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Posted Date: 4 June 2024

doi: 10.20944/preprints202406.0065.v1

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Article

Sharp Coefficient Bounds for Starlike Functions Associated with Cosine Function[†]

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[†] Dedicated to the memory of Professor Lawrence Allen Zalcman (1943–2022)

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Abstract: Let $\mathcal{S}_{\cos}^*$ denote the class of normalized analytic functions f in the open unit disk $\mathbb{D}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \cos z$. In the second section of this article we find the sharp upper bounds for the initial coefficients a_3 , a_4 and a_5 and the sharp upper bound for module of the Hankel determinant $|H_{2,3}(f)|$ for the functions from the class $\mathcal{S}_{\cos}^*$. The first result of the next section deals with the sharp upper bounds of the logarithmic coefficients γ_3 and γ_4 and we found in addition the sharp upper bound for $|H_{2,2}(F_f/2)|$. For obtaining these results we used the very useful and appropriate Lemma 2.4 of N.E. Cho et al. [Filomat 34(6) (2020), 2061–2072], and the technique for finding the maximum value of a three variable function on a closed cuboid. All the maximum found values were checked by using MAPLE™ computare software, and we also found the extremal functions in each cases. All of our present results are the best ones and give sharp versions of those recently published in [Hacet. J. Math. Stat. 52, 596–618, 2023].

Keywords: starlike functions; subordination; cosine functions; initial coefficient bounds; Hankel determinants; logarithmic coefficients

MSC: 30C45; 30C50; 30C80

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all analytic and normalized functions f having the Maclaurin series expansion as follows

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, z \in \mathbb{D}, \quad (1)$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ represents the open unit disk in the complex plane $\mathbb{C}$, and let consider a subclass of $\mathcal{S} \subset \mathcal{A}$ containing all the univalent functions of $\mathcal{A}$.

The subclass of $\mathcal{A}$ defined by

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, z \in \mathbb{D} \right\}$$

is called the class of *starlike functions* (with respect to the origin) in $\mathbb{D}$, and it's well-known that $\mathcal{S}^* \subset \mathcal{S}$. Let $\mathcal{B}$ denote the family of functions w which are analytic in $\mathbb{D}$, such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$. Such functions are referred to as *Schwarz functions*.

Consider two analytic functions f and g in $\mathbb{D}$. Then we say that f is subordinated to g , written as $f(z) \prec g(z)$ if there exists a Schwarz function w such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. If $f(z) \prec g(z)$ then $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$, and if g is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Based on the geometric properties of the image of $\mathbb{D}$ by some analytic functions, the functions can be categorized into different families. Thus, in 1992 Ma and Mind a [19] introduced a generalized subclass of $\mathcal{S}^*$ denoted by $\mathcal{S}^*(\phi)$ which is defined in terms of the subordination as follows:

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

where ϕ satisfies the conditions $\phi(0) = 1$ and $\operatorname{Re} \phi'(z) > 0$ in $\mathbb{D}$ and ϕ maps the unit disk $\mathbb{D}$ onto a star-shaped domain. Several subclasses of $\mathcal{S}^*$ can be obtained by varying the function ϕ . For example, if we choose $\phi(z) = \frac{1+Lz}{1+Mz}$, $-1 \leq M < L \leq 1$, then we get the class $\mathcal{S}^*[L, M] := \mathcal{S}^*\left(\frac{1+Lz}{1+Mz}\right)$ which is called the *Janowski starlike functions class* and was investigated in [12]. Sokół and Stankiewicz [30] defined and studied the class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$, where the function $\phi(z) = \sqrt{1+z}$ maps $\mathbb{D}$ onto the image domain bounded by $|w^2 - 1| < 1$. The class $\mathcal{S}_C^* := \mathcal{S}^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right)$ was studied by Sharma [29] and it's related to the *cardioid function*.

In case of $\phi(z) = \cosh z$ the class $\mathcal{S}_{\cosh}^*$ was defined and examined in [1], whereas the class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ is presented and studied in [21] by Mendiratta et al. For $\phi(z) = 1 + \sin z$ the class $\mathcal{S}_{\sin}^*$ was introduced and investigated in [2,10], while if $\phi(z) = 1 + \tanh z$ the class $\mathcal{S}_{\tanh}^*$ was defined and studied in [31]. Recently, the class $\mathcal{S}_{\cos}^*$ defined by

$$\mathcal{S}_{\cos}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cos z \right\}$$

was introduced and investigated in [5], and to determine sharp results connected with those of [20] represent the main goal of the present paper.

The following part of this section is necessary for the motivation of this work. Thus, in 1916 Bieberbach presented the well-known *Bieberbach conjecture* which was proved by de Branges [7] in 1985. Prior to de Branges proof of this conjecture numerous mathematicians exerted considerable effort to prove it, leading to the establishment of coefficient bounds for some remarkable subclasses of the class $\mathcal{S}$. They also developed new inequalities related to coefficient bounds for some subclasses of univalent functions, and those related to the *Fekete-Szegő functional*, that is $|a_3 - \lambda a_2^2|$, is one of those inequalities. Another coefficient problem closely related to Fekete-Szegő functional is the *Hankel determinant* as we will see below.

Like we could see in Pommerenke's [24], for a function $f \in \mathcal{A}$ the Hankel determinant $H_{m,n}(f)$ is as follows:

$$H_{m,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix},$$

where $m, n \geq 1$. We remark that

$$H_{2,1}(f) = a_3 - a_2^2, \quad H_{2,2}(f) = a_2a_4 - a_3^2, \quad H_{2,3}(f) = a_3a_5 - a_4^2, \quad (2)$$

$$H_{3,1}(f) = 2a_2a_3a_4 - a_4^2 - a_3^3 - a_2^2a_5 + a_3a_5, \quad (3)$$

where $|H_{2,1}(f)|$ is the classical Fekete-Szegő functional obtained for $\lambda = 1$. Several authors have determined the maximum value of $|H_{2,1}(f)|$ and the upper bound for $|H_{2,2}(f)|$ for various subclasses of $\mathcal{A}$, see for example [13,14,22]. Results regarding the second Hankel determinant $H_{2,3}(f)$ could be found also in the recent papers like [23,28,32]. The determinant $H_{2,3}(f)$ is not much studied in the literature. Babalola [3] studied first time a non-sharp bounds for the determinant $H_{3,1}(f)$ for various subclasses of $\mathcal{S}$, while in 2017 Zaprawa [33] improved the results of Babalola by using a new technique. We mention that the sharp bounds of the modulus of $H_{3,1}(f)$ for the class $\mathcal{S}^*$ are recently obtained

by Kowalczyk et al. [17], whereas for the class $\mathcal{C}$ and bounded turning functions sharp bounds were obtained in [15,16], respectively. For sharp inequalities results of the determinant $H_{3,1}(f)$ for some subclasses of $\mathcal{S}^*$ we refer to [4,18,26–28].

Very recently Marimuthu et al. [20] have determined the coefficient bounds, the upper bounds for the second, third and fourth order Hankel determinants for the functions of the class $\mathcal{S}_{\cos}^*$, but most of the results presented in this paper are not sharp.

Motivated from the aforementioned study, in this paper we have established the sharp results for the upper bounds of the coefficients, the inverse coefficients and the logarithmic coefficients of the functions of the class $\mathcal{S}_{\cos}^*$. Also we have developed the sharp upper bounds for the modulus of the second and third order Hankel determinants for the functions of this class.

The well-known *Carathéodory class* $\mathcal{P}$ is the family of holomorphic functions h in $\mathbb{D}$ which satisfy the condition $\operatorname{Re} h(z) > 0$, $z \in \mathbb{D}$, and having the power series expansion of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \quad (4)$$

The study of some coefficient problems in different classes of analytic functions revolves around the idea of expressing function coefficients in a given class by function coefficients that have a positive real part. Thus, inequalities known for the class $\mathcal{P}$ can be used to study coefficient functionals. We require the following results about the class $\mathcal{P}$ for our discussions.

We will recall the well-known *Carathéodory lemma* [8] (see also [25, Corollary 2.3, p. 41], [11, Carathéodory's Lemma, p. 41]):

Lemma 1. *If $h \in \mathcal{P}$ has the form (4), then*

$$|c_n| \leq 2, \quad n \geq 1. \quad (5)$$

The inequality holds for all $n \geq 1$ if and only if $h(z) = \frac{1 + \lambda z}{1 - \lambda z}$, $|\lambda| = 1$.

The next result represents the relations (2.7), (2.8) and (2.9) of Lemma 2.4. from [6]:

Lemma 2. [6, Lemma 2.4] *Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk, and $h \in \mathcal{P}$ be given by (4). Then,*

$$\begin{aligned} c_1 &= 2t_1, \\ c_2 &= 2t_1^2 + 2(1 - |t_1|^2)t_2, \\ c_3 &= 2t_1^3 + 4(1 - |t_1|^2)t_1 t_2 - 2(1 - |t_1|^2)\overline{t_1} t_2^2 + 2(1 - |t_1|^2)(1 - |t_2|^2)t_3, \end{aligned}$$

for some $t_k \in \overline{\mathbb{D}}$ and $k \in \{1, 2, 3\}$.

Note that the extension of this lemma for the coefficients c_4 and c_5 may also be found in Lemma 2.1. of [9].

The next lemma represents the first part of the result from [19, Remark p. 162]:

Lemma 3. *If $h \in \mathcal{P}$ is given by (4), then*

$$|c_2 - \nu c_1^2| \leq 2 - \nu |c_1|^2, \quad 0 < \nu \leq \frac{1}{2}.$$

2. Initial Coefficients Sharp Upper Bounds

The next main results gives us the sharp upper bounds for the initial coefficients of the functions from the class $\mathcal{S}_{\cos}^*$.

Theorem 1. Let $f \in \mathcal{S}_{\cos}^*$ be given by (1). Then,

$$|a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{2}{27}\sqrt{3}, \quad |a_5| \leq \frac{1}{8},$$

and these bounds are sharp.

Proof. If $f \in \mathcal{S}_{\cos}^*$, then by the definition of subordination there exists a Schwarz function w that is analytic in $\mathbb{D}$ and satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = \cos w(z), \quad z \in \mathbb{D}. \quad (6)$$

Therefore, the function h defined by

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots, \quad z \in \mathbb{D}, \quad (7)$$

has the property $h \in \mathcal{P}$.

Using the relations (6) and (7), by equating the first four coefficients we get

$$a_2 = 0, \quad a_3 = -\frac{1}{16}c_1^2, \quad a_4 = \frac{1}{24}c_1^3 - \frac{1}{12}c_1c_2, \quad (8)$$

$$a_5 = -\frac{1}{48}c_1^4 - \frac{1}{32}c_2^2 + \frac{3}{32}c_1^2c_2 - \frac{1}{16}c_1c_3. \quad (9)$$

(i) From the second relation of (8), according to the inequality (5) it follows that

$$|a_3| \leq \frac{1}{4},$$

and this inequality is attained for the function f_* from the Remark 2.2. of [20]. Thus, the above upper bound is sharp, that is the best possible.

(ii) To find the upper bound of $|a_4|$, we see that the third equality of (8) could be written in the form

$$a_4 = -\frac{c_1}{12} \left(c_2 - \frac{c_1^2}{2} \right),$$

and using Lemma 3 for $\nu = \frac{1}{2}$ we obtain

$$|a_4| \leq \frac{|c_1|}{12} \left(2 - \frac{|c_1|^2}{2} \right).$$

Denoting $c := |c_1|$, from Lemma 1 we have $0 \leq c \leq 2$, hence

$$|a_4| \leq \frac{c}{12} \left(2 - \frac{c^2}{2} \right) =: F(c), \quad c \in [0, 2]. \quad (10)$$

It's easy to check that the function F attained the maximum value at $c = \frac{2}{\sqrt{3}}$, and according to the above inequality we get

$$|a_4| \leq F\left(\frac{2}{\sqrt{3}}\right) = \frac{2}{27}\sqrt{3}.$$

For the prove the sharpness of this upper bound, let $\hat{t} = \frac{2}{\sqrt{3}}$, $h_1(z) = \frac{1-z^2}{1-\hat{t}z+z^2}$ and

$$w_1(z) = \frac{h_1(z)-1}{h_1(z)+1} = \frac{z(3z-\sqrt{3})}{\sqrt{3}z-3}, \quad z \in \mathbb{D},$$

thus $w_1(0) = 0$. To prove that $|w_1(z)| < 1$ in $\mathbb{D}$, we remarks that the function w_1 could be written in the form $w_1(z) = z\varphi(z)$, where

$$\varphi(z) = \frac{3z-\sqrt{3}}{\sqrt{3}z-3}, \quad z \in \mathbb{D}.$$

Since $|\varphi(z)| < 1$, $z \in \mathbb{D}$, if and only if $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1-\varphi(z)}{1+\varphi(z)} = \frac{(\sqrt{3}-3)(z+1)}{(\sqrt{3}+3)(z-1)}, \quad z \in \mathbb{D},$$

is a circular transform. It's easy to check that

$$H(0) = \frac{3-\sqrt{3}}{\sqrt{3}+3} > 0, \quad H(-1) = 0, \quad H(i) = \frac{(3-\sqrt{3})i}{\sqrt{3}+3}, \quad H(-i) = \frac{(\sqrt{3}-3)i}{\sqrt{3}+3}.$$

Using the fact that every circular transform maps the circles (in the large sense, that are circles or lines) of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ into circles of $\mathbb{C}_\infty$, from the above values of H it follows that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, which implies $|\varphi(z)| < 1$ in $\mathbb{D}$, hence

$$|w_1(z)| = |z| |\varphi(z)| < 1, \quad z \in \mathbb{D}.$$

Therefore, the function

$$f_1(z) := z \exp\left(\int_0^z \frac{\cos w_1(t)-1}{t} dt\right) = z - \frac{1}{12}z^3 + \frac{2}{27}\sqrt{3}z^4 + \dots, \quad z \in \mathbb{D}, \quad (11)$$

belongs to the class $\mathcal{S}_{\cos}^*$, hence $|a_4| = \frac{2}{27}\sqrt{3}$ for the above function f_1 that proves the sharpness of the second inequality of this theorem.

(iii) To determine the upper bound of $|a_5|$, using the relation (9) combined with Lemma 2 we obtain

$$a_5 = \frac{1}{96} \left[4t_1^4 + 24|t_1|^2 t_2^2 (1-|t_1|^2) - 12t_2^2 (1-|t_1|^2)^2 - 24t_1 (1-|t_1|^2) (1-|t_2|^2) t_3 \right].$$

Setting $x := |t_1| \in [0, 1]$, $y := |t_2| \in [0, 1]$ and $u := |t_3| \in [0, 1]$, from the triangle's inequality we obtain

$$\begin{aligned} |a_5| &\leq \frac{1}{24} \left[x^4 + 6x^2(1-x^2)y^2 + 3(1-x^2)^2 y^2 + 6x(1-x^2)(1-y^2)u \right] \\ &= \frac{1}{24} F(x, y, u), \quad x, y, u \in [0, 1], \end{aligned} \quad (12)$$

where

$$F(x, y, u) := x^4 + 6x^2(1 - x^2)y^2 + 3(1 - x^2)^2y^2 + 6x(1 - x^2)(1 - y^2)u. \quad (13)$$

Denoting by $\Omega := [0, 1] \times [0, 1] \times [0, 1]$ the closed unit cuboid, we have to find the maximum value of F in Ω .

I. First, let's consider that (x, y, u) belongs to the interior of Ω , denoted by $\text{int } \Omega$. Differentiating (13) with respect to u we obtain

$$\frac{\partial F(x, y, u)}{\partial u} = 6x(1 - x^2)(1 - y^2) \neq 0, \quad (x, y, u) \in \text{int } \Omega,$$

therefore the function F have no extremal values in $\text{int } \Omega$.

II. Next we will discuss the existence of the maximum value for F on the open six faces of Ω , as follows.

(i) On the face $x = 0$ the next inequality holds:

$$F(0, y, u) = 3y^2 < 3, \quad y, u \in (0, 1). \quad (14)$$

(ii) On the face $x = 1$ we have the equality

$$F(1, y, u) = 1, \quad y, u \in (0, 1). \quad (15)$$

(iii) On $y = 0$, let denote

$$\kappa_1(x, u) := F(x, 0, u) = x^4 + 6(1 - x^2)xu, \quad x, u \in (0, 1), \quad (16)$$

and because

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 6x(1 - x^2) \neq 0, \quad x \in (0, 1),$$

it follows that the function κ_1 has no extremal values on $(0, 1) \times (0, 1)$.

(iv) On the open face $y = 1$ the function F reduces to

$$s_1(x) := F(x, 1, u) = x^4 + 6x^2(1 - x^2) + 3(1 - x^2)^2, \quad x \in (0, 1). \quad (17)$$

Since $s_1'(x) = -8x^3 < 0$, $x \in (0, 1)$, therefore s_1 is a strictly decreasing function on $(0, 1)$, hence

$$F(x, 1, u) < 3, \quad x, u \in (0, 1). \quad (18)$$

(v) For the open face $u = 0$ we get

$$\kappa_2(x, y) := F(x, y, 0) = x^4 + 6x^2(1 - x^2)y^2 + 3(1 - x^2)^2y^2, \quad x, y \in (0, 1),$$

therefore

$$\frac{\partial \kappa_2(x, y)}{\partial x} = -4x^3(3y^2 - 1), \quad \frac{\partial \kappa_2(x, y)}{\partial y} = -6(x - 1)(x + 1)y(x^2 + 1),$$

hence the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

(vi) On the open face $u = 1$ the function F becomes

$$\kappa_3(x, y) := x^4 + 6x^2(1 - x^2)y^2 + 3y^2(1 - x^2)^2 + 6x(1 - x^2)(1 - y^2), \quad x, y \in (0, 1),$$

and it's easy to check that the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω .

(i) From (13) we obtain

$$F(x, 0, 0) = x^4 \leq 1, \quad x \in [0, 1].$$

(ii) Also, from (16) we get $s_2(x) := F(x, 0, 1) = x^4 + 6x(1 - x^2)$. We may easily see that the zero of s_2' in $[0, 1]$ given by $x_0 = 0.621924 \dots$ satisfies $s_2''(x_0) < 0$, thus

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 2.437828 \dots, \quad x \in [0, 1].$$

(iii) Putting $x = 0$ in (16) we have

$$F(0, 0, u) = 0, \quad u \in [0, 1].$$

(iv) Since (17) is independent of u , similarly as we obtained the inequality (18) we deduce

$$F(x, 1, 1) = F(x, 1, 0) \leq 3, \quad x \in [0, 1],$$

and putting $x = 0$ in (17) we obtain

$$F(0, 1, u) = 3, \quad u \in [0, 1].$$

(v) The function given by (15) is independent on the variables y and u , thus

$$F(1, y, 0) = F(1, y, 1) = F(1, 0, u) = F(1, 1, u) = 1, \quad y, u \in [0, 1].$$

(vi) Since the function defined by (14) is independent of the variable u , we have

$$F(0, y, 0) = F(0, y, 1) \leq 3, \quad y \in [0, 1].$$

From all the above reasons we conclude that

$$\max\{F(x, y, u) : (x, y, u) \in \Omega\} = F(0, 1, u) = 3,$$

and according to (12) we finally obtain that $|a_5| \leq \frac{1}{8}$. This upper bound for $|a_5|$ is sharp for the function

$$f_2(z) := z \exp\left(\int_0^z \frac{\cos(t^2) - 1}{t} dt\right) = z - \frac{1}{8}z^5 + \dots, \quad z \in \mathbb{D}, \quad (19)$$

that completes our proof. $\square$

Remark 1. 1. In [20] it is proved that $|a_4| \leq \frac{1}{3}$ and $|a_5| \leq \frac{11}{24}$ and the authors gave in the Remark 2.2. a function of the class $S_{\cos}^*$ for which $|a_4| = 0$ and $|a_5| = \frac{1}{24}$. In the above theorem the results are sharp by giving the best upper bounds.

2. The maximum value of the function F defined by (13) could be easily found using the MAPLE™ software with the code

```
[> maximize(F,x=0..1,y=0..1,u=0..1,location=true);
```

that gives the same result as above.

In the following two theorems we determined the sharp upper bounds for the Hankel determinants $|H_{3,1}|$ and $|H_{2,3}|$, respectively, over the class $\mathcal{S}_{\cos}^*$.

Theorem 2. If $f \in \mathcal{S}_{\cos}^*$ is given by (1), then

$$|H_{3,1}(f)| \leq -\frac{3115}{164268} + \frac{671}{657072}\sqrt{1342} = 0.018446\dots,$$

and this result is sharp.

Proof. Using the relations (8)–(9) in (3) we obtain

$$H_{3,1}(f) = -\frac{7}{36864}c_1^6 + \frac{5}{4608}c_1^4c_2 + \frac{1}{256}c_1^3c_3 - \frac{23}{4608}c_1^2c_2^2,$$

and replacing all the variables of the above relation with those of Lemma 2 it follows that

$$H_{3,1}(f) = \frac{1}{576} \left[3t_1^6 - 36t_1^2|t_1|^2t_2^2(1-|t_1|^2) - 46t_1^2t_2^2(1-|t_1|^2)^2 + 36t_1^3(1-|t_1|^2)(1-|t_2|^2)t_3 \right].$$

If we set $x := |t_1| \in [0, 1]$, $y := |t_2| \in [0, 1]$ and $u := |t_3| \in [0, 1]$, using the above relation and the triangle's inequality we get

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{1}{576} \left[3x^6 + 36x^4(1-x^2)y^2 + 46x^2(1-x^2)^2y^2 + 36x^3(1-x^2)(1-y^2)u \right] \\ &= \frac{1}{576}F(x, y, u), \quad x, y, u \in [0, 1], \end{aligned} \quad (20)$$

where

$$F(x, y, u) := 3x^6 + 36x^4(1-x^2)y^2 + 46x^2(1-x^2)^2y^2 + 36x^3(1-x^2)(1-y^2)u. \quad (21)$$

With the same notations and method like in the proof of Theorem 1, next we will find the maximum value of F in Ω .

I. If we consider that $(x, y, u) \in \text{int } \Omega$, differentiating (21) with respect to u we obtain

$$\frac{\partial F(x, y, u)}{\partial u} = 36x^3(1-x^2)(1-y^2) \neq 0, \quad (x, y, u) \in \text{int } \Omega,$$

it follows that the function F have no maximum value in $\text{int } \Omega$.

II. In the sequel we will study the existence of the maximum value of F in the interior of six faces of Ω .

(i) On the face $x = 0$ we have

$$F(0, y, u) = 0, \quad y, u \in (0, 1). \quad (22)$$

(ii) On the face $x = 1$ we get

$$F(1, y, u) = 3, \quad y, u \in (0, 1). \quad (23)$$

(iii) On $y = 0$, the function F can be written as

$$\kappa_1(x, u) := F(x, 0, u) = 3x^6 + 36x^3(1-x^2)u, \quad x, u \in (0, 1). \quad (24)$$

Since

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 36x^3(1 - x^2) \neq 0, \quad x \in (0, 1),$$

it implies that the function κ_1 has no maximum point in this face of Ω .

(iv) On $y = 1$, the function F reduces to

$$s_1(x) := F(x, 1, u) = 3x^6 + 36x^4(1 - x^2) + 46x^2(1 - x^2)^2, \quad x \in (0, 1). \quad (25)$$

Since $s'_1(x) = 78x^5 - 224x^3 + 92x$, $x \in (0, 1)$, it follows that $s'_1(x) = 0$ has the zero $x_0 = \frac{\sqrt{2184} - 39\sqrt{1342}}{39} = 0.704685 \dots \in (0, 1)$, that satisfy the inequality $s''_1(x_0) < 0$, we obtain

$$F(x, 1, u) \leq -\frac{49840}{4563} + \frac{2684}{4563}\sqrt{1342} = 10.625427 \dots, \quad x, u \in (0, 1). \quad (26)$$

(v) On the face $u = 0$, the function F becomes

$$\kappa_2(x, y) := F(x, y, 0) = 3x^6 + 36x^4(1 - x^2)y^2 + 46x^2(1 - x^2)^2y^2, \quad x, y \in (0, 1),$$

thus

$$\frac{\partial \kappa_2(x, y)}{\partial x} = (60y^2 + 18)x^5 - 224x^3y^2 + 92x^2, \quad \frac{\partial \kappa_2(x, y)}{\partial y} = 4x^2y(5x^4 - 28x^2 + 23).$$

It's easy to see that the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

(vi) On $u = 1$, the function F takes the form

$$\kappa_3(x, y) = 3x^6 + 36x^4(1 - x^2)y^2 + 46x^2(1 - x^2)^2y^2 + 36x^3(1 - x^2)(1 - y^2), \quad x, y \in (0, 1).$$

Similarly, the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω .

(i) From (24) we obtain

$$F(x, 0, 0) = 3x^6 \leq 3, \quad x \in [0, 1].$$

(ii) The relation (24) at $u = 1$ becomes $s_2(x) := F(x, 0, 1) = 3x^6 + 36x^3(1 - x^2)$, $x \in [0, 1]$. The solution of $s'_2(x) = 18x^2(x^3 - 10x^2 + 6) = 0$ is $x_0 = 0.807920 \dots \in [0, 1]$ and satisfy $s''_2(x_0) < 0$, hence

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 7.427102 \dots, \quad x \in [0, 1].$$

(iii) Putting $x = 0$ in (24) we have

$$F(0, 0, u) = 0, \quad u \in [0, 1]$$

(iv) Since (25) is independent of u , similarly as above we obtain

$$F(x, 1, 1) = F(x, 1, 0) \leq -\frac{49840}{4563} + \frac{2684}{4563}\sqrt{1342}, \quad x \in [0, 1],$$

while if we take $x = 0$ in (25) it follows

$$F(0, 1, u) = 0, \quad u \in [0, 1].$$

(v) The relation (23) is independent on the variables $y, u \in [0, 1]$, hence

$$F(1, y, 0) = F(1, y, 1) = F(1, 0, u) = F(1, 1, u) = 3, \quad y, u \in [0, 1].$$

(vi) Finally, since (22) is independent on the variable $u \in [0, 1]$, we have

$$F(0, y, 0) = F(0, y, 1) = 0, \quad y \in [0, 1].$$

All the inequalities we obtained above show that

$$\begin{aligned} \max\{F(x, y, u) : (x, y, u) \in \Omega\} &= F\left(\frac{\sqrt{2184 - 39\sqrt{1342}}}{39}, 1, u\right) \\ &= -\frac{49840}{4563} + \frac{2684}{4563}\sqrt{1342} = 10.625427\dots, \end{aligned}$$

and using (20) our inequality is proved.

For proving the sharpness of this inequality, let consider $\tilde{t} = \frac{\sqrt{2184 - 39\sqrt{1342}}}{39}$, $h_3(z) = \frac{1 + (1+i)\tilde{t}z + iz^2}{1 - (1-i)\tilde{t}z - iz^2}$ and

$$w_3(z) = \frac{h_2(z) - 1}{h_2(z) + 1} = \frac{z\left(\sqrt{2184 - 39\sqrt{1342}} + 39iz\right)}{39 + \sqrt{2184 - 39\sqrt{1342}}iz}, \quad z \in \mathbb{D}.$$

We can see that $w_3(0) = 0$, and let's write the function w_3 as $w_3(z) = z\psi(z)$, where

$$\psi(z) = \frac{\sqrt{2184 - 39\sqrt{1342}} + 39iz}{39 + \sqrt{2184 - 39\sqrt{1342}}iz}, \quad z \in \mathbb{D}.$$

From the same reasons regarding the circular transforms like in the proof of the sharpness of Theorem 1(ii), we will show that $|\psi(z)| < 1$ in $\mathbb{D}$ by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1 - \psi(z)}{1 + \psi(z)} = \frac{\left(\sqrt{2184 - 39\sqrt{1342}} - 39\right)(z + i)}{\left(\sqrt{2184 - 39\sqrt{1342}} + 39\right)(-z + i)}, \quad z \in \mathbb{D},$$

is a circular transform. Since

$$\begin{aligned} H(0) &= \frac{-\sqrt{2184 - 39\sqrt{1342}} + 39}{\sqrt{2184 - 39\sqrt{1342}} + 39} = 0.173236\dots > 0, \quad H(-i) = 0, \\ H(-1) &= \frac{-\left(\sqrt{2184 - 39\sqrt{1342}} - 39\right)i}{\sqrt{2184 - 39\sqrt{1342}} + 39}, \quad H(1) = \frac{\left(\sqrt{2184 - 39\sqrt{1342}} - 39\right)i}{\sqrt{2184 - 39\sqrt{1342}} + 39}, \end{aligned}$$

like in the above mentioned proof, these values of H lead us to $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, which implies $|\psi(z)| < 1$ in $\mathbb{D}$, therefore

$$|w_3(z)| = |z| |\psi(z)| < 1, \quad z \in \mathbb{D}.$$

Thus, the function

$$f_3(z) = z \exp\left(\int_0^z \frac{\cos w_3(t) - 1}{t} dt\right) = z + \left(-\frac{14}{39} + \frac{\sqrt{1342}}{156}\right)z^3 - \frac{1}{4563}(-17 + \sqrt{1342})\sqrt{2184 - 39\sqrt{1342}}z^4 + \left(\frac{23135}{36504} - \frac{163\sqrt{1342}}{9126}\right)z^5 + \dots, \quad z \in \mathbb{D},$$

belongs to the class $\mathcal{S}_{\cos}^*$, with the initial coefficients

$$a_2 = 0, \quad a_3 = -\frac{14}{39} + \frac{\sqrt{1342}}{156}, \\ a_4 = -\frac{1}{4563}\left((-17 + \sqrt{1342})\sqrt{2184 - 39\sqrt{1342}}\right)i, \quad a_5 = \frac{23135}{36504} - \frac{163\sqrt{1342}}{9126}.$$

From the relation (3) we get $|H_{3,1}(f_3)| = -\frac{3115}{164268} + \frac{671}{657072}\sqrt{1342} = 0.018446\dots$ and the proof is complete. $\square$

Remark 2. In [20] it is proved that $|H_{3,1}(f)| \leq \frac{139}{576} = 0.241319\dots$ for all $f \in \mathcal{S}_{\cos}^*$, but that result wasn't the best possible. If we compare the upper bounds of $|H_{3,1}(f)|$ for $f \in \mathcal{S}_{\cos}^*$, obtained here with those of [20], the result of Theorem 2 is a significant improvement of the previous one. Moreover, the inequality obtained in above theorem is sharp, thus the found upper bound for $|H_{3,1}(f)|$ if $f \in \mathcal{S}_{\cos}^*$ cannot be improved.

Theorem 3. If $f \in \mathcal{S}_{\cos}^*$ has the form (1), then

$$|H_{2,3}(f)| \leq \frac{77}{15552} + \frac{29}{15552}\sqrt{58} = 0.019152\dots,$$

and the result is sharp.

Proof. Replacing in (2) the values of a_2 , a_3 , a_4 , and a_5 given by (8)–(9) we obtain

$$H_{2,3}(f) = -\frac{1}{2304}c_1^6 + \frac{5}{4608}c_1^4c_2 + \frac{1}{256}c_1^3c_3 - \frac{23}{4608}c_1^2c_2^2.$$

According to the Lemma 2, from the above relation we deduce that

$$H_{2,3}(f) = \frac{1}{288}\left[-3t_1^6 - 18t_1^2|t_1|^2t_2^2(1 - |t_1|^2) - 23t_1^2t_2^2(1 - |t_1|^2)^2 + 18t_1^3(1 - |t_1|^2)(1 - |t_2|^2)t_3\right].$$

Setting $x := |t_1| \in [0, 1]$, $y := |t_2| \in [0, 1]$ and $u := |t_3| \in [0, 1]$, then using the triangle's inequality, the above relation lead us to

$$|H_{2,3}(f)| \leq \frac{1}{288}\left[3x^6 + 18x^4(1 - x^2)y^2 + 23x^2(1 - x^2)^2y^2 + 18x^3(1 - x^2)(1 - y^2)u\right] \\ = \frac{1}{288}F(x, y, u), \quad x, y, u \in [0, 1], \quad (27)$$

where

$$F(x, y, u) := 3x^6 + 18x^4(1 - x^2)y^2 + 23x^2(1 - x^2)^2y^2 + 18x^3(1 - x^2)(1 - y^2)u. \quad (28)$$

With the same notations like in the proofs of the two previous theorems, we have to find the maximum value of F on $\text{int } \Omega$, on the six faces, and on the twelve edges of Ω .

I. First we consider the arbitrary interior point $(x, y, u) \in \text{int } \Omega$. Differentiating (28) with respect to u we obtain

$$\frac{\partial F(x, y, u)}{\partial u} = 18x^3(1-x^2)(1-y^2) \neq 0, \quad (x, y, u) \in \text{int } \Omega,$$

therefore we have no maximum value of F in $\text{int } \Omega$.

II. Next we will study the existence of the maximum value of the function F in the interior of six faces of Ω .

(i) On the face $x = 0$ the function F reduces to

$$F(0, y, u) = 0, \quad y, u \in (0, 1). \quad (29)$$

(ii) On the face $x = 1$ it takes the form

$$F(1, y, u) = 3, \quad y, u \in (0, 1). \quad (30)$$

(iii) On $y = 0$, the function F can be written as

$$\kappa_1(x, u) := F(x, 0, u) = 3x^6 + 18x^3(1-x^2)u, \quad x, u \in (0, 1). \quad (31)$$

Since

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 18x^3(1-x^2) \neq 0, \quad x \in (0, 1),$$

it follows that κ_1 has no maximum point in this face of Ω .

(iv) On $y = 1$, the function F reduces to

$$s_1(x) := F(x, 1, u) = 3x^6 + 18x^4(1-x^2) + 23x^2(1-x^2)^2, \quad x \in (0, 1), \quad (32)$$

thus

$$s_1'(x) = 48x^5 - 112x^3 + 46x, \quad x \in (0, 1).$$

Hence $s_1'(x) = 0$ has the zero $x_0 = \frac{\sqrt{42-3\sqrt{58}}}{6} = 0.729396 \dots \in (0, 1)$ and $s_1''(x_0) < 0$. Therefore,

$$F(x, 1, u) \leq F(x_0, 1, u) = \frac{77}{54} + \frac{29}{54}\sqrt{58} = 5.515878 \dots, \quad x, u \in (0, 1). \quad (33)$$

(v) On the face $u = 0$ the function F becomes

$$\kappa_2(x, y) := F(x, y, 0) = 3x^6 + 18x^4(1-x^2)y^2 + 23x^2(1-x^2)^2y^2, \quad x, y \in (0, 1).$$

Therefore,

$$\frac{\partial \kappa_2(x, y)}{\partial x} = 2x(15x^4y^2 + 9x^4 - 56x^2y^2 + 23y^2), \quad \frac{\partial \kappa_2(x, y)}{\partial y} = 2x^2y(5x^4 - 28x^2 + 23),$$

and the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

(vi) On $u = 1$, the function F takes the form

$$\kappa_3(x, y) = 3x^6 + 18x^4(1-x^2)y^2 + 23x^2(1-x^2)^2y^2 + 18x^3(1-x^2)(1-y^2), \quad x, y \in (0, 1),$$

hence

$$\begin{aligned}\frac{\partial \kappa_3(x, y)}{\partial x} &= 6(5y^2 + 3)x^5 - 90(1 - y^2)x^4 - 112x^3y^2 + 54(1 - y^2)x^2 + 46xy^2, \\ \frac{\partial \kappa_3(x, y)}{\partial y} &= 10(x - 1)^2(x + 1)y \left(x + \frac{23}{5}\right)x^2.\end{aligned}$$

Therefore, the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

III. Now we investigate the maximum of F on the edges of Ω .

(i) From (31) we obtain

$$F(x, 0, 0) = 3x^6 \leq 3, \quad x \in [0, 1].$$

(ii) Also, from (31) at $u = 1$ we get $s_2(x) := F(x, 0, 1) = 3x^6 + 18x^3(1 - x^2)$. The solution in $[0, 1]$ of the equation $s_2'(x) = 18x^5 - 90x^4 + 54x^2 = 0$ for which $s_2''(x) < 0$ is $x_0 = 0.850256 \dots \in [0, 1]$, thus

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 4.199005 \dots, \quad x \in [0, 1].$$

(iii) Putting $x = 0$ in (31) we have

$$F(0, 0, u) = 0, \quad u \in [0, 1].$$

(iv) Since (32) is independent on u , according to (33) we obtain

$$F(x, 1, 1) = F(x, 1, 0) \leq \frac{77}{54} + \frac{29}{54}\sqrt{58}, \quad x \in [0, 1].$$

(v) If we take $x = 0$ in (32) we get

$$F(0, 1, u) = 0, \quad u \in [0, 1].$$

Using the fact that the relation (30) is independent on the variables y and u , we deduce

$$F(1, y, 0) = F(1, y, 1) = F(1, 0, u) = F(1, 1, u) = 3, \quad y, u \in [0, 1].$$

(vi) Since the relation (29) is independent of the variable u , we have

$$F(0, y, 0) = F(0, y, 1) = 0, \quad y \in [0, 1].$$

Consequently, from the above reasons we conclude that

$$\begin{aligned}\max\{F(x, y, u) : (x, y, u) \in \Omega\} &= F\left(\frac{\sqrt{42 - 3\sqrt{58}}}{6}, 1, u\right) \\ &= \frac{77}{54} + \frac{29}{54}\sqrt{58} = 5.515878 \dots,\end{aligned}$$

and combining with (27) it follows

$$|H_{2,3}(f)| \leq \frac{77}{15552} + \frac{29}{15552}\sqrt{58} = 0.019152 \dots$$

To prove the sharpness of the above result, let us consider $t^* = \frac{1}{3}\sqrt{42 - 3\sqrt{58}}$, $h_4(z) = \frac{1 + t^*z + z^2}{1 - z^2}$ and

$$w_4(z) = \frac{h_4(z) - 1}{h_4(z) + 1} = \frac{z(\sqrt{42 - 3\sqrt{58}} + 6z)}{6 + \sqrt{42 - 3\sqrt{58}}z}, \quad z \in \mathbb{D}.$$

First, $w_4(0) = 0$, and we will write $w_4(z) = z\chi(z)$, where

$$\chi(z) = \frac{\sqrt{42 - 3\sqrt{58}} + 6z}{6 + \sqrt{42 - 3\sqrt{58}}z}, \quad z \in \mathbb{D}.$$

Using the same property of the circular transforms like in the proof of the sharpness of Theorem 1(ii) and Theorem 2, we will show that $|\chi(z)| < 1$ in $\mathbb{D}$ by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1 - \chi(z)}{1 + \chi(z)} = \frac{(\sqrt{42 - 3\sqrt{58}} - 6)(z - 1)}{(\sqrt{42 - 3\sqrt{58}} + 6)(z + 1)},$$

is a circular transform. Computing the below values

$$\begin{aligned} H(0) &= -\frac{\sqrt{42 - 3\sqrt{58}} - 6}{\sqrt{42 - 3\sqrt{58}} + 6} = 0.156472 \dots > 0, \quad H(1) = 0, \\ H(i) &= \frac{(\sqrt{42 - 3\sqrt{58}} - 6)i}{\sqrt{42 - 3\sqrt{58}} + 6}, \quad H(-i) = -\frac{(\sqrt{42 - 3\sqrt{58}} - 6)i}{\sqrt{42 - 3\sqrt{58}} + 6}, \end{aligned}$$

from the similar reasons as in the above mentioned proofs, these values of H implies $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, that yields $|\chi(z)| < 1$ in $\mathbb{D}$, therefore

$$|w_4(z)| = |z| |\chi(z)| < 1, \quad z \in \mathbb{D}.$$

Consequently, the function

$$\begin{aligned} f_4(z) &= z \exp\left(\int_0^z \frac{\cos w_4(t) - 1}{t} dt\right) = z + \left(-\frac{7}{24} + \frac{1}{48}\sqrt{58}\right)z^3 \\ &\quad - \frac{1}{216}(-2 + \sqrt{58})\sqrt{42 - 3\sqrt{58}}z^4 + \left(-\frac{7}{54} + \frac{5}{216}\sqrt{58}\right)z^5 + \dots, \quad z \in \mathbb{D}, \end{aligned}$$

belongs to the class $\mathcal{S}_{\cos}^*$. In the above power series expansion we have $a_3 = -\frac{7}{24} + \frac{1}{48}\sqrt{58}$, $a_4 = -\frac{1}{216}(-2 + \sqrt{58})\sqrt{42 - 3\sqrt{58}}$ and $a_5 = -\frac{7}{54} + \frac{5}{216}\sqrt{58}$, hence

$$|H_{2,3}(f_4)| = |a_3a_5 - a_4^2| = \frac{77}{15552} + \frac{29}{15552}\sqrt{58},$$

that completes our proof. $\square$

Remark 3. The maximum values of the functions F defined by (21) and (28) could be also found by using the MAPLE™ computer software codes like in the Remark 1 2., and we obtain the same values like in both of the above two theorems.

3. Logarithmic Coefficients Sharp Upper Bounds

The logarithmic coefficients $\gamma_n := \gamma_n(f)$, $n \in \mathbb{N}$, for the function $f \in \mathcal{S}$ are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \quad (34)$$

Since the function $\phi(z) := \cos z$ has positive real part in $\mathbb{D}$, and moreover

$$\operatorname{Re}(\cos z) > \frac{1}{2}, \quad z \in \mathbb{D},$$

it follows that $\mathcal{S}_{\cos}^* \subset \mathcal{S}^* \subset \mathcal{S}$ (see [20, p. 610]). Therefore, it is possible to define the logarithmic coefficients for the functions $f \in \mathcal{S}_{\cos}^*$.

In this section we give the sharp upper bounds estimates for the third and fourth logarithmic coefficients of the functions that belong to the class $\mathcal{S}_{\cos}^*$.

Theorem 4. *If $f \in \mathcal{S}_{\cos}^*$ is given by (1), then*

$$|\gamma_3| \leq \frac{\sqrt{3}}{27}, \quad |\gamma_4| \leq \frac{1}{16}.$$

These bounds are sharp.

Proof. If $f \in \mathcal{S}_{\cos}^*$ has the form (1), then

$$\log \frac{f(z)}{z} = a_2 z + \left(-\frac{a_2^2}{2} + a_3 \right) z^2 + \left(-a_2 a_3 + a_4 + \frac{a_2^3}{3} \right) z^3 + \dots, \quad z \in \mathbb{D}. \quad (35)$$

and equating the first four coefficients of (34) with those of (35) we get

$$\gamma_1 = \frac{a_2}{2}, \quad \gamma_2 = \frac{1}{4}(2a_3 - a_2^2), \quad \gamma_3 = \frac{1}{6}(a_2^3 - 3a_2 a_3 + 3a_4), \quad (36)$$

$$\gamma_4 = \frac{1}{8}(-a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5). \quad (37)$$

With the same notation like in the proof of Theorem 1, replacing in (36) and (37) the values of a_2, a_3, a_4 and a_5 from the relations (8) and (9), we obtain

$$\gamma_1 = 0, \quad \gamma_2 = -\frac{1}{32}c_1^2, \quad \gamma_3 = \frac{1}{48}c_1^3 - \frac{1}{24}c_1 c_2, \quad (38)$$

$$\gamma_4 = -\frac{35}{3072}c_1^4 - \frac{1}{64}c_1^2 c_2 + \frac{3}{64}c_1^2 c_2 - \frac{1}{32}c_1 c_3 \quad (39)$$

For the upper bound of $|\gamma_3|$, using (38) we write

$$\gamma_3 = -\frac{c_1}{24} \left(c_2 - \frac{c_1^2}{2} \right),$$

and according to Lemma 3 for $\nu = \frac{1}{2}$ we obtain

$$|\gamma_3| \leq \frac{|c_1|}{24} \left(2 - \frac{|c_1^2|}{2} \right).$$

Denoting $c := |c_1|$, from Lemma 1 we have

$$|\gamma_3| \leq \frac{c}{24} \left(2 - \frac{c^2}{2} \right) = F(c), \quad c \in [0, 2].$$

Using the result we got for the computation of the maximum F given by (10) we get $|\gamma_3| \leq \frac{\sqrt{3}}{27}$.

To prove the sharpness of this bound let consider the function f_1 given by (11), were $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{12}$ and $a_4 = \frac{2}{27}\sqrt{3}$. Therefore, for this function, by using the last of the relations from (36) we obtain $\gamma_3 = \frac{\sqrt{3}}{27}$.

To find the upper bound of $|\gamma_4|$, from (37) combined with Lemma 2 we can write

$$\gamma_4 = \frac{1}{192} \left[t_1^4 + 24|t_1|^2 t_2^2 (1 - |t_1|^2) - 12t_2^2 (1 - |t_1|^2)^2 - 24t_1 (1 - |t_1|^2) (1 - |t_2|^2) t_3 \right],$$

and setting $x := |t_1| \in [0, 1]$, $y := |t_2| \in [0, 1]$ and $u := |t_3| \in [0, 1]$, using the triangle's inequality we obtain

$$\begin{aligned} |\gamma_4| &\leq \frac{1}{192} \left[x^4 + 24x^2 (1 - x^2) y^2 + 12 (1 - x^2)^2 y^2 + 24x (1 - x^2) (1 - y^2) u \right] \\ &= \frac{1}{192} F(x, y, u), \quad x, y, u \in [0, 1], \end{aligned} \quad (40)$$

where

$$F(x, y, u) := x^4 + 24x^2 (1 - x^2) y^2 + 12 (1 - x^2)^2 y^2 + 24x (1 - x^2) (1 - y^2) u. \quad (41)$$

Using the notations and the technique from the proofs of the previous theorems, we will determine the maximum of F on Ω as follows.

I. In the points $(x, y, u) \in \text{int } \Omega$, differentiating (41) with respect to u we obtain

$$\frac{\partial F(x, y, u)}{\partial u} = 24x (1 - x^2) (1 - y^2) \neq 0, \quad (x, y, u) \in \text{int } \Omega,$$

therefore the function F doesn't attained its maximum value in $\text{int } \Omega$.

II. In the next items we will discuss the existence of the maximum value of F in the interior of six faces of Ω .

(i) On the face $x = 0$ we get

$$F(0, y, u) = 12y^2 \leq 12, \quad y, u \in (0, 1). \quad (42)$$

(ii) On the face $x = 1$ the function F takes the form

$$F(1, y, u) = 1, \quad y, u \in (0, 1). \quad (43)$$

(iii) On $y = 0$, the function can be written as

$$\kappa_1(x, u) := F(x, 0, u) = x^4 + 24x (1 - x^2) u, \quad x, u \in (0, 1), \quad (44)$$

and because

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 24x (1 - x^2) \neq 0, \quad x \in (0, 1),$$

it implies that the function κ_1 has no maximum in $(0, 1) \times (0, 1)$.

(iv) On $y = 1$, the function F reduces to

$$s_1(x) := F(x, 1, u) = -11x^4 + 12, \quad x \in (0, 1). \quad (45)$$

Since $s_1'(x) = -44x^3 < 0$, $x \in (0, 1)$, the function s_1 is strictly decreasing on $(0, 1)$, hence

$$F(x, 1, u) \leq F(0, 1, u) = 12, \quad x, u \in (0, 1). \quad (46)$$

(v) On the face $u = 0$, F will have the form

$$\kappa_2(x, y) := F(x, y, 0) = x^4 + 24x^2(1 - x^2)y^2 + 12(1 - x^2)^2y^2, \quad x, y \in (0, 1).$$

Therefore

$$\frac{\partial \kappa_2(x, y)}{\partial x} = -4x^3(12y^2 - 1), \quad \frac{\partial \kappa_2(x, y)}{\partial y} = -24y(x - 1)(x + 1)(x^2 + 1),$$

thus the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

(vi) On $u = 1$, the function F will be

$$\kappa_3(x, y) := -12x^4y^2 + 24x^3y^2 + x^4 - 24x^3 - 24xy^2 + 12y^2 + 24x, \quad x, y \in (0, 1).$$

Similarly, it's easy to check that the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ have no solutions in $(0, 1) \times (0, 1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω .

(i) From (44) at $u = 0$ we have

$$F(x, 0, 0) = x^4 \leq 1, \quad x \in [0, 1].$$

(ii) Using (44) at $u = 1$, we get $s_2(x) := F(x, 0, 1) = x(x^3 - 24x^2 + 24)$, hence $s_2'(x) = 4x^3 - 72x^2 + 24$. The solution of $s_2'(x) = 0$ in $[0, 1]$ is $x_0 = 0.587000 \dots \in [0, 1]$ and $s_2''(x_0) < 0$ it follows that

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 9.352439 \dots, \quad x \in [0, 1].$$

(iii) Putting $x = 0$ in (44) we get

$$F(0, 0, u) = 0, \quad u \in [0, 1],$$

(iv) Since (45) is independent on u we obtain

$$F(x, 1, 1) = F(x, 1, 0) \leq 12, \quad x \in [0, 1],$$

while for $x = 0$ in (45) we get

$$F(0, 1, u) = 12.$$

(v) The relation (43) is independent with respect to the variables y and u , thus

$$F(1, y, 0) = F(1, y, 1) = F(1, 0, u) = F(1, 1, u) = 1, \quad y, u \in [0, 1].$$

(vi) Finally, since (42) is independent on the variable u , consequently

$$F(0, y, 0) = F(0, y, 1) \leq 1, \quad z \in [0, 1].$$

The above computations lead to

$$\max\{F(x, y, u) : (x, y, u) \in \Omega\} = F(0, 1, u) = 12,$$

and from (40) we conclude that $|\gamma_4| \leq \frac{1}{16}$.

For proving the sharpness of the above inequality, we consider the function f_2 given by (19). In this case $a_1 = 1, a_2 = a_3 = a_4 = 0$ and $a_5 = -\frac{1}{8}$, and from (37) we get $|\gamma_4| = \frac{1}{16}$, that completes our proof. $\square$

Remark 4. In [20] it is proved that $|\gamma_3| \leq \frac{1}{6}$ and $|\gamma_4| \leq \frac{85}{384}$, while our above results are sharp and give the best upper bounds for these coefficients.

Theorem 5. Let $f \in \mathcal{S}_{\cos}^*$ be given by (1). Then, for the function F_f given by (34) the next inequality holds

$$\left| H_{2,2}(F_f/2) \right| \leq -\frac{10577}{1028376} + \frac{1549}{4113504} \sqrt{1549} = 0.015678 \dots,$$

and this result is sharp.

Proof. Replacing the values of (38) and (39) in the relation 2 we obtain

$$H_{2,2}(F_f/2) = \gamma_2 \gamma_4 - \gamma_3^2 = -\frac{23}{294912} c_1^6 + \frac{5}{18432} c_1^4 c_2 + \frac{1}{1024} c_1^3 c_3 - \frac{23}{18432} c_1^2 c_2^2.$$

Using the Lemma 2 we obtain

$$H_{2,2}(F_f/2) = \frac{1}{4608} \left[-3t_1^6 - 72t_1^2 |t_1|^2 t_2^2 (1 - |t_1|^2) - 92t_1^2 t_2^2 (1 - |t_1|^2)^2 + 72t_1^3 (1 - |t_1|^2) (1 - |t_2|^2) t_3 \right],$$

and denoting $x := |t_1| \in [0, 1], y := |t_2| \in [0, 1], u := |t_3| \in [0, 1]$, using the triangle's inequality the above relation yields

$$\left| H_{2,2}(F_f/2) \right| \leq \frac{1}{4608} \left[3x^6 + 72x^4 (1 - x^2) y^2 + 92x^2 (1 - x^2)^2 y^2 + 72x^3 (1 - x^2) (1 - y^2) u \right] = \frac{1}{4608} F(x, y, u), \quad x, y, u \in [0, 1], \quad (47)$$

where

$$F(x, y, u) := 3x^6 + 72x^4 (1 - x^2) y^2 + 92x^2 (1 - x^2)^2 y^2 + 72x^3 (1 - x^2) (1 - y^2) u. \quad (48)$$

With the notations used in the proofs of the previous three theorems we will determine the maximum value of F in Ω .

I. For all the interior points $(x, y, u) \in \text{int } \Omega$, differentiating (48) with respect to u we obtain

$$\frac{\partial F(x, y, u)}{\partial u} = 72x^3 (1 - x^2) (1 - y^2) \neq 0, \quad (x, y, u) \in \text{int } \Omega,$$

hence the function doesn't get its maximum in $\text{int } \Omega$.

II. Next we will study if it's possible to obtain the maximum value of F in the interior of six faces of Ω .

(i) On the face $x = 0$ we have

$$F(0, y, u) = 0, \quad y, u \in (0, 1). \quad (49)$$

(ii) On the face $x = 1$ the function F takes the form

$$F(1, y, u) = 3, \quad y, u \in (0, 1). \quad (50)$$

(iii) On $y = 0$ the function F can be written as

$$\kappa_1(x, u) := F(x, 0, u) = 3x^6 + 72x^3(1 - x^2)u, \quad x, u \in (0, 1). \quad (51)$$

Since

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 72x^3(1 - x^2) \neq 0, \quad x \in (0, 1),$$

it follows that the function κ_1 has no maximum in $(0, 1) \times (0, 1)$.

(iv) On $y = 1$, the function reduces to

$$s_1(x) := F(x, 1, u) = 3x^6 + 72x^4(1 - x^2) + 92x^2(1 - x^2)^2, \quad x \in (0, 1). \quad (52)$$

Since $s_1'(x) = 138x^5 - 448x^3 + 184x = 0$ has on $(0, 1)$ the root $x_0 = \frac{\sqrt{7728 - 138\sqrt{1549}}}{69} = 0.694547 \dots \in (0, 1)$ and $s_1''(x_0) < 0$, we deduce that

$$F(x, 1, u) \leq F(x_0, 1, u) = -\frac{676928}{14283} + \frac{24784}{14283}\sqrt{1549} = 20.899268 \dots, \quad x \in (0, 1). \quad (53)$$

(v) On the face $u = 0$ the function F reduces to

$$\kappa_2(x, y) := F(x, y, 0) = 3x^6 + 72x^4(1 - x^2)y^2 + 92x^2(1 - x^2)^2y^2, \quad x, y \in (0, 1),$$

therefore

$$\frac{\partial \kappa_2(x, y)}{\partial x} = 2x(60x^4y^2 + 9x^4 - 224x^2y^2 + 92y^2), \quad \frac{\partial \kappa_2}{\partial y} = 8x^2(x^2 - 1)y(5x^2 - 23).$$

Thus, the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

(vi) On $u = 1$ the function F takes the form

$$\begin{aligned} \kappa_3(x, y) := F(x, y, 1) &= 3x^6 + 72x^4(1 - x^2)y^2 + 92x^2(1 - x^2)^2y^2 \\ &\quad + 72x^3(1 - x^2)(1 - y^2), \quad x, y \in (0, 1). \end{aligned}$$

We get

$$\frac{\partial \kappa_3(x, y)}{\partial x} = (120y^2 + 18)x^5 + (360y^2 - 360)x^4 - 448x^3y^2 + (-216y^2 + 216)x^2 + 184xy^2,$$

and

$$\frac{\partial \kappa_3}{\partial y} = 8yx^2(5x + 23)(x + 1)(x - 1)^2,$$

hence the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

III. Now we will investigate the maximum of F on the edges of Ω .

(i) From (51) for $u = 0$ we have

$$F(x, 0, 0) = 3x^6 \leq 3, \quad x \in [0, 1].$$

(ii) From (51) for $u = 1$, let $s_2(x) := F(x, 0, 1) = 3x^6 - 72x^5 + 72x^3$. The solutions in $[0, 1]$ of the equation $s_2'(x) = 18x^5 - 360x^4 + 216x^2 = 0$ are $x_0 = 0.790371 \dots \in [0, 1]$ and $x_1 = 0$. Since $s_2''(x_0) < 0$ and $s_2(0) = 0$ we deduce that

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 14.073287 \dots, \quad x \in [0, 1].$$

(iii) Putting $x = 0$ in (51) we have

$$F(0, 0, u) = 0, \quad u \in [0, 1].$$

(iv) Since the relation (52) is independent on u , from (53) we obtain

$$F(x, 1, 1) = F(x, 1, 0) \leq -\frac{676928}{14283} + \frac{24784}{14283}\sqrt{1549}, \quad x \in [0, 1].$$

(v) If we take $x = 0$ in (52), we obtain

$$F(0, 1, u) = 0, \quad u \in [0, 1]$$

(vi) The relation (50) is independent on the variables y and u , thus

$$F(1, y, 0) = F(1, y, 1) = F(1, 0, u) = F(1, 1, u) = 3, \quad y, u \in [0, 1].$$

Similarly, since the formula (49) is independent on the variable u we have

$$F(0, y, 0) = F(0, y, 1) = 0, \quad y \in [0, 1].$$

From the above reasons it follows

$$\max\{F(x, y, u) : (x, y, u) \in \Omega\} = F(x_0, 1, u) = -\frac{676928}{14283} + \frac{24784}{14283}\sqrt{1549},$$

and using (47) we conclude that

$$\left| H_{2,2}(F_f/2) \right| \leq -\frac{10577}{1028376} + \frac{1549}{4113504}\sqrt{1549} = 0.015678 \dots$$

To prove the sharpness of the above inequality, we denote $t_\Delta = \frac{2}{69}\sqrt{7728 - 138\sqrt{1549}}$ and $h_5(z) = \frac{1 + t_\Delta z + z^2}{1 - z^2}$ such that

$$w_5(z) = \frac{h_5(z) - 1}{h_5(z) + 1} = \frac{z(\sqrt{7728 - 138\sqrt{1549}} + 69z)}{69 + \sqrt{7728 - 138\sqrt{1549}}z}, \quad z \in \mathbb{D}.$$

It is easy to see that $w_7(0) = 0$ and to prove that $|w_7(z)| < 1$ in $\mathbb{D}$ we will write the function w_5 like $w_5(z) = z\rho(z)$, where

$$\rho(z) = \frac{\sqrt{7728 - 138\sqrt{1549}} + 69z}{69 + \sqrt{7728 - 138\sqrt{1549}}z}, \quad z \in \mathbb{D}.$$

According to the same reasons regarding the circular transforms like in the proofs of the sharpness of Theorem 1(ii) and Theorem 2, we will show that $|\rho(z)| < 1$, $z \in \mathbb{D}$, by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1 - \rho(z)}{1 + \rho(z)} = \frac{(\sqrt{7728 - 138\sqrt{1549}} - 69)(z - 1)}{(\sqrt{7728 - 138\sqrt{1549}} + 69)(z + 1)}, \quad z \in \mathbb{D},$$

is a circular transform. It's easy to compute

$$H(0) = \frac{\sqrt{7728 - 138\sqrt{1549}} - 69}{\sqrt{7728 - 138\sqrt{1549}} + 69} = 0.1802561912 \dots > 0, \quad H(1) = 0,$$

$$H(i) = \frac{(\sqrt{7728 - 138\sqrt{1549}} - 69)i}{\sqrt{7728 - 138\sqrt{1549}} + 69}, \quad H(-i) = -\frac{(\sqrt{7728 - 138\sqrt{1549}} - 69)i}{\sqrt{7728 - 138\sqrt{1549}} + 69},$$

and these values of the circular transform H implies that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, which leads us to $|\rho(z)| < 1$ in $\mathbb{D}$, hence

$$|w_5(z)| = |z| |\rho(z)| < 1, \quad z \in \mathbb{D}.$$

Consequently, the function

$$f_5(z) = z \exp\left(\int_0^z \frac{\cos w_5(t) - 1}{t} dt\right) = z + \left(-\frac{28}{69} + \frac{1}{138}\sqrt{1549}\right)z^3$$

$$- \frac{1}{14283}(-43 + 2\sqrt{1549})\sqrt{7728 - 138\sqrt{1549}}z^4$$

$$+ \left(-\frac{71467}{114264} + \frac{241}{14283}\sqrt{1549}\right)z^5 + \dots, \quad z \in \mathbb{D},$$

belongs to the class $\mathcal{S}_{\cos}^*$. The initial coefficients of f_5 are

$$a_3 = -\frac{28}{69} + \frac{1}{138}\sqrt{1549}, \quad a_4 = -\frac{1}{14283}(-43 + 2\sqrt{1549})\sqrt{7728 - 138\sqrt{1549}},$$

$$a_5 = -\frac{71467}{114264} + \frac{241}{14283}\sqrt{1549},$$

and from (36) and (37) we obtain

$$\gamma_2 = -\frac{14}{69} + \frac{1}{276}\sqrt{1549}, \quad \gamma_3 = -\frac{(-43 + 2\sqrt{1549})\sqrt{7728 - 138\sqrt{1549}}}{28566},$$

$$\gamma_4 = -\frac{42761}{114264} + \frac{283}{28566}\sqrt{1549}.$$

Hence,

$$\left|H_{2,2}(F_{f_5}/2)\right| = \left|\gamma_2\gamma_4 - \gamma_3^2\right| = -\frac{10577}{1028376} + \frac{1549}{4113504}\sqrt{1549},$$

that proves the sharpness of our estimation. $\square$

Remark 5. Like we already mentioned in the Remark 3, the same maximum values of the functions F defined by (41) and (48) could be also found by using the MAPLE™ computer software codes like in the Remark 1 2.

4. Conclusions

In this article we have obtained the sharp coefficient bounds for the starlike functions that are connected to the cosine function, and we have also obtained the sharp coefficient bounds for the logarithmic coefficients of such functions.

The main tools of our results was those of N.E. Cho et al. [6] that seems to be a very efficient for the estimation of the first coefficients of Carathéodory's function, and in addition we have determined the sharp bound for the fifth coefficient. The technique of this paper can be used to determine the sharp upper bound of the initial coefficients for a variety of classes of analytic functions.

Moreover, we have used these results to determine the sharp upper bounds for the Hankel determinants up to order three, and we emphasize that all the results we obtained are the best possible, so there cannot be improved.

Author Contributions: Conceptualization, R.A., M.R. and T.B.; methodology, R.A., M.R. and T.B.; software, R.A., M.R. and T.B.; validation, R.A., M.R. and T.B.; formal analysis, R.A., M.R. and T.B.; investigation, R.A., M.R. and T.B.; resources, R.A., M.R. and T.B.; data curation, R.A., M.R. and T.B.; writing—original draft preparation, M.R. and T.B.; writing—review and editing, M.R. and T.B.; visualization, R.A., M.R. and T.B.; supervision, M.R. and T.B.; project administration, R.A., M.R. and T.B.. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikeness associated with cosine hyperbolic function. *Mathematics* **2020**, *8*(7), 1118.
2. Arif, M.; Raza, M.; Tang, H.; Hussain, S.; Khan, H. Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Math.* **2019**, *17*(1), 1615–1630.
3. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. In *Inequality Theory and Applications*; Cho, Y.J. (ed.); Nova Science Publishers, New York, 2010, vol. 6, pp. 1–7.
4. Banga, S.; Kumar, S.S. The sharp bounds of the second and third Hankel determinants for the class $S\mathcal{L}^*$. *Math. Slovaca* **2020**, *70*(4), 849–862.
5. Bano, K.; Raza, M. Starlike functions associated with cosine functions. *Bull. Iranian Math. Soc.* **2021**, *47*(5), 1513–1532.
6. Cho, N.E.; Kowalczyk, B.; Lecko, A. Sharp bounds of some coefficient functionals over the class of functions convex in the direction of the imaginary axis. *Bull. Aust. Math. Soc.* **2019**, *100*(1), 86–96.
7. De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*(1-2), 137–152.
8. Carathéodory, C. Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. *Math. Ann.* **1907**, *64*(1), 95–115.
9. Cho, N.E.; Kowalczyk, B.; Lecko, A.; Śmiarowska, B. On the Fourth and Fifth coefficients in the Carathéodory class. *Filomat* **2020**, *34*(6), 2061–2072.
10. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iranian Math. Soc.* **2019**, *45*(1), 213–232.
11. Duren, P.L. *Univalent Functions*. Grundlehren der mathematischen Wissenschaften, vol. 259, Springer New York, NY, 2001.
12. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*(2), 159–177.
13. Janteng, A.; Halim, S.A.; Darus, M. Coefficient inequality for a function whose derivative has a positive real part. *J. Inequal. Pure Appl. Math.* **2006**, *7*(2), 50.
14. Janteng, A.; Halim, S.A.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math. Anal.* **2007**, *1*(13), 619–625.
15. Kowalczyk, B.; Lecko, A. The sharp bound of the third Hankel determinant for functions of bounded turning. *Bol. Soc. Mat. Mex.* **2021**, *27*(3), 69.
16. Kowalczyk, B.; Lecko, A.; Sim, Y.J. The sharp bound for the Hankel determinant of the third kind for convex functions. *Bull. Aust. Math. Soc.* **2018**, *97*(3), 435–445.
17. Kowalczyk, B.; Lecko, A.; Thomas, D.K. The sharp bound of the third Hankel determinant for starlike functions. *Forum Math.* **2022**, *34*(5), 1249–1254.
18. Lecko, A.; Sim, Y.J.; Śmiarowska, B. The sharp bound of the Hankel determinant of the third kind for starlike functions of order $1/2$. *Complex Anal. Oper. Theory* **2019**, *13*(5), 2231–2238.
19. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), pp. 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.
20. Marimuthu, K.; Uma, J.; Bulboacă, T. Coefficient estimates for starlike and convex functions associated with cosine function. *Hacet. J. Math. Stat.* **2023**, *52*(3), 596–618.
21. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*(1), 365–386.

22. Obradović, M. Tuneski, N. Hankel determinants of second and third order for the class $\mathcal{S}$ of univalent functions. *Math. Slovaca* **2021**, *71*(3), 649–654.
23. Obradović, M. Tuneski, N. Two types of the second Hankel determinant for the class $\mathcal{U}$ and the general class $\mathcal{S}$. *Acta Comment. Univ. Tartu. Math.* **2023**, *27*(1), 59–67.
24. Pommerenke, Ch. On the coefficients and Hankel determinants of univalent functions. *J. London Math. Soc.* **1966**, *41*(1), 111–122.
25. Pommerenke, Ch. *Univalent functions*. Vandenhoeck and Ruprecht, Göttingen, 1975.
26. Riaz, A.; Raza, M. The third Hankel determinant for starlike and convex functions associated with lune. *Bull. Sci. Math.* **2023**, *187*, 103289.
27. Riaz, A.; Raza, M.; Thomas, D.K. Hankel determinants for starlike and convex functions associated with sigmoid functions. *Forum Math.* **2022**, *34*(1), 137–156.
28. Riaz, A. Raza, M.; Binyamin, M.A.; Saliu, A. The second and third Hankel determinants for starlike and convex functions associated with Three-Leaf function. *Heliyon* **2023**, *9*(1), 12748.
29. Sharma, K., Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Mat.* **2016**, *27*(5-6), 923–939.
30. Sokół, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz., Mat. Fiz.* **1996**, *19*, 101–105.
31. Ullah, K.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Arjika, S. A study of sharp coefficient bounds for a new subfamily of starlike functions. *J. Inequal. Appl.* **2021**, *1*(2021), 194.
32. Zaprawa, P. On Hankel determinant $H_2(3)$ for univalent functions. *Results Math.* **2018**, *73*(3), 89.
33. Zaprawa, P. Third Hankel determinants for subclasses of univalent functions. *Mediterr. J. Math.* **2017**, *14*(1), 19.

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