n\binom{n+M}{M}=q^N\sum_{g=0}^{M}(-1)^g\frac{\binom{N+M-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}}{(q-1)^{M+1}}$

[Proof]
$M=0,\sum_{n=0}^{N-1}q^n\binom{n}{0}=\frac{q^N-1}{q-1}=\frac{q^N\binom{N-1}{0}}{q-1}-\frac{1}{q-1}$, holds
$M=1,\sum_{n=0}^{N-1}q^n\binom{n+1}{1}=\sum_{n=0}^{N-1}q^n(n+1)=\sum_{n=0}^{N-1}q^n+\sum_{n=1}^{N-1}q^n+\dots+\sum_{n=N-1}^{N-1}q^n$
$=N\sum_{n=0}^{N-1}q^n-(\sum_{n=0}^{0}q^n+\sum_{n=0}^{1}q^n+\dots+\sum_{n=0}^{N-2}q^n)$
$=N\frac{q^N-1}{q-1}-\left(\frac{q-1}{q-1}+\frac{q^2-1}{q-1}+\dots+\frac{q^{N-1}-1}{q-1}\right)=N\frac{q^N-1}{q-1}-\frac{(1+q+q^2+\dots+q^{N-1})-N}{q-1}=\frac{q^N\binom{N}{1}}{q-1}-\frac{q^N\binom{N-1}{0}}{(q-1)^2}+\frac{1}{(q-1)^2}$, holds

When $N=1$, right $=\sum_{g=0}^{M}(-1)^g\frac{1}{(q-1)^{g+1}}+\frac{(-1)^{M+1}}{(q-1)^{M+1}}=\frac{q}{q-1}\sum_{g=0}^{M}\frac{(-1)^g}{(q-1)^g}+\frac{(-1)^{M+1}}{(q-1)^{M+1}}=1=left$
$Use\ \binom{n+M}{M}=\binom{n+M-1}{M}+\binom{n+M-1}{M-1}, inductive\ proof.$
q.e.d.

$11.2)\sum_{n=0}^{N-1}q^n\binom{n+K}{M}=q^N\sum_{g=0}^{M}(-1)^g\frac{\binom{N+K-1-g}{M-g}}{(q-1)^{g+1}}+\frac{(-1)^{M+1}q^{M-K}}{(q-1)^{M+1}}$

Define $A_q^M=\sum_{k=0}^{M}(1-q)^{M-k}q^kS_2(M,k)k!\,, q\neq 0,1, M\in\mathbb{N}, A_q^0=1, A_q^1=q$

Table 11. 1 : A_q^M.

	M=0	M=1	M=2	M =3	M=4	M=5	M=6	OEIS
A_2^M	1	2	6	26	150	1082	9366	$A000629$
A_3^M	1	3	12	66	480	4368	47712	$A123227$
A_4^M	1	4	20	132	1140	12324	160020	$A201355$

11.3) $A_q^M=q\sum_{k=0}^{M}(q-1)^{M-k}S_2(M,k)k!$

[Proof]

$$A_q^M\xrightarrow{4.4)}\sum_{k=0}^{M}(1-q)^{M-k}q^k\sum_{g=0}^{M}(-1)^{M-g}g!\,S_2(M,g)\binom{g-1}{k-1}$$
$$=\sum_{g=0}^{M}(-1)^{M-g}g!\,S_2(M,g)\sum_{k=0}^{M}(1-q)^{M-k}q^k\binom{g-1}{k-1}$$
$$=\sum_{g=0}^{M}(-1)^{M-g}g!\,S_2(M,g)(1-q)^{M-g}q\sum_{k=0}^{g}(1-q)^{g-k}q^{k-1}\binom{g-1}{k-1}\xrightarrow{\sum_{k=0}^{g}(1-q)^{g-k}q^{k-1}\binom{g-1}{k-1}=1}11.3)$$

q.e.d.

$$n^M=\nabla SUM(n,[1\ldots 1],[2\ldots M])=\sum_{g=0}^{M}S_2(M,g)g!\binom{n}{g}=\sum_{g=0}^{M}(-1)^{M-g}S_2(M,g)g!\binom{n+g-1}{g}=\sum_{g=0}^{M}\left\langle{M\atop g}\right\rangle\binom{n+g}{M}$$

$$\sum_{n=0}^{N-1}q^nn^M=\sum_{g=0}^{M}S_2(M,g)g!\sum_{n=0}^{N-1}q^n\binom{n}{g}=\sum_{g=0}^{M}S_2(M,g)g!\left\{q^N\sum_{k=0}^{g}(-1)^k\frac{\binom{N-1-k}{g-k}}{(q-1)^{k+1}}+\frac{(-1)^{g+1}q^g}{(q-1)^{g+1}}\right\}$$
$$=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M}S_2(M,g)g!\left\{\sum_{k=0}^{M}(-1)^k\binom{N-1-k}{g-k}(q-1)^{M-k}\right\}+\frac{(-1)^{M+1}\sum_{g=0}^{M}S_2(M,g)g!(1-q)^{M-g}q^g}{(q-1)^{M+1}}$$
$$=\frac{q^N}{(q-1)^{M+1}}\sum_{k=0}^{M}(q-1)^{M-k}(-1)^k\{\sum_{g=0}^{M}S_2(M,g)g!\binom{N-1-k}{g-k}\}+\frac{(-1)^{M+1}A_q^M}{(q-1)^{M+1}}$$
$$=\frac{q^N}{(q-1)^{M+1}}\sum_{k=0}^{M}(q-1)^{M-k}(-1)^k\nabla^k(N-1)^M+\frac{(-1)^{M+1}A_q^M}{(q-1)^{M+1}}\quad(*)$$

$$\sum_{n=0}^{N-1}q^nn^M=\sum_{g=0}^{M}(-1)^{M-g}S_2(M,g)g!\sum_{n=0}^{N-1}q^n\binom{n+g-1}{g}$$
$$=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M}(-1)^{M-g}S_2(M,g)g!\left\{\sum_{k=0}^{g}(-1)^k\binom{N+g-2-k}{g-k}(q-1)^{M-k}\right\}+\frac{(-1)^{M+1}\sum_{g=0}^{M}S_2(M,g)g!(1-q)^{M-g}q}{(q-1)^{M+1}}$$
$$=\frac{q^N}{(q-1)^{M+1}}\sum_{k=0}^{M}(q-1)^{M-k}(-1)^k\nabla^k(N-1)^M+\frac{(-1)^{M+1}\{\ldots\}}{(q-1)^{M+1}}\xrightarrow{compare\ with\ (*)}11.3)$$

$$\sum_{n=0}^{N-1}q^nn^M=\sum_{g=0}^{M}\left\langle{M\atop g}\right\rangle\sum_{n=0}^{N-1}q^n\binom{n+g}{M}\rightarrow$$

11.4) $A_q^M=\sum_{g=0}^{M}\left\langle{M\atop g}\right\rangle q^{M-g}=\sum_{g=0}^{M}\left\langle{M\atop g}\right\rangle q^{1+g}$

$$n^M=\nabla SUM(n,[1,1\ldots 1],[1,2\ldots M])=\sum_{g=0}^{M}S_2(M,g+1)(g+1)!\binom{n-1}{g}\rightarrow$$

11.5) $A_q^M=\sum_{k=0}^{M}(1-q)^{M-k}q^{K+1}S_2(M+1,k+1)k!\,, M>0$

$$S_2(M+1,k+1)=\frac{1}{(k+1)!}\sum_{j=0}^{k+1}(-1)^{j+k+1}\binom{k+1}{j}j^M=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}(j+1)^{M-1}$$

$$\nabla^k(N-1)^M\xrightarrow{By\ definition}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(N-1-j)^M=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{g=0}^{M}\binom{M}{g}N^g(j+1)^{M-g}(-1)^{M-g}$$
$$=\sum_{g=0}^{M}(-1)^{M-g}\binom{M}{g}N^g\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+1)^{M-g}\right)=\sum_{g=0}^{M}(-1)^{M-g-k}\binom{M}{g}N^g(\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}(j+1)^{M-g})$$
$$=\sum_{g=0}^{M}(-1)^{M-g-k}\binom{M}{g}N^gS_2(M-g+1,k+1)k!$$

$f(N) = \sum_{n=0}^{N-1} q^n n^M = (*) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (q-1)^{M-k} (-1)^k \nabla^k (N-1)^M + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (q-1)^{M-k} (-1)^k \{\sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1, k+1) k!\} + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (q-1)^g (-1)^{M-g} \binom{M}{g} N^g \{\sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1, k+1) k!\} + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$ (*
*)

$f(0) = 0 \to g = 0 \to \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}} = -\frac{(-1)^M}{(q-1)^{M+1}} \{\sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1, k+1) k!\} \to$

11.6) $A_q^M = \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1, k+1) k!$

$A_q^{M-g} = \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1, k+1) k! \to From\ (**) \to$

11.7) $\sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (q-1)^g (-1)^{M-g} \binom{M}{g} A_q^{M-g} N^g + \frac{(-1)^{M+1} A_q^M}{(q-1)^{M+1}}$

Table 11. 1 : $\sum_{n=0}^{N-1} q^n n^M$

$\sum_{n=0}^{N-1} q^n i = \frac{q^N((q-1)N-q)+q}{(q-1)^2}$
$\sum_{i=0}^{N-1} 2^i = 2^N - 1$
$\sum_{i=0}^{N-1} 2^i i = 2^N(N-2) + 2$
$\sum_{i=0}^{N-1} 2^i i^2 = 2^N(N^2 - 4N + 6) - 6$
$\sum_{i=0}^{N-1} 2^i i^3 = 2^N(N^3 - 6N^2 + 18N - 26) + 26$
$\sum_{i=0}^{N-1} 2^i i^4 = 2^N(N^4 - 8N^3 + 36N^2 - 104N + 150) - 150$
$\sum_{i=0}^{N-1} 2^i i^5 = 2^N(N^5 - 10N^4 + 60N^3 - 260N^2 + 750N - 1082) + 1082$
$\sum_{i=0}^{N-1} 3^i = \frac{3^N - 1}{2}$
$\sum_{i=0}^{N-1} 3^i i = \frac{3^N(2N-3)+3}{4}$
$\sum_{i=0}^{N-1} 3^i i^2 = \frac{3^N(4N^2 - 2\times3\times2N+12)-12}{8}$,$2\times3\times2 = (3-1)^1 A_3^1\binom{2}{1}$, $+12 = (-1)^2(3-1)^0 A_3^2\binom{2}{0}$
$\sum_{i=0}^{N-1} 3^i i^3 = \frac{3^N(8N^3 - 4\times3\times3N^2 + 2\times12\times3N - 66)+66}{16}$, $4\times3\times3 = (3-1)^2 A_3^1\binom{3}{2}$
$\sum_{i=0}^{N-1} 3^i i^4 = \frac{3^N(16N^4 - 8\times3\times4N^3 + 4\times12\times6N^2 - 2\times66\times4N + 480)-480}{32}$
$\sum_{i=0}^{N-1} 4^i = \frac{4^N - 1}{3}$
$\sum_{i=0}^{N-1} 4^i i = \frac{4^N(3N-4)+4}{9}$
$\sum_{i=0}^{N-1} 4^i i^2 = \frac{4^N(9N^2 - 3*4*2N + 20)-20}{27}$
$\sum_{i=0}^{N-1} 4^i i^3 = \frac{4^N(27N^3 - 9*4*3N^2 + 3*20*3N - 132)+132}{81}$

$A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^k S_2(M,k) k! = q \sum_{k=0}^{M} (q-1)^{M-k} S_2(M,k) k! = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle q^{1+g} =$
$\sum_{k=0}^{M} (1-q)^{M-k} q^{K+1} S_2(M+1, k+1) k!, M > 0 = \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1, k+1) k!$

$A_2^M = \sum_{k=0}^{M} (-1)^{M-k} 2^k S_2(M,k) k! = 2 \sum_{k=0}^{M} S_2(M,k) k! = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle 2^{1+g} =$
$\sum_{k=0}^{M} (-1)^{M-k} 2^{K+1} S_2(M+1, k+1) k!, M > 0 = \sum_{k=0}^{M} S_2(M+1, k+1) k!$
These expressions can validate 2.2).2.3).2.4).2.8)

We know that Eulerian polynomials $A_M(t)$: $\sum_{i=0}^{\infty} t^i i^M = \frac{t A_M(t)}{(1-t)^{M+1}}$; $A_M(t) = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle t^g$

$0 < q < 1, \lim_{N\to\infty} \sum_{n=0}^{N-1} q^n n^M = \frac{A_q^M}{(1-q)^{M+1}} \to$
11.8) $A_t^M = t A_M(t)$
So there are five expressions for $A_M(t)$*. This can also be obtained from* 11.4).

To calculate $\sum_{n=0}^{N-1} q^n (a + (n-1)d)^M$ *, investigation* $(a + (n-1)d)^M = \nabla$
$SUM(n, [a, a \dots a]: d, [1,2 \dots M])$
$= \sum_{g=0}^{M} H_1(g) \binom{n-1}{g} = \sum_{g=0}^{M} H_2(g) \binom{n+g-1}{g} = \sum_{g=0}^{M} H_3(g) \binom{n+M-g-1}{M} = \sum_{g=0}^{M} H_3(M-g) \binom{n+g-1}{M}$

$\sum_{\mathrm{n}=0}^{\mathrm{N}-1} \mathrm{q}^{n}(a+(n-1)d)^{M}=\sum_{g=0}^{M} H_1(g)\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\binom{\mathrm{n}-1}{\mathrm{g}}=\sum_{g=0}^{M} H_1(g)\left\{q^{N}\sum_{\mathrm{k}=0}^{\mathrm{g}}(-1)^{k}\frac{\binom{\mathrm{N}-2-\mathrm{k}}{\mathrm{g}-\mathrm{k}}}{(q-1)^{k+1}}+\frac{(-1)^{g+1}\mathrm{q}^{g+1}}{(q-1)^{g+1}}\right\}$

$=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M} H_1(g)\left\{\sum_{\mathrm{k}=0}^{M}(-1)^{k}\binom{\mathrm{N}-2-\mathrm{k}}{\mathrm{g}-\mathrm{k}}(q-1)^{M-k}\right\}+\frac{(-1)^{M+1}\sum_{g=0}^{M}H_1(g)(1-q)^{M-g}\mathrm{q}^{g+1}}{(q-1)^{M+1}}$

$=\frac{q^N}{(q-1)^{M+1}}\sum_{\mathrm{k}=0}^{\mathrm{M}}(q-1)^{M-k}(-1)^{k}\{\sum_{g=0}^{M} H_1(g)\binom{\mathrm{N}-2-\mathrm{k}}{\mathrm{g}-\mathrm{k}}\}+\frac{(-1)^{M+1}A_{\mathrm{q}}^{M}(a,d)}{(q-1)^{M+1}}$

$=\frac{q^N}{(q-1)^{M+1}}\sum_{\mathrm{k}=0}^{\mathrm{M}}(q-1)^{M-k}(-1)^{k}\,\nabla^{k}\,(a+(N-2)d)^{M}+\frac{(-1)^{M+1}A_{\mathrm{q}}^{M}(a,d)}{(q-1)^{M+1}}\quad(*)$

$A_{\mathrm{q}}^{M}(a,d)\ is\ defined\ as\ \sum_{g=0}^{M} H_1(g,[a,a\dots a]:d,[1,2\dots M])(1-q)^{M-g}\mathrm{q}^{g+1}.Similarly:$

11.9) $A_{\mathrm{q}}^{M}(a,d)=\sum_{g=0}^{M} H_1(g)(1-q)^{M-g}\mathrm{q}^{g+1}=\sum_{g=0}^{M} H_2(g)(1-q)^{M-g}q=\sum_{g=0}^{M} H_3(g)q^{g+1}$

$\nabla\ SUM(n,[d,d\dots d]:d,[1,2\dots M])=(nd)^{M}=\sum_{g=0}^{M} H_1(g,[d,d\dots d]:d,[1,2\dots M])\binom{\mathrm{n}-1}{\mathrm{g}}$

$\mathrm{H}_1(\mathrm{g},[d,d\dots d]:d,[1,2\dots M])=g!\,d^{g}f(M,g)\xrightarrow{Matrix\ rule}\sum_{\mathrm{j}=1}^{\mathrm{g}+1}(-1)^{\mathrm{g}+1+\mathrm{j}}\binom{\mathrm{g}}{\mathrm{j}-1}\nabla\ \mathrm{SUM(j)}$

$=\sum_{\mathrm{j}=1}^{\mathrm{g}+1}(-1)^{\mathrm{g}+1+\mathrm{j}}\binom{\mathrm{g}}{\mathrm{j}-1}(jd)^{M}=\sum_{\mathrm{j}=0}^{\mathrm{g}}(-1)^{\mathrm{g}+\mathrm{j}}\binom{\mathrm{g}}{\mathrm{j}}[(j+1)d]^{M}$

$f(M,g)\ can\ be\ defined\ as\ some\ kind\ of\ General\ Stirling\ Numbers\ of\ the\ second\ kind.$

$\nabla^{k}\,(a+(N-2)d)^{M}\xrightarrow{By\ definition}\sum_{j=0}^{k}(-1)^{j}\binom{k}{\mathrm{j}}(a+(N-2-j)d)^{M}$

$=\sum_{j=0}^{k}(-1)^{j}\binom{k}{\mathrm{j}}\sum_{g=0}^{M}\binom{\mathrm{M}}{\mathrm{g}}(a+(\mathrm{N}-1)\mathrm{d})^{g}[(\mathrm{j}+1)\mathrm{d}]^{M-g}(-1)^{M-g}$

$=\sum_{g=0}^{M}(-1)^{M-g-k}\binom{\mathrm{M}}{\mathrm{g}}(a+(\mathrm{N}-1)\mathrm{d})^{g}(\sum_{j=0}^{k}(-1)^{j+k}\binom{\mathrm{k}}{\mathrm{j}}[(\mathrm{j}+1)\mathrm{d}]^{M-g})$

$=\sum_{g=0}^{M}(-1)^{M-g-k}\binom{\mathrm{M}}{\mathrm{g}}(a+(\mathrm{N}-1)\mathrm{d})^{g}\mathrm{H}_1(\mathrm{k},[d,d\dots d]:d,[1,2\dots M-g])$

11.10) $\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}(a+(n-1)d)^{M}=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M}(-1)^{M-g}\binom{\mathrm{M}}{\mathrm{g}}(a+(\mathrm{N}-1)\mathrm{d})^{g}\{\sum_{\mathrm{k}=0}^{\mathrm{M}}(q-1)^{M-k}\mathrm{H}_1(k,[d,d\dots d]:d,[1,2\dots M-g])\}+\frac{(-1)^{M+1}A_{\mathrm{q}}^{M}(a,d)}{(q-1)^{M+1}}$

11.11) $\sum_{i=1}^{\infty}\mathrm{t}^{i}\,(a+(i-1)d)^{M}=\frac{A_{\mathrm{t}}^{M}(a,d)}{(1-t)^{M+1}}$

11.12) $A_{\mathrm{q}}^{M}(d,d)=\sum_{k=0}^{M}(q-1)^{M-k}\,\mathrm{H}_1(k,[d,d\dots d]:d,[1,2\dots M])=d^{M}A_{\mathrm{q}}^{M}$

11.13) $\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}(nd)^{M}=\frac{q^N}{(q-1)^{M+1}}\sum_{g=0}^{M}(q-1)^{g}(-1)^{M-g}\binom{\mathrm{M}}{\mathrm{g}}A_{\mathrm{q}}^{M-g}(d,d)(\mathrm{N}d)^{g}+\frac{(-1)^{M+1}A_{\mathrm{q}}^{M}(d,d)}{(q-1)^{M+1}}$

General Eulerian Numbers and Polynomials [11] *are similar to* $H_3(g)$ *and* $A_{\mathrm{q}}^{M}(a,d)$ of $H_3(g)$

We can handle $\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\,\nabla^{p}\,SUM(n+Y,\mathrm{PS},\mathrm{PT})\,,\mathrm{X}=\mathrm{T}_{M}-\mathrm{M}-\mathrm{P}$

$=\sum_{g=0}^{M} H_1(g)\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\binom{\mathrm{n}+\mathrm{Y}+X}{X+1+g}=\sum_{g=0}^{M} H_1(g)\left\{q^{N}\sum_{\mathrm{k}=0}^{\mathrm{X}+1+\mathrm{g}}(-1)^{k}\frac{\binom{\mathrm{n}+\mathrm{Y}+\mathrm{X}-1-\mathrm{k}}{X+1+g-\mathrm{k}}}{(q-1)^{k+1}}+\frac{\mathrm{q}^{1+g-Y}}{(1-q)^{X+2+g}}\right\}$

$=\sum_{g=0}^{M} H_2(g)\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\binom{\mathrm{n}+\mathrm{Y}+X+g}{X+1+g}=\sum_{g=0}^{M} H_2(g)\left\{q^{N}\sum_{\mathrm{k}=0}^{\mathrm{X}+1+\mathrm{g}}(-1)^{k}\frac{\binom{\mathrm{n}+\mathrm{Y}+\mathrm{X}+\mathrm{g}-1-\mathrm{k}}{X+1+g-\mathrm{k}}}{(q-1)^{k+1}}+\frac{\mathrm{q}^{1-Y}}{(1-q)^{X+2+g}}\right\}$

$=\sum_{g=0}^{M} H_3(g)\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\binom{\mathrm{n}+\mathrm{Y}+X+M-g}{X+1+M}=\sum_{g=0}^{M} H_3(g)\left\{q^{N}\sum_{\mathrm{k}=0}^{\mathrm{X}+1+\mathrm{M}}(-1)^{k}\frac{\binom{\mathrm{n}+\mathrm{Y}+\mathrm{X}+\mathrm{M}-\mathrm{g}-1-\mathrm{k}}{X+1+M-\mathrm{k}}}{(q-1)^{k+1}}+\frac{\mathrm{q}^{1+g-Y}}{(1-q)^{X+2+M}}\right\}$

11.14) $\mathrm{Y}=0\rightarrow\sum_{g=0}^{M} H_1(g)(1-q)^{M-g}\mathrm{q}^{g+1}=q\sum_{g=0}^{M} H_2(g)(1-q)^{M-g}=\sum_{g=0}^{M} H_3(g)q^{g+1}\stackrel{\mathrm{def}}{=}\mathrm{A}_{q}(PS,PT)$

Here q can take any value, which is magical. $\mathrm{q}=0.5\rightarrow 2.4)\,;q=2\rightarrow 4.7)$

11.15) $\sum_{\mathrm{n}=0}^{\mathrm{N}-1}\mathrm{q}^{n}\,\nabla\,\mathrm{SUM}(\mathrm{n},\mathrm{PS},[1,2\dots\mathrm{M}])=\frac{q^N}{(q-1)^{M+1}}\sum_{k=0}^{M}(q-1)^{M-k}(-1)^{k}\,\nabla^{k}\,\mathrm{SUM}(\mathrm{N}-1)+\frac{(-1)^{M+1}\mathrm{A}_{q}(PS,PT)}{(q-1)^{M+1}}$

11.16) $\sum_{i=0}^{\infty}\mathrm{t}^{i}\,\nabla^{P}\,\mathrm{SUM}(i+Y,\mathrm{PS},\mathrm{PT})=\frac{\mathrm{A}_{q}(PS,PT)t^{-Y}}{(1-t)^{\mathrm{T}_{M}-\mathrm{P}+2}}$

$\sum_{i=0}^{\infty}\mathrm{t}^{i}\,\nabla^{p}\,\mathrm{SUM}(i+Y,[1],[1])=\sum_{i=0}^{\infty}\mathrm{t}^{i}\binom{i+Y+1-p}{2-p}=\frac{\mathrm{t}}{t^{Y}(1-t)^{3-\mathrm{P}}}\xrightarrow{M=2-p}$

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11.17) $\sum_{i=0}^{\infty} t^i \binom{i+Y+M-1}{M} = \frac{t}{t^Y(1-t)^{M+1}} \to \sum_{i=0}^{\infty} t^i \binom{i+K}{M} = \frac{t^{M-K}}{(1-t)^{M+1}}, match\ 11.2)$
$\frac{1}{M!}\sum_{i=0}^{\infty} t^i\ \nabla^1\ \mathrm{SUM}(i+Y,[1,2 \dots M],[1,2 \dots M]) =$
$\sum_{i=0}^{\infty} t^i \binom{i+Y+M-1}{M+1-1}$ *can also reach the same conclusion.*

$\sum_{n=0}^{\infty} q^n\ \nabla^1\ \mathrm{SUM}(i,[1,1 \dots 1,2,2 \dots 2,k,k \dots k],[1,2 \dots .kM]) = \sum_{n=0}^{\infty} q^n\,(n+1)^M (n+2)^M \dots (n+k)^M$
$H_3(g)$ *solved with matrix is the General Eulerian Numbers of* [12], $\frac{A_q(PS,PT)}{(1-q)^{kM+1}}$ *is its limit value.*

11. Generalization of Wolstenholme Theorem

In this section, p is a prime number,p>3.
$(*)Wolstenholme\ Theorem\colon (p-1)!\sum_{n=1}^{p-1}\frac{1}{n} \equiv 0 MOD\ p^2$
$(p-1)(p-2)\dots(p-p+1) = p^{p-1} - A_1p^{p-2} + \dots + A_{p-3}p^2 - A_{p-2}p^1 + A_{p-1} = (p-1)!$
$\xrightarrow{A_{p-1}=(p-1)!} p^{p-2} - A_1p^{p-3} + \dots + A_{p-3}p^1 - A_{p-2} = 0 \xrightarrow{p|A_1,p|A_2 \dots p|A_{p-3}\ and\ A_{p-2}=(p-1)!\sum_{n=1}^{p-1}\frac{1}{n}} (*)$[13]

12.1) $\sum_{n=1}^{p-1} n^{p-2} \equiv 0\ \mathrm{MOD}\ p^2$
[Proof]
$p^{p-2} + \sum_{n=1}^{p-1} n^{p-2} = \sum_{n=1}^{p} n^{p-2} = \mathrm{SUM}(P,[1,1 \dots 1],[1,2 \dots p-2]) = \sum_{g=0}^{p-2} g!\,S_2(p-1,g+1)\binom{p}{g+1}$
$= \sum_{g=1}^{p-1}(g-1)!\,S_2(p-1,g)\binom{p}{g} = p\sum_{g=1}^{p-1}\frac{g!S_2(p-1,g)}{g^2}\binom{p-1}{g-1} \xrightarrow{\frac{g!S_2(p-1,g)}{g^2}\binom{p-1}{g-1}\ is\ an\ integer}$
$\frac{\sum_{n=1}^{p} n^{p-2}}{p} = \sum_{g=1}^{p-1}\frac{g!S_2(p-1,g)}{g^2}\binom{p-1}{g-1} \xrightarrow{3.8)} \equiv \sum_{g=1}^{p-1}\frac{(-1)^{g+1}}{g^2}\binom{p-1}{g-1} \xrightarrow{\binom{p-1}{g-1}\equiv(-1)^{g-1}} \equiv \sum_{g=1}^{p-1}\frac{1}{g^2} \equiv \sum_{g=1}^{p-1} g^2 \equiv 0\mathrm{MOD}p$
q.e.d.
(*)=SUM(2,[1,2…p-2],[1,3…2(p-2)-1]); 12.1)=SUM(p-1,[1,1…1],[1,2…p-2])
They are two extremes and their proof methods are special.Reconsidering shape:
PS starts from $[1,1 \dots 1]$, ends at $[1,2 \dots p-2]$, $K_1 = 1, \begin{cases} K_i = K_{i+1}, \text{continuity} \\ K_i + 1 = K_{i+1}, \text{discontinuity} \end{cases}$
PT starts from $[1,2 \dots p-2]$, ends at $[1,3 \dots 2p-5]$, $T_1 = 1, \begin{cases} T_i + 1 = T_{i+1}, \text{continuity} \\ T_i + 2 = T_{i+1}, \text{discontinuity} \end{cases}$
12.2) $\sum_{c_1+c_2+\dots+c_q=P-2,c_i>0} k_1^{c_1}k_2^{c_2} \dots k_q^{c_q} \equiv 0\ \mathrm{MOD}\ p^2, K_i \neq K_j 1 \leq K_i \leq P-1$
[Proof]
If $A \equiv 0\mathrm{MOD}\ p^2$ *and* $A + B \equiv 0\mathrm{MOD}\ p^2$, *then* $B \equiv 0\mathrm{MOD}\ p^2$.
The Sum has symmetry. For $\sum A^1B^{P-3}$, pair each product A^1B^{P-3} with another $(p-A)^1B^{P-3}$ to pxB^{P-3}
$(p-A)^1B^{P-3} \in \sum A^1B^{P-3}$ *or* $\sum B^{P-2} \xrightarrow{symmetry} \sum(\text{Paired products}) = x\sum A^1B^{P-3} + y\sum B^{P-2} = p\sum B^{P-3}$
$\sum B^{P-3} \equiv 0\mathrm{MOD}\ p \to p\sum B^{P-3} \equiv 0\mathrm{MOD}\ p^2 \xrightarrow{obviously:0<x,y<p} \sum A^1B^{P-3} \equiv 0\mathrm{MOD}\ p^2$
For $\sum A^2B^{P-4}$, pair A^2B^{P-4} with $(p-A)^1AB^{P-4}$ to $pxAB^{P-4}$, $(p-A)^1AB^{P-4} \in \sum A^2B^{P-4}$ *or* $\sum AB^{P-3}$
$\sum(\text{Paired products}) = x\sum A^2B^{P-4} + y\sum AB^{P-3} = p\sum AB^{P-4} \equiv 0\mathrm{MOD}\ p^2 \to \sum A^2B^{P-4} \equiv 0\mathrm{MOD}\ p^2$
Use the same way $\to \sum_{a+b=P-2,A\neq B,1\leq A,B\leq P-1} A^aB^b \equiv 0\mathrm{MOD}\ p^2$

$\sum A^1B^bC^c$, pair with $(p-A)^1B^bC^c$ to pxB^bC^c, $(p-A)^1B^bC^c \in \sum A^1B^bC^c$ *or* $\sum B^{b+1}C^c$ *or* $\sum B^bC^{c+1}$
Use the same way $\to$ *conclusion*
q.e.d.

12.3) $\sum_{same\ shape,M=p-2} SUM(p-1-PB(PT),PS,PT) \equiv 0\ \mathrm{MOD}\ p^2, 0 \leq PB \leq p-3$
SUM(3,[1,1,2],[1,2,4])+SUM(3,[1,2,2],[1,3,4])
=2(1²)+3(1²+2²)+4(1²+2²+3²)+ 2² (1)+ 3² (1+2)+ 4² (1+2+3)=200≡0MOD25

(#) SUM(5,[1,1,1,1,2],[1,2,3,4,6]) + SUM(5,[1,2,2,2,2],[1,3,4,5,6])=
$2(1^4)+3(1^4+2^4)+4(1^4+2^4+3^4)+5(1^4+2^4+3^4+4^4)+6(1^4+2^4+3^4+4^4+5^4)+$
$2^4(1)+3^4(1+2)+4^4(1+2+3)+5^4(1+2+3+4)+6^4(1+2+3+4+5)=35574\equiv 0 MOD 49$

(##) SUM(5,[1,1,1,2,2],[1,2,3,5,6]) + SUM(5,[1,1,2,2,2],[1,2,4,5,6]) =
$2^2(1^3)+3^2(1^3+2^3)+4^2(1^3+2^3+3^3)+5^2(1^3+2^3+3^3+4^3)+6^2(1^3+2^3+3^3+4^3+5^3)$
$2^3(1^2)+3^3(1^2+2^2)+4^3(1^2+2^2+3^2)+5^3(1^2+2^2+3^2+4^2)+6^3(1^2+2^2+3^2+4^2+5^2)=27930\equiv 0 MOD 49$
(#)+(##)=12.3)

12.4)$E_{P-2}^{P-1}=S_2(2p-3,p-1)\equiv 0 \text{ MOD } p^2; E_{p-2}^{p}=S_2(2p-2,p)\equiv 0 \text{ MOD } p^2$
$S_2(7,4)=350$ and $S_2(8,5)=1050\equiv 0 \text{ MOD } 5^2; S_2(11,6)=179487$ and $S_2(12,7)=627396\equiv 0 \text{ MOD } 7^2$

Generally, PS starts from $[K_2,K_3 \dots K_M]$, ends at $[K_2+1,K_3+2 \dots K_M+M-1], K_i\in\mathbb{N}$
$\mathrm{PS}=[K_2+I_2,K_3+I_3 \dots K_M+I_M], \mathrm{I}_1=0,\begin{cases}\mathrm{I}_i=\mathrm{I}_{i+1}, \text{continuity}\\ \mathrm{I}_i+1=\mathrm{I}_{i+1}, \text{discontinuity}\end{cases}$
PT starts from $[K_2+1,K_3+2 \dots K_M+M-1]$, ends at $[K_2+2,K_3+4 \dots K_M+2(M-1)]$
$\mathrm{PT}=[K_2+J_2,K_3+J \dots K_M+J_M], \mathrm{J}_1=0,\begin{cases}\mathrm{J}_i+1=\mathrm{J}_{i+1}, \text{continuity}\\ \mathrm{J}_i+2=\mathrm{J}_{i+1}, \text{discontinuity}\end{cases}$
$F(PB)\stackrel{\text{def}}{=}\sum_{same\ shape,PB(PTx)=PB,0\le PB\le M-1} SUM(N,PSx,PTx)=$
$\sum_{same\ shape}\sum_{g=0}^{M-1} H_1(g,PSx,PTx)\binom{N+X}{X+1+g}$
$=\sum_{g=0}^{M-1}\sum_{same\ shape} H_1(g,PSx,PTx)\binom{N+X}{X+1+g}\stackrel{\text{def}}{=}\sum_{g=0}^{M-1} H_1(g,PB)\binom{N+X}{X+1+g}$
12.5)$H_1(g,PB)=H_1(PB,g)$
[Proof]
$K_1\times H_1(g,0)\xrightarrow{2.14)}\mathrm{MIN}_g([K_1,K_2 \dots K_M])\xrightarrow{\text{def}}K_1\times H_1(0,g)$
$K_1\times H_1(g,M-1)\xrightarrow{2.14)}\mathrm{MIN}_g([K_1,K_2+1 \dots K_M+M-1])\xrightarrow{\text{def}}K_1\times H_1(M-1,g)$
$K_1\times H_1(g,PB)=\sum_{K_1\times H_1(g,0)=MIN_{PB}([K_1,K_2\dots K_M])} H_1(g)=\sum_{K_1\times H_1(g,0)=MIN_{PB}([K_1,K_2\dots K_M])} MIN_g(PS)$
$K_1\times H_1(PB,g)=\sum_{K_1\times H_1(g,0)=MIN_g([K_1,K_2\dots K_M])} MIN_{PB}(PS)$
Their quantity is $\binom{M-1}{PB}\binom{M-1}{g}$ *and they are paired one by one*
q.e.d.

g from M-1 to 0, list H_1(PB,g) of F(M-1),F(M-2)…F(0) row by row, the matrix will be symmetrical diagonally.

Table 12. 1 :H_1(PB,g) of PS=[1,1,1],PT=[2,3,4].

	g=3	g=2	g=1	g=0
SUM(N,[2,3,4],[3,5,7])	3×5×7	3×5×6+ 3×4×6+2×4×6	2×3×5+ 2×4×5+3×4×5	2×3×4
SUM(N,[1,2,3],[2,4,6])+ SUM(N,[2,2,3],[3,4,6])+ SUM(N,[2,3,3],[3,5,6])	3×5×6+ 3×4×6+ 2×4×6		1×2×4+1×3×4+ 2×3×4+2×2×4+ 2×3×4+3×3×4+ 2×3×4+2×4×4+ 3×4×4=224	1×2×3+ 2×2×3+ 2×3×3=36
SUM(N,[1,1,2],[2,3,5])+ SUM(N,[1,2,2],[2,4,5])+ SUM(N,[2,2,2],[3,4,5])	2×3×5+ 2×4×5+ 3×4×5	2×3×4+2×2×4+ 1×2×4+2×4×4+2×3×4 +1×3×4+ 3×4×4 +3×3×4+ 2×3×4=224		1×1×2+ 1×2×2+ 2×2×2=14
SUM(N,[1,1,1],[2,3,4])	2×3×4	36	14	1

12. Possible future research directions

The proof process of formal calculation needs to meet f(a+b)=f(a)f(b), so it cannot be further generalized through the method of this article.

1) $Can\ Formal\ Calculation\ be\ extended\ to\ other\ functions$?

Examining various products,SUM(3,[1,1],[1,4])=1×1+2×{(1+2)+1} +3×{(1+2+3)+(1+2)+1}

2)$PT = [1,4,7 \dots.], PT = [1,5,9 \dots]\ are\ rarely\ studied.$

3)$Find\ some\ calculations\ to\ extend\ PT\ to\ \mathbb{C}, especially\ to\ rational\ fractions.$

4)$Other\ expressions\ for\ E_p^q \odot ([T_1, T_2 \dots], C).$

$From\ 2.2)\ and\ 2.3), there\ exists\ \ formulas\ between\ S_{1,2}(M,g) and\ S_{2,2}(M,g)$

5)$Are\ there\ any\ formulas\ between\ S_{1,r}(M,g)\ and\ S_{2,r}(M,g)$

$From\ 3.4)\ and\ 3.5), there\ is\ \binom{rN+M}{rg+M}\ in\ the\ formulas\ , but\ there\ is\ also\ \frac{1}{r^{N-g}}\frac{(rN-rg)!}{(N-g)!}\ or\ \frac{1}{r!^{N-g}}\frac{(rN-rg)!}{(N-g)!}$

6) $Can\ Formal\ Calculation\ be\ extended\ to\ \binom{rN+Y}{X}$?

7) $Find\ a\ relationship\ similar\ to\ 2.2)\ in\ general\ H^q(g)$

$From\ 10.1), W(g_1, g_2 \dots g_p, [T_1, T_2 \dots T_M]) = \prod_{i=1}^{M} T_i \binom{M}{g_1, g_2 \dots g_p}.$

$We\ want\ to\ use\ it\ , which\ leads\ to\ the\ generation\ of\ Multiparameter\ Formal\ Calcution. But\ it\ was\ still\ not\ used$

8)$Find\ some\ calculations\ to\ use\ W(g_1, g_2 \dots g_p, [T_1, T_2 \dots T_M])$

Even 2-parameters Formal Calcution are complex.

9)$Using\ Multiparameter\ Formal\ Calcution\ for\ analysis$

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