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Posted Date: 16 July 2025

doi: 10.20944/preprints202507.1321.v1

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BT Inverse and the Generalized BT Inverse in a Ring

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Abstract

In this paper, we introduce and investigate the BT inverse for a ring element. The binary relation BT order has also been studied. Furthermore, we introduce the generalized BT inverse and present some presentations of this new generalized inverse. It also be characterized by using system of equations and Pierce decompositions of ring elements. Many properties of BT inverse for complex matrices are generalized to a broader context within a ring.

Keywords: core inverse; Moore-Penrose inverse; BT inverse; generalized BT inverse; BT order; ring

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1. Introduction

An associative ring with an identity 1 is called a *-ring if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (xy)^* = y^*x^*, (x^*)^* = x$ for all $x, y \in R$. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^\#$, and called the group inverse of a. As is well known, a square complex matrix A has group inverse if and only if $rank(A) = rank(A^2)$ (see [13]). An element $a \in \mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$. Let $R(X)$ represent the range space of a complex matrix X. A square complex matrix A has core inverse $A^{\textcircled{\#}}$ if and only if $AA^{\textcircled{\#}} = P_A$ is a projection and $R(A^{\textcircled{\#}}) \subseteq R(A)$ (see [3,16]). Group and core inverses are extensively studied by many authors from very different points of view, e.g., [1,3,7,10,13,16].

An element $a \in R$ has Moore-Penrose inverse if there exists $x \in R$ such that

$$xax = x, (ax)^* = ax, (xa)^* = xa, axa = a.$$

The preceding x is unique if it exists, and we denote it by $a^\dagger$. The set of all Moore-Penrose invertible elements in R is denoted by $R^\dagger$. Evidently, every square complex matrix has the Moore-Penrose inverse. In [2], Baksalary and Trenkler extended core inverse and introduced a generalized core inverse. Recently, Ferreyra and Malik studied such generalized core inverse and call it the introduced BT inverse for a complex matrix. The matrix $A^\diamond = (AP_A)^\dagger$ is called the BT inverse of A. Many elementary properties of this new generalized inverse are established in [5]. For additional references on the BT inverse, we refer the reader to [6,8,9,17].

The motivation of this paper is to extend the proposed generalized inverse for complex matrices to a more general setting within a ring. We introduce the BT inverse for an element in a ring R. Furthermore, we establish and prove several fundamental properties of the BT inverse within this ring.

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In Section 2, we introduce a new generalized inverse as the generalization of BT inverse of a complex matrix.

Definition 1. *An element $a \in R^{\dagger}$ has BT inverse if there exists $x \in im(a)$ such that*

$$xax = x, (ax)^* = ax, axa^2 = a^2, (a^*xa^2)^* = a^*xa^2.$$

If such x exists, it is unique, and denote it by $a^{\diamondsuit}$. The set of all BT invertible elements in R is denoted by $R^{\diamondsuit}$.

In [15], based on the Hartwig-Spindelbock decomposition of a complex matrix, Wang characterize the BT inverse of a complex matrix by using the system by equations. Replacing the Hartwig-Spindelbock decomposition, we employ the Pierce representation of a ring element as a tool to extend the characterization of BT inverse of a complex matrix to a broader context within a ring. We prove that $a \in R^{\dagger}$ has BT inverse if and only if $a^2a^{\dagger} \in R^{\dagger}$. In this case, $a^{\diamondsuit} = [a^2a^{\dagger}]^{\dagger}$. In Section 3, we investigate the order relation induced by BT inverse. Many characterizations of the BT order are obtained by using Pierce decomposition for a ring element.

Let

$$R^{qnil} = \{x \in R \mid 1 - xr \in R \text{ is invertible for any } r \in R, xr = rx\}.$$

Let $\mathcal{A}$ be a Banach algebra. Evidently,

$$\begin{aligned} \mathcal{A}^{qnil} &= \{x \in \mathcal{A} \mid \lim_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0\} \\ &= \{x \in \mathcal{A} \mid 1 + \lambda x \in \mathcal{A} \text{ is invertible for any } \lambda \in \mathbb{C}\}. \end{aligned}$$

Definition 2. *An element $a \in \mathcal{A}$ has generalized BT inverse if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = xy^* = 0, x \in \mathcal{A}^{\diamondsuit}, y \in \mathcal{A}^{qnil}.$$

We denote $x^{\diamondsuit}$ by $a_g^{\diamondsuit}$ and call it the generalized BT inverse of a. The set of all generalized BT invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}_g^{\diamondsuit}$.

Recall that an element $a \in R$ has generalized Moore-Penrose inverse if there exists $x \in R$ such that

$$x = xax, (ax)^* = ax, (xa)^* = xa, a - axa \in R^{qnil}.$$

The preceding x is denoted by $a^{\oplus}$. The set of all generalized Moore-Penrose invertible elements in R is denoted by $R^{\oplus}$.

In Section 4, we prove that $a \in \mathcal{A}_g^{\diamondsuit}$ if and only if $a \in \mathcal{A}^{\oplus}$ and $a^2a^{\oplus} \in R^{\dagger}$. We further characterize the generalized BT inverse by using the system of equations.

Finally, in Section 5, we present certain characterizations of the generalized BT inverse for a geometrical point of view.

Throughout the paper, all rings are associative *-rings with an identity. $R^{\#}, R^{\dagger}$ and $R^{\oplus}$ denote the sets of all group invertible, More-Penrose invertible and generalized Moore-Penrose invertible elements in R, respectively. Let $a \in R$. Set $im(a) = \{ax \mid x \in R\}$ and $ker(a) = \{x \in R \mid ax = 0\}$. Let $a \in R^{\dagger}$. Then $p_a = aa^{\dagger}$. We use $p_{im(a),ker(b)}$ to denote the projection p such that $im(p) = im(a)$ and $ker(p) = ker(b)$.

2. BT Inverse

The purpose of this section is to investigate the elementary properties of the BT inverse, which will be frequently utilized in subsequent sections. Our starting point is as follows.

Theorem 1. *Let $a \in R^{\dagger}$. Then the following are equivalent:*

(1) $a \in R^{\diamondsuit}$.

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(2) $a^2a^\dagger \in R^\dagger$.

In this case, $a^\diamondsuit = [a^2a^\dagger]^\dagger$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists $x \in im(a)$ such that $xax = x, (ax)^* = ax$, $axa^2 = a^2$, $(a^*xa^2)^* = a^*xa^2$. Write $x = az$ for a $z \in R$. We directly verify that

$$\begin{aligned} x[a^2a^\dagger] &= az[a^2a^\dagger] = aa^\dagger az[a^2a^\dagger] \\ &= aa^\dagger x[a^2a^\dagger] = [aa^\dagger]^* x[a^2a^\dagger] \\ &= [a^\dagger]^*[a^*xa^2]a^\dagger, \\ (a^2a^\dagger)x &= (a^2a^\dagger)az = a(aa^\dagger a)z = a(az) = ax, \\ x[a^2a^\dagger]x &= x(ax) = xax = x, \\ (a^2a^\dagger)x(a^2a^\dagger) &= (ax)(a^2a^\dagger) = (axa^2)a^\dagger = a^2a^\dagger. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} \left(x(a^2a^\dagger)\right)^* &= \left((a^\dagger)^*(a^*xa^2)a^\dagger\right)^* = (a^\dagger)^*(a^*xa^2)^*a^\dagger \\ &= (a^\dagger)^*(a^*xa^2)a^\dagger = x(a^2a^\dagger), \\ \left((a^2a^\dagger)x\right)^* &= (ax)^* = ax = (a^2a^\dagger)x. \end{aligned}$$

Therefore $a^2a^\dagger \in R^\dagger$ and $[a^2a^\dagger]^\dagger = x$. This implies that x is unique, as required.
(2) $\Rightarrow$ (1) Let $x = (a^2a^\dagger)^\dagger$. Then

$$\begin{aligned} x &= (a^2a^\dagger)^\dagger = (a^2a^\dagger)^\dagger(a^2a^\dagger)(a^2a^\dagger)^\dagger \\ &= [(a^2a^\dagger)^\dagger(a^2a^\dagger)]^*(a^2a^\dagger)^\dagger \\ &= (aa^\dagger)[(a^2a^\dagger)^\dagger a]^*(a^2a^\dagger)^\dagger \in im(a). \end{aligned}$$

We further check that

$$\begin{aligned} xax &= (a^2a^\dagger)^\dagger a(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)^\dagger a(a^2a^\dagger)^\dagger(a^2a^\dagger)(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)^\dagger a[aa^\dagger][(a^2a^\dagger)^\dagger a]^*(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)^\dagger a[aa^\dagger]^2[(a^2a^\dagger)^\dagger a]^*(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)^\dagger a^2a^\dagger(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)^\dagger = x, \end{aligned}$$

$$\begin{aligned} ax &= a(a^2a^\dagger)^\dagger \\ &= a(a^2a^\dagger)^\dagger(a^2a^\dagger)(a^2a^\dagger)^\dagger \\ &= a[aa^\dagger]\left((a^2a^\dagger)^\dagger a\right)^*(a^2a^\dagger)^\dagger \\ &= (a^2a^\dagger)(a^2a^\dagger)^\dagger, \\ (ax)^* &= ax, \\ axa^2 &= (a^2a^\dagger)(a^2a^\dagger)^\dagger a^2 \\ &= (a^2a^\dagger)(a^2a^\dagger)^\dagger(a^2a^\dagger)a \\ &= (a^2a^\dagger)a = a^2, \end{aligned}$$

$$\begin{aligned} a^*xa^2 &= a^*(a^2a^\dagger)^\dagger a^2 \\ &= a^*(a^2a^\dagger)^\dagger(a^2a^\dagger)a \\ &= a^*(a^2a^\dagger)^\dagger(a^2a^\dagger)a, \\ (a^*xa^2)^* &= a^*xa^2. \end{aligned}$$

Therefore $x \in im(a)$ is the solution of the system of equations:

$$xax = x, (ax)^* = ax, axa^2 = a^2, (a^*xa^2)^* = a^*xa^2,$$

as required. □

Corollary 1. *Let $a \in R^{\dagger} \bigcap R^{\circledast}$. Then $a \in R^{\diamondsuit}$ and $a^{\diamondsuit} = a^{\circledast}$.*

Proof. In view of [16, Theorem 2.6], $a^{\circledast} = a^{\#}aa^{\dagger}$.

Then we have $a^2a^{\circledast} = a^2[a^{\#}aa^{\dagger}] = a^2a^{\dagger}$. It is easy to verify that

$$[a^2a^{\circledast}]a^{\circledast} = a^{\circledast}[a^2a^{\circledast}] = aa^{\circledast}.$$

Then $[a^2a^{\circledast}]^{\dagger} = a^{\circledast}$. This implies that $a^2a^{\dagger} \in R^{\dagger}$. In this case, $a^{\diamondsuit} = a^{\circledast}$. □

Let $a \in R^{\dagger}$ and $p_a = aa^{\dagger}$. Then we have

$$\begin{aligned} a^2 &= \begin{pmatrix} a^3a^{\dagger} & a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p, \\ a^3 &= \begin{pmatrix} a^4a^{\dagger} & a^3(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p, \\ (a^2)^{\dagger} &= \begin{pmatrix} aa^{\dagger}(a^2)^{\dagger} & 0 \\ (1-aa^{\dagger})(a^2)^{\dagger} & 0 \end{pmatrix}_p. \end{aligned}$$

Lemma 1. *Let $a \in R^{\diamondsuit}$. Then $a^2 \in R^{\dagger}$ and $a^{\diamondsuit} = a(a^2)^{\dagger}$.*

Proof. It is easy to verify that

$$\begin{aligned} a^2[a^{\dagger}(a^2a^{\dagger})^{\dagger}] &= [a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}, \\ (a^{\dagger}(a^2a^{\dagger})^{\dagger})a^2 &= a^{\dagger}((a^2a^{\dagger})^{\dagger})(a^2a^{\dagger})a, \\ a^2[a^{\dagger}(a^2a^{\dagger})^{\dagger}]a^2 &= [a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}a^2 = a^2, \\ (a^{\dagger}(a^2a^{\dagger})^{\dagger})a^2(a^{\dagger}(a^2a^{\dagger})^{\dagger}) &= a^{\dagger}(a^2a^{\dagger})^{\dagger}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left(a^2[a^{\dagger}(a^2a^{\dagger})^{\dagger}]\right)^* &= a^2[a^{\dagger}(a^2a^{\dagger})^{\dagger}], \\ \left([a^{\dagger}(a^2a^{\dagger})^{\dagger}]a^2\right)^* &= [a^{\dagger}(a^2a^{\dagger})^{\dagger}]a^2. \end{aligned}$$

Thus, $a^{\dagger}(a^2a^{\dagger})^{\dagger} = (a^2)^{\dagger}$.

In light of Theorem 2.1, we derive that

$$\begin{aligned} a^{\diamondsuit} &= (a^2a^{\dagger})^{\dagger} \\ &= [(a^2a^{\dagger})^{\dagger}(a^2a^{\dagger})(aa^{\dagger})]^*(a^2a^{\dagger})^{\dagger} \\ &= (aa^{\dagger})^*[(a^2a^{\dagger})^{\dagger}(a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}] \\ &= [aa^{\dagger}][a^2a^{\dagger}]^{\dagger} = a[a^{\dagger}(a^2a^{\dagger})^{\dagger}] = a(a^2)^{\dagger}, \end{aligned}$$

as asserted. □

We come now to establish the representation of the BT inverse by using certain projections.

Theorem 2. *Let $a \in R^{\diamondsuit}$. Then $a^{\diamondsuit} = p_aa^*(ap_aa^* + 1 - p_{a^2})^{-1}$.*

Proof. By virtue of Lemma 2.3, $a^2 \in R^{\dagger}$. Further, we verify that

$$\begin{aligned} ap_aa^* + 1 - p_{a^2} &= a^2a^{\dagger}a^* + 1 - a^2(a^2)^{\dagger} \\ &= [a^2a^{\dagger}][aa^{\dagger}a^*] + 1 - aa^{\diamondsuit} \\ &= [a^2a^{\dagger}][aa^{\dagger}a^*] + 1 - a[a^2a^{\dagger}]^{\dagger} \\ &= [a^2a^{\dagger}][a^2a^{\dagger}]^* + 1 - a(aa^{\dagger})[a^2a^{\dagger}]^{\dagger} \\ &= [a^2a^{\dagger}][a^2a^{\dagger}]^* + 1 - [a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger} \in \mathcal{A}^{-1}. \end{aligned}$$

 doi:10.20944/preprints202507.1321.v1

Furthermore, we check that

$$\begin{aligned} a^{\diamondsuit}[ap_a a^* + 1 - p_{a^2}] &= a^{\diamondsuit}[a^2a^{\dagger}][a^2a^{\dagger}]^* + a^{\diamondsuit}[1 - aa^{\diamondsuit}] \\ &= [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}][a^2a^{\dagger}]^* \\ &= \left([a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\right)^*[a^2a^{\dagger}]^* \\ &= \left([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\right)^* \\ &= [a^2a^{\dagger}]^* = [a(aa^{\dagger})]^* = aa^{\dagger}a^* = p_a a^*. \end{aligned}$$

Therefore $a^{\diamondsuit} = p_a a^*(ap_a a^* + 1 - p_{a^2})^{-1}$. □

Corollary 2. *Let $a \in R^{\diamondsuit}$. The system given by*

$$ax = a(ap_a)^{\dagger}, im(x) \subseteq im(p_a a^*)$$

is consistent and its unique solution is $x = a^{\diamondsuit}$.

Proof. Clearly, we have $aa^{\diamondsuit} = a[a^2a^{\dagger}]^{\dagger} = a(ap_a)^{\dagger}$. In view of Theorem 2.4, $a^{\diamondsuit} = p_a a^*(ap_a a^* + 1 - p_{a^2})^{-1}$. Hence, $a^{\diamondsuit} \in im(p_a a^*)$. We infer that $im(a^{\diamondsuit}) \subseteq im(p_a a^*)$.

Suppose that

$$ax = a(ap_a)^{\dagger}, im(x) \subseteq im(p_a a^*)$$

for some $x \in R$. As $p_a a^* = a^{\diamondsuit}(ap_a a^* + 1 - p_{a^2})$, we write $x = (ap_a)^{\dagger}(z)$ for some $z \in R$. Then

$$\begin{aligned} x &= (ap_a)^{\dagger}z = (ap_a)^{\dagger}ap_a(ap_a)^{\dagger}z \\ &= (ap_a)^{\dagger}ap_a x = (ap_a)^{\dagger}(ax) \\ &= (ap_a)^{\dagger}[a(ap_a)^{\dagger}] \\ &= (ap_a)^{\dagger}(ap_a)(ap_a)^{\dagger}] \\ &= (ap_a)^{\dagger} = a^{\diamondsuit}, \end{aligned}$$

as required. □

Theorem 3. *Let $a \in R$. The system given by*

$$p_a x = (ap_a)^{\dagger}, im(x) \subseteq im(a)$$

is consistent and its unique solution is $x = a^{\diamondsuit}$.

Proof. One easily checks that

$$\begin{aligned} p_a a^{\diamondsuit} &= aa^{\dagger}[a^2a^{\dagger}]^{\dagger} = [a^2a^{\dagger}]^{\dagger} = (ap_a)^{\dagger}, \\ a^{\diamondsuit} &= [a^2a^{\dagger}]^{\dagger} = aa^{\dagger}[a^2a^{\dagger}]^{\dagger} \subseteq im(a). \end{aligned}$$

Suppose that $p_a x_i = (ap_a)^{\dagger}, im(x_i) \subseteq im(a)$ for $i = 1, 2$. Then $p_a x_1 = p_a x_2$; hence, $x_1 - x_2 \in ker(p_a) \bigcap im(a) = 0$. Therefore $x_1 = x_2$, as asserted. □

Let $a, b, c \in R$. An element a has (b, c)-inverse provide that there exists $x \in R$ such that

$$xab = b, cax = c \text{ and } x \in bRx \bigcap xRc.$$

If such x exists, it is unique and denote it by $a^{(b,c)}$ (see [4]).

Theorem 4. *Let $a \in R^{\diamondsuit}$. Then $a^{\diamondsuit} = a^{\left(aa^{\dagger}a^*, aa^{\dagger}a^*\right)}$.*

Proof. Obviously, $a \in R^{\dagger}$. Let $x = a^{\diamondsuit}$. We verify that

$$\begin{array}{rcl} x & = & [a^2a^{\dagger}]^{\dagger} = [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger} \\ & = & [aa^{\dagger}a^*]\big([a^2a^{\dagger}]^{\dagger}\big)^*x \in [aa^{\dagger}a^*]Rx, \\ x & = & x\big([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}\big)^* = x\big([a^2a^{\dagger}]^{\dagger}\big)^*[aa^{\dagger}a^*] \\ & \in & xR[aa^{\dagger}a^*], \\ xa[aa^{\dagger}a^*] & = & [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]a^* = [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]a^* \\ & = & \big([a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\big)^*a^* = \big(a[a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\big)^* \\ & = & \big(a[a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}aa^{\dagger}]\big)^* = \big(a([a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}aa^{\dagger}])^*\big)^* \\ & = & \big(a(aa^{\dagger})^*([a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}])^*\big)^* = \big([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\big)^* \\ & = & [a^2a^{\dagger}]^* = [a(aa^{\dagger})]^* = aa^{\dagger}a^*, \\ aa^{\dagger}a^*ax & = & aa^{\dagger}a^*a[a^2a^{\dagger}]^{\dagger} \\ & = & [(aa^{\dagger})^*a^*][a(aa^{\dagger})][a^2a^{\dagger}]^{\dagger} \\ & = & [a^2a^{\dagger}]^*[a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger} \\ & = & [a^2a^{\dagger}]^*\big([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}\big)^* \\ & = & \big([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\big)^* \\ & = & [a^2a^{\dagger}]^* = aa^{\dagger}a^*. \end{array}$$

Therefore $a^{\diamondsuit} = a^{\left(aa^{\dagger}a^*, aa^{\dagger}a^*\right)}$, as asserted. □

Let $a \in R$ and $T, S \subseteq R$. We say that a has $\{2\}$-inverse x provided that $xax = x$, $im(a) = T$, $ker(a) = S$. We denote x by $a^{(2)}_{T,S}$. We next consider the relation between the BT inverse and $\{2\}$-inverse in a ring. .

Theorem 5. *Let $a \in R^{\diamondsuit}$. Then $a^{\diamondsuit} = a^{(2)}_{im\left(aa^{\dagger}a^*\right), ker(a^*)^2}$.*

Proof. Let $x = a^{\diamondsuit}$. Clearly, we have $x = xax$.

Step 1. $im(x) = im\big(aa^{\dagger}a^*\big)$. In view of Theorem 2.1, we have

$$\begin{array}{rcl} x & = & [a^2a^{\dagger}]^{\dagger} \\ & = & [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger} \\ & = & \big([a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}]\big)^*[a^2a^{\dagger}]^{\dagger} \\ & = & [aa^{\dagger}a^*]\big([a^2a^{\dagger}]^{\dagger}\big)^*[a^2a^{\dagger}]^{\dagger} \\ & \in & im\big(aa^{\dagger}a^*\big), \end{array}$$

$$\begin{array}{rcl} [aa^{\dagger}a^*]^* & = & a^2a^{\dagger} = [a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}] \\ & = & [a^2a^{\dagger}]x[a^2a^{\dagger}] = [a^2a^{\dagger}][a^2a^{\dagger}]^*x^*. \end{array}$$

Hence, $aa^{\dagger}a^* = x[a^2a^{\dagger}][a^2a^{\dagger}]^* \in im(x)$. Therefore $im(x) = im\big(aa^{\dagger}a^*\big)$.

Step 2. $ker(x) = ker(a^*)^2$. If $(a^*)^2r = 0$ for some $r \in R$, then

$$\begin{array}{rcl} xr & = & [a^2a^{\dagger}]^{\dagger}r \\ & = & [a^2a^{\dagger}]^{\dagger}[a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}r \\ & = & [a^2a^{\dagger}]^{\dagger}\big([a^2a^{\dagger}][a^2a^{\dagger}]^{\dagger}\big)^*r \\ & = & [a^2a^{\dagger}]^{\dagger}\big([a^{\dagger}[a^2a^{\dagger}]^{\dagger}\big)^*(a^2)^*r = 0. \end{array}$$

Hence, $r \in ker(x)$.

If $x(r) = 0$, then

$$\begin{array}{rcl} (a^*)^2r & = & (a^2)^*r = [(a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}(a^2a^{\dagger})a]^*r \\ & = & [a^2a^{\dagger}a]^*[(a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}]^*r \\ & = & [a^2a^{\dagger}a]^*[a^2a^{\dagger}](xr) = 0. \end{array}$$

Thus, $r \in ker(a^*)^2$, as required.

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Therefore we complete the proof. □

3. BT Order

This section is devoted to the BT order for two elements is a ring. Let $a, b \in R$ and $a \in R^{\diamondsuit}$.

Definition 3. *We say that $a \leq^{\diamondsuit} b$ if and only if $aa^{\diamondsuit} = ba^{\diamondsuit}, a^{\diamondsuit}a = a^{\diamondsuit}b$.*

Let $p = aa^{\dagger}$. Then we have

$$a = \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p, a^{\dagger} = \begin{pmatrix} a(a^{\dagger})^2 & 0 \\ (1-aa^{\dagger})a^{\dagger} & 0 \end{pmatrix}_p.$$

Moreover, we compute that

$$a^{\diamondsuit} = (a^2a^{\dagger})^{\dagger} = \begin{pmatrix} (a^2a^{\dagger})^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p.$$

Theorem 6. *Let $a \in R^{\diamondsuit}, b \in R$. Then the following are equivalent:*

(1) $a \leq^{\diamondsuit} b$.
(2) There exist $x, y, z \in \mathcal{A}$ such that a and b are represented by

$$\begin{aligned} a &= \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p, \\ b &= \begin{pmatrix} a^2a^{\dagger} & [(1-(a^2a^{\dagger})(a^2a^{\dagger})^{\dagger})x + (a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}a](1-aa^{\dagger}) \\ y[1-(a^2a^{\dagger})^{\dagger}(a^2a^{\dagger})] & z \end{pmatrix}_p, \end{aligned}$$

where $p = aa^{\dagger}$.

Proof. (1) ⇒ (2) Since $a^{\diamondsuit} = \begin{pmatrix} (a^2a^{\dagger})^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p$, we see that

$$\begin{aligned} aa^{\diamondsuit} &= \begin{pmatrix} (a^2a^{\dagger})(a^2a^{\dagger})^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p, \\ a^{\diamondsuit}a &= \begin{pmatrix} (a^2a^{\dagger})^{\dagger}(a^2a^{\dagger}) & (a^2a^{\dagger})^{\dagger}a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p. \end{aligned}$$

Write $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p$. Then

$$\begin{aligned} ba^{\diamondsuit} &= \begin{pmatrix} b_1(a^2a^{\dagger})^{\dagger} & 0 \\ b_3(a^2a^{\dagger})^{\dagger} & 0 \end{pmatrix}_p, \\ a^{\diamondsuit}b &= \begin{pmatrix} (a^2a^{\dagger})^{\dagger}b_1 & (a^2a^{\dagger})^{\dagger}b_2 \\ 0 & 0 \end{pmatrix}_p. \end{aligned}$$

By hypothesis, we have

$$\begin{aligned}(a^2a^\dagger)(a^2a^\dagger)^\dagger &= b_1(a^2a^\dagger)^\dagger,\\ b_3(a^2a^\dagger)^\dagger &= 0,\\ (a^2a^\dagger)^\dagger(a^2a^\dagger) &= (a^2a^\dagger)^\dagger b_1,\\ (a^2a^\dagger)^\dagger a(1-aa^\dagger) &= (a^2a^\dagger)^\dagger b_2.\end{aligned}$$

Hence, $b_1 = [(a^2a^\dagger)^\dagger]^\dagger = a^2a^\dagger, b_3 = b_3[1-(a^2a^\dagger)^\dagger(a^2a^\dagger)], b_2 = [1-(a^2a^\dagger)(a^2a^\dagger)^\dagger]b_2 + (a^2a^\dagger)(a^2a^\dagger)^\dagger a(1-aa^\dagger)$.

(2) $\Rightarrow$ (1) Since $a^\diamondsuit = \begin{pmatrix} (a^2a^\dagger)^\dagger & 0 \\ 0 & 0 \end{pmatrix}_p$, we directly verify that

$$\begin{aligned}aa^\diamondsuit &= \begin{pmatrix} (a^2a^\dagger)(a^2a^\dagger)^\dagger & 0 \\ 0 & 0 \end{pmatrix}_p\\ &= \begin{pmatrix} a^2a^\dagger & [1-(a^2a^\dagger)(a^2a^\dagger)^\dagger]x + (a^2a^\dagger)(a^2a^\dagger)^\dagger a(1-aa^\dagger) \\ y[1-(a^2a^\dagger)^\dagger(a^2a^\dagger)] & z \end{pmatrix}_p\\ &\quad \begin{pmatrix} (a^2a^\dagger)^\dagger & 0 \\ 0 & 0 \end{pmatrix}_p = ba^\diamondsuit,\\ a^\diamondsuit a &= \begin{pmatrix} (a^2a^\dagger)^\dagger(a^2a^\dagger) & (a^2a^\dagger)^\dagger a(1-aa^\dagger) \\ 0 & 0 \end{pmatrix}_p = \begin{pmatrix} (a^2a^\dagger)^\dagger & 0 \\ 0 & 0 \end{pmatrix}_p\\ &\quad \begin{pmatrix} a^2a^\dagger & [1-(a^2a^\dagger)(a^2a^\dagger)^\dagger]x + (a^2a^\dagger)(a^2a^\dagger)^\dagger a(1-aa^\dagger) \\ y[1-(a^2a^\dagger)^\dagger(a^2a^\dagger)] & z \end{pmatrix}_p\\ &= a^\diamondsuit b.\end{aligned}$$

Therefore $a \leq^\diamondsuit b$, as desired. □

Corollary 3. *Let $a \in R^\diamondsuit, b \in R$ and $aa^\diamondsuit a = a$. Then the following are equivalent:*

(1) $a \leq^\diamondsuit b$.
(2) There exists $z \in \mathcal{A}$ such that a and b are represented by

$$\begin{aligned}a &= \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & 0 \end{pmatrix}_p,\\ b &= \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & z \end{pmatrix}_p,\end{aligned}$$

where $p = aa^\dagger$.
(3) $aa^\dagger = ba^\dagger, a^2 = ba$.

Proof. (1) $\Leftrightarrow$ (2) This is obvious by Theorem 3.2.

(2) $\Rightarrow$ (3) By hypothesis, we have $aa^\diamondsuit a = a$; hence, $a(a^2a^\dagger)^\dagger a = a$. This implies that $(a^2a^\dagger)(a^2a^\dagger)^\dagger a = a$. Since $a^\dagger = \begin{pmatrix} a(a^\dagger)^2 & 0 \\ (1-aa^\dagger)a^\dagger & 0 \end{pmatrix}_p$, we verify that

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$$\begin{aligned}
a^{\dagger}a &= \begin{pmatrix} a(a^{\dagger})^2 & 0 \\ (1-aa^{\dagger})a^{\dagger} & 0 \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\
&= \begin{pmatrix} a(a^{\dagger})^2a^2a^{\dagger} & a(a^{\dagger})^2a(1-aa^{\dagger}) \\ (1-aa^{\dagger})a^{\dagger}a^2a^{\dagger} & (1-aa^{\dagger})a^{\dagger}a(1-aa^{\dagger}) \end{pmatrix}_p \\
&= \begin{pmatrix} a(a^{\dagger})^2 & 0 \\ (1-aa^{\dagger})a^{\dagger} & 0 \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & (a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}a(1-aa^{\dagger}) \\ 0 & z \end{pmatrix}_p \\
&= a^{\dagger}b, \\
a^2 &= \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p^2 = \begin{pmatrix} (a^2a^{\dagger})^2 & a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\
&= \begin{pmatrix} a^2a^{\dagger} & (a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}a(1-aa^{\dagger}) \\ 0 & z \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\
&= ba,
\end{aligned}$$

as desired.

(3) $\Rightarrow$ (1) Write $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p$. By hypothesis, we have

$$\begin{aligned}
a^{\dagger}a &= \begin{pmatrix} a(a^{\dagger})^2 & 0 \\ (1-aa^{\dagger})a^{\dagger} & 0 \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\
&= \begin{pmatrix} a(a^{\dagger})^2a^2a^{\dagger} & a(a^{\dagger})^2a(1-aa^{\dagger}) \\ (1-aa^{\dagger})a^{\dagger}a^2a^{\dagger} & (1-aa^{\dagger})a^{\dagger}a(1-aa^{\dagger}) \end{pmatrix}_p \\
&= \begin{pmatrix} a(a^{\dagger})^2b_1 & a(a^{\dagger})^2b_2 \\ (1-aa^{\dagger})a^{\dagger}b_1 & (1-aa^{\dagger})a^{\dagger}b_2 \end{pmatrix}_p \\
&= \begin{pmatrix} a(a^{\dagger})^2 & 0 \\ (1-aa^{\dagger})a^{\dagger} & 0 \end{pmatrix}_p \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p = a^{\dagger}b, \\
a^2 &= \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p^2 \\
&= \begin{pmatrix} (a^2a^{\dagger})^2 & a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\
&= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p = ba.
\end{aligned}$$

Hence,

$$\begin{aligned}
a(a^{\dagger})^2b_2 &= a(a^{\dagger})^2a(1-aa^{\dagger}), \\
(1-aa^{\dagger})a^{\dagger}b_2 &= (1-aa^{\dagger})a^{\dagger}a(1-aa^{\dagger}).
\end{aligned}$$

This implies that $a^{\dagger}b_2 = a(a^{\dagger})^2a(1-aa^{\dagger}) + (1-aa^{\dagger})a^{\dagger}a(1-aa^{\dagger})$; and then

$$\begin{aligned}
b_2 &= aa^{\dagger}b_2 \\
&= a^2(a^{\dagger})^2a(1-aa^{\dagger}) + a(1-aa^{\dagger})a^{\dagger}a(1-aa^{\dagger}) \\
&= [a^2(a^{\dagger})^2 + a(1-aa^{\dagger})a^{\dagger}]a(1-aa^{\dagger}) \\
&= aa^{\dagger}a(1-aa^{\dagger}) = a(1-aa^{\dagger}).
\end{aligned}$$

Since $b_3a^2a^{\dagger} = 0$, we deduce that $b_3(a^2a^{\dagger})(a^2a^{\dagger})^{\dagger}a = 0$. This implies that $b_3a = 0$, and so $b_3 = (b_3a)a^{\dagger} = 0$.

Moreover, we have

$$\begin{aligned}
a(a^{\dagger})^2b_1 &= a(a^{\dagger})^2a^2a^{\dagger}, \\
(1-aa^{\dagger})a^{\dagger}b_1 &= (1-aa^{\dagger})a^{\dagger}a^2a^{\dagger}.
\end{aligned}$$

 doi:10.20944/preprints202507.1321.v1

Hence $a^\dagger b_1 = a(a^\dagger)^2a^2a^\dagger + (1-aa^\dagger)a^\dagger a^2a^\dagger$; and so

$$\begin{aligned} b_1 &= aa^\dagger b_1 \\ &= a^2(a^\dagger)^2a^2a^\dagger + a(1-aa^\dagger)a^\dagger a^2a^\dagger \\ &= aa^\dagger a^2a^\dagger = a^2a^\dagger. \end{aligned}$$

Therefore $b = \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & b_4 \end{pmatrix}_p$, as required. □

Corollary 4. *Let $a \in R^\diamondsuit, b \in R$ and $aa^\diamondsuit a = a$. Then the following are equivalent:*

(1) $a \leq^\diamondsuit b$.
(2) $b - a \in (1-aa^\dagger)R(1-aa^\dagger)$.
(3) $b - a \in (1-aa^\diamondsuit)R(1-a^\diamondsuit a)$.

Proof. (1) $\Rightarrow$ (2) In view of Corollary 3.3,

$$b - a = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}_p,$$

where $p = aa^\dagger$. Thus, $b - a \in (1-p)R(1-p)$, as required.

(2) $\Rightarrow$ (1) In view of Theorem 3.2, $a = \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & 0 \end{pmatrix}_p$. Set $c = b - a$. Then $c \in (1-p)R(1-p)$, and so

$$b = a + c = \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & c \end{pmatrix}_p.$$

By using Corollary 3.3, $a \leq^\diamondsuit b$.

(1) $\Rightarrow$ (3) By hypothesis, we have

$$aa^\diamondsuit = ba^\diamondsuit, a^\diamondsuit a = a^\diamondsuit b.$$

Hence, $(b-a)a^\diamondsuit a = 0, aa^\diamondsuit(b-a) = 0$. Thus $b - a = (1-aa^\diamondsuit)(b-a)(1-a^\diamondsuit a)$, as required.

(3) $\Rightarrow$ (1) Since $b - a \in (1-aa^\diamondsuit)R(1-a^\diamondsuit a)$ and $a^\diamondsuit aa^\diamondsuit = a^\diamondsuit$, we derive that $aa^\diamondsuit = ba^\diamondsuit, a^\diamondsuit a = a^\diamondsuit b$, as desired. □

We are ready to prove:

Theorem 7. *Let $a \in R^\diamondsuit, b \in R, aa^\diamondsuit a = a, ab = ba$ and $a \leq^\diamondsuit b$. If $a^2 \in R^\diamondsuit$, then $a^2 \leq^\diamondsuit b^2$.*

Proof. In view of Theorem 3.2, we have

$$\begin{aligned} a &= \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & 0 \end{pmatrix}_p, \\ b &= \begin{pmatrix} a^2a^\dagger & a(1-aa^\dagger) \\ 0 & z \end{pmatrix}_p, \end{aligned}$$

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where $p = aa^{\dagger}$. Since $ab = ba$, we see that

$$\begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & z \end{pmatrix}_p = \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & z \end{pmatrix}_p \begin{pmatrix} a^2a^{\dagger} & a(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p.$$

This implies that $az = a(1-aa^{\dagger})z = 0$.
In view of Lemma 2.3, $aa^{\diamondsuit} = a^2(a^2)^{\dagger}$. Therefore

$$\begin{aligned} (a^2)^{\diamondsuit} &= [a^4(a^2)^{\dagger}]^{\dagger} = [a^3a^{\diamondsuit}]^{\dagger} \\ &= [\begin{pmatrix} a^4a^{\dagger} & a^3(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} (a^2a^{\dagger})^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p]^{\dagger} \\ &= \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p. \end{aligned}$$

By hypothesis, we compute that

$$\begin{aligned} a^2(a^2)^{\diamondsuit} &= \begin{pmatrix} a^3a^{\dagger} & a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} a^3a^{\dagger}[a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p, \\ b^2(a^2)^{\diamondsuit} &= \begin{pmatrix} a^3a^{\dagger} & a^2(1-aa^{\dagger})+az \\ 0 & z^2 \end{pmatrix}_p \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} a^3a^{\dagger}[a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p, \end{aligned}$$

$$\begin{aligned} (a^2)^{\diamondsuit}a^2 &= \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} a^3a^{\dagger} & a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}a^3a^{\dagger} & [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p, \\ (a^2)^{\diamondsuit}b^2 &= \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} a^3a^{\dagger} & a^2(1-aa^{\dagger}) \\ 0 & z^2 \end{pmatrix}_p \\ &= \begin{pmatrix} [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}a^3a^{\dagger} & [a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}a^2(1-aa^{\dagger}) \\ 0 & 0 \end{pmatrix}_p. \end{aligned}$$

Therefore

$$a^2(a^2)^{\diamondsuit} = b^2(a^2)^{\diamondsuit}, (a^2)^{\diamondsuit}a^2 = (a^2)^{\diamondsuit}b^2.$$

Accordingly, $a^2 \leq^{\diamondsuit} b^2$. □

Corollary 5. *Let $a \in R^{\diamondsuit} \bigcap R^{\#}, b \in R$ and $a \leq^{\diamondsuit} b$. If $a^2 \leq^{\diamondsuit} b^2$ if and only if $ab = ba$.*

Proof. $\Longleftarrow$ Since $a^2 \leq^{\diamondsuit} b^2$, as in the proof in Theorem 3.5, we deduce that

$$[a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}az = 0;$$

hence, $[a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}][a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}az = 0$. This implies that

$$(az)^*[a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}][a^4a^{\dagger}(a^2a^{\dagger})^{\dagger}]^{\dagger}] = 0.$$

 doi:10.20944/preprints202507.1321.v1

It follows that $(az)^*[a^4a^\dagger] = 0$. Thus $(az)^*a^4 = 0$. Since $a \in R^\#$, we have $(az)^*a = 0$; whence, $a^*az = 0$. Thus $a(1 - aa^\dagger)z = az = aa^\dagger az = (aa^\dagger)^*az = (a^\dagger)^*(a^*az) = 0$. Accordingly, $ab = ba$, as required.

$\Longleftarrow$ This is proved in Theorem 3.5. □

4. Generalized BT Inverse

The aim of this section is to introduce the notion of the generalized BT inverse in a ring. For further use, we formally establish the following lemma:

Lemma 2. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\textcircled{\dagger}}$.
(2) There exist $x, y \in R$ such that

$$a = x + y, x^*y = yx^* = 0, x \in R^\dagger, y \in R^{qnil}.$$

Proof. $(1) \Rightarrow (2)$ By hypotheses, there exists $z \in R$ such that

$$z = zaz, (az)^* = az, (za)^* = za, a - aza \in R^{qnil}.$$

Set $x = aza$ and $y = a - aza$. Then $a = x + y$. We claim that x has Moore-Penrose inverse. Evidently, we verify that

$$\begin{aligned} xzx &= (aza)z(aza) = a(zaz)aza = a(zaz)a = aza = x, \\ zxz &= z(aza)z = za(zaz) = zaz = z, \\ xz &= azaz = az, zx = zaza = za, \\ (xz)^* &= (az)^* = az = xz, \\ (zx)^* &= (za)^* = za = zx. \end{aligned}$$

Therefore $x \in R^\dagger$ and $z = x^\dagger$.
Moreover, we see that

$$\begin{aligned} x^*y &= (aza)^*(1 - az)a = a^*(az)^*(1 - az)a \\ &= a^*(az)^*(1 - az)a = 0. \end{aligned}$$

Since $(za)^* = za$, we have

$$\begin{aligned} yx^* &= (a - aza)(aza)^* = a(1 - za)(aza)^* \\ &= a[(1 - za)]^*(aza)^* = a[(aza)(1 - za)]^* = 0. \end{aligned}$$

By hypothesis, we get $y = a - aza \in R^{qnil}$. Therefore there exists the Moore-Penrose decomposition $a = x + y$, as required.

$(2) \Rightarrow (1)$ By hypothesis, there exist $z, y \in R$ such that

$$a = z + y, z^*y = yz^* = 0, z \in R^\dagger, y \in R^{qnil}.$$

Set $x = z^\dagger$. One easily checks that

$$\begin{aligned} xy &= z^\dagger(zz^\dagger)y = z^\dagger(zz^\dagger)^*y \\ &= z^\dagger(z^\dagger)^*(z^*y) = 0, \\ xa &= z^\dagger z + xy = z^\dagger z, \\ xax &= z^\dagger zz^\dagger = z^\dagger = x, \end{aligned}$$

 doi:10.20944/preprints202507.1321.v1

Moreover, we check that

$$yz^\dagger = yz^\dagger zz^\dagger = y[z^\dagger z]z^\dagger = (yz^*)[z^\dagger)^* z^\dagger = 0,$$

and then $ax = (z+y)z^\dagger = zz^\dagger + yz^\dagger = zz^\dagger$. Then

$$(ax)^* = (zz^\dagger)^* = zz^\dagger = ax,$$
$$(xa)^* = (z^\dagger z)^* = z^\dagger z = xa.$$

Since $yz^\dagger = 0$, we see that

$$a - axa = a(1-xa) = (y+z)(1-z^\dagger z) = y(1-z^\dagger z) = y \in R^{qnil}.$$

Therefore $a^{\oplus} = x^\dagger = z$, as asserted. □

Theorem 8. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R_g^{\diamondsuit}$.
(2) $a \in R^{\oplus}$ and $a^2 a^{\oplus} \in R^\dagger$.

In this case, $a_g^{\diamondsuit} = [a^2 a^{\oplus}]^\dagger$.

Proof. (1) ⇒ (2) Since $a \in R_g^{\diamondsuit}$, there exist $x, y \in R$ such that

$$a = x + y, x^* y = xy^* = 0, x \in R^{\diamondsuit}, y \in R^{qnil}.$$

Clearly, $x \in R^\dagger$. In light of Lemma 4.1, $a \in R^{\oplus}$ and $a^{\oplus} = x^\dagger$. It is easy to verify that

$$\begin{aligned} a^2 a^{\oplus} &= (x+y)^2 x^\dagger = (x+y)^2 x^\dagger x x^\dagger \\ &= (x+y)^2 (x^\dagger x)^* x^\dagger = x^2 x^\dagger \in R^\dagger. \end{aligned}$$

Moreover, we check that $a_g^{\diamondsuit} = [x^2 x^{\oplus}]^\dagger = [a^2 a^{\oplus}]^\dagger$, as required.

(2) ⇒ (1) Since $a \in R^{\oplus}$, by virtue of Lemma 4.1, there exist $x, y \in R$ such that

$$a = x + y, x^* y = yx = 0, x \in R^\dagger, y \in R^{qnil}.$$

In this case, $a^{\oplus} = x^\dagger$. Moreover, we have $x^2 x^\dagger = (x+y)^2 x^\dagger = a^2 a^{\oplus} \in R^\dagger$. Therefore $x \in R^{\diamondsuit}$. Accordingly, $a \in R_g^{\diamondsuit}$. □

As an immediate consequence, we derive

Corollary 6. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R_g^{\diamondsuit}$.
(2) The system of conditions

$$x = xax, (ax)^* = ax, (xa)^* = xa, a^2 x \in R^\dagger, a - axa \in R^{qnil}$$

is consistent and it has the unique solution.

In this case, $a_g^{\diamondsuit} = (a^2 x)^\dagger$.

Corollary 7. *Let $a, b \in R_g^{\diamondsuit}$. If $ab = ba = 0, a^* b = 0$, then $a + b \in R_g^{\diamondsuit}$. In this case,*

$$(a+b)_g^{\diamondsuit} = a_g^{\diamondsuit} + b_g^{\diamondsuit}.$$

 doi:10.20944/preprints202507.1321.v1

Proof. Since $a, b \in R_g^{\diamondsuit}$, it follows by Theorem 4.2 that $a, b \in R^{\oplus}, a^2a^{\oplus}, b^2b^{\oplus} \in R^{\dagger}$ and

$$a_g^{\diamondsuit} = (a^2a^{\oplus})^{\dagger}, b_g^{\diamondsuit} = (b^2b^{\oplus})^{\dagger}.$$

Since $ab = ba = 0, a^*b = b^*a = 0$, we verify that $a + b \in R^{\oplus}$ and $(a + b)^{\oplus} = a^{\oplus} + b^{\oplus}$. It is easy to verify that

$$\begin{array}{rl} & ((a+b)^2(a+b)^{\oplus} \\ = & (a+b)^2[a^{\oplus}+b^{\oplus}] \\ = & a^2a^{\oplus}+b^2b^{\oplus}. \end{array}$$

By hypothesis, we verify that

$$\begin{array}{rcl} (a^2a^{\oplus})^*[b^2b^{\oplus}] & = & [aa^{\oplus}][a^*b][bb^{\oplus}] = 0, \\ (b^2b^{\oplus})^*[a^2a^{\oplus}] & = & [bb^{\oplus}][b^*a][aa^{\oplus}] = 0. \end{array}$$

In light of Theorem 4.2,

$$\begin{array}{rcl} (a+b)_g^{\diamondsuit} & = & [a^2a^{\oplus}+b^2b^{\oplus}]^{\dagger} \\ & = & [a^2a^{\oplus}]^{\dagger}+[b^2b^{\oplus}]^{\dagger} \\ & = & a_g^{\diamondsuit}+b_g^{\diamondsuit}, \end{array}$$

as asserted. □

We are ready to prove:

Theorem 9. *Let $a \in R$. Then $a \in R_g^{\diamondsuit}$ if and only if*

(1) $a \in R^{\oplus}$;.
(2) there exists $x \in im(aa^{\oplus})$ such that

$$xax = x, (ax)^* = ax, (axa^2)a^{\oplus} = a^2a^{\oplus}, (a^*xa^2)^* = a^*xa^2.$$

In this case, $a_g^{\diamondsuit} = x$.

Proof. $\Longrightarrow$ Let $x = (a^2a^{\oplus})^{\dagger}$. Then $x = (aa^{\oplus})x \in (aa^{\oplus})R$. We verify that

$$\begin{array}{rcl} ax & = & a(a^2a^{\oplus})^{\dagger} = (a^2a^{\oplus})(a^2a^{\oplus})^{\dagger}, \\ xax & = & x(a^2a^{\oplus})(a^2a^{\oplus})^{\dagger} = x, \\ (ax)^* & = & ((a^2a^{\oplus})(a^2a^{\oplus})^{\dagger})^* \\ & = & (a^2a^{\oplus})(a^2a^{\oplus})^{\dagger} = ax, \\ a^*xa^2 & = & a^*(a^2a^{\oplus})^{\dagger}a^2 = a^*[(a^2a^{\oplus})^{\dagger}(a^2a^{\oplus})]a, \\ (a^*xa^2)^* & = & a^*xa^2. \end{array}$$

Furthermore, we have

$$\begin{array}{rcl} (axa^2)a^{\oplus} & = & a(a^2a^{\oplus})^{\dagger}a^2a^{\oplus} \\ & = & a(a^2a^{\oplus})^{\dagger}(a^2a^{\oplus})(a^2a^{\oplus})^{\dagger}a^2a^{\oplus} \\ & = & a(aa^{\oplus})(a^2a^{\oplus})^{\dagger}a^2a^{\oplus} \\ & = & a^2a^{\oplus}, \end{array}$$

as required.

$\Longleftarrow$ By hypothesis, there exists $x \in im(aa^{\oplus})$ such that

$$xax = x, (ax)^* = ax, (axa^2)a^{\oplus} = a^2a^{\oplus}, (a^*xa^2)^* = a^*xa^2.$$

 doi:10.20944/preprints202507.1321.v1

Write $x = aa^{\oplus}az$ for a $z \in R$. Then we check that

$$\begin{aligned} x[a^2a^{\oplus}] &= aa^{\oplus}az[a^2a^{\oplus}] \\ &= aa^{\oplus}x[a^2a^{\oplus}] = [aa^{\oplus}]^*x[a^2a^{\oplus}] \\ &= [a^{\oplus}]^*[a^*xa^2]a^{\oplus}, \\ (a^2a^{\oplus})x &= (a^2a^{\oplus})az = a(aa^{\oplus}az = ax, \\ x[a^2a^{\oplus}]x &= x(ax) = xax = x, \\ (a^2a^{\oplus})x(a^2a^{\oplus}) &= (ax)(a^2a^{\oplus}) = (axa^2)a^{\oplus} = a^2a^{\oplus}. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} (x(a^2a^{\oplus}))^* &= ((a^{\oplus})^*(a^*xa^2)a^{\oplus})^* = (a^{\oplus})^*(a^*xa^2)^*a^{\oplus} \\ &= (a^{\oplus})^*(a^*xa^2)a^{\oplus} = x(a^2a^{\oplus}), \\ ((a^2a^{\oplus})x)^* &= (ax)^* = ax = (a^2a^{\oplus})x. \end{aligned}$$

Therefore $a^2a^{\oplus} \in R^{\dagger}$ and $[a^2a^{\oplus}]^{\dagger} = x$. This completes the proof. □

Corollary 8. *Let $a \in R$. Then $a \in R^{\diamondsuit}$ if and only if $a \in R^{\dagger} \bigcap R_g^{\diamondsuit}$. In this case, $a^{\diamondsuit} = a_g^{\diamondsuit}$.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Since $a \in R^{\dagger}$, we have $a^{\oplus} = a^{\dagger}$. In view of Theorem 4.5, there exists $x \in im(aa^{\oplus})$ such that

$$xax = x, (ax)^* = ax, (axa^2)a^{\oplus} = a^2a^{\oplus}, (a^*xa^2)^* = a^*xa^2.$$

Hence,

$$axa^2 = [(axa^2)a^{\oplus}]a = [a^2a^{\oplus}]a = a^2.$$

Therefore $a \in R^{\diamondsuit}$. In this case, $a^{\diamondsuit} = x = a_g^{\diamondsuit}$, as asserted. □

5. Characterizations of the Generalized BT-Inverse

The main purpose of this section is to provide new properties of the generalized BT-inverse in a ring. Consider the system given by

$$xax = x, xa = [a^2a^{\oplus}a]^{\dagger}a, ax = a[a^2a^{\oplus}]^{\dagger} \qquad (5.1).$$

Lemma 3. *If the system (5.1) of equations has a solution, then it is unique.*

Proof. Assume that x_1, x_2 satisfy (2.1). Then

$$x_iax_i = x_i, x_ia = [a^2a^{\oplus}]^{\dagger}a, ax_i = a[a^2a^{\oplus}]^{\dagger}$$

for $i = 1, 2$. Therefore

$$\begin{aligned} x_1 &= (x_1a)x_1 = [a^2a^{\oplus}]^{\dagger}ax_1 \\ &= (x_2a)x_1 = x_2(ax_1) = x_2a[a^2a^{\oplus}]^{\dagger} \\ &= x_2ax_2 = x_2, \end{aligned}$$

as desired. □

Theorem 10. *Let $a, x \in R$. Then the following are equivalent:*

(1) $a_g^{\diamondsuit} = x$.
(2) The system (5.1) of equations is consistent and it has the unique solution x.

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Proof. (1) $\Rightarrow$ (2) Taking $x = a_g^{\Diamond}$. In view of Theorem 4.2, $x = [a^2a^{\oplus}]^{\dagger}$. Then

$$\begin{array}{rcl} xax & = & [a^2a^{\oplus}]^{\dagger}a[a^2a^{\oplus}]^{\dagger} \\ & = & [a^2a^{\oplus}]^{\dagger}a[a^2a^{\oplus}]^{\dagger}[a^2a^{\oplus}][a^2a^{\oplus}]^{\dagger} \\ & = & [a^2a^{\oplus}]^{\dagger}a^2a^{\oplus}[a^2a^{\oplus}]^{\dagger} \\ & = & [a^2a^{\oplus}]^{\dagger} = x, \\ xa & = & [a^2a^{\oplus}a]^{\dagger}a, \\ ax & = & a[a^2a^{\oplus}]^{\dagger}. \end{array}$$

By virtue of Lemma 5.1, x is the unique solution of the preceding equations, as required.

(2) $\Rightarrow$ (1) By the argument above, we have $x = [a^2a^{\oplus}]^{\dagger}$. Therefore $a_g^{\Diamond} = x$ by Theorem 4.2. $\square$

We are ready to prove:

Theorem 11. *Let $a, x \in R$. Then the following are equivalent:*

(1) $a_g^{\Diamond} = x$.
(2) $ax = a(ap_a)^{\oplus}, im(x) \subseteq im(ap_a)^*$.
(3) $aa^{\oplus}x = [a^2a^{\oplus}]^{\dagger}, im(x) \subseteq im(aa^{\oplus})$.
(4) $xa = (ap_a)^{\oplus}a, ker(ap_a)^* \subseteq ker(x)$.
(5) $xa = (ap_a)^{\oplus}a, ker(a^2)^* \subseteq ker(x)$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 5.2, $ax = a(ap_a)^{\oplus}$. We verify that

$$\begin{array}{rcl} x & = & [a^2a^{\oplus}]^{\oplus} \\ & = & [a^2a^{\oplus}]^{\oplus}[a^2a^{\oplus}][a^2a^{\oplus}]^{\oplus} \\ & = & \left([a^2a^{\oplus}]^{\oplus}[a^2a^{\oplus}]\right)^*[a^2a^{\oplus}]^{\oplus} \\ & = & (ap_a)^*\left([a^2a^{\oplus}]^{\oplus}\right)^*[a^2a^{\oplus}]^{\oplus} \\ & = & p_aa^*\left([a^2a^{\oplus}]^{\oplus}\right)^*[a^2a^{\oplus}]^{\oplus} \\ & \in & im(p_aa^*), \end{array}$$

as required.

(2) $\Rightarrow$ (1) Write $x = (ap_a)^*z$ for a $z \in R$. Then

$$\begin{array}{rcl} x & = & \left(ap_a(ap_a)^{\oplus}ap_a\right)^*z \\ & = & \left((ap_a)^{\oplus}ap_a\right)^*(ap_a)^*z \\ & = & (ap_a)^{\oplus}ap_a(ap_a)^*z = (ap_a)^{\oplus}a[p_a(ap_a)^*z] \\ & = & (ap_a)^{\oplus}a[(ap_a)^*z] = (ap_a)^{\oplus}(ax) \\ & = & (ap_a)^{\oplus}a(ap_a)^{\oplus} = (ap_a)^{\oplus}ap_a(ap_a)^{\oplus} \\ & = & (ap_a)^{\oplus} = [a^2a^{\oplus}]^{\oplus} = a_g^{\Diamond}. \end{array}$$

as required.

(1) $\Rightarrow$ (3) In view of Theorem 4.2, $x = [a^2a^{\oplus}]^{\dagger}$. Then we easily check that $aa^{\oplus}x = [a^2a^{\oplus}]^{\dagger}, im(x) \subseteq im(aa^{\oplus})$.

(3) $\Rightarrow$ (1) Write $x = aa^{\oplus}z$ for a $z \in R$. Then

$$x = aa^{\oplus}[aa^{\oplus}z] = aa^{\oplus}x = [a^2a^{\oplus}]^{\dagger}.$$

By virtue of Theorem Theorem 4.2, $a_g^{\Diamond} = x$, as desired.

(1) $\Rightarrow$ (4) Obviously, $xa = (ap_a)^{\oplus}a$. If $z \in ker(p_aa^*)$, then $(ap_a)^*z = 0$. Hence, $xz = (ap_a)^{\oplus}z = (ap_a)^{\oplus}(ap_a)(ap_a)^{\oplus}z = (ap_a)^{\oplus}[(ap_a)^{\oplus}]^*(ap_a)^*z = 0$. Thus $z \in ker(x)$. That is, $ker(p_aa^*) \subseteq ker(x)$, as desired.

(4) $\Rightarrow$ (5) We directly verify that $ker(a^2)^* \subseteq ker(p_aa^*)$, as required.

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(5) $\Rightarrow$ (1) As $(ap_a)(ap_a)^\dagger(ap_a) = ap_a$, we get $(ap_a)(ap_a)^\dagger a^2 = (ap_a)(ap_a)^\dagger(ap_a a) = ap_a a = a^2$. Hence, $(a^2)^*[1-(ap_a)(ap_a)^\dagger] = 0$. Since $ker(a^2)^* \subseteq ker(x)$, we have $x[1-(ap_a)(ap_a)^\dagger] = 0$. Therefore

$$\begin{aligned} x &= x(ap_a)(ap_a)^\dagger = xa(ap_a)^\dagger \\ &= [(ap_a)^{\oplus}a](ap_a)^\dagger \\ &= (ap_a)^{\oplus}ap_a(ap_a)^\dagger \\ &= (ap_a)^{\oplus} = [a^2a^\dagger]^{\oplus}. \end{aligned}$$

Therefore we complete the proof by Theorem 4.2. $\square$

Corollary 9. *Let $a, x \in R$. Then the following are equivalent:*

(1) $a_g^{\Diamond} = x$.
(2) $xax = x, ax = p_{ap_a}, xa = p_{(ap_a)^*,(ap_a)^{\oplus}}a$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 5.1, $xax = x$. Moreover, we have

$$\begin{aligned} ax &= a[a^2a^{\oplus}]^{\oplus} \\ &= a^2a^{\oplus}[ap_a]^{\oplus} \\ &= ap_a[ap_a]^{\oplus} = p_{ap_a}, \end{aligned}$$

It is easy to verify that

$$\begin{aligned} x &= [a^2a^{\oplus}]^{\oplus} \\ &= [a^2a^{\oplus}]^{\oplus}[a^2a^{\oplus}][a^2a^{\oplus}]^{\oplus} \\ &= (ap_a)^*x^*x, \\ (ap_a)^* &= \left([a^2a^{\oplus}][a^2a^{\oplus}]^{\oplus}[a^2a^{\oplus}]\right)^* \\ &= [a^2a^{\oplus}]^{\oplus}[a^2a^{\oplus}]\left([a^2a^{\oplus}]\right)^* \\ &= x[a^2a^{\oplus}]\left([a^2a^{\oplus}]\right)^*. \end{aligned}$$

Hence, $im(x) = im(ap_a)^*$. Obviously, $ker(x) = (ap_a)^{\oplus}$. Thus, we have

$$xa = [a^2a^{\oplus}]^{\oplus}a = p_{(ap_a)^*,(ap_a)^{\oplus}}a,$$

as required.

(2) $\Rightarrow$ (1) By hypothesis, $xax = x, ax = p_{ap_a}, xa = p_{(ap_a)^*,(ap_a)^{\oplus}}a$. Then $ax = p_{ap_a} = ap_a(ap_a)^\dagger = a^2a^\dagger[a^2a^\dagger]^\dagger = a[a^2a^\dagger]^\dagger = a(ap_a)^{\oplus}$. Moreover, we have $x = (xa)x = p_{(ap_a)^*,(ap_a)^{\oplus}}ax \in im(ap_a)^* \subseteq im(ap_a)^*$. According to Theorem 5.3, we complete the proof. $\square$

Theorem 12. *Let $a \in R^{\Diamond}$. Then $a_g^{\Diamond} = a^{(2)}_{im(ap_a)^*,ker(ap_a)^*} = a^{(2)}_{im(ap_a)^*,ker(a^2)^*}$.*

Proof. Set $x = a_g^{\Diamond}$. In view of Theorem 5.2, we have $x = xax$.

Step 1. $im(x) = im(ap_a)^*$. In view of Theorem 4.2, we have

$$\begin{aligned} x &= [a^2a^{\oplus}]^\dagger \\ &= [a^2a^{\oplus}]^\dagger[a^2a^{\oplus}][a^2a^{\oplus}]^\dagger \\ &= \left([a^2a^{\oplus}]\right)^*\left([a^2a^{\oplus}]^\dagger\right)^*[a^2a^{\oplus}]^\dagger \\ &= [aa^{\oplus}a^*]\left([a^2a^{\oplus}]^\dagger\right)^*[a^2a^{\oplus}]^\dagger \in im(ap_a)^*, \end{aligned}$$

$$\begin{aligned} (ap_a)^* &= [a^2a^{\oplus}]^* \\ &= \left([a^2a^{\oplus}]x[a^2a^{\oplus}]\right)^* \in im(x). \end{aligned}$$

Accordingly, we have $im(x) = im(ap_a)^*$.

 doi:10.20944/preprints202507.1321.v1

Step 2. $ker(x) = ker(ap_a)^*$. If $(ap_a)^* r = 0$ for some $r \in R$, then

$$\begin{aligned} xr &= [(ap_a)^{\oplus}]^{\dagger} r \\ &= [(ap_a)^{\oplus}]^{\dagger}[(ap_a)^{\oplus}][(ap_a)^{\oplus}]^{\dagger} r \\ &= [(ap_a)^{\oplus}]^{\dagger}\left([(ap_a)^{\oplus}]^{\dagger}\right)^{*}(p_a a^*) r = 0. \end{aligned}$$

Thus $r \in ker(x)$.
If $x(r) = 0$, then

$$\begin{aligned} (ap_a)^* r &= [(a^2 a^{\oplus}) x (a^2 a^{\oplus})]^* r \\ &= [a^2 a^{\oplus}]^* [a^2 a^{\oplus}](xr) = 0. \end{aligned}$$

Thus, $r \in (ap_a)^*$. As a result, we have $ker(x) = ker(ap_a)^*$.

Therefore $x = a^{(2)}_{im(ap_a)^*, ker(ap_a)^*}$. By the similar way, we check that $x = a^{(2)}_{im(ap_a)^*, ker(a^2)^*}$. This completes the proof. □

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