

Relating a System's Hamiltonian to its Entropy Production Using a Complex-Time Approach

Michael Charles Parker¹ and Chris Jeynes^{2,*}

¹ School of Computer Sciences & Electronic Engineering, University of Essex, Colchester CO4 3SQ; mcpark@essex.ac.uk

² University of Surrey Ion Beam Centre, Guildford GU2 7XH

* Correspondence: c.jeynes@surrey.ac.uk

Abstract: We exploit the properties of complex time to obtain an analytical relationship between the two conserved quantities of the (complexified) Entropy Production and the (complexified) system Hamiltonian. In natural units, the one is simply the Wick-rotated complex-conjugate of the other. A Hilbert transform relation is constructed in the formalism of Quantitative Geometrical Thermodynamics which enables system irreversibility to be handled analytically within a framework that unifies both the microscopic and macroscopic scales, and which also unifies the treatment of both reversibility and irreversibility as complementary parts of a single physical description. In particular the thermodynamics of two unitary entities are considered: the alpha particle which is absolutely stable (that is, trivially reversible with zero entropy production), and the black hole whose absolute disequilibrium (unconditional irreversibility) is characterized by a non-zero entropy production for which we show an alternate derivation confirming our previous one (*Universe* 7, 2021, 325). In this treatment the complexification of time also enables a meaningful physical interpretation of both “imaginary time” and “imaginary energy”.

Keywords: QGT; entropic Hamiltonian; Bekenstein-Hawking relation; analytical continuation; Wick rotation; Loschmidt Paradox; Riemannian geometry; Kramers-Kronig relations

1. Introduction

The physical interpretation of time, and the peculiar characteristics of the Arrow of Time, have long excited scientific curiosity and generated perplexing puzzles. The Second Law of Thermodynamics has a problematical status: is it fundamental or emergent? is irreversibility an emergent property determined at the microscopic level? (We could also ask, again related to the intrinsic nature of time, is the universe essentially non-local?) Currently, the fundamental equations of quantum mechanics (QM) are time reversible and the elusiveness of an *emergent* irreversibility (generating the Second Law) is known as the Loschmidt Paradox. On the other hand, most physicists assert that the Second Law is fundamental. How is this state of affairs to be squared? It transcends the other well-known problem: the inherent incompatibility between the realms of QM (the physics of the very small) and general relativity (GR, the physics of the very large). Here we show how complexifying time may help resolve both these problems.

Imaginary time is invoked in discussions of quantum mechanical tunneling times (see for example McGlynn & Simenel 2020¹), although its physical interpretation remains obscure. But it is surprising that the concept of a complex temporal *plane* has not been suggested until very recently², perhaps because it seems intuitive that time is essentially uni-dimensional, in contrast to the obvious multi-dimensionality of space with its rich geometry. It should be noted that the complexification of time is well-known: Minkowski found that a complex description for spacetime elegantly satisfies the geometrical requirements of special relativity: he defined time as imaginary, in contrast to real space (with unit vectors $it \equiv ix_0/c, x_1, x_2, x_3$ where c is the speed of light, such that the squares of the unit vectors are, respectively, $-1, +1, +1, +1$; or simply “ $-+++$ ”). However, the *inverse* metric for spacetime (real time and imaginary space: “ $+---$ ”) is also valid, and we will show where each metric is physically appropriate.

We also note the clear isomorphism between the Hamiltonian H of a system and its entropy production Π . The Hamiltonian of a system describes its energy content, and on the assumption of a lossless (reversible) physical process the 1st Law of Thermodynamics trivially applies and the Hamiltonian is conserved. This is in accordance with Noether's theorem based on the Euler-Lagrange equation for the Principle of Least Action, exploiting *time* as the key symmetry. In the same way we have previously shown³ that the entropy production must also be a conserved quantity by Noether's theorem, this time applied to the *entropic* Euler-Lagrange equation associated with the isomorphic Principle of Least Exertion⁴ (again using the time symmetry). Thus it is clear that the Hamiltonian and the Entropy Production are isomorphic to each other, both being Noether-conserved quantities. We will show here that they are also directly related, physically.

2. Complex Time Descriptions for Reversibility and Irreversibility

The fundamental kinematic equations:

$$\frac{\partial}{\partial t}\Phi = 2kD\nabla^2\Phi \quad \text{and} \quad -i\frac{\partial}{\partial \tau}\Psi = \frac{h}{4\pi m}\nabla^2\Psi \quad (1)$$

are respectively the diffusion equation (describing the *irreversible* evolution of the probability amplitude Φ , and involving Boltzmann's constant k and the diffusion coefficient D), and Schrödinger's equation (describing the *reversible* evolution of the quantum wave function Ψ , and involving Planck's constant h and the mass m of the particle); note both equations are temporal in character, but we employ different symbols for their respective time variables, since Eq.1a is essentially real in nature, whereas Eq.1b is imaginary. Although different, using the diffusion coefficient definitions of Mita⁵ and Nelson⁶ the above two equations are related by:

$$t \leftrightarrow i\tau \quad \text{and} \quad \frac{h}{4\pi m} \leftrightarrow 2kD \equiv \gamma \quad (2)$$

suggesting that due to the isomorphism between Schrödinger's equation and the diffusion equation (Eqs.1) the real time t is isomorphic to the imaginary time τ . Equivalently, in Eq.2a the factor i ($\equiv \sqrt{-1}$) looks like a Wick rotation (see O'Brien's helpful discussion⁷); while Eq.2b reinterprets the diffusion coefficient D as being like an inverse mass, $D \leftrightarrow 1/2m$ (interpreting the 4π as the appropriate solid angle of the full sphere). The factor 2 that appears in Eqs.1a,2b in front of the Boltzmann constant k is a reflection of an explicitly entropic version of the Partition Function being composed of probabilities that are analogous to the modulus-squared of Schrödinger's equation (see Parker & Jeynes 2021⁸ Eq.14d), as is also described by Córdoba *et al.* 2013⁹ (see their Eqs.29,30).

The Wick rotation was implicitly described (in terms of Hodge duals) in Appendix 3 of Parker & Jeynes 2019 [ref.4], while the notation $\hat{\eta} = 2k\nabla$ was employed by Velazquez¹⁰ to provide a commutator-like derivation of the thermodynamics uncertainty relations (again, notice the presence of the factor of 2 in Velazquez's expression for $\hat{\eta}$). This example illustrates the Wick rotation appearing to map a reversible dynamics into an irreversible one, as already emphasized by Córdoba *et al.* [ref.9].

Thus, the Schrödinger equation can be considered either as a non-dissipative wave equation (reversible), or as a diffusion equation (irreversible). The key distinction between whether the Schrödinger equation describes a reversible or irreversible process lies in whether the associated diffusion coefficient $\gamma \equiv h/2m$ (inversely related to the friction coefficient, and now using the reduced Planck constant $\hbar \equiv h/2\pi$ for convenience) is real or imaginary. In a conventional diffusion process (such as involving Brownian motion) the diffusion coefficient is a real quantity, as described by Nelson [ref.6] for example (see also Fritsche & Haugk¹¹). However, Nelson shows that if the diffusion constant were to be imaginary, then the average diffusion velocity would become imaginary, and a Brownian motion with zero friction is described. As a phenomenological description, the diffusion coefficient is conventionally described using the reduced Planck constant. However, exploiting the isomorphism between Eqs.1,2 (where the Planck and Boltzmann constants

are related to each other via a Wick rotation, $-i\hbar \leftrightarrow 2k$) it is clear that an imaginary diffusion coefficient can also be described: $\gamma = ik/m$. Of course, this begs the question of what determines whether the “diffusion coefficient” is real (describing an irreversible process) or imaginary (reversible).

Taken together, we therefore define physical time as a *complex dynamical variable*: That is to say, rather than considering the (imaginary) parameter τ simply as isomorphic to the real time variable t , we now define τ as the *analytic continuation* of real time t into the complex plane. Since it is convenient (following Minkowski) to assume that conventional (reversible) time t is imaginary, we also rotate the complex time plane by a factor i :

$$z \equiv i(t + i\tau) = -\tau + it \quad (3)$$

In this definition of complex time z , we have therefore *analytically continued* time t into an overall *complex time* measure (using the time parameter τ) followed with an overall rotation by a factor $i \equiv \sqrt{-1}$. Whereas in Eq.(2a) we suggested the isomorphism $t \leftrightarrow i\tau$, in Eq.3 we now explicitly distinguish the real and imaginary components of complex time z , such that t and τ appear essentially independent of each other.

Eq.15 of Velazquez *et al.* 2023 [ref.2] specifies an effective equivalence between the normal Hamiltonian (representing the energy of the system) and the “entropic Hamiltonian”, both of them *complex valued*. It is tempting to identify the “entropic Hamiltonian” with the entropy production, but there is some subtlety since an unconditionally *stable* system (such as an alpha particle, for which the entropy production is identically zero) would appear to be a counter-example, since the Hamiltonian of the alpha is non-zero.

Nevertheless, we will show in what way the (complexified) entropic Hamiltonian of the previous treatment of ref.[2] can be identified with the (complexified) entropy production, and in what way the (energy) Hamiltonian can be identified with the entropy production. This demonstration depends on the properties of analytical continuation, and is therefore a consequence of the systematic complexification of the formalism. It is a deeply surprising result intimately related to the fact that these properties are built on a holomorphic representation of the Maximum Entropy state. Analytical continuation is a powerful technique used in conjunction with Riemannian geometry and holomorphic functions, both of which are essential to this formalism [4].

We will also show that an unconditionally *unstable* system far from equilibrium (such as a black hole, for which the entropy production is non-zero and positive) is related. Quantitative Geometrical Thermodynamics (QGT; Parker & Jaynes 2019 [ref.4]) treatments are available for both the alpha particle (Parker *et al.* 2022¹²) and the black hole (Parker & Jaynes 2021 [ref.3]): both are unitary entities (in QGT terms) being specified by only 4 numbers (mass, spin, charge and a length scale factor).

The (energy) Hamiltonian and the entropy production are associated with both real and imaginary components, each associated with the respective temporal domains of the complex time plane ($z \equiv -\tau + it$):

$$H_z = -\frac{\partial S_{cl}}{\partial t} + i\frac{\partial S_{cl}}{\partial \tau} \equiv H + iH_\tau \quad (4a)$$

$$\Pi_z = \frac{\partial S_{th}}{\partial \tau} + i\frac{\partial S_{th}}{\partial t} \equiv \Pi + i\Pi_t \quad (4b)$$

$$\Pi = \frac{\partial S_{th}}{\partial \tau} \leftrightarrow H = -\frac{\partial S_{cl}}{\partial t} \quad (4c)$$

repeating from ref.[2] for convenience their Eqs.A57 & A58 (Eqs.4a,4b here) and their Eqs.4 (Eqs.4c here), where S_{cl} is the classical action, and S_{th} is the entropy. We emphasise that the symbol H represents the real (reversible) part of the complex Hamiltonian H_z (where the subscript z indicates the complex temporal plane z) and Π is the real (irreversible) part of the complex entropy production Π_z . Note that where Velazquez *et al.* [ref.2] use the term “entropic Hamiltonian”, we use instead the term “entropy production” as a synonym for “rate of entropy increase”, $\Pi \equiv \partial S_{th} / \partial \tau$.

The quantities defining the complex Hamiltonian and entropy production are also related to each other via a pair of Cauchy-Riemann relations ref.[2] Eqs.14, which for convenience we reproduce here:

$$\frac{1}{2k} \frac{\partial S_{th}}{\partial \tau} = -\frac{1}{\hbar} \frac{\partial S_{cl}}{\partial t} \quad \text{and} \quad \frac{1}{2k} \frac{\partial S_{th}}{\partial t} = \frac{1}{\hbar} \frac{\partial S_{cl}}{\partial \tau} \quad (4d)$$

The role of the (complex) Hamiltonian H_z and (complex) Entropy Production Π_z in any physical process must be understood from the trajectory across complex time taken by the physical phenomenon. That is, whether the process is reversible or irreversible (or a mixture) will be determined by how the real and imaginary components are combined at each point in time. We point out parenthetically that the representations we use seem ambiguous at this stage as to whether reversibility is indicated by the real or by the imaginary components: we will clarify this later. However (and separately), it is convenient to follow the usual convention of regarding the imaginary temporal t -axis as the reversible one, in the context of a complex temporal plane providing the comprehensive framework in which both reversible and irreversible processes can be consistently and completely described.

In Eqs.2 the complex expressions for H_z and Π_z are comparable to the expressions used in signal processing for analytic quantities, e.g. photons. QGT [ref.4] shows how meromorphic functions are used to express information, and how holomorphic functions express Maximum Entropy systems. John Toll¹³ explicitly gives a rigorous proof that strict causality is logically equivalent to the existence of the “dispersion relations”, which are best known as practical constraints in signal processing, so that in optics (for example) the refractive index has an imaginary component in the presence of absorption. However, since any particle can be represented as a wave, any scattering process must have a representation in terms of a “frequency distribution”, with the corresponding “group” and “phase” velocities. Toll has shown how the real and imaginary properties of the dispersion are mutually related via the Kramers-Kronig relations, using the properties of the Hilbert transform. Toll further points out that exactly the same formalism is applicable generally: not only to optics but also to high energy particle scattering (citing the “excellent discussion” of the so-called “R-matrix” representation by Wigner¹⁴).

In the optics example, the real dielectric component is associated with zero absorption (a thermodynamically reversible phenomenon) and the imaginary component (absorption, or indeed, amplification) is associated with thermodynamic irreversibility. To be more specific, we note that Fourier theory requires the association of the complex time plane $z \equiv -(\tau - it) = i(t + i\tau)$ with its counterpart (complex-conjugated) complex frequency plane $\hat{\omega} = -i(\omega - iv) = -(v + i\omega)$.

3. Fourier and Hilbert Transform Relations

We define the conjugate frequencies of the respective real and imaginary temporal components consistently using the Fourier transform definition:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega \cdot t} dt \rightarrow F(\hat{\omega}) = \int_{-\infty}^{\infty} f(z) e^{i\hat{\omega} \cdot z} dz \quad (5)$$

That is, both the time t and frequency ω are respectively analytically continued into their appropriate complex planes, with the complex time z and complex frequency $\hat{\omega}$ representing the appropriate conjugate pair: $\hat{\omega} \rightleftharpoons z$, with the consistent Fourier transform relation given above.

For the case when the function $H(t)$ is causal (that is, $H(t)=0$ for $t < t_0$, where t_0 is chosen as a convenient point in time to express the causality of the system) and is a physically realisable (square-integrable) function, then Cauchy’s theorem applies, and the Hamiltonian is holomorphic in the required (upper, as appropriate) half-plane such that it obeys the dispersion relations. Following Toll [ref.6] Eqs.2.5 we can write the dispersion of the

complex Hamiltonian (using the terms of Eq.4a above) in terms of the component ω of the complex frequency $\hat{\omega}$:

$$H(\omega) = +\frac{1}{\pi}P \int_{-\infty}^{\infty} \frac{H_{\tau}(\omega')}{\omega' - \omega} d\omega' \quad (6a)$$

$$H_{\tau}(\omega) = -\frac{1}{\pi}P \int_{-\infty}^{\infty} \frac{H(\omega')}{\omega' - \omega} d\omega' \quad (6b)$$

where “P” is the principal part to be taken at the point $\omega'=\omega$. Note that Toll explicitly emphasises that since the real and imaginary parts of H_z (see Eq.4a) are Hilbert transforms of each other (Eqs.6) they are indeed *causal*. The corollary is that Eq.6a implies Eq.6b and *vice versa*. The integration is done parallel to the ω -axis and the analytical continuation into the upper half-plane exists.

The Cauchy-Riemann relations of Velazquez *et al.* (Eqs.14 of ref.[2], written as functions on complex time) clearly imply complementary relations in terms of the complex frequency. Since frequency is essentially inverse time, there is a very close relationship between the entropy production and (energy) Hamiltonian, where the entropy production (being most closely associated with non-reversible, dissipative processes) is aligned with the irreversible (real) temporal τ -axis, and the Hamiltonian (being most closely associated with reversible, non-dissipative processes) is aligned with the reversible (imaginary) temporal t -axis. Equivalently for the conjugate frequency axes: the entropy production is intrinsically associated with the real frequency ν -axis, while the Hamiltonian is intrinsically associated with the imaginary frequency ω -axis.

When we analytically continue the entropy production or the Hamiltonian from one axis across the complex plane to the orthogonal axis, we exploit the symmetries manifest in the mathematics. The key measurables are the energy (where the Hamiltonian is associated with the reversible axis) and the real part of the entropy production (associated with the irreversible axis). From this perspective, when transforming a quantity from one axis to the other, that is, finding the Hilbert transforms of the entropy production and the (energy) Hamiltonian (such transforms can also be regarded as a kind of Wick rotation), the result is particularly useful for interpreting what might be considered as the ‘cross-axial’ terms (Π_t and H_{τ} ; and we shall see that Parseval’s Theorem as applied to the respective Hilbert transform components also provides additional useful insight into their physical properties). Thus, exploiting the mathematical properties associated with the process of analytical continuation, we may write two symmetrical pairs of expressions for how the complex entropy production function and the complex Hamiltonian function relate along the two conjugate frequency axes forming the complex frequency plane:

$$\Pi_t(\omega) = -i\Pi(\nu) \quad (7a)$$

$$\Pi(\omega) = i\Pi_t(\nu) \quad (7b)$$

$$H_{\tau}(\nu) = -iH(\omega) \quad (7c)$$

$$H(\nu) = iiH_{\tau}(\omega) \quad (7d)$$

Using the Cauchy-Riemann relations of Eq.4d (Eqs.14 in ref.[2]), we can now relate the entropy production values on the real frequency ν -axis to the appropriate energy Hamiltonian values on the imaginary frequency ω -axis:

$$\frac{1}{\hbar}H(\omega) = -i\frac{1}{2k}\Pi(\nu) \quad (8a)$$

$$\frac{1}{\hbar}H(\nu) = i\frac{1}{2k}\Pi(\omega) \quad (8b)$$

$$\frac{1}{\hbar}H_{\tau}(\omega) = i\frac{1}{2k}\Pi_t(\nu) \quad (8c)$$

$$\frac{1}{\hbar} H_{\tau}(v) = -i \frac{1}{2k} \Pi_t(\omega) \quad (8d)$$

Applying Eq.8c and Eq.7b to Eq.6b immediately shows that the Entropy Production component as observed on the (reversible time) t -axis is the Hilbert transform of the corresponding Hamiltonian:

$$\frac{1}{2k} \Pi(\omega) = -\frac{1}{\hbar} \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{H(\omega')}{\omega' - \omega} d\omega' \quad (9)$$

Using Eqs.4a,6b,9 this allows us to express the complex-Hamiltonian $H_z(\omega)$ along the same reversible time axis:

$$\frac{1}{\hbar} H_z(\omega) = \frac{1}{\hbar} H(\omega) + i \frac{1}{2k} \Pi(\omega) \quad (10)$$

$$H_z(\omega) = H(\omega) - i \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{H(\omega')}{\omega' - \omega} d\omega' \quad (11)$$

That is to say, the components of the Hamiltonian and Entropy Production quantities along the (reversible time) t -axis respectively represent the real and imaginary components of an overall *causal* expression (both components representing Noether-conserved quantities).

An exactly similar treatment (compare Eqs.6 with Eqs.12, and Eqs.9,10,11 with Eqs.10,11,12) applies along the (irreversible time) τ -axis (although noting that its 'causal' properties are now in the *negative* temporal direction, as per Eq.3, so that the signs of Eqs.12 are inverse to that of Eqs.6) using the associated (conjugate) real frequency component, v , of the complex frequency plane (where "P" is the principal part to be taken at the point $v'=v$):

$$\Pi(v) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Pi_t(v')}{v' - v} dv' \quad (12a)$$

$$\Pi_t(v) = +\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Pi(v')}{v' - v} dv' \quad (12b)$$

Continuing the aside initiated above, it is also interesting to note that by applying the mathematical argumentation of the Hilbert transform to the *negative* temporal direction of the irreversible τ -axis, inverts the conventional cause and effect relationship that occurs in the forward time direction. That is to say, for the irreversible τ -axis, from the perspective of the normal (forward-propagating) direction in time the ordering is inverted: the effect occurs before the cause. Arguably, this is empirically equivalent (or indistinguishable) to what is observed for apparently spontaneous or random phenomena. That is to say, our treatment of the irreversible τ -axis offers a phenomenologically consistent description with what is conventionally associated with entropy-increasing (irreversible) processes such as radioactivity, decoherence, spontaneous emission and other noise-like phenomena, all of which are ubiquitous in the physical realm. The epistemological implications of this approach (that is, the relationship between what are, in effect, advanced waves and random phenomena) continue to be the topic of future investigation. Thus, we find:

$$-\frac{1}{\hbar} H(v) = \frac{1}{2k} \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Pi(v')}{v' - v} dv' \quad (13)$$

$$\frac{1}{2k} \Pi_z(v) = \frac{1}{2k} \Pi(v) - i \frac{1}{\hbar} H(v) \quad (14)$$

$$\Pi_z(v) = \Pi(v) + i \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi(v')}{v' - v} dv' \quad (15)$$

From Eq.4a we have Eqs.16 (below), and from Eqs.4b,7 we have Eqs.17 (below):

$$H_z(\omega) \equiv H(\omega) + iH_\tau(\omega) \quad (16a)$$

$$H_z(v) \equiv H(v) + iH_\tau(v) \quad (16b)$$

$$\Pi_z(v) \equiv \Pi(v) + i\Pi_t(v) \quad (17a)$$

$$\Pi_z(\omega) \equiv \Pi(\omega) + i\Pi_t(\omega) \quad (17b)$$

From Eqs.7,8 we have

$$H(\omega) = \frac{\hbar}{2k} \Pi_t(\omega) \quad (18a)$$

$$H_\tau(\omega) = \frac{\hbar}{2k} \Pi(\omega) \quad (18b)$$

Hence:

$$H_z^*(\omega) \equiv H(\omega) - iH_\tau(\omega) = \frac{\hbar}{2k} \Pi_t(\omega) - i \frac{\hbar}{2k} \Pi(\omega) \quad (19a)$$

$$H_z^*(\omega) = -i \frac{\hbar}{2k} (\Pi(\omega) + i\Pi_t(\omega)) = \frac{\hbar}{i} \frac{\Pi_z(\omega)}{2k} \quad (19b)$$

Thus we can see that a fundamental identity between the *complex-valued* Hamiltonian and the *complex-valued* Entropy Production is yielded by analytically continuing the expressions of Eqs.10,11,14,15 into the complex-frequency plane $\hat{\omega} = -(v + i\omega)$, and using the Cauchy-Riemann relations: the Hamiltonian and the Entropy Production are related by a Wick rotation and complex conjugation. Similarly, note that the Hamiltonian is usually associated with the reversible (imaginary time) t -axis, whereas the Entropy Production is usually associated with the irreversible (real time) τ -axis. In addition, we see that the complex Entropy Production (associated with thermodynamic irreversibility) is conjugated and Wick rotated with respect to the complex Hamiltonian (associated with thermodynamic reversibility). Thus, in slightly more compact form we can write:

$$H_z^* = -i \frac{h}{4\pi k} \Pi_z \quad (19c)$$

where it is the (Wick rotated) *complex conjugate* of the “entropic Hamiltonian” of ref.[2] that is here reinterpreted (in holographic natural units – that is, over the whole 4π sphere) as simply the Entropy Production.

This symmetrical (conjugate) relationship between the Hamiltonian and Entropy Production underlines again the unity of the physical phenomena of thermodynamic reversibility and irreversibility when viewed from the perspective of complex time. Both processes (mutually being Hilbert transforms of each other) are now seen to exhibit fundamental (Noether) conservation properties based on an equivalent variational calculus principle. Using such methods, we also expect to obtain new insights into the acausal (random) phenomena associated with entropy production such as radioactive decay and the apparent indeterminism of quantum mechanical measurement.

Equation 19c also indicates that although the complex time plane is defined by the (reversible) t -axis and the orthogonal (irreversible) τ -axis, yet these axes are not independent. That is to say, the Hamiltonian (usually defined along the reversible t -axis) and the Entropy Production (defined in effect, empirically, along the irreversible τ -axis) are essentially two sides of the same coin, being Hilbert transform related (see Eqs.10,11,14,15). What this means is that the choice of which axis to use to *fully* describe any physical phenomenon is somewhat arbitrary: depending only on the choice of metric. Either metric (+— or +++, when considering the whole of 1+3 Minkowski spacetime) may be used; provided it is consistently applied. Conventionally (and here), the reversible metric for

time, the first component in $(-+++)$, is employed, indicating its imaginary nature, where the energy Hamiltonian is the physical quantity (with real measurable values) that is used to quantify the phenomenon, and the entropy production (emerging now as the conjugate physical quantity of interest) is treated as an imaginary-valued quantity (when viewed from the reversible t -axis). However, if the inverse metric $(+---)$ is used for time, then the entropy production becomes the 'real-valued' physical quantity to be measured, with the energy Hamiltonian now imaginary. Both descriptions are equally valid, but our analysis shows that once a metric is adopted then the primary associated temporal axis is thereby inevitably defined; and either Eq.10 or Eq.14 may be employed, but never both in the same analysis.

That is, we can now simplify the relevant (cross-axial) components of the complex Hamiltonian and Entropy Production in Eqs.4:

$$H_z = -\frac{\partial S_{cl}}{\partial t} + i \frac{\partial S_{cl}}{\partial \tau} \equiv H + i \frac{\hbar}{2k} \Pi \quad (20a)$$

$$\Pi_z = \frac{\partial S_{th}}{\partial \tau} + i \frac{\partial S_{th}}{\partial t} \equiv \Pi + i \frac{2k}{\hbar} H \quad (20b)$$

Similarly, we express the Hilbert transform relationships between the Hamiltonian and Entropy Production, simplifying Eqs.9,13, and for convenience employing the more conventional (familiar), imaginary (reversible) ω -component of the complex frequency:

$$\Pi(\omega) = -\frac{2k}{\pi\hbar} \mathcal{P} \int_{-\infty}^{\infty} \frac{H(\omega')}{\omega' - \omega} d\omega' \quad (21a)$$

$$H(\omega) = \frac{\hbar}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi(\omega')}{\omega' - \omega} d\omega' \quad (21b)$$

Similar relations exist for the real (irreversible) ν -component of the complex frequency:

$$\Pi(\nu) = -\frac{2k}{\pi\hbar} \mathcal{P} \int_{-\infty}^{\infty} \frac{H(\nu')}{\nu' - \nu} d\nu' \quad (21c)$$

$$H(\nu) = \frac{\hbar}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi(\nu')}{\nu' - \nu} d\nu' \quad (21d)$$

Thus, our analysis interprets the concept of imaginary energy as the imaginary component to the Hamiltonian, equivalent to (a real) entropy production; similarly, any imaginary component to the entropy production can simply be regarded as an energy term.

4. Application: the Alpha Particle

A system such as the alpha particle which is unconditionally stable (it doesn't decay; the alpha has a QGT treatment [ref.12]) comes into existence at some time in the past (at its creation) and then has a *constant* (non-time-varying) energy Hamiltonian H_α . It is also independent of frequency so that its associated entropy production Π_α must also be zero (as expected for an unconditionally stable system: see the Hilbert transform of Eq.21a). Note that being reversible, the Lagrangian line integral (in the z -plane) for the alpha remains firmly parallel to the temporal t -axis associated with reversibility. In the case of a freely-moving alpha particle, its Hamiltonian (in the absence of any potential fields) is given below by Eq.22a, where m is the alpha particle mass and p is its momentum, such that its associated entropy production is given by Eq.22b (using Eq.21a):

$$H_\alpha = \frac{p^2}{2m} = -\left(\frac{\hbar^2}{2m}\right) \nabla^2 \quad (22a)$$

$$\Pi_{\alpha}(\omega) = -\frac{2k}{\pi\hbar} \mathcal{P} \int_{-\infty}^{\infty} \frac{H_{\alpha}}{\omega' - \omega} d\omega' = -\frac{2kH_{\alpha}}{\pi\hbar} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{\omega' - \omega} d\omega' = 0 \quad (22b)$$

since the Hilbert transform of a constant quantity is simply zero according with the unconditional stability of the alpha particle. The inverse Hilbert transform can also be undertaken, to yield the alpha Hamiltonian. However, the inverse Hamiltonian of zero is also zero. But in this case, similar to the case of the real part of the refractive index, the Hilbert transform (Kramers-Kronig) relations only refer to the *variable* part of the overall physical expression and ignore any constant aspects to the physical quantity. In which case, the more general expressions for the Hamiltonian and entropy production of a process should also include constant (d.c.) components thus:

$$\Pi(\omega) = \Pi_0 - \frac{2k}{\pi\hbar} \mathcal{P} \int_{-\infty}^{\infty} \frac{H(\omega')}{\omega' - \omega} d\omega' \quad (23a)$$

$$H(\omega) = H_0 + \frac{\hbar}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi(\omega')}{\omega' - \omega} d\omega' \quad (23b)$$

Where the quantities Π_0 and H_0 (the subscript '0' signifying the zero-frequency or d.c. value) are constants independent of any frequency variation. In the case for the alpha particle which is unconditionally stable we must also have $\Pi_0=0$. In which case, the inverse Hilbert transform of $\Pi_{\alpha}(\omega)$ is zero (i.e. the 2nd term associated with the RHS of Eq.23b is zero) so that the Hamiltonian of the alpha is simply given by the constant H_0 which is independent of frequency, and which we therefore simply set to $H_0=H_{\alpha}=p^2/2m=-(\hbar^2/2m)\nabla^2$ (see Eq.22a).

5. Application: the Black Hole

The conjugate system (featuring an unconditionally *irreversible* physical phenomenon with a constant entropy production, see Eq.21b) must have a Hamiltonian with zero variation with frequency. Such a physical system is represented by a black hole of mass M_{BH} , which has a constant entropy production associated with the Hawking radiation (see Parker & Jeynes [ref.3] Eq.25):

$$\Pi_0 = \frac{c^3 k}{2GM_{\text{BH}}} \quad (24)$$

where G is the gravitational constant and c is the speed of light. Substituting Eq.24 into Eq.23b yields, as for Eq.22b:

$$H(\omega) = H_0 + \frac{\hbar}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi}{\omega' - \omega} d\omega' = H_0 + \frac{\hbar\Pi_0}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{\omega' - \omega} d\omega' = H_0 \quad (25)$$

The Hamiltonian for a stationary black hole is conventionally given by its mass M_{BH} within the Schwarzschild radius; yet, as is well known, any Hamiltonian can also be offset by a constant quantity (there being no absolute value for the energy, see Caticha 2021¹⁵) so that the value of the associated Hamiltonian as determined by Eq.23b can therefore be straightforwardly offset, in effect by $H_0=M_{\text{BH}}c^2$ as required. Note, such an offset is similarly seen in the Kramers-Kronig expression for the real part of the refractive index, which is always offset by unity in order to attain the correct Hilbert Transform relationships (see as an example the unity offset in Eq.1.1 of Toll [ref.5]). That is to say, we can now rewrite Eq.25 as:

$$H = M_{\text{BH}}c^2 + \frac{\hbar}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Pi}{\omega' - \omega} d\omega' = M_{\text{BH}}c^2 + \frac{\hbar\Pi_0}{2\pi k} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{\omega' - \omega} d\omega' = M_{\text{BH}}c^2 \quad (26)$$

We note that that Parseval's theorem applies to the two Hilbert transform terms that vary with frequency:

$$\frac{1}{\hbar^2} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{1}{4k^2} \int_{-\infty}^{\infty} |\Pi(\omega)|^2 d\omega \quad (27)$$

Since it can be shown that for the black hole both the Hamiltonian and Entropy Production are essentially independent of frequency (and using Eqs.18), Eq.27 simplifies:

$$\frac{H^2}{\hbar^2} = \frac{\Pi^2}{4k^2} \quad (28)$$

Eq.28 appears to be inconsistent with Eqs.24,26, but the Hilbert transform is irregular for constant terms: the inverse Hilbert transform cannot recover the constant term since the Hilbert transform of a constant (including zero) is zero. Moreover, frequency-independency was also assumed for Eq.28 but the Hawking radiation follows a black-body frequency distribution.

However, consideration of the earlier Eq.18a suggests that the following, related, identity is also true:

$$H^2 \equiv \frac{\hbar^2}{4k^2} \Pi_t^2 \quad (29)$$

That is to say, considering the black hole as a frequency-independent phenomenon, a 'trans-axial' entropy production Π_t would be more appropriate for the Hamiltonian from Eq.26 ($H=M_{\text{BH}}c^2$). In other words, the *other* Hamiltonian and Entropy Production terms (see Eqs.18) associated with a black hole should also be explicitly considered. This is particularly appropriate considering the extreme physical conditions existing at the black hole Schwarzschild radius, especially since this event horizon is associated with a hyperbolic boundary of spacetime beyond which the metric is assumed to invert. Then from Eqs.28,24:

$$H = \frac{\hbar}{2k} \Pi = \frac{c^3 \hbar}{4GM_{\text{BH}}} \quad (30)$$

We now have two (different) evaluations of H from Eqs.26,30: taking their product (Eq.29) yields an expression featuring the cross-axial entropy production:

$$H(M_{\text{BH}}c^2) = \frac{c^5 \hbar}{4G} = \frac{\hbar^2}{4k^2} \Pi_t^2 \quad (31)$$

and hence:

$$\Pi_t = \sqrt{\frac{c^5}{\hbar G}} k \quad (32)$$

which is the same as the imaginary (cross-axial) entropy production term given previously (Parker & Jeynes [ref.4] Eq.31: the factor 2π comes from an ambiguity in the definition of wavenumber). Thus we find (as already discussed in [ref.4]) that associated with a black hole there exists another very large entropy production term Π_t (Eq.32) that is 46 orders of magnitude larger than the term associated with the Hawking radiation Π_0 (Eq.24), and which can be understood as being related to the highly energetic processes seen occurring in the accretion disk surrounding a black hole.

Finally, it is also of interest to note that Eq.18b additionally suggests the existence of a cross-axial Hamiltonian (energy) term (i.e. an alternative rendering of Eq.30) given by:

$$H_\tau = \frac{\hbar}{2k} \Pi = \frac{c^3 \hbar}{4GM_{\text{BH}}} \quad (33)$$

Such an energy term is of a similar (very small) size to the Hawking radiation, and is insignificant compared to the Hamiltonian of the black hole given by its mass (Eq.26).

6. Discussion

Previously the Entropy Production has been simply identified with the Hamiltonian (in natural units): see Eq.15 in [ref.2]. In this complexified analysis we now identify the Entropy Production with the (Wick-rotated) complex-conjugate of the Hamiltonian (Eq.19c). The purpose in the present analysis is to make full use of the power of the Hilbert transform, which requires a fully complexified formalism.

We note that in order for the Entropy Production to be simply the Wick-rotated complex-conjugate of the Hamiltonian (in natural units: Eq.19c) the Planck constant must be reduced by 4π (rather than the conventional 2π). On the one hand, reducing by 4π is appropriately suggestive of the holographic basis of the QGT analysis (and would also avoid the apparent 'paradox' of half-integer spins in a quantum theory); however, rejigging quantum theory to use a 4π -reduced Planck constant in order to achieve consistent natural units between the kinematic and entropic domains would cause confusion in the conventional (historical and widely accepted) formalisms of kinematic quantum theory. Rather, the factor of two must therefore be simply understood and accepted as a historical fact from the development of quantum theory, but which doesn't undermine the assertion of the natural relationship between the entropy production and the conventional Hamiltonian. The alpha particle and the black hole each represent unitary physical systems (both requiring only four parameters for their full description), yet the former is unconditionally reversible while the latter is unconditionally irreversible. It is interesting to note that such complementary physical phenomena both lend themselves to the Hilbert transform analysis of the expressions for their Hamiltonian and entropy production (provided of course these are complexified).

The well-known Kramers-Kronig dispersion relations are directly derived from the Hilbert transform pairs but based on additional symmetry constraints (the real function being even for real frequencies, whereas the imaginary function is odd; the so-called *crossing conditions* in quantum mechanical scattering). With a minimum of assumptions about the physical process under investigation, the Kramers-Kronig relations provide a complete description between two conjugate phenomena using only the empirically accessible positive frequencies, and their application to the system Hamiltonian and entropy production is therefore the subject of future work.

The metric of complex time also has interesting implications in the context of black hole thermodynamics. In particular, it is clear that 'conventional' reversible (imaginary) time has a metric of negative sign ($-+++$), whereas the irreversible (real) time is associated with a metric of positive sign ($++--$). Normally, the sign of the metric is indicative of whether the dimension is timelike (negative) or spacelike (positive) in character. However, here we see that the sign of the metric is also indicative of whether a physical phenomenon is thermodynamically reversible or irreversible. It is well known that the metric changes sign either side of the vicinity of the event horizon of a black hole. Misner *et al.*¹⁶ in interpreting the sign change pose the question: 'What does it mean for r [radius] to "change in character from a spacelike coordinate to a timelike one"?' and offer the most obvious answer as "the reversal there of the roles of t [time] and r as timelike and spacelike coordinates." But now we see an alternative interpretation as the metric changes sign: the spacetime geometry changes from an intrinsically reversible to a thermodynamically irreversible geometry. Interestingly, this is still consistent with the conventional interpretation of the unforgiving 'point of no return' beyond the event horizon. But now we couch it in specifically thermodynamic terms, highlighting the intrinsic irreversibility of the geometry beyond the event horizon (with the inevitable increase in entropy according to the 2nd Law). This has the interesting corollary that the role reversal of the spacelike and timelike coordinates may no longer be strictly necessary. However it is clear that inverting the metric across the event horizon can now be simply interpreted as the transformation of what were reversible processes into irreversible processes (and *vice versa*), with the additional interesting implication that what were irreversible phenomena this side of the event horizon become

reversible ones beyond it. The implications of such a *thermodynamic inversion* at the event horizon lie beyond the scope of this paper.

Phenomenologically, the conventional Hamiltonian of a system (representing the energy) is assumed to be positive-definite (like the inertial mass) because it is considered from the perspective of the real (reversible) time axis. Similarly, the entropy production (the rate of increase in entropy) considered from the perspective of the imaginary (irreversible) time axis is also positive-definite (as required by the 2nd Law). Conversely: considered from the reversible time axis the entropy production is imaginary, and considered from the irreversible time axis the energy is imaginary. Switching between the two axes defining the complex temporal plane is effected by a Wick rotation; as represented by the factor i in Eq.19c. And to obtain the correct relationship between the system's Hamiltonian and its entropy production (two equivalent descriptions, in natural units) a complex conjugation is also needed. In effect, Eq.19c expresses an intrinsic redundancy: either the (complexified) Hamiltonian or the (complexified) entropy production can be used to fully describe the evolution of the system; and each is sufficient by itself to completely define its evolution in time.

Note that if purely real definitions of the Hamiltonian and entropy production are used, then *both* are required to fully describe a system. But by allowing them to be complex-valued, all the information of the 'real' entropy production becomes embedded into the imaginary component of the Hamiltonian; and *vice-versa*, all the information of the 'real' Hamiltonian is to be found in the imaginary component of the entropy production. Together, this now allows us to interpret 'imaginary energy' as simply the entropy production associated with a process, and conversely, the 'imaginary entropy production' of a process is simply its energy content. The Hamiltonian (energy) of a system is usually defined as the total energy of an essentially lossless system (that is, all energy is explicitly accounted for and conserved according to the 1st Law) such that the energy is calculated to be purely real in nature. For such lossless (and therefore reversible) systems, the entropy stays constant and the entropy production is zero. Allowing energy to be lost from a system due to dissipative processes requires a complex Hamiltonian with an associated finite entropy production.

We are so familiar with the (generally implicit) assumption of reversibility as the underlying description of a process that we automatically employ the 'reversible' (and, incidentally, imaginary; although this is generally neither explicitly acknowledged nor recognised) temporal axis to describe physical processes (together with real energies); perhaps this is why it has taken so long to identify the "imaginary" counterpart of energy, the entropy production (also conserved). However, this is just convention. Had the physicists of the 19th century been able to explicitly quantify the dissipations of all parts of a process (rather than merely reasoning qualitatively that dissipation was simply an undesired result of "*not fulfilling the criterion of a 'perfect thermodynamic engine'*"¹⁷), they might have identified the law of conservation of entropy production as the 1st Law of the Thermodynamics first, and then puzzled about the physical meaning of an imaginary aspect of the entropy production... We underline here the importance of the Minkowski space metric $(-+++)$ and its implicit assumption of thermodynamic reversibility. The alternate metric $(+---)$ has the implicit assumption of thermodynamic irreversibility. Note that Roger Penrose¹⁸ (2004, §17.8) also points out that the choice of metric depends on one's purpose.

It is clear that the physical time associated with any physical phenomenon can only be uni-valued as it evolves over time (monotonically increasing) in its trajectory across the complex temporal plane. That is to say, even though here we invoke the concept of a complex temporal plane, the real and imaginary components are not independent of each other but must also obey the familiar Hilbert transform relations described here in detail. Any real time τ can only ever be identified with a single imaginary time t . Complex time (real or imaginary depending on the reversibility/irreversibility of the process) must increase inexorably according to the 2nd Law. However, in addition to describing time as complex, considerations of chirality cannot be neglected since trajectories (not in a straight line) across the temporal plane must locally tend to have either a clockwise or

anticlockwise character (however weak), according to their local gradient or indeed inflection. A natural chirality to physical geometry has already been noticed from a QGT perspective, since Parker & Walker already showed in 2010¹⁹ that natural DNA must be right-handed (see also the more rigorous treatment of Parker & Jeynes [ref.4] Appendix A) according to the 2nd Law. It is becoming clear that the complex conjugate operation can be thought to build chirality into the formalism (see for example the “Berry phase”²⁰ analysis by Laha *et al.*²¹ of chiral waveguide responses). How such chirality in the complex time plane physically manifests itself (and allows itself to be observed) must continue to be a topic of further investigation.

Finally, in applying our approach to the alpha particle (associated with physics on the microscopic scale) and to a black hole (a macroscopic phenomenon) shows how QGT can successfully handle physical phenomena at all scales. In previous work [ref.3,4] we have already indicated how QGT’s underlying basis in hyperbolic spacetime implicitly allows phenomena over all scales to be appropriately treated, although the hyperbolic (essentially, logarithmic) nature of entropic processes hasn’t needed to be explicitly invoked in this paper. However, we see here that the physics of reversible and irreversible processes can now be elegantly integrated via the use of complex time into the microscopic scale; yet without a noticeable micro/macro boundary, such that macroscopic scale processes are also seamlessly handled. Together, this continues to indicate the unity of description for physical phenomena that geometrical thermodynamics (QGT) has to offer, particularly with regard to a theory of scale relativity²².

7. Conclusions

Just as energy and entropy are closely related, so the Hamiltonian and the entropy production are also closely related. However, whereas the conventional thermodynamics considers energy and entropy to be the most-closely related (and thereby quasi-isomorphic) quantities, simply related to each other via the temperature acting as a coupling coefficient, in this work we find that it is the entropy production (rate of increase in entropy) that is the actual isomorph to energy.

Using analytical continuation into a fully complexified representation (in which time also is complex), we show that the natural (Hilbert transform) expressions of the various quantities display these close relations very clearly. This has allowed a detailed calculation of the entropy production of black holes (consistent with a previous such calculation [ref.3] but apparently independent of it), as a simple example of a system with a non-zero entropy production.

Fully complexifying the formalism with appropriate consideration of causal properties immediately allows the application of the Hilbert transform, which in turn enables very simple relationships to be displayed. In particular, irreversible systems are treated in exactly the same way as reversible ones: this allows irreversible systems to be treated as such (invoking their own conservation law based on the application of Noether’s theorem to quantities related to the entropy production) rather than having to use perturbation theory on the formalisms for reversible systems. (And of course, it is well known that application of perturbation theory, on its own, can never recapture the full richness of mathematical behaviours associated with analytic functions, which feature local and global (essentially, non-local) characteristics (holomorphism) and allow the process of analytic continuation.) Understanding the intrinsic role of thermodynamic irreversibility along with its conservation law in the physics of the universe, over and above the conventional understandings relating to energy and its conservation, will help provide critical insights into the resolution of at least some of the paradoxes currently recognized within contemporary physics.

Acknowledgments: The authors acknowledge extensive and helpful conversations with Prof. Luis-beris Velazquez during the preparation of this paper.

References

- ¹ Patrick McGlynn, Cédric Simenel, Imaginary-time mean-field method for collective tunnelling, *Physical Review C* **102** (2020) 064614 (9pp); <https://doi.org/10.1103/PhysRevC.102.064614>
- ² Luisberis Velazquez, Michael Charles Parker, Chris Jeynes, Irreversibility in Geometrical Thermodynamics (submitted to *Entropy*), *Preprints* 2022, 2022070096; <http://dx.doi.org/10.20944/preprints202207.0096.v1>
- ³ Michael C. Parker, Chris Jeynes, A Relativistic Entropic Hamiltonian–Lagrangian Approach to the Entropy Production of Spiral Galaxies in Hyperbolic Spacetime, *Universe* **7** (2021) 325 (15pp); <https://doi.org/10.3390/universe7090325>
- ⁴ M.C.Parker, C.Jeynes, Maximum Entropy (Most Likely) Double Helical and Double Logarithmic Spiral Trajectories in Space-Time, *Scientific Reports* **9** (2019) 10779 (10 pp, 44 pp Appendices); <http://dx.doi.org/10.1038/s41598-019-46765-w>
- ⁵ K. Mita, Schrödinger's equation as a diffusion equation, *American Journal of Physics* **89** (2021) 500-510; <http://dx.doi.org/10.1119/10.0002765>
- ⁶ Edward Nelson, Derivation of the Schrödinger equation from Newtonian mechanics, *Physical Review* **150** (1966) 1079-1085; <http://dx.doi.org/10.1103/PhysRev.150.1079>
- ⁷ D.M.O'Brien, The Wick Rotation, *Australian Journal of Physics* **28** (1975) 7-13; <https://doi.org/10.1071/PH750007>
- ⁸ M. Parker, C. Jeynes, Entropic uncertainty principle, partition function and holographic principle derived from Liouville's Theorem, *Physics Open* **7** (2021) 100068 (11pp); <https://doi.org/10.1016/J.PHYSO.2021.100068>
- ⁹ P Fernández de Córdoba, J M Isidro, Milton H Perea, Emergence from irreversibility, *Journal of Physics: Conference Series* **442** (2013) 012033 (8pp); <http://dx.doi.org/10.1088/1742-6596/442/1/012033>
- ¹⁰ Luisberis Velazquez Abad, Principles of classical statistical mechanics: A perspective from the notion of complementarity, *Annals of Physics* **327** (2012) 1682-1693; <https://doi.org/10.1016/j.aop.2012.03.002>
- ¹¹ L. Fritsche, M. Haugk, A new look at the derivation of the Schrödinger equation from Newtonian mechanics, *Annalen der Physik* **12** (2003) 371–403; <https://doi.org/10.1002/andp.200310017>
- ¹² M.C.Parker, C.Jeynes, W.N.Catford, Halo Properties in Helium Nuclei from the Perspective of Geometrical Thermodynamics, *Annalen der Physik* **534** (2022) 2100278 (11pp); <http://dx.doi.org/10.1002/andp.202100278>
- ¹³ John S. Toll, Causality and the Dispersion Relation: Logical Foundations, *Physical Review*, **104** (1956) 1760-1770; <https://doi.org/10.1103/PhysRev.104.1760>
- ¹⁴ E.P. Wigner, On the Development of the Compound Nucleus Model, *American Journal of Physics* **23** (1955) 371-380; <http://dx.doi.org/10.1119/1.1934007>
- ¹⁵ A. Caticha, Entropy, Information, and the Updating of Probabilities, *Entropy* **23** (2021) 895 (20pp); <https://doi.org/10.3390/e23070895>
- ¹⁶ Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, (Princeton University Press, 2017), §31.3: Schwarzschild Geometry
- ¹⁷ W. Thomson, On a Universal Tendency in Nature to the Dissipation of Mechanical Energy, *Proc. Roy. Soc. Edinburgh* **3** (1857) 139-142; <https://doi.org/10.1017/S0370164600027541>
- ¹⁸ Roger Penrose, *The Road to Reality* (Jonathan Cape, 2004)
- ¹⁹ Michael C. Parker, Stuart D. Walker, A Dynamic Model of Information and Entropy, *Entropy* **12** (2010) 80-88; <https://doi.org/10.3390/e120100802>
- ²⁰ C.Jeynes, How “Berry Phase” Analysis of Non-Adiabatic Non-Hermitian Systems Reflects Their Geometry, *Entropy* **25** (2023) 390 (10pp); <https://doi.org/10.3390/e25020390>
- ²¹ Arnab Laha, Sibnath Dey, and Somnath Ghosh, Reverse-chiral response of two T-symmetric optical systems hosting conjugate exceptional points, *Phys. Rev. A* **105** (2022) 022203; <https://doi.org/10.1103/PhysRevA.105.022203>

-
- ²² C. Auffray, L. Nottale, Scale relativity theory and integrative systems biology: 1 Founding principles and scale laws, *Prog. Biophys. Mol. Biol.* **97** (2008) 79-114; <https://doi.org/10.1016/j.pbiomolbio.2007.09.002>