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The Component Connectivity of Leaf-Sort Graphs

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Abstract: In large-scale parallel computing and communication systems, an interconnect network structure is usually modeled as a graph, in which vertices and edges correspond to processors and communication links, respectively. In a graph, the vertices and edges are likely to fail, so we must think about the fault tolerance of a graph. Connectivity is an important parameter in the study of faulty tolerance for a graph. In this paper, we study a special class of connectivity: m-component connectivity. Let F is a vertex set of G ($i.e, F \subseteq V(G)$), if the following conditions are satisfied, we say F is a m-component cut: (1) $G-F$ is disconnected; (2) the number of components in $G-F$ is greater than or equal to m. The m-component connectivity $c\kappa_m(G)$ is defined as $min\{|F| \mid F \subseteq V(G)$ and F is a m-component cut$\}$. Meanwhile, we can get the values: $c\kappa_3(CF_n) = 3n-6$ (n is odd) and $c\kappa_3(CF_n) = 3n-7$ (n is even) for $n \geq 3$; $c\kappa_4(CF_n) = \frac{9n-21}{2}$ (n is odd) and $c\kappa_4(CF_n) = \frac{9n-24}{2}$ (n is even) for $n \geq 4$; $c\kappa_5(CF_n) = 6n-16$ (n is odd) and $c\kappa_5(CF_n) = 6n-18$ (n is even) for $n \geq 5$. Leaf-sort graph is a special Cayley graph, it has many special properties. So we have to pay attention to some of it's special properties when we study it.

Keywords: component connectivity; cayley graphs; leaf-sort graph; fault-tolerance

MSC: 68R10; 05C40

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1. Introduction

With the rapid development of technologies, interconnection networks play an important role in large multiprocessor system. An interconnection network is usually modeled as an undirected graph $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. In this graph, the vertices and edges correspond to the process and communication links respectively. In the large-scale interconnected network, the failure of vertices or edges is inevitable. Therefore, in order to have an unimpeded interconnection network, we must think about the fault tolerance of a graph. Connectivity is an important parameter to measure the fault tolerance of an interconnection network, so the research of connectivity is very important. Connectivity can be divided into many kinds: maximal connectivity, local connectivity, maximal local connectivity, generalized connectivity and so on. In this paper, we study a special class of connectivity: m-component connectivity. Next, we firstly introduce the traditional connectivity of a graph G.

Let $G = (V(G), E(G))$ be a simple connected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Let $F \subseteq V(G)$, we use $G-F$ to denote the subgraph of G with vertex set $V(G)-F$ and edge set $E(G) - \{(u,v) \in E(G) | \{u,v\} \cap F \neq \emptyset\}$. Let x and y be any two distinct vertices, a path P between them is a sequence of adjacent vertices $< x, w_1, w_2, \ldots, w_k, y >$, where $w_1, w_2, \ldots, w_k$ are distinct ones. The vertices w_i, $i = 1, 2, \ldots, k$ are called internal vertices of the path P. For any two vertices $\{u,v\} \subseteq V(G)$, if there exists a uv-path, we say G is connected. Furthermore, if $G-F$ is disconnected or has only one vertex, we called F is a vertex cut. Meanwhile, we call the biggest component in $G-F$ a big component. It's well know that, when G is not a complete graph, the traditional connectivity $\kappa(G)$ is defined as $min\{|F| \mid F \subseteq V(G)$ and F is a vertex cut $\}$. Otherwise, we say $\kappa(G) = n-1$, where n is the number of vertices in G. A graph G is k-connected if $\kappa(G) \geq k$.

As an extension of traditional connectivity, let's look at the m-component connectivity of a graph G. By referring to the relevant literature, we can find that the notion concerning the number of components in $G-F$ was first introduced by Chartrand et al. [1] and Sampathkumar [2]. Furthermore, the definition of $c\kappa_m(G)$ was first proposed by Hsu et al. [3]. Let F is a vertex set of G $(i.e, F \subseteq V(G))$, if the following conditions are satisfied, we say F is a m-component cut: (1) $G-F$ is disconnected; (2) the number of components in $G-F$ is greater than or equal to m. The m-component connectivity $c\kappa_m(G)$ is defined as $min\{|F| \mid F \subseteq V(G)$ and F ia a m-component cut$\}$. By the above definition, we can easily get that $c\kappa_2(G) = \kappa(G)$ and $c\kappa_{m+1}(G) \geq c\kappa_m(G)$. By referring to the relevant literature, we can also get there exists a certain relationship between m-component connectivity $c\kappa_m(G)$ and m-extra connectivity. The m-extra connectivity, denoted by $\kappa_m(G)$, is defined as the minimum number of vertices whose removal from G results in every component in $G-F$ has at least $(m+1)$ vertices [4]. Li et al. [5], Hao et al. [6] and Guo et al. [7] have studied the relationship between extra (edge) connectivity and component (edge) connectivity in networks. So far, the m-component (edge) connectivity of many graphs has been studied [8–17]. However, these results most are about small m. If we want to get a result about a bigger m, we must expend greater effort. Next, we give some definitions that will be used in the following sections.

For a vertex $v \in V(G)$, $N_G(v) = \{u|(u,v) \in E(G)\}$ is the set of neighbors of v. We let $deg_G(v) = |N_G(v)|$ be the degree of v and $\delta(G) = \min\{deg_G(v)|v \in V(G)\}$ be the minimum degree of G. If $deg_G(v) = k$ for every $v \in V(G)$, then G is k-regular. A singleton of G is a vertex v with $deg_G(v) = 0$. For a vertex set X, $N_G(X) = \{\cup_{x\in X} N_G(x)\} - X$ is the neighbor of X. The distance between any two vertices u and v, denoted by $d_G(u,v)$, is the length of the shortest path from u to v. G is bipartite if there exist two vertex subsets V_1, V_2 with $V_1 \cap V_2 = \emptyset$ such that $V(G) = V_1 \cup V_2$ and for each edge $(u,v) \in E(G)$, $|\{u,v\} \cap V_1| = |\{u,v\} \cap V_2| = 1$. It is well known that bipartite graphs contain no odd cycles. We use Bondy and Murty [18] for terminology and notation not defined here.

In this paper, we study the m-component connectivity of CF_n, prove that $c\kappa_3(CF_n) = 3n-6$ (n is odd) and $c\kappa_3(CF_n) = 3n-7$ (n is even) for $n \geq 3$; $c\kappa_4(CF_n) = \frac{9n-21}{2}$ (n is odd) and $c\kappa_4(CF_n) = \frac{9n-24}{2}$ (n is even) for $n \geq 4$; $c\kappa_5(CF_n) = 6n-16$ (n is odd) and $c\kappa_5(CF_n) = 6n-18$ (n is even) for $n \geq 5$.

The detailed arrangement of the paper ia as follows: Section 2 introduces the definition of CF_n and gives some properties of CF_n. In Section 3, we discuss $c\kappa_3(CF_n)$. In Section 4, we prove the value of $c\kappa_4(CF_n)$. In Section 5, we prove some useful lemmas firstly, and then calculate the value of $c\kappa_5(CF_n)$. Section 6 concludes this paper. Next, let's first introduce the Leaf-sort graph.

2. Preliminaries

Let l_1, l_2 be two integers with $1 \leq l_1 \leq l_2$. Set $[l_1, l_2] = \{l | l$ is an integer with $l_1 \leq l \leq l_2\}$. In the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$, $i \to p_i$. For the convenience, we denote the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ by $p_1p_2\cdots p_n$. In [19], every permutation can be denoted by a product of disjoint cycles. For example, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$. Specially, $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = (1)$. The product $\sigma\tau$ of two permutations is the composition function τ followed by σ, e.g., $(12)(13) = (132)$. For terminology and notation not defined here, we follow [19]. Now we give the definition of n-dimensional leaf-sort graphs CF_n.

Let $[1,n] = \{1,2,\ldots,n\}$, and let S_n be the symmetric group on $[1,n]$ containing all permutations $p = p_1p_2\cdots p_n$ of $[1,n]$. It is well known that $\{(1i) : i = 2,3,\ldots,n\}$ is a generating set for S_n. So $\{(1,i) : i = 2,3,\ldots,n\} \cup \{(j,j+1) : j = 2,4,\ldots,n-1\}$ (n is odd.) is also a generating set for S_n and $\{(1,i) : i = 2,3,\ldots,n\} \cup \{(j,j+1) : j = 2,4,\ldots,n-2\}$ (n is even.) is also a generating set for S_n. The n-dimensional leaf-sort graph CF_n is the graph with vertex set $V(CF_n) = S_n$ in which two vertices u, v are adjacent if and only if $u = v(1,i), 2 \leq i \leq n$, or $u = v(j,j+1), 2 \leq j \leq n-1$ when j is even and n is odd, or $u = v(j,j+1), 2 \leq j \leq n-2$ when j and n are even. By the definition, we can get CF_n is

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a $\frac{3n-3}{2}$-regular graph on $n!$ vertices for odd n, and $\frac{3n-4}{2}$-regular graph on $n!$ vertices for even n. The graphs CF_2, CF_3 and CF_4 are depicted in Figure 1.

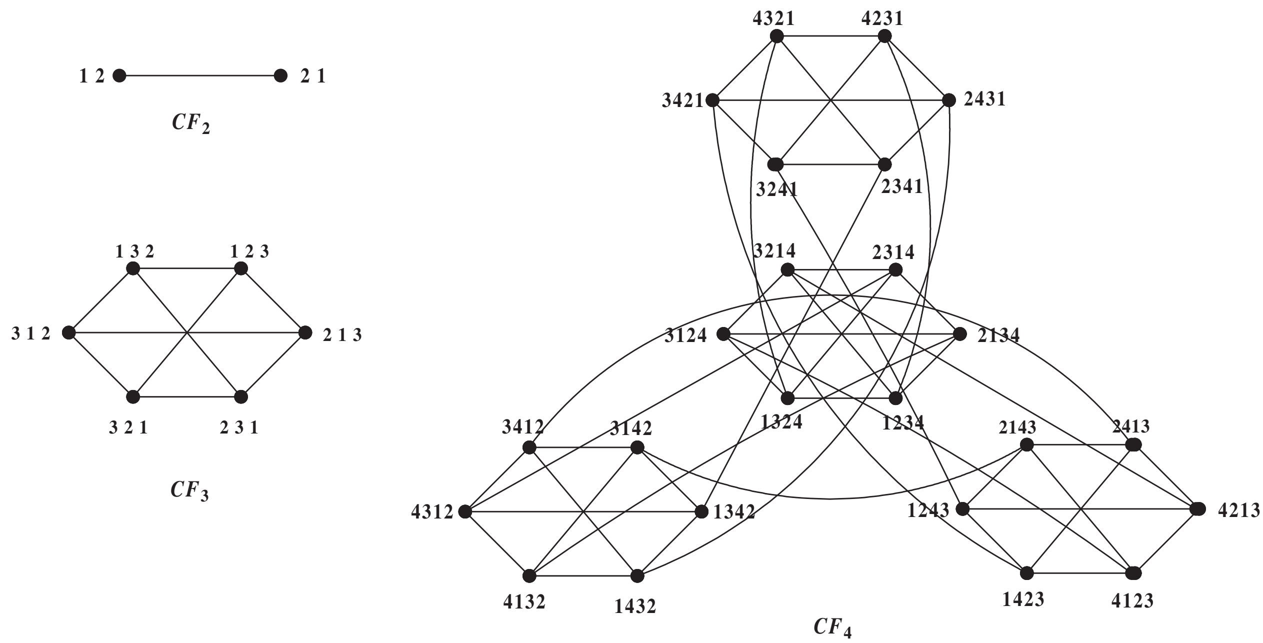

Figure 1. The leaf-sort graphs CF_2, CF_3 and CF_4.

We can partition CF_n into n subgraphs $CF_n^1, CF_n^2, \ldots, CF_n^n$ where every vertex $u = x_1x_2\cdots x_n \in CF_n^i$ has a fixed integer i in the last position x_n for $i \in [1,n]$. It is obvious that CF_n^i is isomorphic to CF_{n-1} for $i \in [1,n]$ [20]. The edges whose end vertices in different CF_n^i are called cross-edges. Two edges are said to be independent if they are not adjacent. For any $u \in CF_n^i$, we denote $u^+ = u(1,n)$, $u^- = u(n-1,n)$, and $N_u^+ = \{u^+, u^-\}$, which are called outgoing neighbors of u. Denote $E_{i,j}(CF_n) = E_{CF_n}(V(CF_n^i), V(CF_n^j))$ for $i,j \in [1,n]$.

Proposition 2.1. ([20]) Let CF_n^i $(1 \le i \le n)$ be defined as above. Then there are $2(n-2)!$ independent cross-edges between two different CF_n^i when n is odd; there are $(n-2)!$ independent cross-edges between two different CF_n^i when n is even.

Proposition 2.2. ([20]) Let $v \in V(CF_n^i)$ $(1 \le i \le n)$ which be defined as above. Then $v(1,n)$ and $v(n-1,n)$ belong to two different CF_n^j's $(j \ne i)$ when n is odd; $v(1,n)$ belong to CF_n^j $(j \ne i)$ when n is even.

Proposition 2.3. For any $u, v \in V(CF_n^i)$, $N_u^+ \cap N_v^+ = \emptyset$ when n is odd; $u^+ \ne v^+$ when n is even.
Proof. Let $u = u_1u_2\cdots u_{n-1}i$ and $v = v_1v_2\cdots v_{n-1}i$, where $u_j \ne v_j$ for some $j \in [1, n-1]$. Then $u^+ = iu_2\cdots u_{n-1}u_1 \ne iv_2\cdots v_{n-1}v_1 = v^+$, $u^- = u_1u_2\cdots iu_{n-1} \ne v_1v_2\cdots iv_{n-1} = v^-$. Moreover, $u^+ \ne v^-$ and $v^+ \ne u^-$. Hence when n is odd, $N_u^+ \cap N_v^+ = \emptyset$; when n is even, $u^+ \ne v^+$.

Proposition 2.4. ([20]) For any integer $n \ge 2$, CF_n is bipartite.

Proposition 2.5. For any two vertices $x, y \in V(CF_n)$ $(n \ge 3)$, $|N_{CF_n}(x) \cap N_{CF_n}(y)| \le 3$.
Proof. If $d_{CF_n}(x,y) = 1$ or $d_{CF_n}(x,y) \ge 3$, then $|N_{CF_n}(x) \cap N_{CF_n}(y)| = 0$; Otherwise (*i.e.*, $|N_{CF_n}(x) \cap N_{CF_n}(y)| \ge 1$), there will be a 3-circle or $d_{CF_n}(x,y) = 2$, a contradiction. So we consider $d_{CF_n}(x,y) = 2$. Next, we proof this result by induction on n.

For $n = 3$, $|N_{CF_3}(x) \cap N_{CF_3}(y)| = 3$ as $CF_3 \cong K_{3,3}$ and $d_{CF_3}(x,y) = 2$.

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For $n = 4$, if $x, y \in V(CF_4^i)$, $|N_{CF_4}(x) \cap N_{CF_4}(y)| = 3$ as $CF_4^i \cong CF_3$ and $x^+ \neq y^+$. If $x \in V(CF_4^i)$, $y \in V(CF_4^j)$ $(i \neq j)$, let $x = x_1x_2x_3i$ and $y = y_1y_2y_3j$, we know $x^+ \neq y^+$. If $y^+ \in CF_4^i$ or $x^+ \in CF_4^j$, then $N_{CF_4}(x) \cap N_{CF_4}(y) \subseteq \{x^+, y^+\}$. Thus $|N_{CF_4}(x) \cap N_{CF_4}(y)| \leq 2$.

Now we assume $n \geq 5$ and the result holds for CF_{n-1}. If $x, y \in V(CF_n^i)$ for $i \in [1, n]$, then by Proposition 2.3 and the inductive hypothesis, the result holds. So we let $x \in V(CF_n^i)$, $y \in V(CF_n^j)$ $(i \neq j)$, $x = x_1x_2 \cdots x_{n-1}i$, $y = y_1y_2 \cdots y_{n-1}j$. Then $x^+ = ix_2 \cdots x_{n-1}x_1$, $x^- = x_1x_2 \cdots ix_{n-1}$, $y^+ = jy_2 \cdots y_{n-1}y_1$, $y^- = y_1y_2 \cdots jy_{n-1}$. We know $x^+ \neq x^-$ and $y^+ \neq y^-$. Since $i \neq j$, $x^+ \neq y^+$ and $x^- \neq y^-$. As $d_{CF_n}(x, y) = 2$ and $N_{CF_n}(x) \cap N_{CF_n}(y) \subseteq \{x^+, x^-, y^+, y^-\}$, we can assume $y^+ \in N_{CF_n}(x) \cap N_{CF_n}(y)$.

When n is odd. If $y^+ = x^-$, then $jy_2y_3 \cdots y_{n-1}y_1 = x_1x_2 \cdots ix_{n-1}$, $x_1 = j, y_2 = x_2, y_3 = x_3, \cdots, y_{n-2} = x_{n-2}, y_{n-1} = i, y_1 = x_{n-1}$. Thus $x^+ = y_{n-1}y_2y_3 \cdots y_{n-2}y_1j \in CF_n^j$ and $y^- = x_{n-1}x_2x_3 \cdots x_{n-2}x_1i \in CF_n^i$, $yx^+ \in E(CF_n^j)$, $xy^- \in E(CF_n^i)$. So $|N_{CF_n}(x) \cap N_{CF_n}(y)| = 3$. If $y^+ \in CF_n^i$ and adjacent to x, then $y^+ = jy_2 \cdots y_{n-1}i$, $N_{CF_n}(x) \cap N_{CF_n}(y) \subseteq \{x^+, y^+, x^-\}$. Thus $|N_{CF_n}(x) \cap N_{CF_n}(y)| \leq 3$. Furthermore, if and only if $y^- = x^+$, there will be $|N_{CF_n}(x) \cap N_{CF_n}(y)| = 3$, this is similar to the situation $y^+ = x^-$. When $y^- \neq x^+$, $|N_{CF_n}(x) \cap N_{CF_n}(y)| \leq 2$. So when n is odd, this result holds.

When n is even. Note that $x^+ \neq y^+$. If $y^+ \in CF_n^i$ or $x^+ \in CF_n^j$, then $N_{CF_n}(x) \cap N_{CF_n}(y) \subseteq \{x^+, y^+\}$. Thus $|N_{CF_n}(x) \cap N_{CF_n}(y)| \leq 2$.

In summary, this proposition is proven.

Corollary 2.6. When $n \geq 4$ is even, if x and y belong to two different subgraphs in CF_n, then $|N_{CF_n}(x) \cap N_{CF_n}(y)| \leq 2$.

Corollary 2.7. When $n \geq 3$ is odd, for any two vertices $x, y \in V(CF_n)$, where $x \in V(CF_n^i)$, $y \in V(CF_n^j)$ $(i \neq j)$. Then $|N_{CF_n}(x) \cap N_{CF_n}(y)| = 3$ if and only if $y^+ = x^-$ or $y^- = x^+$.

Proposition 2.8. ([20]) Let CF_n be the leaf-sort graph. Then the connectivity $\kappa(CF_n) = \frac{3n-3}{2}$ when n is odd and $\kappa(CF_n) = \frac{3n-4}{2}$ when n is even.

Lemma 1. ([21]) Let $F \subseteq V(CF_n)$ with $|F| \leq 3n - 6$ when n is odd $(n \geq 5)$ and $|F| \leq 3n - 7$ when n is even $(n \geq 4)$. If $CF_n - F$ is disconnected, then $CF_n - F$ satisfies one of the following conditions:

(1) $CF_n - F$ has two components, one of which is a singleton;

(2) $CF_n - F$ has three components, two of which are singletons.

The conclusion of Lemma 1 is closely linked to the proof of m-component connectivity of CF_n, that is why we listed it first. Next, we will discuss the component connectivity of CF_n.

3. The 3-component connectivity of CF_n

In this section, we will discuss the 3-component connectivity of CF_n, and will prove that: when n is odd, $c\kappa_3(CF_n) = 3n - 6$ for $n \geq 3$; When n is even, $c\kappa_3(CF_n) = 3n - 7$ for $n \geq 3$. Before proving our main results, we prove some useful lemmas firstly. Let S is a subset of $V(G)$ (*i.e.*, $S \subseteq V(G)$), if any two vertices x_1 and x_2 in S are nonadjacent, we call S an independent set. For convenience, we can simply write the independent set as *Ind*-set.

Lemma 3.1. When n is odd, if $x_1 \in V(CF_n^i)$, then there exists only $(n-3)$ vertices in CF_n^i, which can satisfy that $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$; When n is even, if $x_1 \in V(CF_n^i)$, then there exists only $(n-2)$ vertices in CF_n^i such that $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$. In addition, these vertices are all regular.

Proof. Note that $CF_n^i \cong CF_{n-1}$. When n is odd, $n-1$ is even, $\{x_1, x_2\} \subseteq V(CF_n^i)$. Since $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$, by Corollary 2.6, we know that x_1, x_2 must belong to a common subgraph in CF_n^i; Otherwise, if x_1, x_2 are in different subgraphs in CF_n^i, then $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| \leq 2$. So we can assume $x_1 = i_1i_2i_3 \cdots i_{n-2}ji$, $x_2 = j_1j_2j_3 \cdots j_{n-2}ji$. Now we let $x_1' = i_1i_2i_3 \cdots i_{n-2}$, $x_2' = j_1j_2j_3 \cdots j_{n-2}$,

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then $\{x_1', x_2'\} \subseteq V(G_1)$, $G_1 \cong CF_{n-2}$, $n-2$ is odd. Since $|N_{G_1}(x_1') \cap N_{G_1}(x_2')| = 3$, by the proof process of Proposition 2.5, we know x_1', x_2' have two different situations:

Case 1. x_1', x_2' belong to two different subgraphs in G_1, and $(x_1')^+ = (x_2')^-$ or $(x_1')^- = (x_2')^+$.

In this case, we have $i_{n-2} \neq j_{n-2}$. If $(x_1')^+ = i_{n-2}i_2i_3\cdots i_{n-3}i_1 = (x_2')^- = j_1j_2j_3\cdots j_{n-2}j_{n-3}$, then $j_1 = i_{n-2}, j_2 = i_2, \ldots, j_{n-4} = i_{n-4}, j_{n-3} = i_1, j_{n-2} = i_{n-3}$. Thus $x_2 = i_{n-2}i_2i_3\cdots i_{n-4}i_1i_{n-3}\,ji$. If $(x_1')^- = i_1i_2i_3\cdots i_{n-2}i_{n-3} = (x_2')^+ = j_{n-2}j_2j_3\cdots j_{n-3}j_1$, then $j_{n-2} = i_1, j_2 = i_2, \ldots, j_{n-4} = i_{n-4}, j_{n-3} = i_{n-2}, j_1 = i_{n-3}$. Thus $x_2 = i_{n-3}i_2i_3\cdots i_{n-4}i_{n-2}i_1ji$.

Case 2. x_1', x_2' belong to the same subgraph in G_1.

In this case, we have $i_{n-2} = j_{n-2}$, $x_1' = i_1i_2i_3\cdots i_{n-2}$, $x_2' = j_1j_2j_3\cdots i_{n-2}$. As $G_1^{i_{n-2}} \cong CF_{n-3}$, $n-3$ is even, $|N_{G_1^{i_{n-2}}}(x_1') \cap N_{G_1^{i_{n-2}}}(x_2')| = 3$, we can get x_1', x_2' belong to a common subgraph in $G_1^{i_{n-2}}$. Then $i_{n-3} = j_{n-3}$. Let $x_1'' = i_1i_2i_3\cdots i_{n-4}$, $x_2'' = j_1j_2j_3\cdots j_{n-4}$, then $\{x_1'', x_2''\} \subseteq V(G_2)$, $G_2 \cong CF_{n-4}$ and $|N_{G_2}(x_1'') \cap N_{G_2}(x_2'')| = 3$. As $n-4$ is odd, there are two different situations:

Subcase 2.1. x_1'', x_2'' belong to two different subgraphs in G_2, and $(x_1'')^+ = (x_2'')^-$ or $(x_1'')^- = (x_2'')^+$.

If $(x_1'')^+ = (x_2'')^-$, we can get $j_1 = i_{n-4}, j_2 = i_2, \ldots, j_{n-6} = i_{n-6}, j_{n-4} = i_{n-5}, j_{n-5} = i_1$. Thus $x_2 = i_{n-4}i_2i_3\cdots i_{n-6}i_1i_{n-5}i_{n-3}i_{n-2}ji$. If $(x_1'')^- = (x_2'')^+$, then $j_1 = i_{n-5}, j_2 = i_2, \ldots, j_{n-4} = i_1, j_{n-5} = i_{n-4}$. Thus $x_2 = i_{n-5}i_2i_3\cdots i_{n-6}i_{n-4}i_1i_{n-3}i_{n-2}ji$.

Subcase 2.2. x_1'', x_2'' belong to a same subgraph in G_2.

This case is similar to case 2, this is a finite cycle process.

Finally, when n is odd, we can get $(n-3)$ vertices such that $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$, they are $i_{n-2}i_2i_3\cdots i_{n-4}i_1i_{n-3}ji$, $i_{n-3}i_2i_3\cdots i_{n-4}i_{n-2}i_1ji$, $i_{n-4}i_2i_3\cdots i_{n-6}i_1i_{n-5}i_{n-3}i_{n-2}ji$, $i_{n-5}i_2i_3\cdots i_{n-4}i_1i_{n-3}i_{n-2}ji$,…,$i_3i_1i_2i_4\cdots i_{n-2}ji$, $i_2i_3i_1i_4\cdots i_{n-2}ji$.

When n is even, $n-1$ is odd, $CF_n^i \cong CF_{n-1}$. Let $x_1 = i_1i_2i_3\cdots i_{n-1}i$, $x_2 = j_1j_2j_3\cdots j_{n-1}i$, $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$. Assume $x_1' = i_1i_2i_3\cdots i_{n-1}$, $x_2' = j_1j_2j_3\cdots j_{n-1}$, then $\{x_1', x_2'\} \subseteq V(G_1)$, $G_1 \cong CF_{n-1}$, similarly, we can also divide it into two different situations:

Case 1. x_1', x_2' belong to two different subgraphs in G_1, and $(x_1')^+ = (x_2')^-$ or $(x_1')^- = (x_2')^+$.

In this case, we have $i_{n-1} \neq j_{n-1}$. If $(x_1')^+ = (x_2')^-$, then $j_1 = i_{n-1}, j_2 = i_2, \ldots, j_{n-1} = i_{n-2}, j_{n-2} = i_1$. Thus $x_2 = i_{n-1}i_2i_3\cdots i_{n-3}i_1i_{n-2}i$. If $(x_1')^- = (x_2')^+$, then $j_1 = i_{n-2}, j_2 = i_2, \ldots, j_{n-2} = i_{n-1}, j_{n-1} = i_1$. Thus $x_2 = i_{n-2}i_2i_3\cdots i_{n-3}i_{n-1}i_1i$.

Case 2. x_1', x_2' belong to a same subgraph in G_1.

In this case, $i_{n-1} = j_{n-1}$. Since $G_1^{i_{n-1}} \cong CF_{n-2}$ and $|N_{G_1^{i_{n-1}}}(x_1') \cap N_{G_1^{i_{n-1}}}(x_2')| = 3$, x_1' and x_2' are in a same subgraph in $G_1^{i_{n-1}}$. So $i_{n-2} = j_{n-2}$. Let $x_1'' = i_1i_2i_3\cdots i_{n-3}$, $x_2'' = j_1j_2j_3\cdots j_{n-3}$, then $\{x_1'', x_2''\} \subseteq V(G_2)$, $G_2 \cong CF_{n-3}$, $n-3$ is odd. As $|N_{G_2}(x_1'') \cap N_{G_2}(x_2'')| = 3$, we can also consider the following two situations:

Subcase 2.1. x_1'' and x_2'' belong to two different subgraphs in G_2, and $(x_1'')^+ = (x_2'')^-$ or $(x_1'')^- = (x_2'')^+$.

If $(x_1'')^+ = (x_2'')^-$, then $j_1 = i_{n-3}, j_2 = i_2, \ldots, j_{n-3} = i_{n-4}, j_{n-4} = i_1$. Thus $x_2 = i_{n-3}i_2i_3\cdots i_{n-5}$ $i_1i_{n-4}i_{n-2}i_{n-1}i$. If $(x_1'')^- = (x_2'')^+$, then $j_1 = i_{n-4}, j_2 = i_2, \ldots, j_{n-3} = i_1, j_{n-4} = i_{n-3}$. Thus $x_2 = i_{n-4}i_2i_3\cdots i_{n-5}i_{n-3}i_1i_{n-2}i_{n-1}i$.

Subcase 2.2. x_1'' and x_2'' belong to a same subgraph in G_2.

This case is similar to case 2, this is also a finite cycle process.

Finally, when n is even, we can get $(n-2)$ vertices such that $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$, they are $i_{n-1}i_2i_3\cdots i_{n-3}i_1i_{n-2}i$, $i_{n-2}i_2i_3\cdots i_{n-3}i_{n-1}i_1i$, $i_{n-3}i_2i_3\cdots i_{n-5}i_1i_{n-4}i_{n-2}i_{n-1}i$, $i_{n-4}i_2i_3\cdots$ $i_{n-5}i_{n-3}i_1i_{n-2}i_{n-1}i$,…,$i_3i_1i_2i_4\cdots i_{n-2}i_{n-1}i$, $i_2i_3i_1i_4\cdots i_{n-2}i_{n-1}i$.

Corollary 3.2. Let CF_n is an n-dimension leaf-sort graph, $\{x_1, x_2\} \subseteq V(CF_n^i)$. If $x_1 = i_1i_2i_3\cdots i_{n-1}i$, $x_2 = j_1j_2j_3\cdots j_{n-1}i$ and $|N_{CF_n^i}(x_1) \cap N_{CF_n^i}(x_2)| = 3$, then $j_1 \neq i_1$.

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Lemma 3.3. For $n \geq 3$, let S is an Ind-set and $|S| = 2$, then when n is odd, $|N_{CF_n}(S)| \geq 3n-6$; when n is even, $|N_{CF_n}(S)| \geq 3n-7$.

Proof. Let $S = \{x_1, x_2\}$, as S is an Ind-set, so x_1 and x_2 are nonadjacent. By Proposition 2.5 and the definition of CF_n, we know that $|N_{CF_n}(x_1) \cap N_{CF_n}(x_2)| \leq 3$ and CF_n is a $\frac{3n-3}{2}$-regular graph (n is odd) or $\frac{3n-4}{2}$-regular graph (n is even). So when n is odd, $|N_{CF_n}(S)| = |N_{CF_n}(x_1)| + |N_{CF_n}(x_2)| - |N_{CF_n}(x_1) \cap N_{CF_n}(x_2)| \geq 2 \times \frac{3n-3}{2} - 3 = 3n-6$. When n is even, $|N_{CF_n}(S)| = |N_{CF_n}(x_1)| + |N_{CF_n}(x_2)| - |N_{CF_n}(x_1) \cap N_{CF_n}(x_2)| \geq 2 \times \frac{3n-4}{2} - 3 = 3n-7$.

Corollary 3.4. For $n \geq 4$, let F is a subset of $V(CF_n)$ (*i.e.*, $F \subseteq V(CF_n)$) and when n is odd, $|F| \leq 3n-7$; when n is even, $|F| \leq 3n-8$, then $CF_n - F$ contains a big component C, which satisfies $|V(C)| \geq n! - |F| - 1$.

Theorem 1. For $n \geq 3$, when n is odd, $c\kappa_3(CF_n) = 3n-6$; when n is even, $c\kappa_3(CF_n) = 3n-7$.

Proof. For $n = 3$, since $c\kappa_{l+1}(CF_n) \geq c\kappa_l(CF_n)$, we can get $c\kappa_3(CF_3) \geq \frac{3n-3}{2} = 3 = 3n-6$ by Proposition 2.8. For $n \geq 4$, by Corollary 3.4, we can also get $c\kappa_3(CF_n) \geq 3n-6$ when n is odd and $c\kappa_3(CF_n) \geq 3n-7$ when n is even. Next, we will prove that: when n is odd, $c\kappa_3(CF_n) \leq 3n-6$ and when n is even, $c\kappa_3(CF_n) \leq 3n-7$. Since $|N_{CF_n}(x_1) \cap N_{CF_n}(x_2)| \leq 3$, we can choose two different vertices $x_1, x_2 \in V(CF_n)$, which can satisfy the condition $|N_{CF_n}(x_1) \cap N_{CF_n}(x_2)| = 3$. From the definition of CF_n, we know that CF_n is a $\frac{3n-3}{2}$-regular (n is odd) and $\frac{3n-4}{2}$-regular graph (n is even), so when n is odd, $|N_{CF_n}(\{x_1, x_2\})| = 2 \times \frac{3n-3}{2} - 3 = 3n-6$; when n is even, $|N_{CF_n}(\{x_1, x_2\})| = 2 \times \frac{3n-4}{2} - 3 = 3n-7$. Thus let $F = N_{CF_n}(\{x_1, x_2\})$, we know $CF_n - F$ contains three components and two of them only has a singleton. So we can get $c\kappa_3(CF_n) \leq 3n-6$ (n is odd) and $c\kappa_3(CF_n) \leq 3n-7$ (n is even).

4. The 4-component connectivity of CF_n

Lemma 4.1. When $n = 4$, let S is an Ind-set and $|S| = 3$, $|N_{CF_4}(S)| \geq 6$.

Proof. Let $S = \{x_1, x_2, x_3\}$, since S is an Ind-set, x_1, x_2, x_3 are nonadjacent with each other. Note that $CF_4^i \cong CF_3$, $CF_3 \cong K_{3,3}$. Next, we will think about the following three cases.

Case 1. x_1, x_2, x_3 belong to the same subgraph CF_4^i.

In this case, since $CF_4^i \cong K_{3,3}$ and S is an Ind-set, $|N_{CF_4^i}(S)| = 3$. By Proposition 2.3, we know the outgoing neighbors of $\{x_1, x_2, x_3\}$ are different. Thus, $|N_{CF_4}(S)| = 3 + 3 = 6$.

Case 2. x_1, x_2, x_3 belong to two different subgraphs CF_4^i, CF_4^j $(i \neq j)$.

In this case, we can let $\{x_1, x_2\} \subseteq V(CF_4^1)$, $x_3 \in V(CF_4^2)$. Since x_1 and x_2 are nonadjacent, we can get $|N_{CF_4^1}(\{x_1, x_2\})| = 3$, $|N_{CF_4^2}(x_3)| = 3$. By the definition of CF_n, we know x_i $(i \in \{1, 2, 3\})$ only has one outgoing neighbor. If the structure shown in Figure 2 (a) exists, $|N_{CF_4}(S)| \geq |N_{CF_4^1}(\{x_1, x_2\})| + |N_{CF_4^2}(x_3)| = 6$. Now we can show that this structure does not exist. As $|N_{CF_4^1}(\{x_1, x_2\})| = 3$ and $\{x_1, x_2\} \subseteq V(CF_4^1)$, we assume $x_1 = 2341$, then $x_2 = 4231$ or $x_2 = 3421$. Thus $x_1^+ \in V(CF_4^2)$, $x_2^+ \notin V(CF_4^2)$, the structure shown in Figure 2 (a) does not exist. Thus $|N_{CF_4}(S)| \geq 7$.

Case 3. x_1, x_2, x_3 belong to three different subgraphs CF_4^i, CF_4^j, CF_4^k (i, j, k are different from each other).

In this case, we can let $x_i \in V(CF_4^i)$. By the definition of CF_n, we know $|N_{CF_4^i}(x_i)| = 3$, thus $|N_{CF_4}(S)| \geq 3 \times 3 = 9$.

Combining the above three situations, we can get $|N_{CF_4}(S)| \geq 6$.

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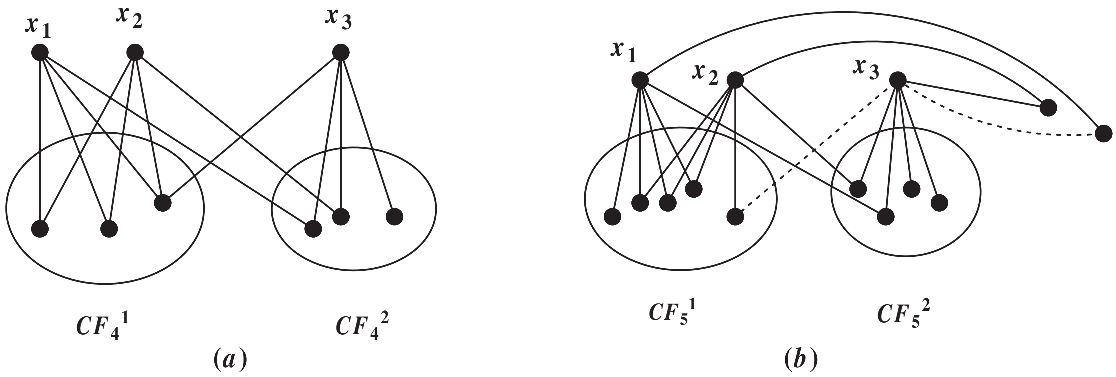

Figure 2. The illustration of Case 2.

Lemma 4.2. When $n = 5$, let S is an Ind-set and $|S| = 3$, $|N_{CF_5}(S)| \geq 12$.

Proof. Let $S = \{x_1, x_2, x_3\}$, since S is an Ind-set, x_1, x_2, x_3 are nonadjacent with each other. Note that $CF_5^i \cong CF_4$ $(1 \leq i \leq 5)$. We think about the following three cases.

Case 1. x_1, x_2, x_3 belong to the same subgraph CF_5^i.

Since $CF_5^i \cong CF_4$, by Lemma 4.1, we can get $|N_{CF_5^i}(S)| \geq 6$. By the definition of CF_n and Proposition 2.3, we know the outgoing neighbors of $\{x_1, x_2, x_3\}$ are different and every vertex has two different outgoing neighbors. Thus $|N_{CF_5}(S)| = |N_{CF_5^i}(S)| + 6 \geq 6 + 6 = 12$.

Case 2. x_1, x_2, x_3 belong to two different subgraphs CF_5^i, CF_5^j $(i \neq j)$.

In this case, we can let $\{x_1, x_2\} \subseteq V(CF_5^1)$, $x_3 \in V(CF_5^2)$. Since $CF_5^1 \cong CF_4$, by Lemma 3.3, we know that $|N_{CF_5^1}(\{x_1, x_2\})| \geq 3(n-1) - 7 = 3 \times 5 - 10 = 5$. By Proposition 2.8, we can get $|N_{CF_5^2}(x_3)| = \frac{3(n-1)-4}{2} = 4$. By Proposition 2.2, we can get x_1 and x_2 have at most two neighbors which belong to CF_5^2. In other words, there are at least two neighbors of x_1 and x_2 belong to $CF_5 - CF_5^1 - CF_5^2$. If the structure shown in Figure 2 (b) exists, then $|N_{CF_5}(S)| \geq |N_{CF_5^1}(\{x_1, x_2\})| + |N_{CF_5^2}(x_3)| + 2 = 5 + 4 + 2 = 11$. Furthermore, $|N_{CF_5}(S)| = 11$ if and only if this structure exists. Now, we can prove this structure does not exist.

Since $\{x_1, x_2\} \subseteq V(CF_5^1)$, $CF_5^1 \cong CF_4$ and $|N_{CF_5^1}(x_1) \cap N_{CF_5^1}(x_2)| = 3$, by Corollary 2.6, we can know that x_1, x_2 must belong to a common subgraph in CF_5^1. So we can assume $x_1 = i_1i_2i_3i_41$, $x_2 = j_1j_2j_3i_41$. As the subgraph of CF_5^1 is isomorphic to CF_3, thus $x_2 = i_3i_1i_2i_41$ or $x_2 = i_2i_3i_1i_41$ $(i_1, i_2, i_3, i_4$ are different from each other$)$.

If $x_2 = i_3i_1i_2i_41$, then $x_2^+ = 1i_1i_2i_4i_3$, $x_2^- = i_3i_1i_21i_4$. Since $x_1^+ = 1i_2i_3i_4i_1$, $x_1^- = i_1i_2i_31i_4$ and one of the two outgoing neighbors of x_1, x_2 belong to CF_5^2, we can get $i_4 = 2$. Thus $\{x_1^-, x_2^-\} \subseteq V(CF_5^2)$. As x_1^-, x_2^- are adjacent to x_3, so $x_3 = i_2i_1i_312$ or $x_3 = i_3i_2i_112$ or $x_3 = i_1i_3i_212$. When $x_3 = i_2i_1i_312$, $x_3^+ = 2i_1i_31i_2 \in V(CF_5^{i_2})$, $x_3^- = i_2i_1i_321 \in V(CF_5^1)$, and x_3^- is adjacent to x_2. Since $x_1^+ = 1i_2i_32i_1 \in V(CF_5^{i_1})$, $x_2^+ = 1i_1i_22i_3 \in V(CF_5^{i_3})$, $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$. When $x_3 = i_3i_2i_112$, $x_3^+ = 2i_2i_11i_3 \in V(CF_5^{i_3})$, $x_3^- = i_3i_2i_121 \in V(CF_5^1)$, and x_3^- is adjacent to x_1. Since $x_1^+ = 1i_2i_32i_1$, $x_2^+ = 1i_1i_22i_3$ and $1 \neq 2$, $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$. When $x_3 = i_1i_3i_212$, $x_3^+ = 2i_3i_21i_1 \in V(CF_5^{i_1})$, $x_3^- = i_1i_3i_221 \in V(CF_5^1)$, and x_3^- is adjacent to x_2. Since $x_1^+ = 1i_2i_32i_1$, $x_2^+ = 1i_1i_22i_3$ and $1 \neq 2$, $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$. So this structure does not exist.

If $x_2 = i_2i_3i_1i_41$, then $x_2^+ = 1i_3i_1i_4i_2$, $x_2^- = i_2i_3i_11i_4$. Since $x_1^+ = 1i_2i_3i_4i_1$, $x_1^- = i_1i_2i_31i_4$ and one of the two outgoing neighbors of x_1, x_2 belong to CF_5^2, we can get $i_4 = 2$. Thus $\{x_1^-, x_2^-\} \subseteq V(CF_5^2)$. As x_1^-, x_2^- are adjacent to x_3, so $x_3 = i_3i_2i_112$ or $x_3 = i_1i_3i_212$ or $x_3 = i_2i_1i_312$. When $x_3 = i_3i_2i_112$, $x_3^+ = 2i_2i_11i_3 \in V(CF_5^{i_3})$, $x_3^- = i_3i_2i_121 \in V(CF_5^1)$, and x_3^- is adjacent to x_2. Since $x_1^+ = 1i_2i_32i_1 \in V(CF_5^{i_1})$, $x_2^+ = 1i_3i_12i_2 \in V(CF_5^{i_2})$, $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$. When $x_3 = i_1i_3i_212$, $x_3^+ = 2i_3i_21i_1 \in V(CF_5^{i_1})$, $x_3^- = i_1i_3i_221 \in V(CF_5^1)$, and x_3^- is adjacent to x_1. Since $x_1^+ = 1i_2i_32i_1$, $x_2^+ = 1i_3i_12i_2$ and $1 \neq 2$,

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$x_3^+ \neq x_1^+, x_3^+ \neq x_2^+$. When $x_3 = i_2 i_1 i_3 12$, $x_3^+ = 2 i_1 i_3 1 i_2 \in V(CF_5^{i_2})$, $x_3^- = i_2 i_1 i_3 21 \in V(CF_5^1)$, and x_3^- is adjacent to x_2. Since $x_1^+ = 1 i_2 i_3 2 i_1$, $x_2^+ = 1 i_3 i_1 2 i_2$ and $1 \neq 2$, $x_3^+ \neq x_1^+, x_3^+ \neq x_2^+$.

Thus the structure shown in Figure 2 (b) does not exist, $|N_{CF_5}(S)| \geq 12$.

Case 3. x_1, x_2, x_3 belong to three different subgraphs CF_5^i, CF_5^j, CF_5^k (i, j, k are different from each other).

Without loss of generality, we can let $x_1 \in V(CF_5^1)$, $x_2 \in V(CF_5^2)$, $x_3 \in V(CF_5^3)$. By the definition of CF_n, we can get $|N_{CF_5^i}(x_i)| = \frac{3(n-1)-4}{2} = 4$ ($i \in \{1, 2, 3\}$), thus $|N_{CF_5}(S)| \geq |N_{CF_5^1}(x_1)| + |N_{CF_5^2}(x_2)| + |N_{CF_5^3}(x_3)| = 12$.

Combining the above three situations, we can get $|N_{CF_5}(S)| \geq 12$.

Lemma 4.3. When n is odd, let $S = \{x_1, x_2, x_3\}$ is an *Ind*-set, where $\{x_1, x_2\} \subseteq V(CF_n^i)$, $x_3 \in V(CF_n^j)$ $(i \neq j)$ and $|N_{CF_n^i}(\{x_1, x_2\})| = 3n - 10$. Then $|N_{CF_n}(S)| \neq \frac{9n-23}{2}$.

Proof. When n is odd, $|N_{CF_n}(S)| = \frac{9n-23}{2}$ occurs if and only if the structure in Figure 3 appears. Next, we will prove that these structures can not appear.

Firstly, we prove that the structure shown in Figure 3 (a) does not exist. Suppose on the contrary, we assume this structure exists and $\{x_1, x_2\} \subseteq V(CF_n^1)$, then $x_1 = i_1 i_2 i_3 \cdots i_{n-2} i_{n-1} 1$, $x_2 = j_1 j_2 j_3 \cdots j_{n-2} j_{n-1} 1$. As $|N_{CF_n^i}(\{x_1, x_2\})| = 3n - 10$, by the proof process of Lemma 3.3, we can know that x_1, x_2 must have three common neighbors in CF_n^1. Note that n is odd, then $n-1$ is even, and $CF_n^i \cong CF_{n-1}$. By Corollary 2.6, we can get x_1, x_2 must belong to a common subgraph in CF_n^1; Otherwise, if x_1, x_2 belong to two different subgraphs in CF_n^1, then $|N_{CF_n^1}(\{x_1, x_2\})| \leq 2$, a contradiction. So $j_{n-1} = i_{n-1}$, $x_1^+ = 1 i_2 i_3 \cdots i_{n-2} i_{n-1} i_1$, $x_1^- = i_1 i_2 i_3 \cdots i_{n-2} 1 i_{n-1}$, $x_2^+ = 1 j_2 j_3 \cdots j_{n-2} i_{n-1} j_1$, $x_2^- = j_1 j_2 j_3 \cdots j_{n-2} 1 i_{n-1}$. By Corollary 3.2, we know $j_1 \neq i_1$. As one of the two outgoing neighbors of x_1 and x_2 belong to a common subgraph with x_3, so $\{x_1^-, x_2^-\} \subseteq V(CF_n^{i_{n-1}})$ and $x_3 \in V(CF_n^{i_{n-1}})$. Now we assume $x_3 = k_1 k_2 k_3 \cdots k_{n-2} k_{n-1} i_{n-1}$. As one of the two outgoing neighbors of x_3 belongs to CF_n^1, so $x_3 = k_1 k_2 k_3 \cdots k_{n-2} 1 i_{n-1}$ or $x_3 = 1 k_2 k_3 \cdots k_{n-2} k_{n-1} i_{n-1}$. When $x_3 = k_1 k_2 k_3 \cdots k_{n-2} 1 i_{n-1}$, $x_3^+ = i_{n-1} k_2 k_3 \cdots k_{n-2} 1 k_1$, $x_3^- = k_1 k_2 k_3 \cdots k_{n-2} i_{n-1} 1$. In this situation, $x_3^- \in V(CF_n^1)$, $x_3^+ \in V(CF_n^{k_1})$, since $i_{n-1} \neq 1$, we have $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$. Thus this structure does not exist. When $x_3 = 1 k_2 k_3 \cdots k_{n-2} k_{n-1} i_{n-1}$, as x_1^-, x_2^- are adjacent to x_3, we can get $x_1^- = x_3(1, n-1)$, $x_2^- = x_3(1, n-1)$, this contradicts to the fact $x_1^- \neq x_2^-$, thus this structure does not exist.

Next we will prove that the structure in Figure 3 (b) does not exist. Similarly, we know that $j_1 \neq i_1$, $j_{n-1} = i_{n-1}$, $\{x_1^-, x_2^-\} \subseteq V(CF_n^{i_{n-1}})$ and $x_3 \in V(CF_n^{i_{n-1}})$. We let $x_3 = k_1 k_2 k_3 \cdots k_{n-2} k_{n-1} i_{n-1}$, then $x_3^+ = i_{n-1} k_2 k_3 \cdots k_{n-2} k_{n-1} k_1$, $x_3^- = k_1 k_2 k_3 \cdots k_{n-2} i_{n-1} k_{n-1}$. Thus $x_3^- \neq x_1^+$ and $x_3^- \neq x_2^+$, the structure in Figure 3 (b) does not exist.

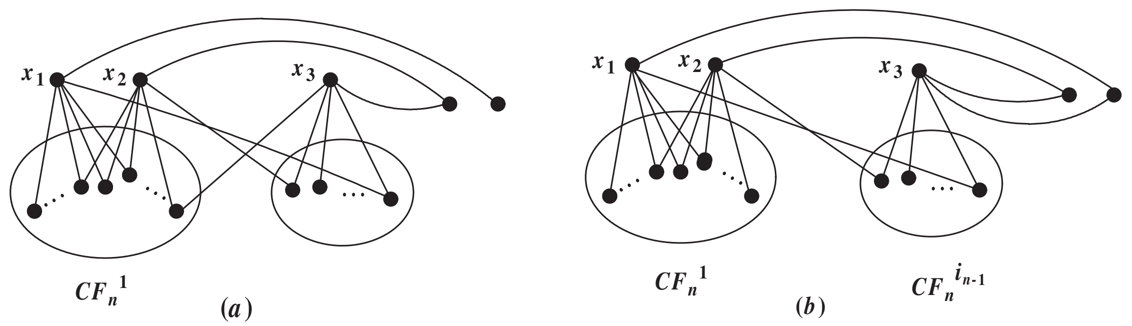

Figure 3. The case of $|N_{CF_n}(S)| = \frac{9n-23}{2}$.

Lemma 4.4. When n is even, let $S = \{x_1, x_2, x_3\}$ is an *Ind*-set, where $\{x_1, x_2\} \subseteq V(CF_n^i)$, $x_3 \in V(CF_n^j)$ $(i \neq j)$ and $|N_{CF_n^i}(\{x_1, x_2\})| = 3n - 9$. Then $|N_{CF_n}(S)| \neq \frac{9n-24}{2}$.

Proof. When n is even, $|N_{CF_n}(S)| = \frac{9n-24}{2}$ occurs if and only if the structure in Figure 4 appears. Next, we will prove that this structure does not exist. Note that $CF_n^i \cong CF_{n-1}$, $n-1$ is odd.

Suppose on the contrary, we assume this structure exists, as $|N_{CF_n^i}(\{x_1,x_2\})| = 3n-9$, we know x_1, x_2 must have three common neighbors in CF_n^i by Lemma 3.3. Now, we let $x_1 = i_1i_2i_3\cdots i_{n-2}i_{n-1}i$, $x_2 = j_1j_2j_3\cdots j_{n-2}j_{n-1}i$. By Corollary 3.2, we know $j_i \in [i_1, i_{n-1}]$ and $j_1 \neq i_1$. Thus $x_1^+ \in V(CF_n^{i_1})$, $x_2^+ \in V(CF_n^{j_1})$, x_1^+ and x_2^+ can not belong to a common subgraph in CF_n, this contradicts to this structure. Thus $|N_{CF_n}(S)| \neq \frac{9n-24}{2}$.

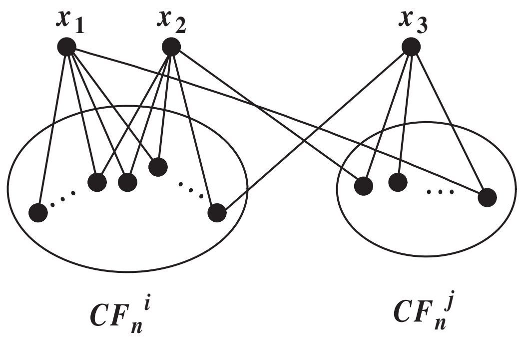

Figure 4. The case of $|N_{CF_n}(S)| = \frac{9n-24}{2}$.

Lemma 4.5. When $n \geq 4$, let S is an Ind-set and $|S| = 3$, then when n is odd, $|N_{CF_n}(S)| \geq \frac{9n-21}{2}$; when n is even, $|N_{CF_n}(S)| \geq \frac{9n-24}{2}$.

Proof. Let $S = \{x_1, x_2, x_3\}$, since S is an Ind-set, x_1, x_2, x_3 are nonadjacent with each other. We proof this result by induction on n. By Lemma 4.1 and Lemma 4.2, we know when $n = 4, 5$, this result holds. Now we assume that $n \geq 6$ and the result holds for CF_{n-1}. Note that $CF_n^i \cong CF_{n-1}$. Next, we think about the following three cases:

Case 1. x_1, x_2, x_3 belong to a same subgraph CF_n^i.

When n is odd, by induction hypothesis, we have $|N_{CF_n^i}(S)| \geq \frac{9(n-1)-24}{2} = \frac{9n-33}{2}$. By Proposition 2.3 and the definition of CF_n, we know the neighbors of x_1, x_2, x_3 in $CF_n - CF_n^i$ are different and every vertex has two outgoing neighbors. Thus $|N_{CF_n}(S)| = |N_{CF_n^i}(S)| + |N_{CF_n - CF_n^i}(S)| \geq \frac{9n-33}{2} + 6 = \frac{9n-21}{2}$.

When n is even, by induction hypothesis, we have $|N_{CF_n^i}(S)| \geq \frac{9(n-1)-21}{2} = \frac{9n-30}{2}$. By Proposition 2.3 and the definition of CF_n, we know the neighbors of x_1, x_2, x_3 in $CF_n - CF_n^i$ are different and every vertex has only one outgoing neighbor. Thus $|N_{CF_n}(S)| = |N_{CF_n^i}(S)| + |N_{CF_n - CF_n^i}(S)| \geq \frac{9n-30}{2} + 3 = \frac{9n-24}{2}$.

Case 2. x_1, x_2, x_3 belong to two different subgraphs CF_n^i, CF_n^j $(i \neq j)$.

In this case, we can let $\{x_1, x_2\} \subseteq V(CF_n^i)$, $x_3 \in V(CF_n^j)$. By Lemma 3.3, we can get: when n is odd, $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3(n-1) - 7 = 3n - 10$; when n is even, $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3(n-1) - 6 = 3n - 9$. By Proposition 2.8, we know when n is odd, $|N_{CF_n^j}(x_3)| = \frac{3(n-1)-4}{2} = \frac{3n-7}{2}$; when n is even, $|N_{CF_n^j}(x_3)| = \frac{3(n-1)-3}{2} = \frac{3n-6}{2}$. When n is odd, by Proposition 2.2, we know x_1, x_2 have at most two outgoing neighbors can belong to CF_n^j, in another word, there are at least two outgoing neighbors of $\{x_1, x_2\}$ can belong to $CF_n - CF_n^i - CF_n^j$. So $|N_{CF_n}(S)| \geq |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(x_3)| + 2 = \frac{9n-23}{2}$. By Lemma 4.3, we know $|N_{CF_n}(S)| \neq \frac{9n-23}{2}$, thus $|N_{CF_n}(S)| \geq \frac{9n-23}{2} + 1 = \frac{9n-21}{2}$. When n is even, if the structure of Figure 4 exist, then $|N_{CF_n}(S)| \geq |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(x_3)| = 3n - 9 + \frac{3n-6}{2} = \frac{9n-24}{2}$. By Lemma 4.4, we know this structure does not exist, so $|N_{CF_n}(S)| \geq \frac{9n-24}{2} + 1 = \frac{9n-22}{2}$.

Case 3. x_1, x_2, x_3 belong to three different subgraphs CF_n^i, CF_n^j, CF_n^k (i, j, k are different from each other).

Without loss of generality, we can let $x_1 \in V(CF_n^1)$, $x_2 \in V(CF_n^2)$, $x_3 \in V(CF_n^3)$. By Proposition 2.8, we have when n is odd, $|N_{CF_n^i}(x_i)| = \frac{3n-7}{2}$ $(i \in \{1,2,3\})$; when n is even, $|N_{CF_n^i}(x_i)| = \frac{3n-6}{2}$

$(i \in \{1,2,3\})$. Thus when n is odd, $|N_{CF_n}(S)| \geq 3 \times \frac{3n-7}{2} = \frac{9n-21}{2}$; when n is even, $|N_{CF_n}(S)| \geq 3 \times \frac{3n-6}{2} = \frac{9n-18}{2}$.

Thus the result holds.

Corollary 4.6. When n is even, let $S = \{x_1, x_2, x_3\}$ is an Ind-set, if $|N_{CF_n}(S)| = \frac{9n-24}{2}$, then x_1, x_2, x_3 belong to a same subgraph in CF_n.

Lemma 4.7. For $n = 5$, if $|F| \leq 11$, then $CF_5 - F$ contains a big component C, which satisfies the result $|V(C)| \geq n! - |F| - 2$.

Proof. We are not going to think about $CF_5 - F$ is connected for the moment, so we assume that $CF_5 - F$ is disconnected. Let $F_i = F \cap V(CF_5^i)$ for $i = 1,2,3,4,5$ with $|F_{i_1}| \geq |F_{i_2}| \geq |F_{i_3}| \geq |F_{i_4}| \geq |F_{i_5}|$, where $i_j \in \{1,2,3,4,5\}$. If $|F_{i_3}| = 0$, then $|F_{i_4}| = |F_{i_5}| = 0$ and $CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}$ is connected. By Proposition 2.2, we know there exists a vertex in $CF_5^{i_1} - F_{i_1}$ $(resp., CF_5^{i_2} - F_{i_2})$, which has neighbor in $CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}$. So $CF_5 - F$ is connected, a contradiction. Hence we consider $|F_{i_3}| \geq 1$. Since $|F| \leq 11$, we have $|F_{i_5}| \leq 2$, $|F_{i_4}| \leq 2$, $1 \leq |F_{i_3}| \leq 3$, $1 \leq |F_{i_2}| \leq 5$, $1 \leq |F_{i_1}| \leq 9$. Firstly, we proof the following Claim is correct.

Claim 1. If $|F_{i_j}| \leq 3$ for some $i_j \in [i_1, i_4]$, then $CF_5^{[i_j,i_5]} - F^{[i_j,i_5]}$ is connected.

Proof of Claim 1. By Proposition 2.8, we can get $CF_5^j - F_j$ is connected for each $j \in [i_j, i_5]$. On the other hand, as $|F_{i_5}| \leq 2$, we can get $|E_{p,i_5}(CF_5)| = 12 > 5 \geq |F_p| + |F_{i_5}|$, which implies $E_{p,i_5}(CF_5 - F) \neq \emptyset$ for $p \in [i_j, i_4]$. Hence $CF_5^{[i_j,i_5]} - F^{[i_j,i_5]}$ is connected.

Since $|F_{i_5}| \leq |F_{i_4}| \leq |F_{i_3}| \leq 3$, by Claim 1, we can get $CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}$ is connected. If $CF_5^{i_1} - F_{i_1}$ and $CF_5^{i_2} - F_{i_2}$ are all connected, we know $CF_5 - F$ is connected. As $|E_{i_2,i_3}(CF_5)| = 12 > 8 \geq |F_{i_2} \cup F_{i_3}|$, $CF^{[i_2,i_5]} - F^{[i_2,i_5]}$ is connected. As $|E_{i_1,i_5}(CF_5)| = 12 > 9 + 2 = 11 \geq |F_{i_1} \cup F_{i_5}|$, we have $CF_5 - F$ is connected. Since $CF_5 - F$ is disconnected, at least one of $CF_5^i - F_i, i \in \{i_1, i_2\}$ is disconnected, which leads to the following two cases.

Note that if $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \leq 1$, by Proposition 2.3, we can get $CF_5^i - F_i$ $(i \in \{i_1, i_2\})$ has a big component C_i and at most two vertices, which has a neighbor in $F_{i_3} \cup F_{i_4} \cup F_{i_5}$. Thus, if $CF_5^{i_1} - F_{i_1}$ or $CF_5^{i_2} - F_{i_2}$ is connected, then $CF_5 - F$ satisfies the condition (1). If $CF_5^{i_1} - F_{i_1}$ and $CF_5^{i_2} - F_{i_2}$ are all disconnected, then $CF_5 - F$ satisfies the condition (2). Hence we only think about this situation: $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \geq 2$.

Case 1. Both $CF_5^{i_1} - F_{i_1}$ and $CF_5^{i_2} - F_{i_2}$ are disconnected.

In this case, we know $|F_{i_1}| \geq |F_{i_2}| \geq 4$. Since $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \geq 2$, $|F_{i_1} \cup F_{i_2}| \leq |F| - |F_{i_3} \cup F_{i_4} \cup F_{i_5}| \leq 11 - 2 = 9$. Hence $|F_{i_2}| = 4$, $4 \leq |F_{i_1}| \leq 5$. By Corollary 3.4, we know $CF_5^{i_2} - F_{i_2}$ has a big component C_2 and a singleton x_2. By Lemma 1, $CF_5^{i_1} - F_{i_1}$ should consider the following two situations: (1) $CF_5^{i_1} - F_{i_1}$ has two components, one of which is a singleton. (2) $CF_5^{i_1} - F_{i_1}$ has three components, two of which are singletons. For (1), let C_1 is the big component and x_1 is the singleton of $CF_5^{i_1} - F_{i_1}$, since $|V(C_1)| = |V(C_2)| = |V(CF_5^i) - F_i - \{x_i\}| \geq 4! - 5 - 1 = 18$ $(i \in \{i_1, i_2\})$ and $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \leq 3$, by Proposition 2.3, we can get $CF_5[V(CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}) \cup V(C_1)]$ is connected. Similarly, we can also get $CF_5[V(CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}) \cup V(C_2)]$ is connected. Thus the result holds. For (2), let C_1 is the big component and x_1, x_3 are singletons of $CF_5^{i_1} - F_{i_1}$. If x_2 is nonadjacent to $\{x_1, x_3\}$, then $\{x_1, x_2, x_3\}$ are three singletons in $CF_5 - F$. By Lemma 4.2, we can get $|F| \geq 12$, this contradicts to the fact $|F| \leq 11$. If x_2 is adjacent to $\{x_1, x_3\}$, say $(x_1, x_2) \in E(CF_5 - F)$, then $CF_5 - F$ only has two components; Otherwise, we let $CF_5 - F$ has three components, then x_3 is a singleton in $CF_5 - F$. By Proposition 2.5, we have $|F| \geq |N_{CF_5}(\{x_1, x_2\}) \cup N_{CF_5}(x_3)| \geq 5 \times 2 + 6 - 3 = 13 > 11$, a contradiction. Thus the result holds.

Case 2. Only $CF_5^{i_2} - F_{i_2}$ is disconnected.

Since $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \geq 2$, $|F_{i_1} \cup F_{i_2}| \leq 9$. As $CF_5^{i_2} - F_{i_2}$ is disconnected, we have $|F_{i_2}| \geq 4$ and then $|F_{i_1}| \leq 5$. If $|F_{i_2}| = 5$, then $|F_{i_1}| \geq 5$, $|F| \geq |F_{i_1}| + |F_{i_2}| + |F_{i_3} \cup F_{i_4} \cup F_{i_5}| \geq 5 + 5 + 2 = 12$, a contradiction.

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Thus $|F_{i_2}| = 4$. Since $|E_{i_1,i_3}(CF_5)| = 12 > 8 \geq |F_{i_1} \cup F_{i_3}|$, $CF_5[V(CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}) \cup V(CF_5^{i_1} - F_{i_1})]$ is connected. As $|F_{i_2}| = 4$, by Corollary 3.4, we know $CF_5^{i_2} - F_{i_2}$ has a big component S and at most one singleton. By the same argument as that of Case 1, we can get $CF_5[V(CF_5^{[i_3,i_5]} - F^{[i_3,i_5]}) \cup V(CF_5^{i_1} - F_{i_1}) \cup V(S)]$ is connected. Then $CF_5 - F$ must satisfies condition (1).

Case 3. Only $CF_5^{i_1} - F_{i_1}$ is disconnected.

In this case, we have $|F_{i_1}| \geq 4$ by Proposition 2.8 and $|F_{i_2}| \leq 4$ since $|F| \leq 11$ and $|F_{i_3} \cup F_{i_4} \cup F_{i_5}| \geq 2$. As $|E_{i_2,i_3}(CF_5)| = 12 > 7 \geq |F_{i_2} \cup F_{i_3}|$, we have $CF_5^{[i_2,i_5]} - F^{[i_2,i_5]}$ is connected.

If $|F_{i_1}| \leq 5$, by Lemma 1, $CF_5^{i_1} - F_{i_1}$ has a big component S and one single and two singletons. By the same argument as that of Case 1, we can get $CF_5[V(CF_5^{[i_2,i_5]} - F^{[i_2,i_5]}) \cup V(S)]$ is connected. Then $CF_5 - F$ must be one of conditions (1) and (2).

Now, we suppose $|F_{i_1}| \geq 6$. Then $|F_{i_2} \cup F_{i_3} \cup F_{i_4} \cup F_{i_5}| \leq 5$. Let W be the union of components of $CF_5 - F$, whose vertices, which are totally contained in $CF_5^{i_1} - F_{i_1}$, are not connected with $CF_5^{[i_2,i_5]} - F^{[i_2,i_5]}$. By Proposition 2.2 and Proposition 2.3, we know $2|W| \leq |F - F_{i_1}| \leq 5$, which implies $|W| \leq 2$. Thus $CF_5 - F$ satisfies (1) or (2).

Combing the above three cases, we know this result holds.

Lemma 4.8. For $n = 6$, if $|F| \leq 14$, then $CF_6 - F$ contains a big component C, which satisfies the result $|V(C)| \geq n! - |F| - 2$.

Proof. Similarly, we do not think about the situation $CF_6 - F$ is connected, so we let $CF_6 - F$ is disconnected. Let $F_i = F \cap V(CF_6^i)$ for $i \in [1,6]$ with $|F_{i_1}| \geq |F_{i_2}| \geq |F_{i_3}| \geq |F_{i_4}| \geq |F_{i_5}| \geq |F_{i_6}|$, where $i_j \in \{1,2,3,4,5,6\}$. If $|F_{i_2}| = 0$, then $|F_{i_3}| = |F_{i_4}| = \cdots = |F_{i_6}| = 0$ and $CF^{[i_2,i_6]} - F^{[i_2,i_6]}$ is connected. By Proposition 2.2, we can get $CF_6 - F$ is connected. So we assume $|F_{i_2}| \geq 1$. Since $|F| \leq 14$, we have $|F_{i_6}| \leq 2$, $|F_{i_5}| \leq 2$, $|F_{i_4}| \leq 3$, $|F_{i_3}| \leq 4$, $1 \leq |F_{i_2}| \leq 7$, $1 \leq |F_{i_1}| \leq 13$. Firstly, we proof the following Claim is correct.

Claim 2. If $|F_{i_j}| \leq 5$, then $CF_6^{[i_j,i_6]} - F^{[i_j,i_6]}$ is connected.

Proof of Claim 2. By Proposition 2.8, we know $CF_6^j - F_j$ is connected for each $j \in [i_j, i_6]$. On the other hand, since $|E_{p,i_6}(CF_6)| = (6-2)! = 24 > 7 \geq |F_p \cup F_{i_6}|$ for $p \in [i_j, i_5]$, we can get $E_{p,i_6}(CF_6 - F) \neq \emptyset$. Thus $CF_6^{[i_j,i_6]} - F^{[i_j,i_6]}$ is connected.

By Claim 2, we know $CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}$ is connected. If both $CF_6^{i_1} - F_{i_1}$ and $CF_6^{i_2} - F_{i_2}$ are connected, we can get $CF_6 - F$ is connected. As $|E_{i_2,i_3}(CF_6)| = (6-2)! = 24 > 11 \geq |F_{i_2} \cup F_{i_3}|$, $|E_{i_1,i_3}(CF_6)| = (6-2)! = 24 > 17 \geq |F_{i_1} \cup F_{i_3}|$ and $CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}$ is connected, thus $CF_6 - F$ is connected. Since $CF_6 - F$ is disconnected, at least one of $CF_6^i - F_i$ $(i \in \{i_1, i_2\})$ is disconnected, which leads to the following cases.

Case 1. Both $CF_6^{i_1} - F_{i_1}$ and $CF_6^{i_2} - F_{i_2}$ are disconnected.

In this case, we know $6 \leq |F_{i_2}| \leq |F_{i_1}| \leq |F| - |F_{i_2}| \leq 8 < 9$. By Corollary 3.4, we know that $CF_6^{i_1} - F_{i_1}$ (*resp.*, $CF_6^{i_2} - F_{i_2}$) has a big component C_1 (*resp.*, C_2) and one singleton x_1 (*resp.*, x_2). As $|E_{CF_6-F}(V(C_1), V(CF_6^{i_3} - F_{i_3}))| \geq 24 - 2 - 8 - 1 = 13 > 1$. Thus $CF_6 - F[V(CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}) \cup V(C_1)]$ is connected. Similarly, we can get $CF_6 - F[V(CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}) \cup V(C_2)]$ is also connected. Thus the result holds.

Case 2. Only $CF_6^{i_2} - F_{i_2}$ is disconnected.

As $CF_6^{i_2} - F_{i_2}$ is disconnected, we have $|F_{i_2}| \geq 6$ and then $|F_{i_1}| \leq 8$. Since $|E_{i_1,i_3}(CF_6)| = 24 > 10 \geq |F_{i_1} \cup F_{i_3}|$, $CF_6[V(CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}) \cup V(CF_6^{i_1} - F_{i_1})]$ is connected. Since $|F_{i_2}| \leq |F_{i_1}| \leq 8$, by Corollary 3.4, $CF_6^{i_2} - F_{i_2}$ has a big component C and one singleton. Since $|E_{i_2,i_3}(CF_6)| = 24 > 11 \geq |F_{i_2} \cup F_{i_3}| + 1$, we can get $CF_6[V(CF_6^{[i_3,i_6]} - F^{[i_3,i_6]}) \cup V(CF_6^{i_1} - F_{i_1}) \cup V(C)]$ is connected. Then $CF_6 - F$ must be one of conditions (1).

Case 3. Only $CF_6^{i_1} - F_{i_1}$ is disconnected.

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In this case, $6 \leq |F_{i_1}| \leq 13$, $|F_{i_2}| \leq 8$. As $|E_{i_2,i_3}(CF_6)| = 24 > 11 > |F_{i_2} \cup F_{i_3}|$, we have $CF_6^{[i_2,i_6]} - F^{[i_2,i_6]}$ is connected.

If $|F_{i_1}| \leq 11$, by Lemma 4.8, $CF_6^{i_1} - F_{i_1}$ has a big component C with $|V(C)| \geq 5! - |F_{i_1}| - 2$. By the same argument as that of Case 2, we can get $CF_6[V(CF_6^{[i_2,i_6]} - F^{[i_2,i_6]}) \cup V(C)]$ is connected. Thus the result holds.

If $|F_{i_1}| \geq 12$, then $|F_{i_2} \cup F_{i_3} \cup F_{i_4} \cup F_{i_5} \cup F_{i_6}| \leq 2$. Let W be the union of components of $CF_6 - F$, whose vertices, which are totally contained in $CF_6^{i_1} - F_{i_1}$, and are not connected with $CF_6^{[i_2,i_6]} - F^{[i_2,i_6]}$. By Proposition 2.2 and Proposition 2.3, we have $|W| \leq |F - F_{i_1}| \leq 2$. Thus the result holds.

Lemma 4.9. Let $|F| \leq \frac{9n-23}{2}$ for odd n $(n \geq 5)$ and $|F| \leq \frac{9n-26}{2}$ for even n $(n \geq 6)$, then $CF_n - F$ contains a big component C, which satisfies that $|V(C)| \geq n! - |F| - 2$.

Proof. By Lemma 4.7 and Lemma 4.8, the result holds for $n = 5, 6$. We proof this result by induction on n. Assume $n \geq 7$ and the result holds for CF_{n-1}. Now we suppose $CF_n - F$ is disconnected for any $F \subseteq V(CF_n)$ with $|F| \leq \frac{9n-23}{2}$ or $|F| \leq \frac{9n-26}{2}$. Let $F_i = F \cap V(CF_n^i)$ for $i \in [1, n]$ with $|F_{i_1}| \geq |F_{i_2}| \geq \cdots \geq |F_{i_n}|$, where $i_j \in [1, n]$.

When n is odd, if $|F_{i_3}| = 0$, then $|F_{i_4}| = |F_{i_5}| = \cdots = |F_{i_n}| = 0$ and $CF^{[i_3,i_n]} - F^{[i_3,i_n]}$ is connected, and thus, by Proposition 2.2, we can get $CF_n - F$ is connected. Now we assume $|F_{i_3}| \geq 1$; When n is even, if $|F_{i_2}| = 0$, then $|F_{i_3}| = |F_{i_4}| = \cdots = |F_{i_n}| = 0$ and $CF^{[i_2,i_n]} - F^{[i_2,i_n]}$ is connected, and thus, by Proposition 2.2, we can get $CF_n - F$ is connected. Now we assume $|F_{i_2}| \geq 1$.

Claim 3. When n is even, if $|F_{i_j}| \leq \frac{3n-8}{2}$ $(i_j \in [i_1, i_{n-1}])$, then $CF_n^{[i_j,i_n]} - F^{[i_j,i_n]}$ is connected; When n is odd, if $|F_{i_j}| \leq \frac{3n-9}{2}$ $(i_j \in [i_1, i_{n-1}])$, then $CF_n^{[i_j,i_n]} - F^{[i_j,i_n]}$ is connected;

Proof of Claim 3. By Proposition 2.8, we know $CF_n^j - F_j$ is connected for each $j \in [i_j, i_n]$. On the other hand, since $|E_{p,i_n}(CF_n)| = (n-2)! > 3n - 8 > |F_p \cup F_{i_n}|$ for $p \in [i_j, i_{n-1}]$ (n is even) and $|E_{p,i_n}(CF_n)| = 2(n-2)! > 3n - 9 > |F_p \cup F_{i_n}|$ for $p \in [i_j, i_{n-1}]$ (n is odd), we can get $E_{p,i_n}(CF_n - F) \neq \emptyset$. Thus $CF_n^{[i_j,i_n]} - F^{[i_j,i_n]}$ is connected.

Since $|F| \leq \frac{9n-26}{2}$ (n is even) and $|F| \leq \frac{9n-23}{2}$ (n is odd), we have $\frac{3n-8}{2} > |F_{i_3}| \geq |F_{i_4}| \geq \cdots \geq |F_{i_n}|$ for even n and $\frac{3n-9}{2} \geq |F_{i_3}| \geq |F_{i_4}| \geq \cdots \geq |F_{i_n}|$ for odd n. By Claim 3, we can get $CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}$ is connected. If $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ are all connected, as $|E_{i_2,i_3}(CF_n)| = (n-2)! > \frac{9n-26}{2} > |F_{i_2} \cup F_{i_3}|$ (n is even) and $|E_{i_2,i_3}(CF_n)| = 2(n-2)! > \frac{9n-23}{2} > |F_{i_2} \cup F_{i_3}|$ (n is odd), then $CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}$ is connected. Similarly, we can also get $CF_n - F$ is connected. So at least one of $CF_n^i - F_i$ $(i \in \{i_1, i_2\})$ is disconnected, which leads to the following cases.

Note that, when n is odd, if $|F_{i_3} \cup F_{i_4} \cup F_{i_5} \cup \cdots \cup F_{i_n}| \leq 1$, by the same argument of Lemma 4.7, we know $CF_n - F$ satisfies condition (1) or (2). Hence we assume that $|F_{i_3} \cup F_{i_4} \cup F_{i_5} \cup \cdots \cup F_{i_n}| \geq 2$.

Case 1. Both $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ are disconnected.

When n is even, we have $\frac{3n-6}{2} \leq |F_{i_2}| \leq |F_{i_1}| \leq |F| - |F_{i_2}| \leq \frac{9n-26}{2} - \frac{3n-6}{2} = 3n - 10$. By Corollary 3.4, we know $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ all have a big component C_1, C_2 and one singleton. As $|E_{CF_n - F}(V(C_1), V(CF_n^{i_3} - F_{i_3}))| \geq (n-2)! - 1 - \frac{9n-26}{2} > 1$, $|E_{CF_n - F}(V(C_2), V(CF_n^{i_3} - F_{i_3}))| \geq (n-2)! - 1 - \frac{9n-26}{2} > 1$, Thus $CF_n - F[V(CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}) \cup V(C_1) \cup V(C_2)]$ is connected, the result holds.

When n is odd, we have $\frac{3n-7}{2} \leq |F_{i_2}| \leq |F_{i_1}| \leq |F| - |F_{i_2}| - |F_{i_3} \cup F_{i_4} \cup \cdots \cup F_{i_n}| \leq \frac{9n-23}{2} - \frac{3n-7}{2} - 2 = 3n - 10$.So by Lemma 1, we would consider the following three subcases: (1) Both $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ have three components, two of which are singletons; (2) Only one of $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ has three components, two of which are singletons; (3) Both $CF_n^{i_1} - F_{i_1}$ and $CF_n^{i_2} - F_{i_2}$ have two components, one of which is singleton. Now, we just proof the first subcase and the other two subcases could be proved by the same argument. Let x_1, y_1, C_1 (resp., x_2, y_2, C_2) be the two singletons and the other big component of $CF_n^{i_1} - F_{i_1}$ (resp., $CF_n^{i_2} - F_{i_2}$). Since $|V(C_1)| = |V(CF_n^{i_1}) - F_{i_1} - \{x_1, y_1\}| \geq (n-1)! - (3n-10) - 2$ and $|F_{i_3} \cup F_{i_4} \cup \cdots \cup F_{i_n}| \leq \frac{3n-9}{2}$, by Proposition 2.3, we know

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$CF_n[V(CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}) \cup V(C_1)]$ is connected. Similarly, we can get $CF_n[V(CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}) \cup V(C_2)]$ is connected.

If x_1, x_2, y_1, y_2 are four singletons in $CF_n - F$, then by Proposition 2.5, we know $|F| \geq |N_{CF_n^{i_1}}(\{x_1, y_1\}) \cup N_{CF_n^{i_2}}(\{x_2, y_2\}) \cup (F_{i_3} \cup F_{i_4} \cup \cdots \cup F_{i_n})| \geq [\frac{3(n-1)-4}{2} \times 2 - 3] \times 2 + 2 = 6n - 18 > \frac{9n-23}{2}$, a contradiction.

If $CF_n - F$ has three singletons, then by Lemma 4.5, we can get $|F| \geq \frac{9n-21}{2}$, this contradicts to the fact $|F| \leq \frac{9n-23}{2}$. So $CF_n - F$ has two singletons or only one singleton.

Claim 4. If $(CF_n - F)[\{x_1, y_1, x_2, y_2\}]$ has at least one edge, say $(x_1, x_2) \in E(CF_n - F)$, then $CF_n - F$ only has two components.

Proof of Claim 4. Suppose $CF_n - F$ has at least three components. Then y_1 is a singleton or $(y_1, y_2) \in E(CF_n - F)$. If y_1 is a singleton, then $|F| \geq |N_{CF_n}(\{x_1, x_2\}) \cup N_{CF_n}(y_1)| \geq (\frac{3n-3}{2} - 1) \times 2 + \frac{3n-3}{2} - 3 = \frac{9n-19}{2} > \frac{9n-23}{2}$, a contradiction. If $(y_1, y_2) \in E(CF_n - F)$, then $|F| \geq |N_{CF_n}(\{x_1, x_2\}) \cup N_{CF_n}(\{y_1, y_2\})| \geq (\frac{3n-3}{2} - 1) \times 4 - 3 \times 2 = 6n - 16 > \frac{9n-23}{2}$, a contradiction.

Thus, by Claim 4, the result holds.

Case 2. Only $CF_n^{i_2} - F_{i_2}$ is disconnected.

As $CF_n^{i_2} - F_{i_2}$ is disconnected, we have $|F_{i_2}| \geq \frac{3n-6}{2}$ for even n and $|F_{i_2}| \geq \frac{3n-7}{2}$ for odd n, then $|F_{i_1}| \leq 3n - 10 < 3n - 9$ (n is even) and $|F_{i_1}| \leq 3n - 10$ (n is odd). Since $|E_{i_1,i_3}(CF_n)| = (n-2)! > \frac{9n-26}{2} \geq |F_{i_1} \cup F_{i_3}|$ (n is even) and $|E_{i_1,i_3}(CF_n)| = 2(n-2)! > \frac{9n-23}{2} \geq |F_{i_1} \cup F_{i_3}|$ (n is odd), $CF_n[V(CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}) \cup V(CF_n^{i_1} - F_{i_1})]$ is connected. Since $|F_{i_2}| \leq |F_{i_1}| \leq 3n - 10$, when n is even, by Corollary 3.4, we know $CF_n^{i_2} - F_{i_2}$ has a big component C and one singleton; When n is odd, by Lemma 1, we know $CF_n^{i_2} - F_{i_2}$ has a big component C and at most two singletons. By the same argument as that of Case 1, we can get $CF_n[V(CF_n^{[i_3,i_n]} - F^{[i_3,i_n]}) \cup V(CF_n^{i_1} - F_{i_1}) \cup V(C)]$ is connected. Then $CF_n - F$ must be one of conditions (1) and (2).

Case 3. Only $CF_n^{i_1} - F_{i_1}$ is disconnected.

In this case, $\frac{3n-6}{2} \leq |F_{i_1}| \leq \frac{9n-28}{2}$ for even n and $\frac{3n-7}{2} \leq |F_{i_1}| \leq \frac{9n-29}{2}$ for odd n. As $|E_{i_2,i_3}(CF_n)| = (n-2)! > \frac{9n-26}{2} \geq |F_{i_2} \cup F_{i_3}|$ (n is even) and $|E_{i_2,i_3}(CF_n)| = 2(n-2)! > \frac{9n-23}{2} \geq |F_{i_2} \cup F_{i_3}|$ (n is odd), we have $CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}$ is connected.

When n is even, if $|F_{i_1}| \leq \frac{9n-32}{2}$, by introduction, we know $CF_n^{i_1} - F_{i_1}$ has a big component C with $|V(C)| \geq (n-1)! - |F_{i_1}| - 2$. By the same argument as that of Case 1, we can get $CF_n[V(CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}) \cup V(C)]$ is connected. Thus the result holds. If $|F_{i_1}| \geq \frac{9n-30}{2}$, then $|F_{i_2} \cup F_{i_3} \cup \cdots \cup F_{i_n}| \leq 2$. Let W be the union of components of $CF_n - F$, whose vertices, which are totally contained in $CF_n^{i_1} - F_{i_1}$, and are not connected with $CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}$. By Proposition 2.2 and Proposition 2.3, we have $|W| \leq |F - F_{i_1}| \leq 2$. Thus the result holds.

When n is odd, if $|F_{i_1}| \leq \frac{9n-35}{2}$, by introduction, $CF_n^{i_1} - F_{i_1}$ has a big component C with $|V(C)| \geq (n-1)! - |F_{i_1}| - 2$. By the same argument as that of Case 1, we can get $CF_n[V(CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}) \cup V(C)]$ is connected. Thus the result holds. If $|F_{i_1}| \geq \frac{9n-33}{2}$, then $|F_{i_2} \cup F_{i_3} \cup \cdots \cup F_{i_n}| \leq 5$. Let W be the union of components of $CF_n - F$, whose vertices, which are totally contained in $CF_n^{i_1} - F_{i_1}$, and are not connected with $CF_n^{[i_2,i_n]} - F^{[i_2,i_n]}$. By Proposition 2.3, $2|W| \leq |F - F_{i_1}| \leq 5$. Then we have $|W| \leq 2$. Thus the result holds.

Theorem 2. For $n \geq 4$, when n is odd, $c\kappa_4(CF_n) = \frac{9n-21}{2}$; when n is even, $c\kappa_4(CF_n) = \frac{9n-24}{2}$.

Proof. By Lemma 1, we have $c\kappa_4(CF_4) \geq 6 = \frac{9\times4-24}{2}$. For $n \geq 5$, by Lemma 4.9, we can get when n is odd, $c\kappa_4(CF_n) \geq \frac{9n-21}{2}$; when n is even, $c\kappa_4(CF_n) \geq \frac{9n-24}{2}$. Next, we will prove that $c\kappa_4(CF_n) \leq \frac{9n-21}{2}$ and $c\kappa_4(CF_n) \leq \frac{9n-24}{2}$. For $n = 4$, if we let $F = \{2314, 3124, 1234, 4213, 4132, 4321\}$, then $CF_n - F$ has three singletons: $x_1 = 3214, x_2 = 2134, x_3 = 1324$. Thus $c\kappa_4(CF_4) \leq 6 = \frac{9\times4-24}{2}$. For $n \geq 5$, when n is odd, let $S = \{x_1, x_2, x_3\}$, where $x_1 = i_1i_2i_3 \cdots i_{n-4}3214$, $x_2 = 2i_2i_3 \cdots i_{n-4}i_1314$, $x_3 = 2i_2i_3 \cdots i_{n-4}3i_141$, then $|N_{CF_n}(S)| = 3n - 10 + \frac{3n-7}{2} + 3 = \frac{9n-21}{2}$ and there are three singletons $\{x_1, x_2, x_3\}$ in $CF_n - N_{CF_n}(S)$. Thus when n is odd, $c\kappa_4(CF_n) \leq \frac{9n-21}{2}$. When n is even, let $S = \{x_1, x_2, x_3\}$, where

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$x_1 = i_1i_2i_3\cdots i_{n-4}3214j$, $x_2 = 2i_2i_3\cdots i_{n-4}i_1314j$, $x_3 = 2i_2i_3\cdots i_{n-4}3i_141j$, then $\{x_1, x_2, x_3\} \subseteq V(CF_n^j)$ and $|N_{CF_n^j}(S)| = \frac{9(n-1)-21}{2} = \frac{9n-30}{2}$. As x_1, x_2, x_3 belong to a common subgraph, by Proposition 2.3, we know x_1, x_2, x_3 have different outgoing neighbors. So $|N_{CF_n}(S)| = \frac{9n-30}{2} + 3 = \frac{9n-24}{2}$ and x_1, x_2, x_3 are three singletons in $CF_n - N_{CF_n}(S)$. Thus when n is even, $c\kappa_4(CF_n) \leq \frac{9n-24}{2}$.

5. The 5-component connectivity of CF_n

Lemma 5.1. For $n = 4$, let S is an Ind-set and $|S| = 4$, then $|N_{CF_4}(S)| \geq 8$.

Proof. Let $S = \{x_1, x_2, x_3, x_4\}$, since S is an Ind-set, x_1, x_2, x_3, x_4 are nonadjacent to each other. As $CF_4^i \cong CF_3$ ($i \in \{1, 2, 3, 4\}$), we know x_1, x_2, x_3, x_4 can not belong to a same subgraph of CF_4. So we need think about the following cases:

Case 1: x_1, x_2, x_3, x_4 belong to two different subgraphs of CF_4.

In this case, we can divide it into two subcases:

Subcase 1.1: There are two subgraphs of CF_4 which contain only two vertices of S. Without loss of generality, we can let $\{x_1, x_2\} \subseteq V(CF_4^1)$, $\{x_3, x_4\} \subseteq V(CF_4^2)$. By the definition of CF_n, $|N_{CF_4^1}(\{x_1, x_2\})| = |N_{CF_4^2}(\{x_3, x_4\})| = 3$. Now we let $x_1 = 2341$, then $x_2 = 4231$ or $x_2 = 3421$. Thus $x_1^+ \in V(CF_4^2)$, $x_2^+ = 1234 \in V(CF_4^4)$ or $x_2^+ = 1423 \in V(CF_4^3)$. Hence x_1^+ and x_2^+ can not belong to a common subgraph of CF_4. Similarly, we know x_3^+ and x_4^+ can not belong to a common subgraph of CF_4. If x_1^+ or x_2^+ belong to CF_4^2 and adjacent to $\{x_3, x_4\}$, meanwhile x_3^+ or x_4^+ belong to CF_4^1 and adjacent to $\{x_1, x_2\}$, then $|N_{CF_4}(S)| = 3 + 3 + 2 = 8$. Now, we illustrate this structure exists. Let $x_3 = 3142$, $x_4 = 1432$, then x_1^+ is adjacent to x_3, x_4^+ is adjacent to x_1. Thus $|N_{CF_4}(S)| \geq 8$.

Subcase 1.2: There is a subgraph of CF_4 which contains three vertices of S. In this subcase, we can let $\{x_1, x_2, x_3\} \subseteq V(CF_4^1)$, $x_4 \in V(CF_4^2)$. We let $x_1 = 2341$, $x_2 = 4231$, $x_3 = 3421$, then $x_1^+ = 1342 \in V(CF_4^2)$, $x_2^+ = 1234 \in V(CF_4^4)$, $x_3^+ = 1423 \in V(CF_4^3)$. Clearly, $|N_{CF_4^1}(\{x_1, x_2, x_3\})| = 3$, $|N_{CF_4^2}(x_4)| = 3$. If $x_4^+ \in V(CF_4^1)$ and is adjacent to $\{x_1, x_2, x_3\}$, meanwhile x_1^+ is adjacent to x_4, then $|N_{CF_4}(S)| = 3 + 3 + 2 = 8$. Let $x_4 = 1432$, then $x_4^+ = 2431$. Thus x_1^+ is adjacent to x_4 and x_4^+ is adjacent to x_1. Thus $|N_{CF_4}(S)| \geq 8$.

Case 2: x_1, x_2, x_3, x_4 belong to three different subgraphs of CF_4.

In this case, there exists a subgraph CF_4^i which must contains two vertices of S. Now we can let $\{x_1, x_2\} \subseteq V(CF_4^i)$, $x_3 \in V(CF_4^j)$, $x_4 \in V(CF_4^k)$. Then $|N_{CF_4^i}(\{x_1, x_2\})| = 3$, $|N_{CF_4^j}(x_3)| = |N_{CF_4^k}(x_4)| = 3$, $|N_{CF_4}(S)| \geq |N_{CF_4^i}(\{x_1, x_2\})| + |N_{CF_4^j}(x_3)| + |N_{CF_4^k}(x_4)| = 9$.

Case 3: x_1, x_2, x_3, x_4 belong to four different subgraphs of CF_4.

In this case, we can let $x_k \in V(CF_n^k)$, then $|N_{CF_n^k}(x_k)| = 3$. Thus $|N_{CF_4}(S)| \geq 4 \times |N_{CF_n^k}(x_k)| = 4 \times 3 = 12$.

Combing the above three cases, we have $|N_{CF_4}(S)| \geq 8$.

Lemma 5.2. When n is odd, let $S = \{x_1, x_2, x_3, x_4\}$ is an Ind-set and $\{x_1, x_2\} \subseteq V(CF_n^i)$, $\{x_3, x_4\} \subseteq V(CF_n^j)$ $(i \neq j)$. If $|N_{CF_n^i}(\{x_1, x_2\})| = |N_{CF_n^j}(\{x_3, x_4\})| = 3n - 10$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Proof. Since $|N_{CF_n^i}(\{x_1, x_2\})| = |N_{CF_n^j}(\{x_3, x_4\})| = 3n - 10$, by the proof process of Lemma 3.3, we know x_1, x_2 (*resp.*, x_3, x_4) have three common neighbors in CF_n^i (*resp.*, CF_n^j). So by Lemma 3.1, we can let $x_1 = i_1i_2\cdots i_{n-2}i_{n-1}i$, $x_2 = k_1k_2\cdots k_{n-2}i_{n-1}i$, where $k_i \in [i_1, i_{n-2}]$ and $k_1 \neq i_1$. Then $x_1^+ = ii_2i_3\cdots i_{n-2}i_{n-1}i_1$, $x_1^- = i_1i_2i_3\cdots i_{n-2}ii_{n-1}$, $x_2^- = k_1k_2\cdots k_{n-2}ii_{n-1}$, $x_2^+ = ik_2k_3\cdots k_{n-2}i_{n-1}k_1$. By Proposition 2.3 and Proposition 2.2, we have $2 \leq |N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2\})| \leq 4$, $2 \leq |N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| \leq 4$. Then we think about the following three cases:

Case 1: $|N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2\})| = 4$.

In this case, we can easily get $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Case 2: $|N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2\})| = 3$.

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In this case, only one of the outgoing neighbors of $\{x_1, x_2\}$ belong to CF_n^j. So $i_1 = j$ or $k_1 = j$. We assume $i_1 = j$, then $x_1^+ \in V(CF_n^j)$. If $|N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| = 4$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. If $|N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| \leq 3$, one of the outgoing neighbors of $\{x_3, x_4\}$ must belong to CF_n^i, we assume x_3^+ or x_3^- belong to CF_n^i, then $x_3 = ij_2j_3 \cdots j_{n-2}j_{n-1}j$ or $x_3 = j_1j_2j_3 \cdots j_{n-2}ij$. When $x_3 = ij_2j_3 \cdots j_{n-2}j_{n-1}j$, then by the proof process of Lemma 3.1, we can let $x_4 = l_1l_2l_3 \cdots l_{n-2}j_{n-1}j$, where $l_i \in ([j_2, j_{n-2}] \cup \{i\})$ and $l_1 \neq i$. Thus $x_3^+ = jj_2j_3 \cdots j_{n-2}j_{n-1}i$, $x_3^- = ij_2j_3 \cdots j_{n-2}jj_{n-1}$, $x_4^+ = jl_2l_3 \cdots l_{n-2}j_{n-1}l_1$, $x_4^- = l_1l_2l_3 \cdots l_{n-2}jj_{n-1}$. Then $x_3^+ \in V(CF_n^i)$. Since $j_{n-1} \neq i$, $x_4^- \notin V(CF_n^i)$. Thus $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$ as $x_4^- \neq x_1^-$, $x_4^- \neq x_2^-$, $x_4^- \neq x_2^+$. When $x_3 = j_1j_2j_3 \cdots j_{n-2}ij$, then by the proof process of Lemma 3.1, we can let $x_4 = l_1l_2l_3 \cdots l_{n-2}ij$, where $l_i \in [j_1, j_{n-2}]$ and $l_1 \neq j_1$. Then $x_3^+ = jj_2j_3 \cdots j_{n-2}ij_1$, $x_3^- = j_1j_2j_3 \cdots j_{n-2}ji$, $x_4^+ = jl_2l_3 \cdots l_{n-2}il_1$, $x_4^- = l_1l_2l_3 \cdots l_{n-2}ji$, $\{x_3^-, x_4^-\} \subseteq V(CF_n^i)$. Since $j_1 \neq i$, $l_1 \neq i$, we know $\{x_3^+, x_4^+\} \subseteq V(CF_n - CF_n^i - CF_n^j)$. As $i \neq j$ and $k_1 \neq j$, then $x_1^- \neq x_3^+$, $x_3^+ \neq x_2^+$, $x_3^+ \neq x_2^-$. Thus $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Case 3: $|N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2\})| = 2$.

In this case, two outgoing neighbors of $\{x_1, x_2\}$ belong to CF_n^j. Thus $\{x_1^-, x_2^-\} \subseteq V(CF_n^j)$ and $\{x_3, x_4\} \subseteq V(CF_n^j)$. Now we let $x_3 = j_1j_2j_3 \cdots j_{n-2}j_{n-1}j$, then $x_4 = u_1u_2u_3 \cdots u_{n-2}j_{n-1}j$, where $u_i \in [j_1, j_{n-2}]$ and $u_1 \neq j_1$. If $|N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| = 4$, clearly $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. If $|N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| = 3$, the proof process is similar to Case 2, we can get $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. So we let $|N_{CF_n - CF_n^i - CF_n^j}(\{x_3, x_4\})| = 2$, then one of the outgoing neighbors of x_3 and x_4 belong to CF_n^i, we assume $x_3 = ij_2j_3 \cdots j_{n-2}j_{n-1}j$ or $x_3 = j_1j_2j_3 \cdots j_{n-2}ij$. When $x_3 = ij_2j_3 \cdots j_{n-2}j_{n-1}j$, we can get $j_{n-1} \neq i$, so $x_4 = iu_2u_3 \cdots u_{n-2}j_{n-1}j$. By Corollary 3.2, we know x_3 and x_4 can not have three common neighbors in CF_n^j, so $x_3 = j_1j_2j_3 \cdots j_{n-2}ij$, $x_4 = u_1u_2u_3 \cdots u_{n-2}ij$. Since $x_4^+ = ju_2u_3 \cdots u_{n-2}iu_1$, $x_3^+ = jj_2j_3 \cdots j_{n-2}ij_1$ and $i \neq j$, we have $x_3^+ \neq x_1^+$, $x_3^+ \neq x_2^+$, $x_4^+ \neq x_1^+$, $x_4^+ \neq x_2^+$. Thus $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Lemma 5.3. When n is odd, let $S = \{x_1, x_2, x_3, x_4\}$ is an Ind-set and $\{x_1, x_2, x_3\} \subseteq V(CF_n^i)$, $x_4 \in V(CF_n^j)$ $(i \neq j)$. If $|N_{CF_n^i}(S)| = \frac{9(n-1)-24}{2} = \frac{9n-33}{2}$, $|N_{CF_n^j}(S)| = \frac{3n-7}{2}$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Proof. Since $\{x_1, x_2, x_3\} \subseteq V(CF_n^i)$, $CF_n^i \cong CF_{n-1}$, $n-1$ is even and $|N_{CF_n^i}(S)| = \frac{9n-33}{2}$, by Corollary 4.6, we know x_1, x_2, x_3 must belong to a common subgraph in CF_n^i. So we let $x_1 = i_1i_2i_3 \cdots i_{n-2}i_{n-1}i$, $x_2 = j_1j_2j_3 \cdots j_{n-2}i_{n-1}i$, $x_3 = k_1k_2k_3 \cdots k_{n-2}i_{n-1}i$. Then $x_1^+ = ii_2i_3 \cdots i_{n-2}i_{n-1}i_1$, $x_1^- = i_1i_2i_3 \cdots i_{n-2}ii_{n-1}$, $x_2^+ = ij_2j_3 \cdots j_{n-2}i_{n-1}j_1$, $x_2^- = j_1j_2j_3 \cdots j_{n-2}ii_{n-1}$, $x_3^+ = ik_2k_3 \cdots k_{n-2}i_{n-1}k_1$, $x_3^- = k_1k_2k_3 \cdots k_{n-2}ii_{n-1}$. If $|N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2, x_3\})| \geq 4$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. Thus we assume $|N_{CF_n - CF_n^i - CF_n^j}(\{x_1, x_2, x_3\})| = 3$, then one of the two outgoing neighbors of x_1, x_2, x_3 must belong to a common subgraph of CF_n, we need to think about the following two situations:

Case 1: $\{x_1^-, x_2^-, x_3^-\} \subseteq V(CF_n^{i_{n-1}})$ and $x_4 \in V(CF_n^{i_{n-1}})$.

In this case, we let $x_4 = l_1l_2l_3 \cdots l_{n-2}l_{n-1}i_{n-1}$, then $x_4^+ = i_{n-1}l_2l_3 \cdots l_{n-2}l_{n-1}l_1$, $x_4^- = l_1l_2l_3 \cdots l_{n-2}i_{n-1}l_{n-1}$. If $x_4^+ \notin V(CF_n^i)$ and $x_4^- \notin V(CF_n^i)$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$ as $i_{n-1} \neq i$ and $x_4^+ \neq x_1^+$, $x_4^+ \neq x_2^+$, $x_4^+ \neq x_3^+$. If one of the two outgoing neighbors of x_4 belong to CF_n^i, we can assume $x_4 = l_1l_2l_3 \cdots l_{n-2}ii_{n-1}$ or $x_4 = il_2l_3 \cdots l_{n-2}l_{n-1}i_{n-1}$. If $x_4 = il_2l_3 \cdots l_{n-2}l_{n-1}i_{n-1}$, since $|N_{CF_n^j}(S)| = \frac{3n-7}{2}$, we know x_1^-, x_2^-, x_3^- are adjacent to x_4, so $x_1^- = x_4(1, n-1)$, $x_2^- = x_4(1, n-1)$, $x_3^- = x_4(1, n-1)$, this contradicts to the fact that x_1^-, x_2^-, x_3^- are different from each other. So $x_4 = l_1l_2l_3 \cdots l_{n-2}ii_{n-1}$, then $x_4^+ = i_{n-1}l_2l_3 \cdots l_{n-2}il_1$, $x_4^- = l_1l_2l_3 \cdots l_{n-2}i_{n-1}i$. Hence $x_4^- \in V(CF_n^i)$, $x_4^+ \neq x_1^+$, $x_4^+ \neq x_2^+$, $x_4^+ \neq x_3^+$. Thus $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Case 2: Let $i_1 = j_1 = k_1$, $\{x_1^+, x_2^+, x_3^+\} \subseteq V(CF_n^{i_1})$ and $x_4 \in V(CF_n^{i_1})$.

In this case, $x_1 = i_1i_2i_3 \cdots i_{n-2}i_{n-1}i$, $x_2 = i_1j_2j_3 \cdots j_{n-2}i_{n-1}i$, $x_3 = i_1k_2k_3 \cdots k_{n-2}i_{n-1}i$. Then $x_1^+ = ii_2i_3 \cdots i_{n-2}i_{n-1}i_1$, $x_2^+ = ij_2j_3 \cdots j_{n-2}i_{n-1}i_1$, $x_3^+ = ik_2k_3 \cdots k_{n-2}i_{n-1}i_1$, $x_1^- = i_1i_2i_3 \cdots i_{n-2}ii_{n-1}$, $x_2^- = i_1j_2j_3 \cdots j_{n-2}ii_{n-1}$, $x_3^- = i_1k_2k_3 \cdots k_{n-2}ii_{n-1}$. We let $x_4 = l_1l_2l_3 \cdots l_{n-2}\ l_{n-1}i_1$, then

$x_4^+ = i_1 l_2 l_3 \cdots l_{n-2} l_{n-1} l_1$, $x_4^- = l_1 l_2 l_3 \cdots l_{n-2} i_1 l_{n-1}$. If $x_4^+ \notin V(CF_n^i)$ and $x_4^- \notin V(CF_n^i)$, then $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$ as $x_4^- \neq x_1^-$, $x_4^- \neq x_2^-$, $x_4^- \neq x_3^-$. If one of the two outgoing neighbors of x_4 belong to CF_n^i, we can assume $x_4 = l_1 l_2 l_3 \cdots i i_1$ or $x_4 = i l_2 l_3 \cdots l_{n-1} i_1$. When $x_4 = l_1 l_2 l_3 \cdots i i_1$, as x_1^+, x_2^+, x_3^+ are adjacent to x_4, so $x_1^+ = x_4(1, n-1)$, $x_2^+ = x_4(1, n-1)$, $x_3^+ = x_4(1, n-1)$, this contracts to the fact x_1^+, x_2^+, x_3^+ are different from each other. So $x_4 = i l_2 l_3 \cdots l_{n-1} i_1$, $x_4^+ = i_1 l_2 l_3 \cdots l_{n-1} i$, $x_4^- = i l_2 l_3 \cdots i_1 l_{n-1}$. Then $x_4^+ \in V(CF_n^i)$, $x_4^- \neq x_1^-$, $x_4^- \neq x_2^-$, $x_4^- \neq x_3^-$. Thus $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$.

Lemma 5.4. When n is odd, let $S = \{x_1, x_2, x_3, x_4\}$ is an Ind-set and $\{x_1, x_2\} \subseteq V(CF_n^i)$, $\{x_3\} \subseteq V(CF_n^j)$, $\{x_3\} \subseteq V(CF_n^k)$ (i, j, k are different from each other). If $|N_{CF_n^i}(\{x_1, x_2\})| = 3n - 10$, then $|N_{CF_n - CF_n^i - CF_n^j - CF_n^k}(S)| \geq 1$.

Proof. If $|N_{CF_n - CF_n^i - CF_n^j - CF_n^k}(S)| = 0$, the structure in Figure 5 must exists. Now we can proof this structure does not exist. As $|N_{CF_n^i}(\{x_1, x_2\})| = 3n - 10$, we can get x_1, x_2 must have three common neighbors in CF_n^i. So we can assume $x_1 = i_1 i_2 i_3 \cdots i_{n-2} j i$, then by Lemma 3.1, we can let $x_2 = j_1 j_2 j_3 \cdots j_{n-2} j i$, where $j_i \in [i_1, i_{n-2}]$ and $j_1 \neq i_1$. Then $x_1^+ = i i_2 i_3 \cdots i_{n-2} j i_1$, $x_2^+ = i j_2 j_3 \cdots j_{n-2} j j_1$, $x_1^- = i_1 i_2 i_3 \cdots i_{n-2} i j$, $x_2^- = j_1 j_2 j_3 \cdots j_{n-2} i j$. Since $i_1 \neq j$ and $i_1 \neq j_1$, x_1^+ can not belong to a common subgraph with x_2^+ or x_2^-. Thus the structure in Figure 5 does not exist, $|N_{CF_n - CF_n^i - CF_n^j - CF_n^k}(S)| \geq 1$.

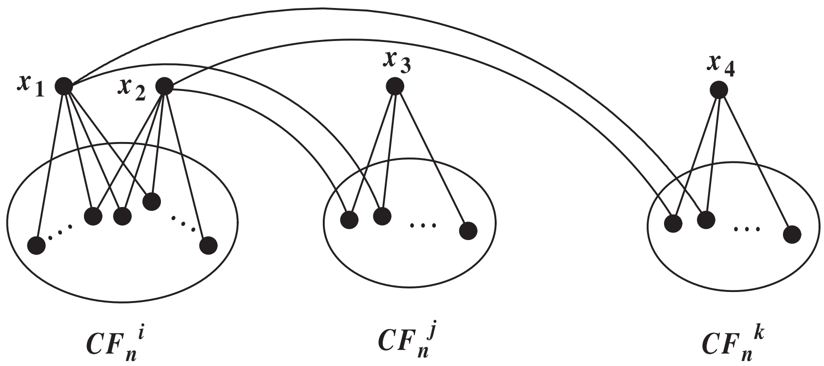

Figure 5. The case of $|N_{CF_n - CF_n^i - CF_n^j - CF_n^k}(S)| = 0$.

Lemma 5.5. For $n = 5$, let S is an Ind-set and $|S| = 4$, then $|N_{CF_5}(S)| \geq 14$.

Proof. Let $S = \{x_1, x_2, x_3, x_4\}$, since S is an Ind-set, x_1, x_2, x_3, x_4 are nonadjacent to each other. Note that $CF_5^i \cong CF_4$. Now we think about the following four cases:

Case 1: $S = \{x_1, x_2, x_3, x_4\}$ belong to a same subgraph CF_5^i.

By Lemma 5.1, we have $|N_{CF_5^i}(S)| \geq 8$. By Proposition 2.3, we know $|N_{CF_5 - CF_5^i}(S)| = 8$. Thus $|N_{CF_5}(S)| = |N_{CF_5^i}(S)| + |N_{CF_5 - CF_5^i}(S)| \geq 8 + 8 = 16$.

Case 2: $S = \{x_1, x_2, x_3, x_4\}$ belong to two different subgraphs CF_5^i, CF_5^j ($i \neq j$).

In this case, we need to think about the following two situations:

Subcase 2.1: $\{x_1, x_2\} \subseteq V(CF_5^i)$, $\{x_3, x_4\} \subseteq V(CF_5^j)$.

By Lemma 3.3, we can get $|N_{CF_5^i}(\{x_1, x_2\})| \geq 3n - 10 = 5$, $|N_{CF_5^j}(\{x_3, x_4\})| \geq 3n - 10 = 5$. By Lemma 5.2, we have $|N_{CF_5}(S)| = |N_{CF_5^i}(\{x_1, x_2\})| + |N_{CF_5^j}(\{x_3, x_4\})| + |N_{CF_5 - CF_5^i - CF_5^j}(S)| \geq 2 \times 5 + 4 = 14$.

Subcase 2.2: $\{x_1, x_2, x_3\} \subseteq V(CF_5^i)$, $x_4 \in V(CF_5^j)$.

By Lemma 4.1, we have $|N_{CF_5^i}(\{x_1, x_2, x_3\})| \geq 6$, $|N_{CF_5^j}(x_4)| = \frac{3 \times 4 - 4}{2} = 4$. By Lemma 5.3, we know $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. Thus $|N_{CF_n}(S)| = |N_{CF_5^i}(\{x_1, x_2, x_3\})| + |N_{CF_5^j}(x_4)| + |N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 6 + 4 + 4 = 14$.

Case 3: $S = \{x_1, x_2, x_3, x_4\}$ belong to three different subgraphs CF_5^i, CF_5^j, CF_5^k (i, j, k are different from each other).

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In this case, there exists a subgraph CF_5^i, which contains two vertices of S, we let $\{x_1, x_2\} \subseteq V(CF_5^i)$. Clearly, $|N_{CF_5^i}(\{x_1, x_2\})| \geq 3n - 10 = 5$, $|N_{CF_5^j}(x_3)| = |N_{CF_5^k}(x_4)| = 4$. Thus, by Lemma 5.4, we have $|N_{CF_5}(S)| = |N_{CF_5^i}(\{x_1, x_2\})| + |N_{CF_5^j}(x_3)| + |N_{CF_5^k}(x_4)| + |N_{CF_5 - CF_5^i - CF_5^j - CF_5^k}(S)| \geq 5 + 4 + 4 + 1 = 14$.

Case 4: $S = \{x_1, x_2, x_3, x_4\}$ belong to four different subgraphs.

In this case, we can let $x_k \in V(CF_5^k)$. Clearly, $|N_{CF_5^k}(x_k)| = 4$, thus $|N_{CF_5}(S)| \geq 4 \times |N_{CF_5^k}(x_k)| = 4 \times 4 = 16 > 14$.

Combing the above four cases, we can get $|N_{CF_5}(S)| \geq 14$.

Lemma 5.6. For $n = 6$, let S is an Ind-set and $|S| = 4$, then $|N_{CF_6}(S)| \geq 18$.

Proof. Let $S = \{x_1, x_2, x_3, x_4\}$, since S is an Ind-set, x_1, x_2, x_3, x_4 are nonadjacent to each other. Note that $CF_6^i \cong CF_5$. Now we think about the following four cases:

Case 1: $S = \{x_1, x_2, x_3, x_4\}$ belong to a same subgraph CF_6^i.

By Lemma 5.5, we can get $|N_{CF_6^i}(S)| \geq 14$. By the definition of CF_n, we know every vertex in CF_6 has only one outgoing neighbor. Thus $|N_{CF_6}(S)| = |N_{CF_6^i}(S)| + 4 \geq 18$.

Case 2: $S = \{x_1, x_2, x_3, x_4\}$ belong to two different subgraphs CF_6^i, CF_6^j $(i \neq j)$.

In this case, we also need to think about the following two situations:

Subcase 2.1: $\{x_1, x_2\} \subseteq V(CF_6^i)$, $\{x_3, x_4\} \subseteq V(CF_6^j)$.

By Lemma 3.3, we can get $|N_{CF_6^i}(\{x_1, x_2\})| \geq 3n - 9 = 9$, $|N_{CF_6^j}(\{x_3, x_4\})| \geq 3n - 9 = 9$. Thus $|N_{CF_6}(S)| \geq |N_{CF_6^i}(\{x_1, x_2\})| + |N_{CF_6^j}(\{x_3, x_4\})| = 9 + 9 = 18$.

Subcase 2.2: $\{x_1, x_2, x_3\} \subseteq V(CF_6^i)$, $x_4 \in V(CF_6^j)$.

By Lemma 4.2, we have $|N_{CF_6^i}(\{x_1, x_2, x_3\})| \geq 12$, $|N_{CF_6^j}(x_4)| = \frac{3\times 5 - 3}{2} = 6$. Thus $|N_{CF_6}(S)| \geq |N_{CF_6^i}(\{x_1, x_2, x_3\})| + |N_{CF_6^j}(x_4)| = 12 + 6 = 18$.

Case 3: $S = \{x_1, x_2, x_3, x_4\}$ belong to three different subgraphs CF_6^i, CF_6^j, CF_6^k $(i, j, k$ are different from each other$)$.

In this case, there exists a subgraph CF_6^i, which contains two vertices of S, we let $\{x_1, x_2\} \subseteq V(CF_6^i)$. Clearly, $|N_{CF_6^i}(\{x_1, x_2\})| \geq 3n - 9 = 9$, $|N_{CF_6^j}(x_3)| = |N_{CF_6^k}(x_4)| = 6$. Thus, $|N_{CF_6}(S)| \geq |N_{CF_6^i}(\{x_1, x_2\})| + |N_{CF_6^j}(x_3)| + |N_{CF_6^k}(x_4)| = 9 + 6 + 6 = 21 > 18$.

Case 4: $S = \{x_1, x_2, x_3, x_4\}$ belong to four different subgraphs.

In this case, we can let $x_k \in V(CF_6^k)$. Clearly, $|N_{CF_6^k}(x_k)| = 6$, thus $|N_{CF_6}(S)| \geq 4 \times |N_{CF_6^k}(x_k)| = 4 \times 6 = 24 > 18$.

Combing the above four cases, we can get $|N_{CF_6}(S)| \geq 18$.

Lemma 5.7. For $n \geq 5$, let S is an Ind-set and $|S| = 4$, then when n is odd, $|N_{CF_n}(S)| \geq 6n - 16$; when n is even, $|N_{CF_n}(S)| \geq 6n - 18$.

Proof. We proof this result by induction on n. By Lemma 5.5 and Lemma 5.6, we know when $n = 5, 6$, this result holds. Now we assume $n \geq 7$ and the result holds for CF_{n-1}. Let $S = \{x_1, x_2, x_3, x_4\}$, since S is an Ind-set, x_1, x_2, x_3, x_4 are nonadjacent to each other. Note that $CF_n^i \cong CF_{n-1}$. Now we think about the following four cases:

Case 1: $S = \{x_1, x_2, x_3, x_4\}$ belong to a same subgraph CF_n^i.

By induction hypothesis, we know when n is odd, $|N_{CF_n^i}(S)| \geq 6(n-1) - 18 = 6n - 24$; when n is even, $|N_{CF_n^i}(S)| \geq 6(n-1) - 16 = 6n - 22$. By Proposition 2.3, we know when n is odd, $|N_{CF_n - CF_n^i}(S)| = 8$; when n is even, $|N_{CF_n - CF_n^i}(S)| = 4$. Thus $|N_{CF_n}(S)| = |N_{CF_n^i}(S)| + |N_{CF_n - CF_n^i}(S)| \geq 6n - 24 + 8 = 6n - 16$ (n is odd) and $|N_{CF_n}(S)| \geq 6n - 22 + 4 = 6n - 18$ (n is even).

Case 2: $S = \{x_1, x_2, x_3, x_4\}$ belong to two different subgraphs CF_n^i, CF_n^j $(i \neq j)$.

In this case, we need to think about two situations:

Subcase 2.1: $\{x_1, x_2\} \subseteq V(CF_n^i)$, $\{x_3, x_4\} \subseteq V(CF_n^j)$.

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By Lemma 3.3, we can get when n is odd, $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3n-10$, $|N_{CF_n^j}(\{x_3,x_4\})| \geq 3n-10$; when n is even, $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3n-9$, $|N_{CF_n^j}(\{x_3,x_4\})| \geq 3n-9$. When n is odd, by Lemma 5.2, we have $|N_{CF_n}(S)| = |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(\{x_3,x_4\})| + |N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 2\times(3n-10)+4 = 6n-16$. When n is even, $|N_{CF_n}(S)| \geq |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(\{x_3,x_4\})| = 2\times(3n-9) = 6n-18$.

Subcase 2.2: $\{x_1,x_2,x_3\} \subseteq V(CF_n^i)$, $x_4 \in V(CF_n^j)$.

When n is odd, by Lemma 4.5, we have $|N_{CF_n^i}(\{x_1,x_2,x_3\})| \geq \frac{9(n-1)-24}{2} = \frac{9n-33}{2}$. By Lemma 5.3, we know $|N_{CF_n - CF_n^i - CF_n^j}(S)| \geq 4$. As $|N_{CF_n^j}(x_4)| = \frac{3\times(n-1)-4}{2} = \frac{3n-7}{2}$, so $|N_{CF_n}(S)| = |N_{CF_n^i}(\{x_1,x_2,x_3\})| + |N_{CF_n^j}(x_4)| + |N_{CF_n - CF_n^i - CF_n^j}(S)| \geq \frac{9n-33}{2} + \frac{3n-7}{2} + 4 = 6n-16$. When n is even, by Lemma 4.5, we have $|N_{CF_n^i}(\{x_1,x_2,x_3\})| \geq \frac{9(n-1)-21}{2} = \frac{9n-30}{2}$. As $|N_{CF_n^j}(x_4)| = \frac{3\times(n-1)-3}{2} = \frac{3n-6}{2}$, so $|N_{CF_n}(S)| \geq |N_{CF_n^i}(\{x_1,x_2,x_3\})| + |N_{CF_n^j}(x_4)| = \frac{9n-30}{2} + \frac{3n-6}{2} = 6n-18$.

Case 3: $S = \{x_1,x_2,x_3,x_4\}$ belong to three different subgraphs CF_n^i, CF_n^j, CF_n^k (i,j,k are different from each other).

In this case, there exists a subgraph CF_n^i, which contains two vertices of S, we let $\{x_1,x_2\} \subseteq V(CF_n^i)$. When n is odd, by Lemma 3.3, we have $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3n-10$. Clearly, $|N_{CF_n^j}(x_3)| = |N_{CF_n^k}(x_4)| = \frac{3n-7}{2}$. Thus, by Lemma 5.4, we have $|N_{CF_n}(S)| = |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(x_3)| + |N_{CF_n^k}(x_4)| + |N_{CF_n - CF_n^i - CF_n^j - CF_n^k}(S)| \geq 3n-10+2\times\frac{3n-7}{2}+1 = 6n-16$. When n is even, by Lemma 3.3, we have $|N_{CF_n^i}(\{x_1,x_2\})| \geq 3n-9$. Clearly, $|N_{CF_n^j}(x_3)| = |N_{CF_n^k}(x_4)| = \frac{3n-6}{2}$. Thus, $|N_{CF_n}(S)| \geq |N_{CF_n^i}(\{x_1,x_2\})| + |N_{CF_n^j}(x_3)| + |N_{CF_n^k}(x_4)| = 3n-9+2\times\frac{3n-6}{2} = 6n-15 > 6n-18$.

Case 4: $S = \{x_1,x_2,x_3,x_4\}$ belong to four different subgraphs.

In this case, we can let $x_i \in V(CF_n^i)$. Clearly, when n is odd, $|N_{CF_n^i}(x_i)| = \frac{3n-7}{2}$; when n is even, $|N_{CF_n^i}(x_i)| = \frac{3n-6}{2}$. Thus when n is odd, $|N_{CF_n}(S)| \geq 4\times|N_{CF_n^i}(x_i)| = 4\times\frac{3n-7}{2} = 6n-14 > 6n-16$; when n is even, $|N_{CF_n}(S)| \geq 4\times|N_{CF_n^i}(x_i)| = 4\times\frac{3n-6}{2} = 6n-12 > 6n-18$.

Combing the above four cases, we know the result holds.

Lemma 5.8. For $n=5$, if F satisfies the condition $|F| \leq 13$, then $CF_5 - F$ contains a big component C with $|V(C)| \geq 5! - |F| - 3$.

Proof. In this Lemma, we do not think about the situation $CF_5 - F$ is connected, so we let $CF_5 - F$ is disconnected. Let $F_i = F \cap CF_n^i$ ($i \in [1,5]$). By Proposition 2.8, we know $\kappa(CF_5^i) = \frac{3(n-1)-4}{2} = \frac{3\times4-4}{2} = 4$. Since $|F| \leq 14$, we can get there exists at most three vertex set F_i, which can satisfies the condition $|F_i| \geq 4$. Now we think about the following situations:

Case 1: $|F_i| \leq 3$ for every $i \in [1,5]$.

By Proposition 2.8, we know $CF_5^i - F_i$ is connected for $i \in [1,5]$. Since $E_{i,j}(CF_5) = 2(n-2)! = 12 > 6 \geq |F_i| + |F_j|$, CF_5^i and CF_5^j are connected. Thus we can get $CF_5 - F$ is connected, this contradicts to the assumption $CF_5 - F$ is disconnected.

Case 2: There exists only one F_i, which can satisfies that $|F_i| \geq 4$.

In this case, we can let $|F_1| \geq 4$, then $|F_i| \leq 3$ for $i \in [2,5]$. Hence by Proposition 2.8, we can get $CF_5^i - F_i$ is connected ($i \in [2,5]$). Let $M = CF_5 - F - CF_5^1$, similarly to the discussion of Case 1, we know M is connected. Now we think about the following two situations:

Subcase 2.1: $|F - F_1| \leq 7$.

By the definition of CF_n and Proposition 2.3, we know every vertex has two outgoing neighbors, and these outgoing neighbors are different from each other. Thus, if $CF_5^1 - F_1$ is connected and $|CF_5^1 - F_1| \geq 4$, there must exists a vertex x_1 in $CF_5^1 - F_1$ such that it has a good neighbor in M. Thus $CF_5 - F$ is connected, this contradicts to the assumption $CF_5 - F$ is disconnected. If $CF_5^1 - F_1$ is connected and $|CF_5^1 - F_1| \leq 3$, the result is certainly true. If $CF_5^1 - F_1$ is disconnected, we can assume

C_1 is the vertex set in $CF_5^1 - F_1$, which has no good neighbors in M. As $|F - F_1| \leq 7$, we can get $|V(C_1)| \leq 3$, so the result holds.

Subcase 2.2: $|F - F_1| \geq 8$.

In this case, if $CF_5^1 - F_1$ is connected, similar to the case 1, we can get $CF_5 - F$ is connected. Now we assume $CF_5^1 - F_1$ is disconnected. Since $|F - F_1| \geq 8$, we have $|F_1| \leq |F| - 8 \leq 13 - 8 = 5$, by Lemma 1, we know $CF_5^1 - F_1$ has a big component C_1 and one singleton or two singletons. Since $|V(C_1)| \geq |V(CF_5^1 - F_1)| - 2 \geq 4! - 5 - 2 = 17 > 13 > |F_2 \cup F_3 \cup F_4 \cup F_5|$, C_1 is connected to M. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq n! - |F| - 2$.

Case 3: There exists two vertex set F_i, F_j, which satisfy that $|F_i| \geq 4$, $|F_j| \geq 4$.

In this case, we can let $|F_1| \geq |F_2| \geq 4$. Then $|F_i| \leq 3$ ($i \in [3,5]$), $|F_1| \leq |F| - |F_2| \leq 13 - 4 = 9$. Let $M = CF_5 - F - CF_5^1 - CF_5^2$, similarly, we can get M is connected. If $CF_5^i - F_i$ ($i \in \{1,2\}$) is connected, by the same argument with Subcase 2.2, we know it connected to M. Thus, if $CF_5^1 - F_1$ and $CF_5^2 - F_2$ are all connected, then $CF_5 - F$ is connected, this contradicts to the assumption that $CF_5 - F$ is disconnected. Hence at least one of $CF_5^i - F_i$ ($i \in \{1,2\}$) is disconnected. Now we think about the following three cases:

Subcase 3.1: $|F_2| = |F_1| = 4$.

By Corollary 3.4, we know if $CF_5^i - F_i$ ($i \in \{1,2\}$) is disconnected, then it has a big component C_i and one singleton. Similarly, we can get C_i is connected to M. Thus $CF_5 - F$ has a component C with $|V(C)| \geq 5! - |F| - 2$.

Subcase 3.2: $|F_1| = 5$.

In this case, we know $4 \leq |F_2| \leq |F_1| = 5$. By Lemma 1, we can get if $CF_5^1 - F_1$ is disconnected, then it has a big component C_1 and one singleton or two singletons. If $|F_2| = 4$, by Corollary 3.4, we can get $CF_5^2 - F_2$ has a big component C_2 and at most one singleton. Similarly, we can also get C_i is connected to M. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$, the result holds. If $|F_2| = 5$, then $|F - F_1 - F_2| \leq 13 - 5 - 5 = 3$. By Lemma 1, we know, if $CF_5^i - F_i$ ($i \in \{1,2\}$) is disconnected, it has a big component C_i and one singleton or two singletons. If one of $CF_5^i - F_i$ is connected or has only one singleton, then the result holds. Now we consider $CF_5^1 - F_1$ and $CF_5^2 - F_2$ are all disconnected and they all have two singletons, we let $\{x_1, x_2\} \subseteq V(CF_5^1 - F_1)$, $\{y_1, y_2\} \subseteq V(CF_5^2 - F_2)$. Similarly, we can know that C_i must connected to M, then $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 4$. Since $|F_1| = |F_2| = 5$ and x_1, x_2 (resp.,y_1, y_2) are two singletons, x_1, x_2 must have three common neighbors in CF_5^1 (resp.,y_1, y_2 must have three common neighbors in CF_5^2). If x_1, x_2, y_1, y_2 are singletons in $CF_5 - F$, then by Lemma 5.2, we can get $|N_{CF_5 - CF_5^1 - CF_5^2}(\{x_1, x_2, y_1, y_2\})| \geq 4$. Since $|F - F_1 - F_2| \leq 3$, we know at least one vertex in $\{x_1, x_2, y_1, y_2\}$ must connected to M. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 3$. If $(CF_5 - F)[\{x_1, x_2, y_1, y_2\}]$ has at least one edge, we assume $(x_2, y_1) \in E(CF_5 - F)$, then $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 3$. Suppose $|V(C)| = 5! - |F| - 4$, then x_1, y_2 are singletons in $CF_5 - F$ or $(x_1, y_2) \in E(CF_5 - F)$. If x_1, y_2 are singletons in $CF_5 - F$, then $|F| \geq |N_{CF_5}(\{x_2, y_1\}) \cup N_{CF_5}(x_1) \cup N_{CF_5}(y_2)| \geq 5 \times 2 + 6 \times 2 - 3 \times 2 = 16 > 13$, a contradiction. If $(x_1, y_2) \in E(CF_5 - F)$, then $|F| \geq |N_{CF_5}(\{x_2, y_1\}) \cup N_{CF_5}(\{x_1, y_2\}) \geq 5 \times 4 - 3 \times 2 = 14 > 13$, a contradiction. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$.

Subcase 3.3: $6 \leq |F_1| \leq 9$.

In this case, we can get $4 \leq |F_2| \leq |F| - |F_1| \leq 13 - 6 = 7$. If $|F_2| = 7$, then $|F_1| \geq 7$, $|F| \geq |F_1| + |F_2| \geq 14$, a contradiction. Thus $4 \leq |F_2| \leq 6$.

If $|F_2| = 4$ and $CF_5^2 - F_2$ is disconnected, then $CF_5^2 - F_2$ has a big component C_2 and one singleton x_4. Furthermore, we have $|F| - |F_1| - |F_2| \leq 13 - 6 - 4 = 3$. When $|F| - |F_1| - |F_2| \leq 2$, since every vertex in CF_5 has two outgoing neighbors, there are at most two vertices in $CF_5^1 - F_1$, which can satisfy that one of the two outgoing neighbors belong to F_2 and the other belongs to $F - F_1 - F_2$. Thus the result holds. When $|F| - |F_1| - |F_2| = 3$, $CF_5^1 - F_1$ has at most three vertices, which can satisfy that one of their outgoing neighbors belongs to F_2 and the other belongs to $F - F_1 - F_2$. We let they are x_1, x_2, x_3. If $CF_5^1 - F_1$ is connected or has at most two vertices, then the result holds. Now we consider there are

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three vertices in $CF_5^1 - F_1$, we can get $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 4$. If $|V(C)| = 5! - |F| - 4$, the structure in Figure 6 must exists. If there exists one edge between $\{x_1, x_2, x_3\}$, then there will be a 5-circle in CF_5, a contradiction. So x_1, x_2, x_3 are three singletons in $CF_5^1 - F_1$. If x_4 is adjacent to x_1, x_2, x_3, then there exists a 3-circle in CF_5. Thus x_1, x_2, x_3, x_4 are four singletons in $CF_5 - F$. By Lemma 5.3, we know $|N_{CF_5 - CF_5^1 - CF_5^2}(S)| \geq 4$, this contradicts to the fact $|F| - |F_1| - |F_2| = 3$, thus $|V(C)| \geq n! - |F| - 3$.

If $5 \leq |F_2| \leq 6$, then $|F| - |F_1| - |F_2| \leq 13 - 5 - 6 \leq 2$. When $|F| - |F_1| - |F_2| \leq 1$, $CF_5 - F$ has a big component and at most two vertices x_1, x_2, where $x_i \in V(CF_5^i - F_i)$ and they have common outgoing neighbor vertex in $F \setminus (F_1 \cup F_2)$ and the other outgoing neighbor vertex belong to CF_5^2 or CF_5^1, so the result holds. When $|F| - |F_1| - |F_2| = 2$, we can get $|F_1| = 6$, $|F_2| = 5$, and if $CF_5^i - F_i$ $(i \in \{1, 2\})$ is disconnected, then it has at most two vertices which has neighbors in $F - F_1 - F_2$. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 4$. Now, we proof $|V(C)| = 5! - |F| - 4$ can not exist. Suppose on the contrary, the structure in Figure 7 exists. Since $|F_2| = 5$, we can get x_1, x_2 are two singletons and have three common neighbors in CF_5^2; Otherwise, if x_1 is adjacent to x_2, then $|F_2| = 6$, a contradiction. So by Lemma 3.1, we let $x_1 = i_1 i_2 i_3 i_4 2$ and $x_2 = j_1 j_2 j_3 i_4 2$, where $j_i \in [i_1, i_3]$ and $j_1 \neq i_1$. Then $x_1^+ = 2 i_2 i_3 i_4 i_1$, $x_1^- = i_1 i_2 i_3 2 i_4$, $x_2^+ = 2 j_2 j_3 i_4 j_1$, $x_2^- = j_1 j_2 j_3 2 i_4$. Since one of the out neighbor vertices of x_1, x_2 belong to CF_5^1 and $j_1 \neq i_1$, $i_1 \neq i_4$, $j_1 \neq i_4$, we have $i_4 = 1$. Thus $x_1^+ = 2 i_2 i_3 1 i_1$, $x_2^+ = 2 j_2 j_3 1 j_1$. From the structure of Figure 7, we have $(x_1^+)^- = 2 i_2 i_3 i_1 1 = x_3$, $(x_2^+)^- = 2 j_2 j_3 j_1 1 = x_4$. Let $x_3' = 2 i_2 i_3 i_1$, $x_4' = 2 j_2 j_3 j_1$, then $\{x_3', x_4'\} \subseteq V(G_1)$, $G_1 \cong CF_4$. Since $j_1 \neq i_1$, x_4' and x_3' belong to different subgraph in G_1. As $(x_3')^+ = i_1 i_2 i_3 2$, $(x_4')^+ = j_1 j_2 j_3 2$, we know $(x_3')^+ \neq (x_4')^+$. Thus x_3 and x_4 have no common neighbors in CF_5^1. Clearly, we have x_3 is nonadjacent to x_4; Otherwise, there is a 7-circle in CF_5 (as shown in Figure 7 by read line). Thus, $|F_1| \geq 8$, this contradicts to the fact $|F_1| = 6$, this structure does not exist. So $CF_5 - F$ has a large component C with $|V(C)| \geq n! - |F| - 3$.

Case 4: There exists three vertex set F_i, F_j, F_k, which can satisfy that $|F_i| \geq 4$, $|F_j| \geq 4$ and $|F_k| \geq 4$ (i, j, k are different from each other).

In this case, we have $|F_i| \leq |F| - |F_j| - |F_k| \leq 13 - 4 - 4 = 5$. Similarly, we can get $|F_j| \leq 5$, $|F_k| \leq 5$. Let $M = CF_5 - CF_5^i - CF_5^j - CF_5^k$, we can get M is connected. If $|F_i| \leq 4$, $|F_j| \leq 4$ and $|F_k| \leq 4$, by Corollary 3.4, we know this result holds. If $|F_i| = 5$, $|F_j| \leq 4$ and $|F_k| \leq 4$, then by Lemma 1, there are at most two singletons in $CF_5^i - F_i$. Thus $CF_5 - F$ has a big component C with $|V(C)| \geq 5! - |F| - 4$. If $|V(C)| = n! - |F| - 4$, then $CF_5^i - F_i$ has two singletons $\{x_1, x_2\}$ and $CF_5^j - F_j$, $CF_5^k - F_k$ only has one singleton. Since $|F_i| = 5$, x_1 and x_2 have three common neighbors in CF_5^i. Since $|F| - |F_i| - |F_j| - |F_k| \leq 13 - 5 - 4 - 4 = 0$, we know x_1^+ belong to a common subgraph with x_2^+ or x_2^-. By the proof process of Lemma 5.4, we know this situation will not exist. So $|V(C)| \neq n! - |F| - 4$, $CF_5 - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$. If $|F_i| = 5$, $|F_j| = 5$, $|F_k| \leq 4$, then $|F| \geq |F_i| - |F_j| - |F_k| \geq 5 + 5 + 4 = 14$, a contradiction. If there are three F_i, F_j, F_k, such that $|F_i| = 5$, $|F_j| = 5$, $|F_k| = 5$, then $|F| \geq |F_i| - |F_j| - |F_k| \geq 5 + 5 + 5 = 15$, a contradiction.

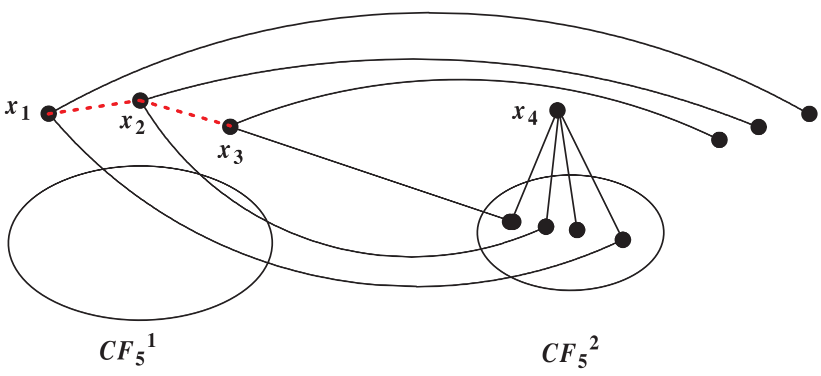

Figure 6. The illustration of Subcase 3.3 ($|F_2| = 4$).

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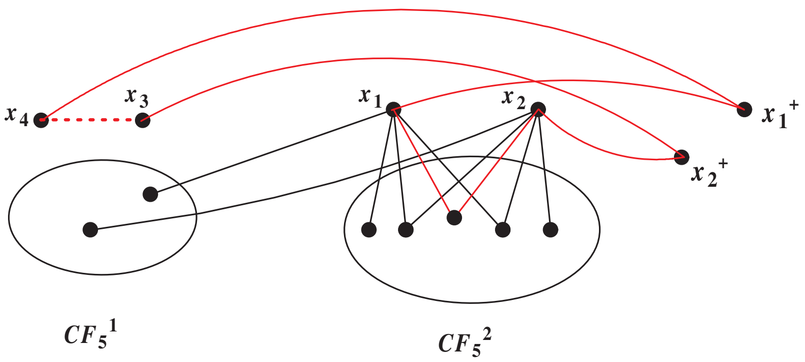

Figure 7. The illustration of Subcase 3.3 ($5 \leq |F_2| \leq 6$).

Lemma 5.9. For $n = 6$, if F satisfies the condition $|F| \leq 17$, then $CF_6 - F$ contains a big component C with $|V(C)| \geq 6! - |F| - 3$.

Proof. In this Lemma, we do not think about $CF_6 - F$ is connected, so we assume that $CF_6 - F$ is disconnected. Let $F_i = F \cap CF_6^i$, note that $CF_6^i \cong CF_5$. By Proposition 2.8, we have $\kappa(CF_6^i) = \frac{3n-6}{2} = 6$. Since $|F| \leq 17$, we can get there exists at most two vertex set F_i, which can satisfy $|F_i| \geq 6$. Next we think about the following three cases:

Case 1: $|F_i| \leq 5$ for every $i \in [1, 6]$.

In this case, we know $CF_6^i - F_i$ is connected. Since there are $(n-2)! = 4! = 24$ cross-edges in different CF_6^i and $24 > 5 + 5 = 10$, $CF_6 - F$ is connected, a contradiction.

Case 2: There exists only one vertex set F_i, which can satisfies the condition $|F_i| \geq 6$.

In this case, we know $|F_j| \leq 5$ $(j \in [6] \setminus \{i\})$. So $CF_6^j - F_j$ is connected. Let $M = CF_6 - F - CF_6^i$, similarly, we can get M is connected. When $CF_6^i - F_i$ is connected, since $24 > 17$, $CF_6 - F$ is connected, this contradicts to the assumption $CF_6 - F$ is disconnected. When $CF_6^i - F_i$ is disconnected, let S be the set of vertices in $CF_6^i - F_i$, which have no good neighbors in M. If $|F| - |F_i| \leq 3$, at most three vertices in $CF_6^i - F_i$ such that their out neighbor vertex belong to $F - F_i$. Thus $|V(S)| \leq 3$. If $|F| - |F_i| \geq 4$, then $|F_i| \leq |F| - 4 = 17 - 4 = 13$. By Lemma 5.8, we know the result holds.

Case 3: There exists two vertex set F_i, F_j, which can satisfy that $|F_i| \geq 6$ and $|F_j| \geq 6$.

In this case, we have $|F_i| \leq |F| - |F_j| \leq 17 - 6 = 11$. Similarly, we can get $|F_j| \leq 11$. By Lemma 4.7, we know $CF_6^i - F_i$ and $CF_6^j - F_j$ contain a big component C_k $(k \in \{i, j\})$ with $|V(C_k)| \geq 5! - |F_k| - 2$. If $6 \leq |F_i| \leq 8$, by Corollary 3.4, we have $CF_6^i - F_i$ has a big component C_i with $|V(C_i)| \geq 5! - |F_i| - 1$. Thus $CF_6 - F$ has a big component C with $|V(C)| \geq 6! - |F| - 3$. If $9 \leq |F_i| \leq 11$, then $|F_j| \leq |F| - |F_i| \leq 17 - 9 = 8$. By Corollary 3.4, we can also get $CF_6^j - F_j$ has a big component C_j with $|V(C_j)| \geq 5! - |F_j| - 1$. Thus $CF_6 - F$ has a big component C with $|V(C)| \geq 6! - |F| - 3$.

Lemma 5.10. For $n \geq 5$, if F satisfies that$|F| \leq 6n - 17$ (n is odd) and $|F| \leq 6n - 19$ (n is even), then $CF_n - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$.

Proof. In this Lemma, we only think about the case $CF_n - F$ is disconnected. Let $F_i = F \cap CF_n^i$, we proof this result by induction on n. By Lemma 5.8 and Lemma 5.9, we know this result holds for $n = 5, 6$. Now we assume $n \geq 7$ and the result holds for CF_{n-1}. Note that $CF_n^i \cong CF_{n-1}$. By Proposition 2.8, we know when n is odd, $\kappa(CF_n^i) = \frac{3n-7}{2}$; when n is even, $\kappa(CF_n^i) = \frac{3n-6}{2}$. Since $|F| \leq 6n - 17$ (n is odd) and $|F| \leq 6n - 19$ (n is even), we can get there exists at most three vertex set F_i, which can satisfy that $|F_i| \geq \frac{3n-7}{2}$ (n is odd) and $|F_i| \geq \frac{3n-6}{2}$ (n is even); Otherwise, for $n \geq 7$, when n is odd, $|F| \geq 4 \times \frac{3n-7}{2} = 6n - 14 > 6n - 17$; when n is even, $|F| \geq 4 \times \frac{3n-6}{2} = 6n - 12 > 6n - 19$. Next, we will think about the following situations:

Case 1. When n is odd, $|F_i| \leq \frac{3n-7}{2} - 1 = \frac{3n-9}{2}$ for every $i \in [1, n]$; when n is even, $|F_i| \leq \frac{3n-6}{2} - 1 = \frac{3n-8}{2}$ for every $i \in [1, n]$.

By Proposition 2.8, we can get $CF_n^i - F_i$ is connected for $i \in [1, n]$. By Proposition 2.1, we know when n is odd, there are $2(n-2)!$ cross-edges between CF_n^i and CF_n^j $(i \neq j)$; when n is even, there are $(n-2)!$ cross-edges between CF_n^i and CF_n^j $(i \neq j)$. Since $n \geq 7$, $2(n-2)! \geq 6n-17$ and $(n-2)! \geq 6n-19$, there are at least one cross-edge between $CF_n^i - F_i$ and $CF_n^j - F_j$. Thus $CF_n - F$ is connected, a contradiction.

Case 2. When n is odd, there exists only one vertex set F_i, which satisfies that $|F_i| \geq \frac{3n-7}{2}$; when n is even, there exists only one vertex set F_i, which satisfies that $|F_i| \geq \frac{3n-6}{2}$ $(i \in [1, n])$.

In this case, we can assume $|F_1| \geq \frac{3n-7}{2}$ (n is odd) and $|F_1| \geq \frac{3n-6}{2}$ (n is even). Then we can get $|F_j| \leq \frac{3n-9}{2}$ (n is odd) and $|F_j| \leq \frac{3n-8}{2}$ (n is even) for every $j \in [n] \setminus \{1\}$. By Proposition 2.8, we know $CF_n^j - F_j$ is connected. Let $M = CF_n - (F \cup CF_n^1)$, by the same argument with Case 1, we know M is connected. Now we think about the following two subcases:

Subcase 2.1: When n is odd, $|F - F_1| \leq 7$; when n is even, $|F - F_1| \leq 3$.

By the definition of CF_n, we know when n is odd, every vertex in CF_n has two outgoing neighbors; when n is even, every vertex in CF_n has only one outgoing neighbor. If $CF_n^1 - F_1$ is connected with $|CF_n^1 - F_1| \geq 4$, since $|F - F_1| \leq 7$ (n is odd) and $|F - F_1| \leq 3$ (n is even), we can get $CF_n^1 - F_1$ is connected to M. Thus we can get $CF_n - F$ is connected, this contradicts to the assumption $CF_n - F$ is disconnected. If $CF_n^1 - F_1$ is connected with $|CF_n^1 - F_1| \leq 3$, the conclusion is certainly true. If $CF_n^1 - F_1$ is disconnected, we can let S is the set of vertices in $CF_n^1 - F_1$ which has no good neighbors in M. Since $|F - F_1| \leq 7$ (n is odd) and $|F - F_1| \leq 3$ (n is even), we can get $|V(S)| \leq 3$, the result holds.

Subcase 2.2: When n is odd, $|F - F_1| \geq 8$; when n is even, $|F - F_1| \geq 4$.

In this case, since $|V(CF_n^1) - F_1| - |F - F_1| \geq (n-1)! - (6n-17) > 1$ (n is odd) and $|V(CF_n^1) - F_1| - |F - F_1| \geq (n-1)! - (6n-19) > 1$ (n is even) for $n \geq 7$, we know if $CF_n^1 - F_1$ is connected, then $CF_n - F$ is also connected, this contradicts to the assumption $CF_n - F$ is disconnected. Now we assume $CF_n^1 - F_1$ is disconnected. Since $|F - F_1| \geq 8$ (n is odd) and $|F - F_1| \geq 4$ (n is even), we can get when n is odd, $|F_1| \leq |F| - 8 \leq 6n - 17 - 8 = 6n - 25 = 6(n-1) - 19$; when n is even, $|F_1| \leq |F| - 4 \leq 6n - 19 - 4 = 6n - 23 = 6(n-1) - 17$. Thus by inductive hypothesis, we know $CF_n^1 - F_1$ has a big component C_1 with $|V(C_1)| \geq (n-1)! - |F_1| - 3$. Next, we will prove that C_1 is connected to M. Clearly, we can get $|F - F_1| \leq 6n - 17 - 8 = 6n - 25$ (n is odd) and $|F - F_1| \leq 6n - 19 - 4 = 6n - 23$ (n is even). As $2|V(C_1)| - (6n-25) \geq 2[(n-1)! - |F_1| - 3] - (6n-25) \geq 1$ (n is odd) and $|V(C_1)| - (6n-13) \geq (n-1)! - |F_1| - 3 - (6n-23) \geq 1$ (n is even) for $n \geq 7$, we know C_1 must has one vertex which must has a good neighbor in M. So the result holds.

Case 3. When n is odd, there exists two vertex set F_i, F_j, which can satisfies that $|F_i| \geq \frac{3n-7}{2}$ and $|F_j| \geq \frac{3n-7}{2}$; when n is even, there exists two vertex set F_i, F_j, which can satisfies that $|F_i| \geq \frac{3n-6}{2}$ and $|F_j| \geq \frac{3n-6}{2}$ $(i \neq j)$.

In this case, we know $|F_k| \leq \frac{3n-9}{2}$ (n is odd) and $|F_k| \leq \frac{3n-8}{2}$ (n is even), where $k \in [n] \setminus \{i, j\}$. By Proposition 2.8, we know $CF_n^k - F_k$ is connected. Let $M = CF_n - (F \cup V(CF_n^i) \cup V(CF_n^j))$, by the same argument with Case 1, we know M is connected.

Firstly, we think about n is odd. Now we can let $|F_1| \geq |F_2| \geq \frac{3n-7}{2}$, then as $|F| \geq |F_1| + |F_2|$, we can get $|F_1| \leq |F| - |F_2| \leq 6n - 17 - \frac{3n-7}{2} = \frac{9n-27}{2}$. Now, we think about the following three cases:

Case 3.1: $\frac{3n-7}{2} \leq |F_2| \leq |F_1| \leq 3n - 11$.

In this case, since $|F| - |F_1| - |F_2| \leq 6n - 17 - (3n-7) = 3n - 10$, $|V(CF_n^1) - F_1| \geq (n-1)! - (3n-11)$, $|V(CF_n^2) - F_2| \geq (n-1)! - (3n-11)$ and $(n-1)! - (3n-11) - (3n-10) > 1$, we can get if $CF_n^1 - F_1$ or $CF_n^2 - F_2$ is connected, then they are connected to M. Hence, if $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected, then $CF_n - F$ is connected, this contradicts to the assumption $CF_n - F$ is disconnected. Now we assume $CF_n^i - F_i$ $(i \in \{1, 2\})$ is disconnected. Since $CF_n^i \cong CF_{n-1}$ and $\frac{3n-7}{2} \leq |F_2| \leq |F_1| \leq 3n - 11 = 3(n-1) - 8$, by Corollary 3.4, we know $CF_n^i - F_i$ has a big component C_i with $|V(C_i)| \geq (n-1)! - |F_i| - 1$. Since $(n-1)! - (3n-11) - 1 - (3n-10) > 1$ for $n \geq 7$, we know C_i must connected to M. Thus $CF_n - F$ contains a big component C with $|V(C)| \geq n! - |F| - 2$.

Case 3.2: $3n - 10 \leq |F_1| \leq \frac{9n-35}{2}$.

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In this case, we can get $\frac{3n-7}{2} \le |F_2| \le |F - F_1| \le 6n - 17 - (3n - 10) = 3n - 7$. If $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected, by the similarly discussion with Case 3.1, we can get $CF_n - F$ is connected, this contradicts to the assumption $CF_n - F$ is disconnected. Thus at least one of $CF_n^1 - F_1$ and $CF_n^2 - F_2$ is disconnected. Without loss of generality, we let $CF_n^1 - F_1$ is disconnected. Since $|F_1| \le \frac{9n-35}{2} = \frac{9(n-1)-26}{2}$, by Lemma 4.9, we can get $CF_n^1 - F_1$ has a big component C_1 with $|V(C_1)| \ge (n-1)! - |F_1| - 2$. Similarly, we can get C_1 is connected to M. If $CF_n^2 - F_2$ is connected, we know $CF_n^2 - F_2$ is connected to M, thus $CF_n - F$ has a big component C with $|V(C)| \ge n! - |F| - 2$, the result holds. Now we consider $CF_n^2 - F_2$ is disconnected. If $|F_2| \le 3n - 11 = 3(n-1) - 8$, by Corollary 3.4, we know $CF_n^2 - F_2$ has a big component and a singleton, the conclusion is certainly true. Now we think about $3n - 10 \le |F_2| \le 3n - 7$, then $|F - F_1 - F_2| \le 6n - 17 - 2(3n - 10) = 3$. If $|F - F_1 - F_2| = 0$, we know the vertices in $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected to M, thus $CF_n - F$ is connected, a contradiction. If $|F - F_1 - F_2| = 1$, then $CF_n - F$ has a big component and at most two vertices u, v, where $u \in V(CF_n^1)$, $v \in V(CF_n^2)$ and they have a common outgoing neighbor in $F \setminus (F_1 \cup F_2)$ and the other outgoing neighbor of u (resp.,v) belong to F_2 (*resp.*, F_1).

If $|F - F_1 - F_2| = 3$, then $|F_1| = |F_2| = 3n - 10$. Since $|F_2| \le |F_1| \le \frac{9n-35}{2} = \frac{9(n-1)-26}{2}$, by Lemma 4.9, we can get $CF_n^1 - F_1$ and $CF_n^2 - F_2$ have a big component C_i with $|V(C_i)| \ge (n-1)! - |F_i| - 2$ ($i \in \{1,2\}$). If one of C_i satisfies $|V(C_i)| \ge (n-1)! - |F_i| - 1$, the conclusion is certainly true. Now we consider $|V(C_i)| = (n-1)! - |F_i| - 2$ for all $i \in \{1,2\}$, in another word, $CF_n^i - F_i$ has a big component and two vertices. Similarly, we can get C_i is connected to M. Since $|F_1| = |F_2| = 3n - 10$, we know $CF_n^i - F_i$ contains two singletons u_i, v_i and these two singletons have three common neighbors in CF_n^i; Otherwise, if u_i is adjacent to v_i, then $|F_i| \ge (\frac{3n-7}{2} - 1) \times 2 = 3n - 9 > 3n - 10$, a contradiction. Thus $CF_n - F$ has a big component C with $|V(C)| \ge n! - |F| - 4$. Let $S = \{u_1, u_2, v_1, v_2\}$. If S is an Ind-set, by Lemma 5.2, we have $|N_{CF_n - CF_n^1 - CF_n^2}(S)| \ge 4$. Since $|F - F_1 - F_2| = 3$, $CF_n - F$ can not has a big component and four vertices. Thus $CF_n - F$ has a big component C with $|V(C)| \ge n! - |F| - 3$. If there exists at least one edge in S, say $(v_1, u_2) \in E(CF_n - F)$, then $|V(C)| \ge n! - |F| - 3$. Suppose on the contrary, we let $|V(C)| = n! - |F| - 4$. If there exists only one edge between S and u_1, v_2 are singletons in $CF_n - F$, then $|F| \ge |N_{CF_n}(\{v_1, u_2\})| + |N_{CF_n}(u_1)| + |N_{CF_n}(v_2)| - 3 \times 2 = (\frac{3n-3}{2} - 1) \times 2 + \frac{3n-3}{2} \times 2 - 3 \times 2 = 6n - 14 > 6n - 17$, a contradiction. If there exists two edges between S in $CF_n - F$, then $(v_1, u_2) \in E(CF_n - F)$, $(v_2, u_1) \in E(CF_n - F)$. Thus $|F| \ge |N_{CF_n}(\{v_1, u_2\})| + |N_{CF_n}(\{v_2, u_1\})| - 3 \times 2 = (\frac{3n-3}{2} - 1) \times 4 - 3 \times 2 = 6n - 16 > 6n - 17$, a contradiction.

If $|F - F_1 - F_2| = 2$, then $|F_1| = 3n - 10$ or $|F_2| = 3n - 10$. We assume $|F_1| = 3n - 10$, then $|F_2| = 3n - 9$. By the same argument with the above discussion, we know if one of C_i satisfies $|V(C_i)| \ge (n-1)! - |F_i| - 1$, the result holds, so we consider $|V(C_i)| = (n-1)! - |F_i| - 2$ for all $i \in \{1,2\}$. Hence $CF_n - F$ has a big component C with $|V(C)| \ge n! - |F| - 4$. If we want $|V(C)| = n! - |F| - 4$, the structure in Figure 8 must exists. Now we proof this structure can not exist. Since $|F_1| = 3n - 10$, $CF_n^1 - F_1$ contains two singletons u_1, v_1 and this two singletons have three common neighbors in CF_n^1. Let $CF_n^2 - F_2$ has a big component and two vertices u_2, v_2, then u_2, v_2 are nonadjacent; Otherwise, there exists a 7-circle (as shown by the red line in Figure 8). By Lemma 3.1, we can let $u_1 = i_1 i_2 i_3 \cdots i_{n-2} i_{n-1} 1$, then $v_1 = j_1 j_2 j_3 \cdots j_{n-2} i_{n-1} 1$, where $j_i \in [i_1, i_{n-2}]$ and $j_1 \ne i_1$. So $u_1^+ = 1 i_2 i_3 \cdots i_{n-2} i_{n-1} i_1$, $u_1^- = i_1 i_2 i_3 \cdots i_{n-2} 1 i_{n-1}$, $v_1^+ = 1 j_2 j_3 \cdots j_{n-2} i_{n-1} j_1$, $v_1^- = j_1 j_2 j_3 \cdots j_{n-2} 1 i_{n-1}$. Since $j_1 \ne i_1$, $i_1 \ne i_{n-1}$ and $i_{n-1} \ne j_1$, we can get $i_{n-1} = 2$, then $u_1^-, v_1^- \in V(CF_n^2)$ and $u_2, v_2 \in V(CF_n^2)$. By the structure in Figure 8, we know $u_2 = 1 i_2 i_3 \cdots i_{n-2} i_1 2$, $v_2 = 1 j_2 j_3 \cdots j_{n-2} j_1 2$. Let $u_2' = 1 i_2 i_3 \cdots i_{n-2} i_1$, $v_2' = 1 j_2 j_3 \cdots j_{n-2} j_1$, then $\{u_2', v_2'\} \subseteq V(G_1)$, $G_1 \cong CF_{n-1}$ and u_2', v_2' belong to different subgraphs in G_1. Since $n-1$ is even and $(u_2')^+ = i_1 i_2 i_3 \cdots i_{n-2} 1$, $(v_2')^+ = j_1 j_2 j_3 \cdots j_{n-2} 1$, we know $(u_2')^+ \ne (v_2')^+$. Thus u_2' and v_2' have no common neighbors in G_1. In other words, u_2 and v_2 have no common neighbors in CF_n^2. So $|F_2| \ge 3n - 7$, this contradicts to the fact that $|F_2| = 3n - 9$. Thus this structure does not exist, $CF_n - F$ has a big component C with $|V(C)| \ge n! - |F| - 3$.

Case 3.3: $\frac{9n-33}{2} \le |F_1| \le \frac{9n-27}{2}$.

By the same argument with Case 3.2, we know if $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected, then $CF_n - F$ is connected, this contradicts to the assumption $CF_n - F$ is disconnected. Thus we assume

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$CF_n^2 - F_2$ is disconnected. In this case, we can get $|F - F_1 - F_2| \leq 6n - 17 - \frac{9n-33}{2} - \frac{3n-7}{2} = 3$, $\frac{3n-7}{2} \leq |F_2| \leq |F| - |F_1| \leq 6n - 17 - \frac{9n-33}{2} = \frac{3n-1}{2}$. Thus if $CF_n^2 - F_2$ is disconnected, then it has a big component and only one singleton; Otherwise, we assume $CF_n^2 - F_2$ has two vertices u_2, v_2. If u_2 is adjacent to v_2, then $|F_2| = 3n - 9 > \frac{3n-1}{2}$, a contradiction. If u_2 is nonadjacent to v_2, then $|F_2| \geq 3n - 10 > \frac{3n-1}{2}$, a contradiction. If $CF_n^1 - F_1$ is connected, the result is certainly true. Now, we assume $CF_n^1 - F_1$ is disconnected. If $|F - F_1 - F_2| \leq 1$, be the similarly discussion with Case 3.2, the conclusion is certainly true. If $|F - F_1 - F_2| = 2$, then $CF_n^1 - F_1$ contains a big component C_1 and at most two vertices, which have no neighbors in $F - F_1 - F_2$, the result holds.

When $|F - F_1 - F_2| = 3$, we have $|F_1| = \frac{9n-33}{2}$ and $|F_2| = \frac{3n-7}{2}$. Thus we can let H is the vertex set in $CF_n^1 - F_1$, which are not adjacent to M, as $|F - F_1 - F_2| = 3$, we can get $|V(H)| \leq 3$. Hence $CF_n - F$ contains a big component C with $|V(C)| \geq n! - |F| - 4$. Furthermore, $|V(C)| = n! - |F| - 4$ if and only if the structure in Figure 9 exists. Now we can proof this structure does not exist. Let $CF_n^1 - F_1$ has a big component and three vertices x_2, x_3, x_4 and $CF_n^2 - F_2$ has a big component and one singleton x_1. Firstly, we can get x_2, x_3, x_4 are three singletons in $CF_n^1 - F_1$; Otherwise, there exist a 5-circle in CF_n (as shown by the red line in Figure 9). Furthermore, we can get x_1 is nonadjacent to $\{x_2, x_3, x_4\}$; Otherwise, there exists a 3-circle in CF_n. Thus $\{x_1, x_2, x_3, x_4\}$ is an Ind-set in $CF_n - F$. We let $x_1 = i_1 i_2 i_3 \cdots i_{n-2} i_{n-1} 2$, then $x_1^+ \in V(CF_n^{i_1})$, $x_1^- \in V(CF_n^{i_{n-1}})$. Since $|F - F_1 - F_2| = 3$, x_1^+ or x_1^- must equal to one of the outgoing neighbors of $\{x_2, x_3, x_4\}$ in $F \setminus (F_1 \cup F_2)$. We assume $x_1^+ = x_3^-$, then $x_3 = 2 i_2 i_3 \cdots i_{n-2} i_1 i_{n-1}$. Thus $i_{n-1} = 1$, $x_1^- \in V(CF_n^1)$. The neighbors of x_1 in CF_n^2 are follows: $a_1 = i_2 i_1 i_3 \cdots i_{n-2} 12, \ldots,$ $a_{n-2} = 1 i_2 i_3 i_4 \cdots i_{n-2} i_1 2$, $b_{n-1} = i_1 i_3 i_3 \cdots i_{n-2} 12, \ldots, b_{\frac{3n-7}{2}} = i_1 i_2 i_3 \cdots i_{n-2} i_{n-3} 12$. In these vertices, we need to choose three vertices such that one of their outgoing neighbor belong to CF_n^1. Then x_2, x_3, x_4 must belong to A, where $A = \{a_1^-, \ldots, a_{n-3}^-, a_{n-2}^+, b_{n-1}^-, \ldots, b_{\frac{3n-7}{2}}^-\}$. Now, we choose a vertex in A such that one of it's outgoing neighbors equal to x_1^+, we assume this vertex is x_3, $x_3 = 2 i_2 i_3 i_4 \cdots i_{n-2} i_1 1$. As $CF_n^1 \cong CF_{n-1}$, x_2, x_3, x_4 are three singletons, and $|F_1| = \frac{9n-33}{2} = \frac{9(n-1)-24}{2}$, by Corollary 4.6, we can get x_2, x_3, x_4 must belong to a common subgraph of CF_n^1. From the choose of x_2, x_3, x_4, we know there have not one vertex which belongs to a common subgraph with x_3 in CF_n^1. Thus this structure does not exist, $CF_n - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$.

When n is even, we can assume $|F_1| \geq |F_2| \geq \frac{3n-6}{2}$. Then $|F_1| \leq |F| - |F_2| \leq 6n - 19 - \frac{3n-6}{2} = \frac{9n-32}{2}$, we think about the following cases:

Case 3.1: $\frac{3n-6}{2} \leq |F_2| \leq |F_1| \leq 3n - 10$.

In this case, since $|F| - |F_1| - |F_2| \leq 6n - 17 - (3n-6) = 3n - 11$, $|V(CF_n^1) - F_1| \geq (n-1)! - (3n-10)$, $|V(CF_n^2) - F_2| \geq (n-1)! - (3n-10)$ and $(n-1)! - (3n-10) - (3n-11) > 1$, we can get if $CF_n^1 - F_1$ or $CF_n^2 - F_2$ is connected, then it is connected to M. If $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected, then $CF_n - F$ is connected, this contradicts to the assumption $CF_n - F$ is disconnected. Thus we assume $CF_n^i - F_i$ $(i \in \{1,2\})$ is disconnected. Since $CF_n^i \cong CF_{n-1}$ and $\frac{3n-6}{2} \leq |F_2| \leq |F_1| \leq 3n - 10 = 3(n-1) - 7$, by Corollary 3.4, we know $CF_n^i - F_i$ $(i \in \{1,2\})$ contains a big component C_i, which satisfies that $|V(C_i)| \geq (n-1)! - |F_i| - 1$. Similarly to the above discussion, we know the big component of $CF_n^i - F_i$ is connected to M. Thus $CF_n - F$ contains a big component C, which can satisfies that $|V(C)| \geq n! - |F| - 2$.

Case 3.2: $3n - 9 \leq |F_1| \leq \frac{9n-32}{2}$.

In this case, we know $|F_2| \leq |F| - |F_1| \leq 6n - 19 - (3n-9) = 3n - 10 = 3(n-1) - 7$. By the same argument with Case 3.1, we know if $CF_n^1 - F_1$ and $CF_n^2 - F_2$ are all connected, then $CF_n - F$ is connected, a contradiction. Thus we assume $CF_n^2 - F_2$ is disconnected. By Corollary 3.4, we know $CF_n^2 - F_2$ contains a big component C_2, which can satisfies that $|V(C_2)| \geq (n-1)! - |F_2| - 1$. If $CF_n^1 - F_1$ is connected, the conclusion is certainly true. Now we think about $CF_n^1 - F_1$ is disconnected. Since $\leq |F_1| \leq \frac{9n-32}{2} = \frac{9(n-1)-23}{2}$, $CF_n^1 - F_1$ has a big component C_1 with $|V(C_1)| \geq (n-1)! - |F_1| - 2$. Thus the result holds.

Case 4. When n is odd, there exists three vertex set F_i, F_j, F_k, which can satisfy that $|F_i| \geq \frac{3n-7}{2}$, $|F_j| \geq \frac{3n-7}{2}$, $|F_k| \geq \frac{3n-7}{2}$; when n is even, there exists three vertex set F_i, F_j, F_k, which can satisfy that $|F_i| \geq \frac{3n-6}{2}$, $|F_j| \geq \frac{3n-6}{2}$, $|F_k| \geq \frac{3n-6}{2}$ (i, j, k are different from each other).

In this case, we let $M = CF_n - (F \cup CF_n^i \cup CF_n^j \cup CF_n^k)$.

When n is even, we have $|F_i| \leq |F| - |F_j| - |F_k| \leq 6n - 19 - 2 \times \frac{3n-6}{2} = 3n - 13 < 3n - 10$, by Corollary 3.4, we know this result holds.

When n is odd, we have $|F_i| \leq |F| - |F_j| - |F_k| \leq 6n - 17 - 2 \times \frac{3n-7}{2} = 3n - 10$. If $|F_i| \leq 3n - 11$, $|F_j| \leq 3n - 11$, $|F_k| \leq 3n - 11$, by Corollary 3.4, we know the result holds. If $|F_i| = 3n - 10$, $|F_j| \leq 3n - 11$ and $|F_k| \leq 3n - 11$. By Lemma 1, we know there exists one singleton or two singletons in $CF_n^i - F_i$. Thus $CF_n - F$ contains a big component C with $|V(C)| \geq n! - |F| - 4$. If $|V(C)| = n! - |F| - 4$, then $CF_n^i - F_i$ has two singletons x_1, x_2 and $CF_n^j - F_j$, $CF_n^k - F_5$ only has one singleton, meanwhile, these singletons are not connected to M, then $|F| - |F_i| - |F_j| - |F_k| \leq 6n - 17 - (3n - 10) - 2 \times \frac{3n-7}{2} = 0$. Since $|F_i| = 3n - 10$, by the proof process of Lemma 5.4, we know x_1^+ can not belong to a common subgraph with x_2^+ and x_2^-. Thus $|F| - |F_i| - |F_j| - |F_k| \geq 1$, this contradicts to the fact $|F| - |F_i| - |F_j| - |F_k| \leq 0$. Thus $|V(C)| \neq n! - |F| - 4$, $CF_n - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$. If $|F_i| = |F_j| = 3n - 10$, then $|F| \geq |F_i| + |F_j| + |F_k| = 2 \times (3n - 10) + \frac{3n-7}{2} = \frac{15n-47}{2} > 6n - 17$, a contradiction. If $|F_i| = |F_j| = |F_k| = 3n - 10$, then $|F| \geq |F_i| + |F_j| + |F_k| = 3 \times (3n - 10) = 9n - 30 > 6n - 17$, a contradiction.

Thus, for $n \geq 5$, if F satisfies the condition $|F| \leq 6n - 17$ (n is odd) and $|F| \leq 6n - 19$ (n is even), then $CF_n - F$ has a big component C with $|V(C)| \geq n! - |F| - 3$.

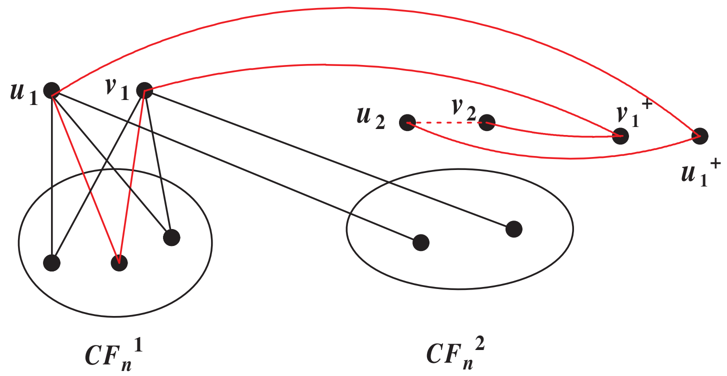

Figure 8. The case of $|V(M)| = n! - |F| - 4$ in Case 3.2.

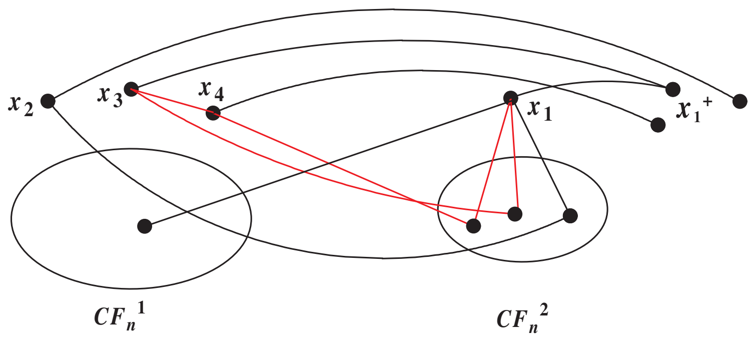

Figure 9. The case of $|V(M)| = n! - |F| - 4$ in Case 3.3.

Theorem 3. For $n \geq 5$, when n is odd, $c\kappa_5(CF_n) = 6n - 16$; when n is even, $c\kappa_5(CF_n) = 6n - 18$.

Proof. By Lemma 5.10, we can get when n is odd, $c\kappa_5(CF_n) \geq 6n - 16$; when n is even, $c\kappa_5(CF_n) \geq 6n - 18$. Next, we will proof $c\kappa_5(CF_n) \leq 6n - 16$ (n is odd) and $c\kappa_5(CF_n) \leq 6n - 18$

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(n is even). When n is odd, let $x_1 = i_1 i_2 i_3 \cdots i_{n-2} ji$, $x_2 = i_{n-2} i_2 i_3 \cdots i_1 i_{n-3} ji$, $x_3 = i_3 i_2 i_1 i_4 \cdots i_{n-2} ij$, $x_4 = i_{n-2} i_2 i_1 i_4 \cdots i_3 i_{n-3} ij$. Then $|N_{CF_n}(\{x_1, x_2, x_3, x_4\})| = 6n - 16$. Let $F = N_{CF_n}(\{x_1, x_2, x_3, x_4\})$, then $CF_n - F$ has a big component and four singletons x_1, x_2, x_3, x_4. Thus $c\kappa_5(CF_n) \le 6n - 16$. When n is even, we let $x_1 = i_1 i_2 i_3 \cdots i_{n-2} jik$, $x_2 = i_{n-2} i_2 i_3 \cdots i_1 i_{n-3} jik$, $x_3 = i_3 i_2 i_1 i_4 \cdots i_{n-2} ijk$, $x_4 = i_{n-2} i_2 i_1 i_4 \cdots i_3 i_{n-3} ijk$. Then $\{x_1, x_2, x_3, x_4\} \subseteq V(CF_n^k)$ and $|N_{CF_n^k}(\{x_1, x_2, x_3, x_4\})| = 6(n-1) - 16 = 6n - 22$. In CF_n^k, if we remove $N_{CF_n^k}(\{x_1, x_2, x_3, x_4\})$, we can get four singletons. From the definition of CF_n, we know any two vertices in CF_n^k have different outgoing neighbors. So, if we remove $N_{CF_n^k}(\{x_1, x_2, x_3, x_4\}) \cup \{x_1^+, x_2^+, x_3^+, x_4^+\}$ in CF_n, we can get a big component and four singletons x_1, x_2, x_3, x_4. Thus $c\kappa_5(CF_n) \le 6n - 22 + 4 = 6n - 18$.

6. Conclusions

It is very useful to study the connectivity of a graph. In this paper, we study the m-component connectivity of CF_n. The Leaf-sort graph is a special Cayley graph, it has many special properties. We have shown that for $n \ge 3$, $c\kappa_3(CF_n) = 3n - 6$ (n is odd) and $c\kappa_3(CF_n) = 3n - 7$ (n is even); for $n \ge 4$, $c\kappa_4(CF_n) = \frac{9n-21}{2}$ (n is odd) and $c\kappa_4(CF_n) = \frac{9n-24}{2}$ (n is even); for $n \ge 5$, $c\kappa_5(CF_n) = 6n - 16$ (n is odd) and $c\kappa_5(CF_n) = 6n - 18$ (n is even). So far, we only get the value of $c\kappa_3(CF_n)$, $c\kappa_4(CF_n)$ and $c\kappa_5(CF_n)$, for $m \ge 6$, this problem is still unsolved. So in the future, this problem is well worth studying. Furthermore, by referring to the references, we can find that: there is a regularity between $c\kappa_m(BS_n)$ and $\kappa^{(m)}(BS_n)$. Similarly, $c\kappa_m(BP_n)$ and $\kappa^{(m)}(BP_n)$ also have regularity. So we can think about, is there some sort of relationship between $c\kappa_m(CF_n)$ and $\kappa^{(m)}(CF_n)$?

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