Article

Not peer-reviewed version

From Littlewood and Fujii to Riemann

Huan Xiao *

Posted Date: 21 January 2026

doi: 10.20944/preprints202501.0292.v2

Keywords: Riemann hypothesis

 doi:10.20944/preprints202501.0292.v2

Article

From Littlewood and Fujii to Riemann

Huan Xiao

School of Artificial Intelligence, Zhuhai City Polytechnic, Zhuhai, China; xiaogo66@outlook.com

Abstract

By using Littlewood's oscillatory theorem and a result of Fujii we disprove the Riemann hypothesis.

Keywords: Riemann hypothesis

---

1. Introduction

The infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$ is a complex number, converges for $\sigma > 1$. The Riemann zeta function is its meromorphic continuation to the whole complex plane. It is well known that the Riemann zeta function $\zeta(s)$ has zeros at negative even integers which are called trivial zeros. The Riemann hypothesis asserts that all zeros in the strip $0 < \sigma < 1$ satisfy $\sigma = 1/2$. For the basic theory of the Riemann zeta function one may refer to [2,4,5,8].

While it is widely believed the Riemann hypothesis might be true, in this note we are going to prove that this is not the case.

Theorem 1. *The Riemann hypothesis is false.*

The main tools in the proof are Littlewood's oscillatory theorem and a result of Fujii [3].

2. Hardy-Littlewood Type Results

Let O be the big O notation and o the little o notation. Let $\Omega = \Omega_{\pm}$ be the big omega notation.

Let $a_1, a_2, \cdots$ be a sequence of real numbers. Delange [[1], p. 60] noticed, by simple arguments, that

$$\sum_{n\le x} a_n = o(x) \ as \ x \to \infty \Longrightarrow \sum_{n=1}^{\infty} a_n x^n = o\left(\frac{1}{1-x}\right) \ as \ x \to 1^-. \tag{1}$$

$$\sum_{n=1}^{\infty} a_n x^n = \Omega_{\pm}\left(\frac{1}{\sqrt{1-x}}\right) \ as \ x \to 1^- \Longrightarrow \sum_{n\le x} a_n = \Omega_{\pm}(\sqrt{x}) \ as \ x \to \infty. \tag{2}$$

The statements of (1) and (2) are results of Hardy-Littlewood type, though they are not explicitly recorded in the book [6].

Using the analogous arguments as in Delange [[1], p. 60] we have

Lemma 1. *Let $a_1, a_2, \cdots$ be a sequence of real numbers.*

$$\begin{aligned}\sum_{n\le x} a_n =& o(x^{3/2}\log\log\log x) \ as \ x \to \infty \\ &\Longrightarrow \sum_{n=1}^{\infty} a_n x^n = o\left(\frac{1}{(1-x)^{3/2}}\log\log\log\frac{1}{1-x}\right) \ as \ x \to 1^-.\end{aligned} \tag{3}$$

doi:10.20944/preprints202501.0292.v2

$$\sum_{n=1}^{\infty} a_n x^n = \Omega_+\left(\frac{1}{(1-x)^{3/2}} \log\log\log \frac{1}{1-x}\right) \text{ as } x \to 1^- \\ \implies \sum_{n \le x} a_n = \Omega_+(x^{3/2} \log\log\log x) \text{ as } x \to \infty. \tag{4}$$

Proof. We first prove the implication (3) and thus suppose $\sum_{n\le x} a_n = o(x^{3/2} \log\log\log x)$. As in [[1], p. 60] we set $A_n = \sum_{k\le n} a_k$ for $n \ge 1$ and $A_0 = 0$, then

$$\sum_{n=1}^{\infty} a_n x^n = (1-x)\sum_{n=1}^{\infty} A_n x^n = (1-x)\cdot o\left(\sum_{n=1}^{\infty}(n^{3/2}\log\log\log n)x^n\right) \\ = o\left(\frac{1}{(1-x)^{3/2}}\log\log\log\frac{1}{1-x}\right) \text{ as } x \to 1^-. \tag{5}$$

As for (4), it is a contrapositive statement of (3). □

Of course one may also show (4) by Delange's argument. That is, by the substitution $x = e^{-u}$ with $u > 0$. We have

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n e^{-nu} = u\int_0^{\infty} e^{-ux} A(x)dx \tag{6}$$

where $A(x) = \sum_{n\le x} a_n$. Then an inequality on limsup leads to (4).

3. Proof of Theorem 1

Let $\Lambda(n)$ be the von Mangoldt function. The following is Littlewood's oscillatory theorem.

Theorem 2 ([7]). *As $x \to \infty$,*

$$\sum_{n\le x} \Lambda(n) = x + \Omega_\pm(\sqrt{x}\log\log\log x). \tag{7}$$

Let

$$r_2(n) = \sum_{\ell+m=n} \Lambda(\ell)\Lambda(m).$$

In his study of the Goldbach conjecture Fujii proved the following result.

Theorem 3 ([3]). *Suppose the Riemann hypothesis is true, then*

$$\sum_{n\le x} r_2(n) = \frac{1}{2}x^2 + O(x^{3/2}). \tag{8}$$

In particular the Riemann hypothesis implies

$$\sum_{n\le x} r_2(n) = \frac{1}{2}x^2 + o(x^{3/2}\log\log\log x). \tag{9}$$

We now return to the proof of Theorem 1.

Proof of Theorem 1. It follows from (7) that as $x \to 1^-$,

$$f(x) := \sum_{n=1}^{\infty} \Lambda(n)x^n = \frac{1}{1-x} + \Omega_+\left(\frac{1}{\sqrt{1-x}}\log\log\log\frac{1}{1-x}\right), \tag{10}$$

doi:10.20944/preprints202501.0292.v2

and thus as $x \to 1^-$,

$$\begin{aligned} f(x)^2 &= \sum_{n=1}^{\infty}\left(\sum_{\ell+m=n}\Lambda(\ell)\Lambda(m)\right)x^n \\ &= \sum_{n=1}^{\infty} r_2(n)x^n \\ &= \frac{1}{(1-x)^2} + \Omega_+\left(\frac{1}{(1-x)^{3/2}}\log\log\log\frac{1}{1-x}\right) + \Omega_+\left(\frac{1}{1-x}\left(\log\log\log\frac{1}{1-x}\right)^2\right) \\ &= \frac{1}{(1-x)^2} + \Omega_+\left(\frac{1}{(1-x)^{3/2}}\log\log\log\frac{1}{1-x}\right). \end{aligned} \tag{11}$$

Thus we have as $x \to 1^-$,

$$\begin{aligned} \sum_{n=1}^{\infty} r_2(n)x^n - \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty}(r_2(n)-(n+1))x^n \\ &= \Omega_+\left(\frac{1}{(1-x)^{3/2}}\log\log\log\frac{1}{1-x}\right). \end{aligned} \tag{12}$$

By (4),

$$\sum_{n\le x}(r_2(n)-(n+1)) = \Omega_+(x^{3/2}\log\log\log x), \tag{13}$$

from which we have

$$\sum_{n\le x} r_2(n) = \frac{1}{2}x^2 + \Omega_+(x^{3/2}\log\log\log x), \tag{14}$$

which contradicts (9). □

Acknowledgments: The first draft of this paper was written during my stay at Nagoya University. Special thanks to the staff of the library of Department of Science of Nagoya University, who kindly allowed me to use this library.

References

1. H. Delange, Sur certaines séries entières particulières, Acta Arith 92, pp. 59-70, 2000.
2. H. M. Edwards, Riemann's Zeta Function. Academic Press. 1974.
3. A. Fujii, An additive problem of prime numbers, Acta Arith 58(2), 173-179, 1991.
4. A. Ivić, The Riemann Zeta Function, New York: John Wiley Sons, 1985.
5. A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function. Berlin: W. de Gruyter.1992.
6. J. Korevaar, Tauberian Theory. A Century of Developments. Grundlehren der Mathematischen Wissenschaften, 329, Springer, Berlin (2004)
7. J. E. Littlewood, Sur la distribution des nombres premiers. C. R. Acad. Sci. Paris Sér. I Math. 158, 1869-1872 (1914)
8. E. C. Titchmarsh, The Theory of the Riemann Zeta Function, 2nd revised edition. Oxford University Press.1986.