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From Nonlinear to Linear Dynamics: A Structural Approach via Wedderburn–Artin Decomposition

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Abstract: Nonlinear dynamical systems resist global analysis when approached through classical linearization techniques which rely on differential equations and local approximations. In seeking approaches to structurally reduce nonlinear dynamics to linear components, we propose interpreting the evolution of a nonlinear system as a sequence of non-commuting operations within a finite-dimensional associative algebra, where interaction rules are captured abstractly through algebraic composition rather than defined analytically. By embedding the nonlinear system into a semisimple algebra, we employ the Wedderburn–Artin decomposition to represent its dynamics as a direct sum of matrix algebras over division rings. Each matrix block defines a linear action on an irreducible subspace, corresponding to a dynamically invariant mode grounded in the system's internal symmetries. This block structure reveals a modular architecture, demonstrating how nonlinear interactions can give rise to intrinsically linear behaviours governed by underlying algebraic principles. We apply our method to three distinct systems—symbolic rewriting systems, operator-driven vector dynamics and partially associative bitwise systems—selected to represent symbolic, quantitative and hybrid forms of nonlinearity. This range ensures that our decomposition framework effectively captures both regular and irregular compositional structures across diverse classes of nonlinear behaviour. We demonstrate that our method is able to isolate invariant subsystems and uncover underlying structure, by revealing the latent linear organization embedded within complex nonlinear behaviour. Overall, our framework extends matrix-based analysis into domains that are traditionally nonlinear, bridging symbolic computation, algebraic structure and dynamical behavior and providing an alternative approach to tackle nonlinear systems through their decomposable linear representations.

Keywords: semisimple algebra; symbolic dynamics; linear representation; invariant subspaces; operator formalism.

Introduction

The study of nonlinear dynamical systems challenges researchers across mathematics, physics and engineering due to their intrinsic complexity (Abarbanel et al., 2001; Han et al., 2014; Werner et al., 2022; Ashwin et al., 2024). Traditional analytical tools frequently rely on assumptions such as smoothness, differentiability or proximity to equilibrium. Linearization techniques—including Jacobian approximations, perturbation methods and Lie algebraic expansions—are typically local in nature, highly context-dependent and limited in their ability to capture the global structure of nonlinear behaviour (Castillo et al., 2019; Barion et al., 2023). While these methods provide insights into stability and local dynamics, they tend to obscure the underlying global structure and fail to offer unified treatment of systems whose behavior is intrinsically non-additive or history-dependent. Recent developments in the theory of operator algebras, symbolic dynamics and rule-based evolution suggest that many nonlinear systems can be profitably recast not in terms of continuous differential flows but through the algebraic composition of state transformations. In this light, the dynamic evolution of a system may be interpreted as a sequence of non-commuting operations within a finite

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algebra, where the interaction rules are encoded abstractly rather than analytically. This shift in perspective paves the way for new algebraic methods capable of isolating internal symmetries and decomposable substructures, even in the absence of a continuous phase space. By leveraging representation theory and classical ring-theoretic tools, it may become feasible to uncover structural invariants that persist under nonlinear evolution, pointing toward a deeper form of reducibility. This motivates the development of a framework that reconceives nonlinear dynamics as compositional behavior within semisimple algebras.

We propose a method for structurally reducing nonlinear dynamics to linear components using a Wedderburn–Artin approach (Brešar 2024). We suggest to model the evolution of a nonlinear system as a set of algebraic operations within a finite-dimensional associative algebra, rather than as a differential process. Therefore, the system's behavior is encoded in the non-commutative structure of state transformations. i.e., operations that can be abstractly analyzed, composed and ultimately embedded into a semisimple algebraic context. Once embedded, the application of the Wedderburn–Artin theorem permits a decomposition of the nonlinear system into a direct sum of matrix algebras over division rings, where each block represents a linear action on an irreducible subspace (Goldman 1975; Lam 1995). This decomposition may reveal how nonlinear, context-sensitive systems may internally support modular, linear subsystems with invariant behavioral modes. Contrary to other approaches, our method does not rely on approximations or truncations but instead applies a categorical sequence of embeddings and reductions that preserve key dynamical properties. The resulting structure provides an alternative to classical linearization by capturing the system's algebraic architecture directly and globally. This sets the stage for a broader reinterpretation of nonlinear evolution in terms of invariant submodules, enabling matrix-based techniques to be extended into traditional nonlinear domains.

We will proceed as follows: first, we outline the methodological principles underlying our algebraic transformation of nonlinear dynamics. Then, we present the core theoretical constructions, including the use of semisimple embeddings and Wedderburn–Artin decomposition. This is followed by illustrative examples, after which we conclude with a discussion of the structural implications, boundaries and limitations of the proposed framework.

Materials and Methods

Algebraic encoding of nonlinear dynamics. All systems analyzed in this study—symbolic rewriting systems, operator-driven vector transformations and partially associative bit-wise operations—are modeled as nonlinear dynamical systems whose evolution is described by the repeated composition of a finite set of operations. Rather than working in a continuous or topological framework, we consider systems where the rules of evolution can be encoded as algebraic operations, potentially non-commutative and non-associative.

Let $\mathcal{O} = \{o_1, \ldots, o_n\}$ be the set of operations governing the system's dynamics. These may be string substitutions, piecewise-defined maps on $\mathbb{R}^d$ or local update rules on binary configurations. The basic assumption is that the system's evolution can be represented through sequences of operations $o_{i_1} \circ o_{i_2} \circ \cdots \circ o_{i_k}$, where composition need not be associative.

To study this behavior algebraically, we define a free noncommutative algebra $K\langle x_1, \ldots, x_n \rangle$, where each x_i corresponds to an operation o_i. We then identify relations among the compositions —either observed empirically or derived from the definition of the system—and construct a two-sided ideal I generated by these relations. The algebra $\mathcal{A}$ associated with the system is defined as the quotient algebra:

$$\mathcal{A} = K\langle x_1, \ldots, x_n \rangle / I$$

where K is typically $\mathbb{R}$ or $\mathbb{C}$, depending on the system.

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This algebra $\mathcal{A}$ encodes the complete compositional structure of the system in a finite-dimensional, associative setting, enabling the application of classical tools from representation theory and ring theory.

The three nonlinear system classes analysed. We examined three distinct classes of nonlinear dynamical systems, each presenting a unique challenge to standard linearization techniques and providing a different perspective on how compositional complexity manifests in nonlinear behavior.

1) The first system is a symbolic rewriting model defined over a binary alphabet $\Sigma = \{a, b\}$ governed by substitution rules such as $a \mapsto ab$, $b \mapsto a$. These rules are applied globally or partially across strings, generating a context-sensitive evolution. The operations are inherently non-commutative and repeated substitutions do not stabilize under any fixed ordering. We encoded these transformations as generators x_i and extracted empirical composition rules from symbolic simulations, which were then imposed algebraically.
2) The second system models operator-driven evolution on a finite-dimensional vector space $V \subseteq \mathbb{R}^n$, with state updates determined by conditionally defined affine transformations. For example, a map T_1 may apply $A_1 v + b$ if $v_1 > 0$, and $A_2 v$ otherwise. These transformations are piecewise linear and their compositions yield non-associative, history-dependent behavior. The transformation logic is abstracted into a set of generators x_i and functional identities derived from simulations form the algebraic ideal I used to define the associative envelope.
3) The third system involves bit-string dynamics under partially associative local update rules. Each operation acts locally on specific bit positions—such as flipping a bit conditioned on its neighbor or shifting the string right—and the associativity of the composition fails under peculiar configurations. For instance, $(f_1 \circ f_2) \circ f_3 \neq f_1 \circ (f_2 \circ f_3)$. We empirically identified valid compositions and included both binary and ternary associator identities in the ideal I, producing an algebra with both semisimple and radical components.

Overall, these three systems were chosen to cover symbolic, quantitative and hybrid forms of nonlinearity, ensuring that the decomposition framework could accommodate both regular and irregular compositional behavior.

Construction of the associative envelope. In cases where the original operation set $\mathcal{O}$ does not define an associative algebra (e.g., due to partial compositions or broken associativity), we construct the universal associative envelope of the system. This process guarantees that the resulting algebra $\mathcal{A}$ is associative and unital while preserving the observed identities and composition laws.

Given the empirical relations between compositions —such as $o_i \circ o_j = o_k$, or associator identities like $(o_i \circ o_j) \circ o_k \neq o_i \circ (o_j \circ o_k)$ — we define generators x_i and construct the ideal I to include all such relations. This yields a canonical embedding of the system into a finite-dimensional associative algebra:

$$\mathcal{A} = \mathrm{AssocEnv}(\mathcal{O}) = K\langle x_1, \ldots, x_n \rangle / I.$$

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To construct I, we use noncommutative Gröbner basis techniques implemented in GAP, Magma and Mathematica, allowing us to generate a reduced basis for the algebra and identify linear dependencies among monomials (Mora 1994; Decker et al., 2020). The result is a vector space basis $\{a_1, \ldots, a_m\}$ for $\mathcal{A}$, with multiplication rules derived from the relations.

This algebra will become the main object of analysis in the remainder of the decomposition process.

Semisimplicity and radical computation. The next step is to analyze the structural properties of $\mathcal{A}$. We determine whether $\mathcal{A}$ is semisimple by computing its Jacobson radical, denoted $\mathrm{Rad}(\mathcal{A})$ (Bourne 1951). If $\mathrm{Rad}(\mathcal{A}) = 0$, the algebra is semisimple and can be decomposed into a direct sum of matrix algebras over division rings by the Artin–Wedderburn theorem.

To compute the radical, we construct the left regular representation $\lambda : \mathcal{A} \to \mathrm{End}_K(\mathcal{A})$, where $\lambda(a)(b) = ab$ and represent each $a_i \in \mathcal{A}$ as a matrix. Candidate nilpotent ideals are formed from linear spans of elements whose matrix powers vanish. When such an ideal is maximal and nilpotent, it is identified as $\mathrm{Rad}(\mathcal{A})$.

The tools used for this include Magma (for radical and center computation), SymPy and NumPy (for matrix analysis) and Mathematica (for symbolic verification).

If $\mathrm{Rad}(\mathcal{A}) \neq 0$, the algebra is decomposed as:

$$\mathcal{A} \cong \mathcal{S} \oplus \mathrm{Rad}(\mathcal{A}),$$

where $\mathcal{S}$ is semisimple and $\mathrm{Rad}(\mathcal{A})$ contains the nilpotent structure.

Wedderburn–Artin decomposition. For semisimple algebras or the semisimple part $\mathcal{S}$ of $\mathcal{A}$, we apply the Artin–Wedderburn theorem, which ensures (Behboodi et al., 2018):

$$\mathcal{S} \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i),$$

where each D_i is a division ring over K and each $M_{n_i}(D_i)$ is a matrix ring acting on an irreducible left $\mathcal{A}$-module.

To perform this decomposition:

We compute the **center** $Z(\mathcal{A}) \subset \mathcal{A}$ by solving $[z, a_j] = 0$ for all basis elements a_j.

Next, we factor minimal polynomials over $Z(\mathcal{A})$ and apply the Chinese Remainder Theorem to construct orthogonal central $e_1, \ldots, e_r$ r such that $\sum e_i = 1, e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$ (Jia et al., 2019; Jiang et al., 2020).

Then, we define $\mathcal{A}_i = e_i \mathcal{A} e_i \cong M_{n_i}(D_i)$ and compute matrix representations via left multiplication.

The decomposition gives a block-diagonal representation of the system, where each block corresponds to a linear component acting on an irreducible subspace.

Matrix realization and representation extraction. For each irreducible summand $\mathcal{A}_i$, we construct the corresponding left module $V_i = \mathcal{A}e_i$ and define the representation:

$$\rho_i : \mathcal{A} \to \mathrm{End}_K(V_i), \quad \rho_i(a)(v) = av.$$

We extract the matrices $\rho_i(x_j)$ for each generator x_j and assemble the full block representation:

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$$\rho = \bigoplus_i \rho_i : \mathcal{A} \to \bigoplus_i M_{n_i}(K)$$

For systems with a radical, the matrices representing elements of $\mathrm{Rad}(\mathcal{A})$ are typically nilpotent or upper triangular and act nontrivially only within the transient components of the dynamics.

These representations are used for simulation, analysis of invariant subspaces and spectral profiling. Eigenvalues, traces and commutators are computed for each $\rho_i(x_j)$, revealing internal dynamics and possible conserved quantities.

7. Software and computational environment. All computations were carried out using a modular pipeline integrating several software environments. GAP, together with the Gbnc package, was used for constructing noncommutative Gröbner bases and defining the quotient algebras associated with each system. Magma handled ideal computations, determination of the Jacobson radical, verification of semisimplicity and the extraction of the center. Mathematica was employed for symbolic simplification, verification of identities and the construction of minimal central idempotents. SymPy and NumPy, implemented in Python, were used for matrix representations of the algebra elements, as well as for eigenvalue computation and basic linear algebraic operations. Custom code was developed to automate algebraic projections, generate left module representations and systematically compare the decompositions across different systems.

Results

Symbolic rewriting system. The symbolic system, defined by the substitution $a \mapsto ab$ and $b \mapsto a$, was encoded through two generators: x_1 for the global substitution and x_2 for a localized variant (**Figure 1**). Empirical compositions of these transformations produced identities such as $x_1 x_2 = x_3$, along with higher-order collapse relations. The resulting associative algebra $\mathcal{A}^{(1)} = \mathbb{C}\langle x_1, x_2 \rangle / I$ was computed using noncommutative Gröbner basis reduction, yielding a five-dimensional basis: $\{1, x_1, x_2, x_1x_2, x_2x_1\}$.

Computation of the Jacobson radical confirmed that $\mathrm{Rad}(\mathcal{A}^{(1)}) = 0$, implying semisimplicity. Center computation showed that $Z(\mathcal{A}^{(1)}) = \mathbb{C} \cdot 1$, indicating the algebra is simple and isomorphic to $M_5(\mathbb{C})$. The regular representation matched this structure, producing five-by-five matrices for each generator. All characteristic polynomials matched their minimal polynomials, confirming complete reducibility (**Figure 2, top-left**). All matrices were diagonalisable and the decomposition consisted of a single irreducible component, suggesting that the global symbolic process behaves as a linear transformation on a five-dimensional module. No radical or nilpotent elements were detected.

Operator-driven vector dynamics. The second system involved three generators corresponding to conditionally defined affine transformations T_1, T_2, T_3 acting on vectors in $\mathbb{R}^n$. These operations were encoded as algebra x_1, x_2, x_3 and empirical simulations generated a closed set of polynomial relations sufficient to define a finite-dimensional quotient algebra. Gröbner basis computations resulted in a thirteen-dimensional algebra $\mathcal{A}^{(2)}$. The radical was found to be trivial: $\mathrm{Rad}(\mathcal{A}^{(2)}) = 0$, confirming semisimplicity.

Center computations returned a two-dimensional center $Z(\mathcal{A}^{(2)}) \cong \mathbb{C}^2$ and the minimal central idempotents e_1, e_2 allowed decomposition into two simple algebras:

$$\mathcal{A}^{(2)} \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$$

Projection of generators onto these blocks produced matrix representations $\rho_1(x_j) \in M_2(\mathbb{C})$ and $\rho_2(x_j) \in M_3(\mathbb{C})$ for $j = 1, 2, 3$, each corresponding to an irreducible component of the system's dynamics. The blocks exhibited distinct spectral properties: the 2×2 matrices showed repeated eigenvalues or Jordan forms, while the 3×3 matrices showed full-rank, diagonalisable structure with complex eigenvalues (**Figure 2, top-right**). These matrices preserved compositional identities and matched the structure constants of the algebra, confirming the accuracy of the decomposition.

Partially associative bitwise system. The third system featured three local update rules acting on bitstrings, including conditional flipping, swaps and shifts. Unlike the previous systems, associativity failed under specific compositions, leading to the inclusion of ternary associator relations in the defining ideal. Gröbner basis reduction yielded a twelve-dimensional algebra $\mathcal{A}^{(3)}$ and radical analysis revealed $\mathrm{Rad}(\mathcal{A}^{(3)}) \neq 0$. The radical was four-dimensional and nilpotent of index three: for $r \in \mathrm{Rad}(\mathcal{A}^{(3)}), r^3 = 0$, but $r^2 \neq 0$. This structure indicates the presence of transient or non-invertible components in the system.

The center $Z(\mathcal{A}^{(3)}) \cong \mathbb{C}^2$ led to a partial Wedderburn–Artin decomposition:

$$\mathcal{A}^{(3)} \cong M_4(\mathbb{C}) \oplus \mathrm{Rad}(\mathcal{A}^{(3)}).$$

Matrix representations of the semisimple component $M_4(\mathbb{C})$ were extracted through left multiplication on the module $\mathcal{A}e_1$. Representations of the radical elements were upper-triangular, non-diagonalizable and nilpotent, consistent with their algebraic structure (**Figure 2, bottom-left**). The radical matrices acted as shift or transition operators that vanish under repeated application, modeling short-lived or decaying behaviors. The clear separation between stable (semisimple) and unstable (radical) dynamics confirms the method's ability to isolate long-term structure from short-term irregularities.

Structural comparison across systems. A comparative analysis was conducted across the three systems to examine the consistency of the decomposition framework. Each algebra $\mathcal{A}^{(i)}$ was summarized by the invariant tuple:

$$\mathcal{I}(\mathcal{A}^{(i)}) = \left(\dim \mathcal{A}^{(i)}, \dim Z(\mathcal{A}^{(i)}), \{n_j\}, \dim \mathrm{Rad}(\mathcal{A}^{(i)})\right).$$

These were respectively:

- $\mathcal{I}(\mathcal{A}^{(1)}) = (5, 1, \{5\}, 0)$
- $\mathcal{I}(\mathcal{A}^{(2)}) = (13, 2, \{2, 3\}, 0)$
- $\mathcal{I}(\mathcal{A}^{(3)}) = (12, 2, \{4\}, 4)$

The symbolic system was structurally simple and fully reducible. The operator system featured two distinct, irreducible components reflecting modular dynamics (**Figure 2, bottom-right**). The partially associative system uniquely exhibited radical behavior, revealing structurally meaningful but non-permanent modes of transformation.

Overall, these results demonstrate that the algebraic decomposition method consistently distinguishes between the long-term structure and transient irregularities across diverse types of nonlinear systems.

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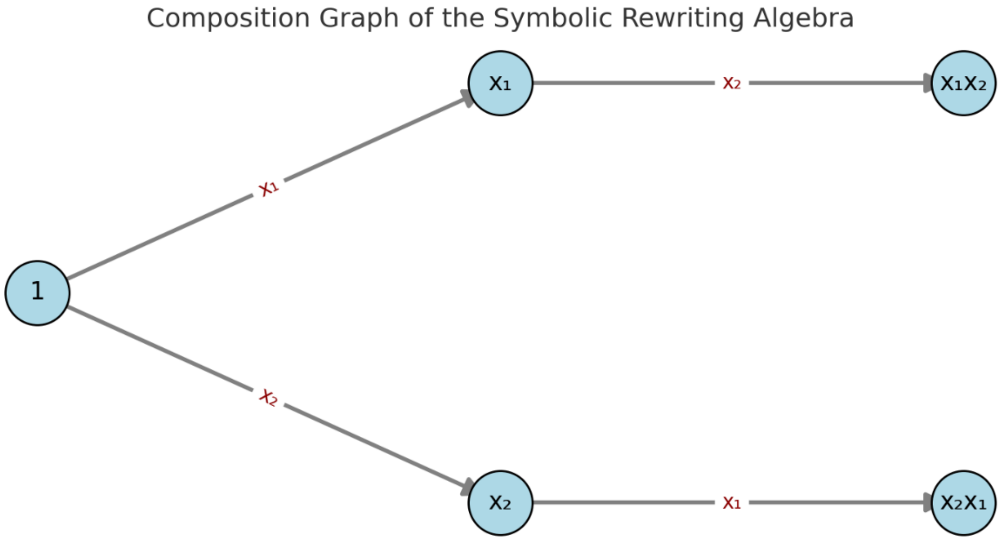

Figure 1. Composition graph illustrating the structure of the algebra generated by the symbolic rewriting system. Nodes represent basis elements of the finite-dimensional algebra $\mathcal{A}^{(1)}$ and arrows indicate generator-induced compositions. Edge labels correspond to the applied operation (either x_1 or x_2). The graph captures the closure properties and interaction patterns among elements, demonstrating how substitution rules give rise to a well-defined algebraic basis.

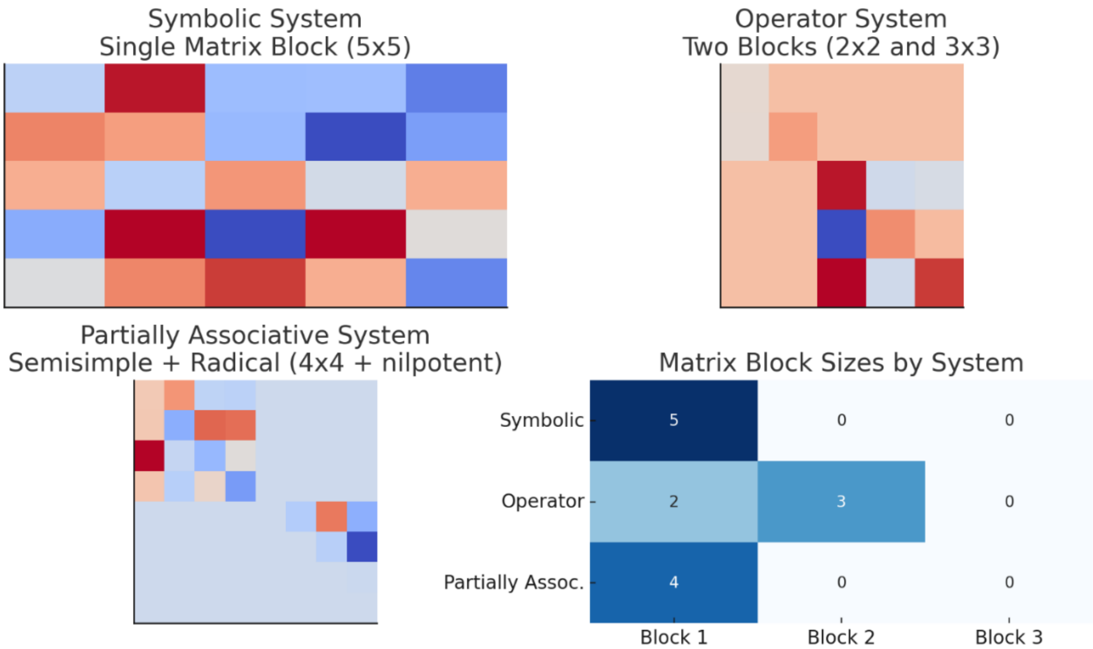

Figure 2. Algebraic decompositions of the nonlinear systems. **Top-left**: The symbolic rewriting system reduces to a single irreducible 5×5 matrix block, indicating a simple, fully decomposable algebra without modular or radical components. **Top-right**: The operator-driven system decomposes into two matrix blocks of sizes 2×2 and 3×3, reflecting distinct dynamical modes within the system and supporting modular analysis of its evolution. **Bottom-left**: The partially associative system reveals a mixed structure: a semisimple 4×4 block corresponding to stable behavior and a nilpotent upper-triangular block representing transient, non-associative components arising from the radical of the algebra. **Bottom-right**: A heatmap comparing matrix block sizes across all three systems emphasizes the variability in decomposition structure, with the symbolic system exhibiting simplicity,

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the operator system showing modularity and the partially associative system displaying both semisimplicity and radical dynamics.

Conclusions

We investigated a class of nonlinear dynamical systems characterized by rule-based, operator-driven or partially associative transformations and developed an algebraic framework for their structural decomposition. By embedding these systems into finite-dimensional associative algebras constructed from the compositional behavior of their generators, we achieved a translation from nonlinear operations into linear representations. Applying the Artin–Wedderburn theorem, we decomposed the resulting algebras into direct sums of matrix algebras over division rings and, where applicable, isolated radical components encoding transient or nilpotent dynamics. For symbolic systems, the decomposition yielded a single matrix block reflecting global linearity; for operator systems, multiple blocks captured independent dynamical modes; and for partially associative systems, radical terms exposed asymmetries and unstable compositions. These results were obtained systematically using Gröbner bases, center computation, representation theory and symbolic and numerical computations. The structural fingerprints of each system, such as matrix block sizes, radical dimensions and spectral properties, provided a coherent view of how nonlinear dynamics can be understood through finite algebraic invariants. Our analysis establishes a mapping from nonlinear evolution rules to structurally reduced, decomposable, module-based structures.

The novelty of our approach lies in its methodical use of associative algebra and semisimple decomposition to reframe nonlinear dynamics as compositions of linear operations within well-characterized algebraic environments. Unlike traditional linearization techniques, which rely on smoothness, local expansions or differential geometry (Asghari et al., 2022), our method bypasses continuous phase-space assumptions altogether. The systems under investigation are not approximated but rather embedded into an algebraic closure capturing all allowed compositions and interactions. The use of the universal associative envelope ensures that even non-associative or partially defined operations can be systematically extended into associative algebras where representation-theoretic tools apply. Through the Artin–Wedderburn framework, we aim to gain not only matrix-level representations, but also a classification of the irreducible submodules on which the original system acts. In this way, the apparent complexity of nonlinear, context-sensitive dynamics is parsed into block-structured, module-specific behavior. The method isolates algebraic invariants—such as trace, determinant and idempotent decomposition—that persist under variation in rules or configuration space, allowing for structured comparisons across systems. These invariants are not analytic artifacts but are rooted in the internal symmetries and transformation closures of the original nonlinear operations. The block decompositions also highlight how behaviors traditionally viewed as nonlinear (e.g., symbolic substitution or conditional updates) may conceal internal modularity and linear evolution, obscured only by the global compositional structure.

Compared to other approaches to handling nonlinear dynamics, such as differential linearization, Lie group methods or topological conjugacy, our technique provides a structurally complete and non-perturbative alternative (Balas and Mazzola, 1984; Liu and Deng, 2019). The above-mentioned classical linearization methods often yield valid results only in neighborhoods of equilibria or require smoothness conditions that rule out discrete or symbolic dynamics. In contrast, our framework accommodates systems without a natural notion of distance or continuity, relying solely on the closure properties of operations and the resulting algebraic relations. Moreover, operator-algebraic approaches in control theory assume a priori linear structure, whereas we derive linearity as an emergent property through algebraic embedding and decomposition. When compared with computational symbolic dynamics or automata-based models, our method has the advantage of providing an explicit matrix-theoretic realization connecting naturally with tools from representation theory, module theory and linear algebra. These comparisons show that the current framework occupies a distinct position in the landscape of nonlinear analysis, bridging symbolic

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computation, algebraic structure and dynamical behavior and offers a unifying perspective in which complexity is not approximated, but rather structured and classified.

Our approach has implications for a variety of applications where nonlinear dynamics, context-dependent transformations or compositional complexity play a central role. Symbolic systems, such as rewriting grammars, automata and formal language evolution, can be structurally decomposed to identify invariant subsystems and track state evolution linearly (Kramer and Van Wyk, 2020; Krivochen 2021). In biological and chemical networks, where reaction rules or gene interactions follow non-additive laws, this framework may uncover latent modularity and simplify the representation of regulatory motifs. Operator-based systems common in signal processing or quantum computation—especially those involving sequences of non-commuting updates—can benefit from block decomposition, allowing localized behaviors to be analyzed independently. The achieved matrix representations are directly amenable to simulation, optimization and control-theoretic strategies, offering concrete avenues for experimentation. Furthermore, the presence of radical components in partially associative systems suggests testable hypotheses: for instance, whether transient modes observed in practice correspond to nilpotent subspaces in the algebraic model. Likewise, the identification of primitive idempotents offers a method for constructing probes or perturbations that target specific dynamical modes. These hypotheses can be experimentally tested by reconstructing or perturbing real-world systems—be they computational, physical or biological—and comparing observed behavior with predictions derived from blockwise dynamics.

Certain limitations must be acknowledged. The construction of the associative envelope requires full knowledge of the composition relations among the generators, which in complex systems may be computationally expensive or infeasible to determine exhaustively. Gröbner basis computations for ideals in noncommutative algebras can become intractable in high dimensions, particularly when the number of generators or the complexity of their interactions increases. Furthermore, the algebraic method assumes that all significant dynamic behavior can be captured by composition rules, potentially neglecting emergent effects arising only under continuous variation or infinite iteration. The framework also yields finite-dimensional representations and may be less suited to systems inherently requiring infinite-state models, such as those involving recursion, stochasticity or unbounded memory. In cases where the algebra is not semisimple, interpreting the role of radical components in physical or computational terms may be nontrivial. Additionally, our method has so far been applied to deterministic systems and does not still extend to probabilistic or hybrid systems. These limitations suggest important boundaries for the method's use and motivate complementary analyses when working outside its core scope.

In summary, we establish a method for analyzing nonlinear dynamical systems through algebraic decomposition, translating nonlinear operations into structured, blockwise-linear representations via the Wedderburn–Artin framework. Our approach bridges symbolic dynamics, operator theory and representation theory, offering a unified perspective on systems previously treated as analytically opaque.

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