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Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality

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Abstract

Massera and Schaffer [*Ann. Math. (2), 1958*] derived a breakthrough upper bound for the Clarkson angle between two nonzero vectors in a normed linear space, which was later improved by Maligranda [*Am. Math. Mon., 2006*]. Pecaric and Rajic [*Math. Inequal. Appl., 2007*] extended Maligranda's inequality to finitely many nonzero vectors. We derive a non-Archimedean version of Massera-Schaffer-Maligranda-Pecaric-Rajic inequality.

Keywords: normed linear space; triangle inequality; non-archimedean linear space; ultra-norm

MSC: 12J25; 46S10

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1. Introduction

Let $\mathcal{X}$ be a normed linear space (NLS). Recall that for $x, y \in \mathcal{X} \setminus \{0\}$, the **Clarkson angle** [1] between x and y is defined as

$$\alpha[x,y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Important Massera-Schaffer inequality [2] says that

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|}, \quad \forall x, y \in \mathcal{X} \setminus \{0\}. \tag{1}$$

Maligranda improved Inequality (1) in 2006.

Theorem 1. *[3,4] (**Massera-Schaffer-Maligranda Inequality**) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,*

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \leq \frac{2\|x-y\|}{\|x\|+\|y\|}. \tag{2}$$

We note that, in 1964 Dunkl and Williams [5] independently showed that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|}, \quad \forall x, y \in \mathcal{X} \setminus \{0\}. \tag{3}$$

Inequality (3) is famously known as Dunkl-Williams inequality in the literature. In 2007, Pecaric and Rajic extended Inequality (2) to finitely many nonzero elements [6].

Theorem 2. *[6] (**Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality**) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$,*

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leq \min_{1 \leq k \leq n} \left\{ \frac{1}{\|x_k\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \right) \right\}.$$

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It is important to ask what are non-Archimedean versions of Theorems 1 and 2? We answer the question by deriving non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality (Theorem 4).

2. Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality

Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \to [0, \infty)$ is said to be a non-Archimedean valuation if following conditions holds.

(i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
(ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
(iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as non-Archimedean valued field [7]. Let $\mathcal{X}$ be a vector space over a non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. A map $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is said to be a non-Archimedean norm if following conditions holds.

(i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
(iii) (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as non-Archimedean linear space (NALS) [8]. Let $\mathcal{X}$ be a NALS. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathcal{X}$. We define the **non-Archimedean Clarkson angle** between x and y as

$$\alpha[x, y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Non-Archimedean version of Theorem 1 now reads as follows.

Theorem 3. *(**Non-Archimedean Massera-Schaffer Inequality**) Let $\mathcal{X}$ be a NALS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathcal{X}$ it holds*

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\max\{\|x - y\|, |\|x\| - \|y\||\}}{\max\{\|x\|, \|y\|\}}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathcal{X}$. Then

$$\begin{aligned}
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \frac{x - y}{\|x\|} + \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) y \right\| \\
&\leq \max\left\{ \left\| \frac{x - y}{\|x\|} \right\|, \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) y \right\| \right\} \\
&= \max\left\{ \frac{\|x - y\|}{\|x\|}, \frac{|\|y\| - \|x\||}{\|x\|} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) x + \frac{x - y}{\|y\|} \right\| \\
&\leq \max\left\{ \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) x \right\|, \left\| \frac{x - y}{\|y\|} \right\| \right\} \\
&= \max\left\{ \frac{|\|y\| - \|x\||}{\|y\|}, \frac{\|x - y\|}{\|y\|} \right\}.
\end{aligned}$$

Therefore

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{1}{\|x\|} \max\{\|x - y\|, |\|x\| - \|y\||\} \tag{4}$$

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and

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq \frac{1}{\|y\|}\max\{\|x-y\|, |\|x\|-\|y\||\}. \tag{5}$$

Inequalities (4) and (5) give

$$\begin{aligned}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &\leq \left(\min\left\{\frac{1}{\|x\|},\frac{1}{\|y\|}\right\}\right)\max\{\|x-y\|, |\|x\|-\|y\||\}\\ &= \frac{\max\{\|x-y\|, |\|x\|-\|y\||\}}{\max\{\|x\|,\|y\|\}}.\end{aligned}$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathcal{X}$ in the previous theorem. This is necessary because, since the norm is a real number, we generally cannot guarantee that it belongs to the given non-Archimedean field. We now derive the non-Archimedean version of Theorem 2.

Theorem 4. *(Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality) Let $\mathcal{X}$ be a NALS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathcal{X}$ it holds*

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \leq \min_{1\leq k\leq n}\left\{\frac{1}{\|x_k\|}\max\left\{\left\|\sum_{j=1}^{n}x_j\right\|, \max_{1\leq j\leq n}|\|x_j\|-\|x_k\||\right\}\right\}.$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathcal{X}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &= \left\|\frac{x_k}{\|x_k\|}+\sum_{j=1, j\neq k}^{n}\frac{x_j}{\|x_j\|}\right\|\\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_k\|}-\sum_{j=1, j\neq k}^{n}\frac{x_j}{\|x_k\|}+\sum_{j=1, j\neq k}^{n}\frac{x_j}{\|x_j\|}\right\|\\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_k\|}-\sum_{j=1, j\neq k}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|\\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_k\|}-\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|\\
&\leq \max\left\{\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_k\|}\right\|, \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|\right\}\\
&= \max\left\{\frac{1}{\|x_k\|}\left\|\sum_{j=1}^{n}x_j\right\|, \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|\right\}\\
&\leq \max\left\{\frac{1}{\|x_k\|}\left\|\sum_{j=1}^{n}x_j\right\|, \max_{1\leq j\leq n}\left\|\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|\right\}\\
&= \frac{1}{\|x_k\|}\max\left\{\left\|\sum_{j=1}^{n}x_j\right\|, \max_{1\leq j\leq n}|\|x_j\|-\|x_k\||\right\}\end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \leq \min_{1\leq k\leq n}\left\{\frac{1}{\|x_k\|}\max\left\{\left\|\sum_{j=1}^{n}x_j\right\|, \max_{1\leq j\leq n}|\|x_j\|-\|x_k\||\right\}\right\}.$$

□

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3. Conclusions

1. In 1958, Massera and Schaffer derived a surprising upper bound for the Clarkson angle between two nonzero elements in a normed linear space [2].
2. In 2006, Maligranda improved Massera-Schaffer inequality [3].
3. In 2007, Pecaric and Rajic extended Maligranda inequality for finitely many nonzero elements [6].
4. In this article, we introduced the notion of non-Archimedean Clarkason angle and derived non-Archimedean version of Massera-Schaffer-Maligranda-Pecaric-Rajic inequality.

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