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Geometric Interpretation of the Minkowski Metric

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In this article it is shown, first, that a change of basis in Minkowski space is equivalent to a change of basis in Euclidean space if a basis element is replaced by its dual element, constituting a mixed basis set.

Second, that such mixed bases can be interpreted as a measurement system used by local, flat observers whose direct distance measurements are restricted to an $(n-1)$-dimensional submanifold in an n-dimensional Euclidean space.

Combining these steps, it is concluded that a local, flat observer in a four-dimensional Euclidean space measures a Minkowski spacetime. This interpretation could be useful for theories in special relativity and related fields that rely on spacetime concepts, as it offers a more intuitive geometric understanding of the Minkowski metric.

I. INTRODUCTION

Relativistic covariant theories contain an intrinsic difficulty in the interpretation of length measurements. Consider the scalar product on a Minkowski space $\mathbb{R}(3,1)$, defined by:

$$\mathbf{dv} \cdot \mathbf{dw} = \sum_{\mu\nu} g^M_{\mu\nu} dv^\mu dw^\nu \quad \text{with} \quad g^M_{\mu\nu} = \text{diag}\,(-1,+1,+1,+1)\,. \tag{1}$$

This measurement rule is not positive definite, which implies that the norm induced by the scalar product $\|\mathbf{dv}\| = \sqrt{\mathbf{dv}\cdot\mathbf{dv}}$ can assume imaginary values (Figure 1). Therefore imaginary distances between two points can occur in this vector space, and there is no intuitive geometric interpretation of such occurrences. A direct trigonometric interpretation in the Euclidean sense is not possible.

This is an unsatisfactory situation for concepts that seek geometric interpretations of spacetime (e.g. as in [1] and references therein, [2], [3], [4], [5]). Moreover, any framework that relies on spacetime concepts could benefit from a new geometric interpretation of the Minkowski metric by making complex notions more accessible and visually understandable.

In order to develop an interpretation of the negative sign in the metric tensor $g^M_{\mu\nu}$ (Eq. 1), the following steps are performed:

It is first shown in Section II that the Minkowski metric could appear when a four-dimensional Euclidean space is measured with a mixed basis $\{\mathbf{e^0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. As a possible interpretation, Section III considers the hypothesis of a flat observer that has no spatial extent in one dimension, but who may still be able to measure the additional dimension indirectly by using a contravariantly transforming scale.

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II. MAPPING BETWEEN SPACES WITH MIXED COVARIANT AND CONTRAVARIANT BASES

A. Basics and notation

The following proof requires a precise notation for co- and contravariant transforming objects. Since there are different conventions, a brief summary of the nomenclature used in this article is given.

Co- and contravariant vectors are written in bold, second order tensors are written in bold and have capital letters; Greek indexes run from 0 to 3, Latin indexes from 1 to 3, unless otherwise noted.

Let $\mathbb{R}^4$ be a four-dimensional Euclidean vector space with states $\mathbf{v} \in \mathbb{R}^4$. These states are expressed as a linear combination of a canonical basis $\{\mathbf{e_\mu}\} \in \mathbb{R}^{4*}$ with elements from the dual space (orig. [6], available at [7]; introductions e.g. in [8], [9]):

$$\mathbf{v} = \sum_\mu \mathbf{e_\mu} x^\mu \qquad \text{where}$$
$$x^\mu = (\mathbf{v}\cdot\mathbf{e^\mu}) = \sum_\nu \left(v_\nu \cdot (e^\mu)^\nu\right) \qquad \mathbf{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{2}$$

The scalar coefficients x^μ are called the coordinates of state $\mathbf{v}$ with respect to the basis $\{\mathbf{e_\mu}\}$. They can be summarized as a coordinate vector $\mathbf{x}$ (Eq. 2 last term). The coordinates are generated by the dual basis $\{\mathbf{e^\mu}\} \in \mathbb{R}^4$, which is defined by the Euclidean metric tensor $g^E_{\nu\xi}$:

$$\left(e_\mu\right)_\nu = \sum_\xi g^E_{\nu\xi}\left(e^\mu\right)^\xi \qquad \text{with} \qquad g^E_{\nu\xi} = \delta_{\nu\xi}. \tag{3}$$

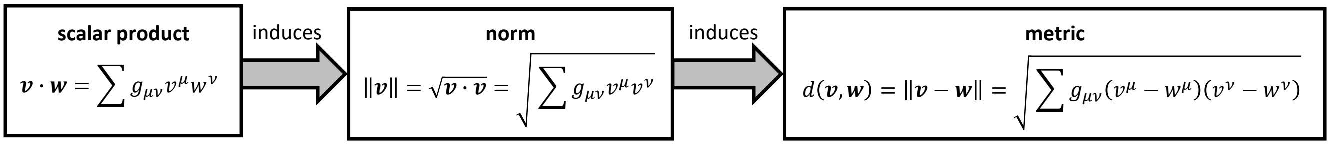

FIG. 1. Relation between scalar product, induced norm and metric.

Analogously, any dual state $\mathbf{w}\in\mathbb{R}^{4*}$ can be expressed with respect to a canonical dual basis:

$$\begin{aligned}\mathbf{w} &= \sum_{\mu} y_{\mu}\mathbf{e}^{\boldsymbol{\mu}} \qquad \text{where}\\ y_{\mu} &= \left(\mathbf{e}_{\boldsymbol{\mu}}\cdot\mathbf{w}\right)=\sum_{\nu}\left(\left(e_{\mu}\right)_{\nu}\cdot w^{\nu}\right) \qquad (4)\\ \mathbf{y} &= \left(y_0\ \ y_1\ \ y_2\ \ y_3\right).\end{aligned}$$

The coefficients y_{μ} are called dual coordinates of the dual state $\mathbf{w}$ with respect to the dual basis $\{\mathbf{e}^{\boldsymbol{\mu}}\}$. They can be consolidated as a dual coordinate vector $\mathbf{y}$ (Eq. 4 last row). Thus, scalar and tensor products can be defined directly:

$$\mathbf{y}\cdot\mathbf{x}=\sum_{\mu} y_{\mu}x^{\mu}\in\mathbb{R} \qquad \mathbf{x}\otimes\mathbf{y}=\mathbf{M}\in\mathbb{R}^{4\times4} \qquad \text{with}\quad {M^{\mu}}_{\nu}=x^{\mu}y_{\nu}\,. \qquad (5)$$

The state $\mathbf{v}$ does not vary when there's a change of basis. For this to be the case, the coordinates $\mathbf{x}$ must change inversely to the basis.

Orientation preserving changes of orthonormal bases in four-dimensional Euclidean space are isomorphic to the Lie-group $SO(4)$, which can be represented by the special orthogonal matrices of fourth order $SO(4)=\{\mathbf{R}\in GL(4)\mid \mathbf{R}^{\mathbf{T}}\mathbf{R}=\mathbb{1},\ \det\mathbf{R}=1\}$. Consider the change of basis from $\{\mathbf{e}_{\boldsymbol{\nu}}\}$ to $\{\mathbf{e}'_{\boldsymbol{\mu}}\}$. The elements of the new basis can be written as a linear combination of the elements of the old basis:

$$\mathbf{e}'_{\boldsymbol{\mu}}=\sum_{\nu}\mathbf{e}_{\boldsymbol{\nu}}\ {R_{\mu}}^{\nu}\,. \qquad (6)$$

This translates into the following transformation rule for coordinate vectors (overall and componentwise):

$$\begin{aligned}\mathbf{x}' &= \mathbf{R}^{-1}\cdot\mathbf{x}\\ x'^{\mu} &= \sum_{\xi}{\left(R^{-1}\right)_{\xi}}^{\mu}\ x^{\xi}\,,\end{aligned} \qquad (7)$$

which in the case of orthogonal bases becomes:

$$\begin{aligned}\mathbf{x}' &= \mathbf{R}^{\mathbf{T}}\cdot\mathbf{x}\\ x'^{\mu} &= \sum_{\xi}{R^{\mu}}_{\xi}\ x^{\xi}\,.\end{aligned} \qquad (8)$$

The transpose of matrix $\mathbf{R}$ is expressed by the index interchange in the last line of Eq. 8. The components of the transformation matrix and its inverse are:

$${R_{\mu}}^{\nu}=\mathbf{e}'_{\boldsymbol{\mu}}\cdot\mathbf{e}^{\boldsymbol{\nu}} \qquad {\left(R^{-1}\right)_{\mu}}^{\nu}={R^{\nu}}_{\mu}=\mathbf{e}_{\boldsymbol{\mu}}\cdot\mathbf{e}'^{\boldsymbol{\nu}}\,. \qquad (9)$$

It is said that the basis $\{\mathbf{e}_{\boldsymbol{\mu}}\}$ transforms covariantly, whereas the coordinates $\mathbf{x}$ transform contravariantly. As required, the state $\mathbf{v}$ remains unchanged:

$$\begin{aligned}\mathbf{v}' &= \sum_{\mu}\mathbf{e}'_{\boldsymbol{\mu}}x'^{\mu}=\sum_{\mu,\nu,\xi}\mathbf{e}_{\boldsymbol{\nu}}\ {R_{\mu}}^{\nu}\ {R^{\mu}}_{\xi}\ x^{\xi}\\ &= \sum_{\nu,\xi}\mathbf{e}_{\boldsymbol{\nu}}\ \delta^{\nu}_{\xi}\ x^{\xi}=\sum_{\nu}\mathbf{e}_{\boldsymbol{\nu}}x^{\nu}=\mathbf{v}\,.\end{aligned} \qquad (10)$$

B. Definitions

The transformation properties of mixed bases consisting of co- and contravariant transforming elements are examined. For this purpose, the following definitions are introduced:

Definition 1 (Mixed basis)**.** *A mixed basis $\{\mathbf{e}^{\mathbf{0}},\mathbf{e}_{\mathbf{1}},\mathbf{e}_{\mathbf{2}},\mathbf{e}_{\mathbf{3}}\}$ is defined as a basis where all elements belong to the canonical basis, except one which is replaced by its dual element, with* $\left(e_0\right)_{\nu}=\sum_{\xi}g^{E}_{\nu\xi}\left(e^0\right)^{\xi}$.

Definition 2 (Mixed change of basis)**.** *Let a mixed change of basis be the change of basis between two mixed bases $\{\mathbf{e}^{\mathbf{0}},\mathbf{e}_{\mathbf{1}},\mathbf{e}_{\mathbf{2}},\mathbf{e}_{\mathbf{3}}\}$ and $\{\mathbf{e}'^{\mathbf{0}},\mathbf{e}'_{\mathbf{1}},\mathbf{e}'_{\mathbf{2}},\mathbf{e}'_{\mathbf{3}}\}$.*

It is clear that mixed bases as in Definition 1 do not change like regular bases in Euclidean space; nor can the scalar or tensor product of the Euclidean space be applied to the mixed bases. The task is to define a vector space in which the mixed bases can be described in a coherent mathematical manner. Hence the following proposition:

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C. Proposition and proof

Proposition 1. *A mixed basis* $\{\mathbf{e^0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ *in Euclidean space as the one in Definition 1 undergoing mixed changes of basis as the ones in Definition 2 behaves just like a regular basis* $\{\mathbf{f_\mu}\}$ *would in Minkowski space, where the scalar product is defined as:*

$$\mathbf{y}\cdot\mathbf{x} = \sum_{\mu\nu} g^M_{\mu\nu} y^\mu x^\nu$$
$$\text{with} \qquad g^M_{\mu\nu} = \operatorname{diag}\left(-1,+1,+1,+1\right).$$

Proof. Let $\mathbb{R}^4$ be an Euclidean space with canonical basis $\{\mathbf{e_\mu}\}$ and dual basis $\{\mathbf{e^\mu}\}$; $\mathbb{R}(3,1)$ a Minkowski space with orthonormalized basis $\{\mathbf{f_\mu}\}$. It is to be shown that, upon a change of basis, the element $\mathbf{e^0}$ of the dual Euclidean basis transforms like the element $\mathbf{f_0}$ of the Minkowski basis.

The orientation preserving changes of basis in Euclidean space $SO(4) = \{\mathbf{R} \in GL(4) \mid \mathbf{R^T R} = \mathbb{1}, \det\mathbf{R} = 1\}$ can be expressed as an exponential series:

$$\mathbf{R} = e^{t\mathbf{A}} \qquad\qquad \mathbf{R^{-1}} = \mathbf{R^T} = e^{-t\mathbf{A}}. \tag{11}$$

Where t is the transformation parameter and $\mathbf{A}$ is a skew symmetric matrix ($\mathbf{A^T} = -\mathbf{A}$). The skew symmetry of these matrices is what leads to the orthogonality of the finite transformations and the coordinate vectors to change with the transposed transformation rule (Eq. 8).

The matrices $\mathbf{A}$ build, together with the commutator $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, the Lie-algebra $so(4)$ of the Lie-group $SO(4)$: the algebra that generates the infinitesimal orientation preserving coordinate transformations in $\mathbb{R}^4$, which through exponential mapping span the whole Lie-group $SO(4)$ (for introductions see e.g. [10], [11]).

In the case of infinitesimal transformations ($t \ll 1$) it is sufficient to only consider the first terms of the exponential series. The special role of the Lie-algebra is seen here:

$$\mathbf{R} = e^{t\mathbf{A}} \approx (\mathbf{1} + t\mathbf{A}) \qquad \mathbf{R^{-1}} = e^{-t\mathbf{A}} \approx (\mathbf{1} - t\mathbf{A}). \tag{12}$$

The elements $\mathbf{A}$ of the Lie-algebra can once more be expressed as a linear combination of a basis, which in the case of $so(4)$ consists of six skew symmetrical matrices, e.g.:

$$\mathbf{L_1} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix} \mathbf{L_2} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix} \mathbf{L_3} = \begin{pmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix} \tag{13}$$

$$\mathbf{K_1} = \begin{pmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \mathbf{K_2} = \begin{pmatrix} 0&0&-1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix} \mathbf{K_3} = \begin{pmatrix} 0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix} \tag{14}$$

$$\begin{aligned} \text{with} \qquad [\mathbf{L_i}, \mathbf{L_j}] &= \varepsilon_{ijk}\mathbf{L_k} \\ [\mathbf{K_i}, \mathbf{K_j}] &= \varepsilon_{ijk}\mathbf{L_k} \\ [\mathbf{L_i}, \mathbf{K_j}] &= \varepsilon_{ijk}\mathbf{K_k}, \end{aligned} \tag{15}$$

and therefore

$$t\mathbf{A} = \sum_{i=1,2,3} \left(t^i\mathbf{L_i} + s^i\mathbf{K_i}\right). \tag{16}$$

To apply a change of the mixed basis, a transformation rule has to be constructed which transforms the element $\mathbf{e_0}$ of the basis with the same coefficients that a normalized coordinate vector $\mathbf{v^0} = (1,0,0,0)^\mathbf{T}$ is transformed with. Using transformation rule Eq. 7 and definition Eq. 14, the infinitesimal change of coordinate vector $\mathbf{v^0}$ can be written as follows:

$$\begin{aligned} \mathbf{v'^0} &= (\mathbf{1} - t\mathbf{A})\,\mathbf{v^0} \\ &= \left(\mathbf{1} - \sum_{i=1,2,3} s^i\mathbf{K_i}\right)\mathbf{v^0} \\ &= \left(\mathbf{1} + \sum_{i=1,2,3} s^i\mathbf{K_i^{-1}}\right)\mathbf{v^0}. \end{aligned} \tag{17}$$

The (pseudo-)inverse $\mathbf{K_i^{-1}} = -\mathbf{K_i}$ denotes the inverse of $\mathbf{K_i}$ on the subspace spanned by $\mathbf{e_i}$ and $\mathbf{e_0}$. In componentwise notation this reads:

$$\begin{aligned} \left(v'^0\right)^\nu &= \left(\delta_0{}^\nu - tA_0{}^\nu\right)\left(v^0\right)^0 \\ &= \left(\delta_0{}^\nu - \sum_{i=1,2,3} s^i (K_i)_0{}^\nu\right)\left(v^0\right)^0 \\ &= \left(\delta_0{}^\nu + \sum_{i=1,2,3} s^i \left(K_i^{-1}\right)_0{}^\nu\right)\left(v^0\right)^0. \end{aligned} \tag{18}$$

Eq. 18 is rearranged to show the change $\mathbf{dv} = (\mathbf{v'^0} - \mathbf{v^0})$ caused by the transformation. It holds for the individual components:

$$\begin{aligned} dv^\nu = \left(v'^0\right)^\nu - \left(v^0\right)^\nu &= \sum_{i=1,2,3} s^i \left(K_i^{-1}\right)_0{}^\nu \left(v^0\right)^0 \\ &= \sum_{i=1,2,3} s^i \left(K_i^{-1}\right)_0{}^\nu. \end{aligned} \tag{19}$$

On the other hand, the transformation rule to obtain $\mathbf{e'_0}$ is given by (from Eq. 6):

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$$\mathbf{e_0'} = \sum_\nu \mathbf{e_\nu} \left(\delta_0{}^\nu + tA_0{}^\nu\right) = \sum_\nu \mathbf{e_\nu} \left(\delta_0{}^\nu + \sum_{i=1,2,3} s^i \left(K_i\right)_0{}^\nu\right). \tag{20}$$

Rearrangement of Eq. 20 provides the change $\mathbf{de} = \mathbf{e_0'} - \mathbf{e_0}$ induced by the transformation:

$$\mathbf{de} = \mathbf{e_0'} - \mathbf{e_0} = \sum_{\nu,i} \mathbf{e_\nu} s^i \left(K_i\right)_0{}^\nu, \tag{21}$$

which results in component notation:

$$de_\nu = \left(e_0' - e_0\right)_\nu = \sum_{i=1,2,3} \left(e_\nu\right)_\nu s^i \left(K_i\right)_0{}^\nu. \tag{22}$$

Sought is an infinitesimal basis transformation with which the individual entries of change dv^ν of the vector (Eq. 19) coincide with the entries of change de_ν of the basis vector (Eq. 22). In order for this to be true for all transformations, the comparison of these two equations yields that the entries of the first row $\left(K_i\right)_0{}^\nu$ of matrices $\mathbf{K_i}$ must be equal to the corresponding entries $\left(K_i^{-1}\right)_0{}^\nu$ of their inverted matrices $\mathbf{K_i^{-1}}$.

As expected, this condition cannot be fulfilled by regular (non-trivial) Euclidean transformations, since in this case the $\mathbf{K_i}$ are skew symmetric and orthogonal, and thus $\mathbf{K_i^{-1}} = -\mathbf{K_i}$. To nonetheless find a change of basis which has the same effect on $\mathbf{e_0}$ as it has on $\mathbf{v^0}$, one must depart from changes of basis in the ordinary Euclidean sense and search for a new set of transformations.

In order for the new base transformations to be useful, they must again function as a Lie algebra, such that finite transformations can be generated from the infinitesimal transformations using the exponential map. The starting point is thus given by a new set of infinitesimal generators $\overline{\mathbf{A}}$ (marked with a bar) of the same form as in Eq. 16:

$$t\overline{\mathbf{A}} = \sum_{i=1,2,3} \left(t^i \mathbf{L_i} + \bar{s}^i \overline{\mathbf{K}}_\mathbf{i}\right). \tag{23}$$

According to the previous considerations, different elements $\overline{\mathbf{K}}_\mathbf{i}$ of the basis are sought, which are their own inverses in their spanned subspace:

$$\left(\overline{K}_i\right)_0{}^\nu \stackrel{!}{=} \left(\overline{K}_i^{-1}\right)_0{}^\nu. \tag{24}$$

Since this condition alone does not give a unique form for the matrices $\overline{\mathbf{K}}_\mathbf{i}$, some additional requirements can be applied. The transformation of the other basis vectors should remain the same as before, thus the coefficients in columns $j \neq 0$ stay unchanged with respect to Eq. 14:

$$\left(\overline{K}_i\right)_j{}^\nu \stackrel{!}{=} \left(K_i\right)_j{}^\nu \qquad j = 1,2,3. \tag{25}$$

Furthermore, volume conservation can be preserved:

$$\left(\overline{K}_i\right)_0{}^0 \stackrel{!}{=} \left(K_i\right)_0{}^0 = 0. \tag{26}$$

Considering conditions Eqs. 24, 25 and 26, the following new transformation matrices can be constructed:

$$\overline{\mathbf{K}}_\mathbf{1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overline{\mathbf{K}}_\mathbf{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overline{\mathbf{K}}_\mathbf{3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

The new elements retain the matrix form of the transformations, yet without orthogonality, since the skew symmetry of the Lie-algebra's elements had to be abandoned. The generators Eq. 27 are inserted back into Eq. 23, which now describes the infinitesimal basis transformations between mixed bases according to Definition 2.

On the other hand, the unchanged matrices $\{\mathbf{L_i}\}$ (Eq. 13) build, together with the new matrices $\{\overline{\mathbf{K}}_\mathbf{i}\}$ (Eq. 27) and the commutator as a Lie bracket, the Lie-algebra $so(3,1)$ with elements $\overline{\mathbf{A}} \in so(3,1)$. By means of the exponential mapping this Lie-Algebra translates into the proper Lorentz group $SO(3,1)$, where:

$$\mathbf{\Lambda} = e^{t\overline{\mathbf{A}}} \qquad \mathbf{\Lambda} \in SO(3,1). \tag{28}$$

Yet the elements of the proper Lorentz group $\mathbf{\Lambda}$ are defined as those orientation preserving changes of basis taking place in the Minkowski space $\mathbb{R}(3,1)$, with metric tensor $g^M_{\mu\nu} = \mathrm{diag}\left(-1,+1,+1,+1\right)$ (see e.g. [12]).

The determined infinitesimal transformation rule according to Eq. 23 for mixed bases in Euclidean space as specified in Definition 1 thus corresponds exactly to the infinitesimal transformation rule for a canonical base $\{\mathbf{f_\mu}\}$ of Minkowski space. Moreover, since only the basis vector $\mathbf{e_0}$ has been changed, the Euclidean space with mixed basis can be directly substituted by a Minkowski space with canonical basis:

$$\begin{aligned} \mathbf{e^0} &\longmapsto \mathbf{f_0} \\ \mathbf{e_i} &\longmapsto \mathbf{f_i} \qquad i = 1,2,3. \end{aligned} \tag{29}$$

$\mathbf{e^0}$ transforms like the element $\mathbf{f_0}$ of the Minkowski basis and vice versa, and the discovered transformation is equivalent to a regular change of basis in Minkowski space. □

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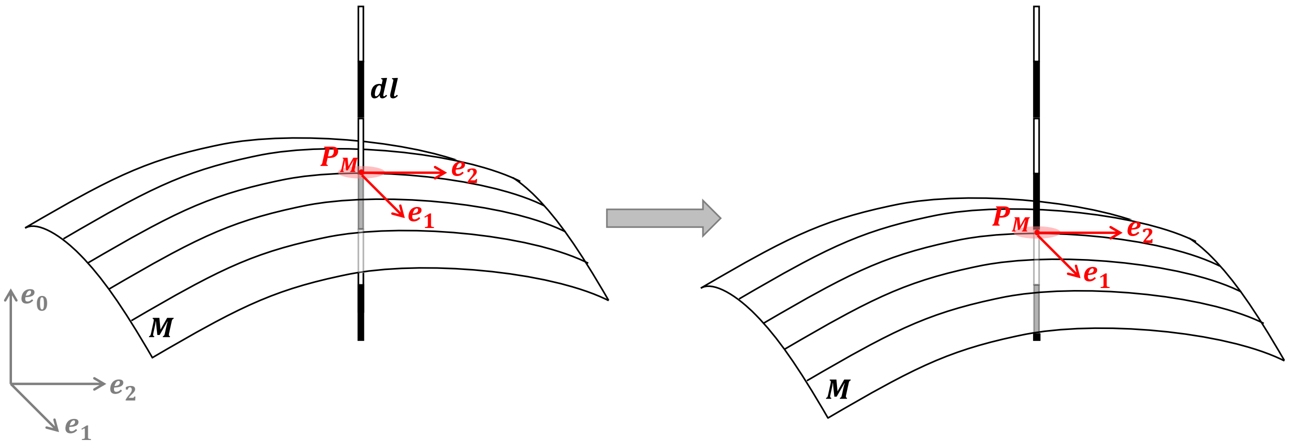

FIG. 2. Illustrative example in three dimensions with a two-dimensional submanifold: an infinitesimally thin rod in $\mathbf{e_0}$ direction, consisting of black and white elements of length dl. The flat observer at point P_M in a submanifold M with basis $\{\mathbf{e_1}, \mathbf{e_2}\}$ (red) measures the thin rod only as a point or very small disk. The observer can determine whether the disk is black or white. Now imagine the flat observer and the rod moving relative to each other along the $\mathbf{e_0}$ direction (situation on the right). By counting the black and white elements that have passed through, the flat observer is given a measuring rule for distances which are not in its submanifold. In this example, the length of a black or white element serves as the standard length, the count corresponds to determining any distance as a multiple of the standard length. On the bottom left in gray the basis system $\{\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}\}$ of a local observer is shown, whose measurement is not limited to a submanifold.

III. INTERPRETATION AS DISTANCE MEASUREMENT BY A FLAT OBSERVER

The transition between co- and contravariant measurement scales outlined in Section II is a tangible geometric operation that can be further interpreted. One possible interpretation is detailed in this section.

A. Definitions

Definition 3 (Local observer)**.** *Let $\mathbb{R}^4$ be a four-dimensional Euclidean vector space with the metric tensor $g^E_{\mu\nu} = \mathrm{diag}\,(+1, +1, +1, +1)$ and the metric*

$$d(\mathbf{dv}, \mathbf{dw}) = \|\mathbf{dv} - \mathbf{dw}\| = \sqrt{\sum_{\mu\nu} g^E_{\mu\nu}(dv^\mu - dw^\mu)(dv^\nu - dw^\nu)}\,. \quad (30)$$

In this space, a local observer is defined as a device that performs distance measurements within the infinitesimal neighborhood ε of a point P using metric Eq. 30.

Definition 4 (Flat observer)**.** *Let M be a be a connected, analytic submanifold of this vector space $\mathbb{R}^4$ with* $\dim(M) = 3$.

A flat observer is defined as a local observer at point P_M according to Definition 3, with the restriction that the flat observer as well as its measurement neighborhood ε_M are elements of M. Thus $P_M \in M$ and $\varepsilon_M \in M$.

Corollary 1. *The measurement neighborhood ε_M of a flat observer is defined through the tangential space of the submanifold M at the position of observer P_M.*

Since M is analytical and connected, a canonical basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ can be found for the flat observer, where $\mathbf{e_0}$ is normal to M.

Definition 5 (Measurable objects)**.** *Measurable objects are constructs which trigger a measurable signal that can be evaluated by a local observer according to Definition 3.*

Corollary 2. *According to Eq. 30, the only quantities which can be inputted in a measurement procedure carried out by a local observer are vectors, thus measurable objects according to Definition 5 must be inherently defined as vectorial quantities.*

B. Proposition and proof

By Definition 4, a flat observer cannot directly take distance measurements along the direction normal to its submanifold parallel to $\mathbf{e_0}$. Under certain conditions, however, indirect distance measurements in this direction may become possible:

Lemma 1. *A flat observer according to Definition 4 can make distance measurements in the $\mathbf{e_0}$ direction if:*

A) A measurable object $\mathbf{O}$ according to Definition 5 with spatial extent in the $\mathbf{e_0}$ direction is present, it intersects the measurement neighborhood ε_M of the flat observer ($\mathbf{O} \cap \varepsilon_M \neq \{\}$), and

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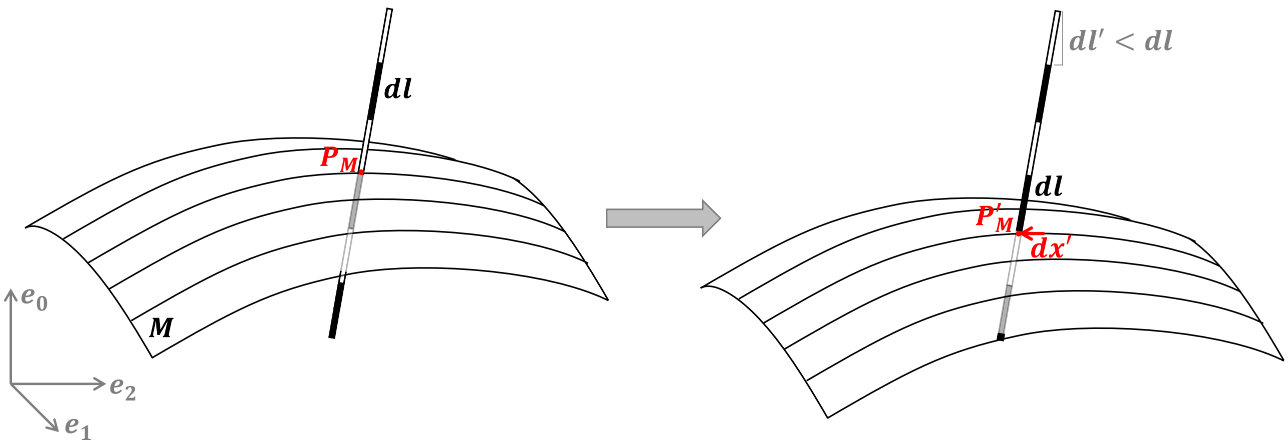

FIG. 3. Illustrative example: an infinitesimally thin rod tilted at a small angle from the $\mathbf{e_0}$ direction. From the situation on the left to the situation on the right there is a relative displacement between the submanifold of the flat observer M and the rod. After defining the distance measurement outside of its submanifold M by counting black and white elements on the rod, the flat observer has to assume that the distance from black (situation on the left) to white (situation on the right) has remained the same as in Figure 2. In addition, the observer has covered the distance dx' within its submanifold to follow the rod. Thus, according to this measuring rule, the total distance covered is given by $ds'^2 = dl^2 + dx'^2$, which can be expected when using a mixed basis and choosing the length element $\mathbf{e^0} = \mathbf{dl}$ as the basis vector for lengths outside of M. In contrast, the local observer can use the basis vector $\mathbf{e_0}$ as a reference and finds that the distance traveled in the $\mathbf{e_0}$ direction is slightly shorter than the one in Figure 2 ($dl' < dl$). The local observer thus makes Euclidean measurements $ds'^2 = dl'^2 + dx'^2 = dl^2$ (indicated in gray).

B) *there is relative displacement between ε_M and the object in the $\mathbf{e_0}$ direction.*

Proof. A) As per Definition 3, to perform any measurement, the metric in Eq. 30 must be used. Measurable objects are defined as the input quantities for this metric. They are also the only way the flat observer can measure distances in the $\mathbf{e_0}$ direction, since a basis vector is not available. Hence, for the measurement in this direction, a measurable object $\mathbf{O}$ must be present.

To measure a distance other than zero in some direction, the two vectors inserted into the metric must differ in the coordinate of interest. A length $d(\mathbf{dv}, \mathbf{dw})$ normal to M can only be defined by two vectors $\mathbf{dv}$, $\mathbf{dw}$ with different values in the $\mathbf{e_0}$ component. This difference between the two vectors $\mathbf{dv} - \mathbf{dw}$ can just as well be defined as a measurable object $\mathbf{O}$ with a non zero spatial extent in the $\mathbf{e_0}$ direction. Finally, a flat observer can only perform measurements in its neighborhood ε_M, thus the measurable object must intersect that neighborhood.

B) Since the flat observer only has the overview of one single coordinate in the $\mathbf{e_0}$ direction, a relative shift between $\mathbf{O}$ and ε_M is necessary to observe at least two different coordinate values.

□

For further illustration of this concept, see the three-dimensional schematic example in Figure 2.

Lemma 2. *The measurement of a distance normal to the submanifold according to Lemma 1 by a flat observer according to Definition 4 produces a contravariantly transforming scale.*

Proof. For a flat observer, the only measurable relation to the outside of its submanifold M is given by the measurable object $\mathbf{O}$. The basis vector $\mathbf{e_0}$ is by Definition 4 not available for measurement.

Since there are no further references, a standard length in the $\mathbf{e_0}$ direction can only be defined using object $\mathbf{O}$, and any additional distance measurement is subsequently quantified as multiples of this standard length. However, the measured standard length used as the basis element and the resulting scale transform contravariantly, since a measurable object $\mathbf{O}$ is a vectorial quantity, as stated in Corollary 2. □

The effect of a contravariant measurement is illustrated in Figure 3 with a three-dimensional example. These considerations lead to the following proposition:

Proposition 2. *It is possible to interpret Minkowski spacetime as a four-dimensional Euclidean space measured by a flat observer according to Definition 4.*

Proof. The proof is divided into two parts. The first part is to show that the Minkowski metric can be understood as a measurement rule in an Euclidean space if a mixed basis $\{\mathbf{e^0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is used when carrying out the measurement. This part was proven in Section II C.

The second part is covered by Lemma 2, which shows that a flat observer uses such a mixed basis, where the scale for distance measurement is covariant within its submanifold and contravariant when normal to its submanifold. Thus, the argumentation is consistent and Proposition 2 is proven. □

IV. CONCLUSION

In this article it is shown that local, flat observers without extension in one direction $\mathbf{e_0}$ of an Euclidean space $\mathbb{R}^4$ can measure the extra dimension, provided there is an interaction with other objects possessing extension in that very direction.

To achieve this, the flat observer employs a measurement rule that uses mixed transforming basis elements $\{\mathbf{e^0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. It is shown that the thereby formed mixed basis transforms like a regular basis of Minkowski space.

This derivation allows the interpretation of the mapping between Minkowski and Euclidean spaces in physical contexts as a change of perspective between flat and local observers.

More generally, this rationale provides a general physical interpretation for the sign inversion of individual or all entries of the metric tensor. The sign inversion corresponds to the role reversal between physical objects (vectors) and the abstract measurement scale (covectors).

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