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Article

A General Tensorial Formulation of Acoustoelasticity and Its Representation in Cylindrical Coordinates

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Abstract

Acoustoelasticity describes the relationship between elastic wave velocities and the initial stress present in media. Most of the current research in this area is focused on elastic waves propagating in naturally isotropic media with initially uniaxial stress and confined the solution of acoustoelastic problem to the Cartesian coordinate system, which limits the application for the acoustoelasticity theory. In this paper, to make up for this deficiency, an acoustoelastic formulation, i.e. the equation of motion for incremental displacement with explicit dependency on initial stress or strain, is developed based on the general theory of incremental deformation. This formulation can be applied to material with any symmetricity, initial stress with any number of principle axis. Also, it can be expressed in any form of coordinate system for its tensorial nature. Furthermore, to the knowledge of authors, there is no integrated cylindrical expression for acoustoelastic theory, an expansion of the formulation in a cylindrical system for an orthotropic material is given in detail for the convenience of application, such as cylindrical waves.

Keywords: acoustoelasticity theory; arbitrary coordinates; continuum mechanics; anisotropy media; tensor analysis

1. Introduction

The speeds of ultrasonic waves can be influenced by the initial stresses in the medium, which is known as acoustoelasticity. It is based on a continuum theory of small disturbances superimposed on an elastically deformed body, formulated by Cauchy in 1829 [1]. A major motivation for developing acoustoelastic formulations lies in the increasing demand for stress detection in engineering and geophysical applications [2]. Typical examples include stress measurement in load-bearing components, structural health monitoring in critical infrastructure, and in-situ earth stress evaluation for ensuring wellbore stability during directional drilling [3]. Another important application is the inverse problem—determining higher-order elastic moduli by applying known stress levels and measuring the resulting changes in wave velocity. These moduli serve as indicators of material degradation, such as metal fatigue [4].

Over the past decades, significant progress has been made in laying the theoretical foundation for acoustoelasticity. Murnaghan described the strain energy function as a third-order polynomial in strains, which led to a second-order constitutive relationship [5] for the theory. By adding terms involving third-order elastic moduli into the constitutive equations, the modem theory of acoustoelasticity was founded by Hughes and Kelly [6]. Toupin and Bernstein [7] considered hyperelastic materials and defined three states for ultrasonic waves in stressed media. Thurston and Brugger [8] derived expressions for the velocity and the stress derivatives in terms of Second Order Elastic Constants (SOECs) and Third Order Elastic Constants (TOECs). Pao and Sachse [9] proposed a distinct formula describing the relation between the velocity and the elastic properties of materials, which laid the basis for a convenient approach for experimental verification and engineering

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application [10][11]. Abiza [12] elucidated in general form the dependence of speeds of body waves on the initial stress, based on a finite deformation elasticity theory founded by Odgen [13] and Shams [14]. By this stage, the classical theory of acoustoelasticity has reached a prototype [15].

Nowadays, several studies have been conducted to further develop the acoustoelastic theory. Tang [16] proposed a novel methodology for investigating acoustoelasticity based on the relationship between second harmonics and acoustoelastic effect. Tian [17] gave an analytical case where the wave velocity has complex non-linear relation with the applied hydrostatic pressure. Kube [18] related velocity change to the initial stress only through the compliance constants by a stress formulation [19] of elastic equation. What's more, there are some studies related to the acoustoelastic effect in anisotropic media. Such as empirical study of velocity variation due to residual stress in an unidirectional composite material [20], evaluation on the effect of the anisotropy generated by rolling on the acoustoelastic effect for aluminum alloy [21], and measurement of velocity variations induced by anisotropy and internal stresses in carbon fiber reinforced plastic composite with acoustic coefficients determined by a series of calibration tests [22]. However, these researches are most experimental work and apply restrictedly to the specific circumstance. Castellano [23] did proposed a universal acoustoelastic relation regardless of the symmetric nature of wave-carrying material, based on an acoustoelastic model [24] developed in another nonlinear acoustics framework [25] pioneered by Biot [26][27] which is different from the one adopted by [6][7][8] and the followers [11][14]. However, as manifested itself in the paper [23], this technique requires immersion ultrasonic tests to achieve the desired resolution of velocity change, which limits its applications.

Moreover, almost all the acoustoelasticity related studies are expressed in the Cartesian coordinate system, which is enough for most application scenario. However, for certain applications [28] , surface wave are more suitable to be expressed in the cylindrical system [29]. Based on the acoustoelastic model [24], Hoger gave an qualitative analysis of residual stress distribution on an cylinder [30], using a set of equations expressed in the cylindrical system, with limitation on the constitutive relation mentioned in their study.

Motivated by the lack of a general, coordinate-independent acoustoelastic formulation applicable to anisotropic materials, the present work develops a theoretical model within the small-on-large framework, following the approach of Haupt [31]. The governing acoustoelastic equations are derived in a fully tensorial form, without assuming material symmetry or a specific coordinate system. To facilitate practical applications, these general equations are further expanded explicitly in the cylindrical coordinate system for orthotropic materials, which serves as a representative example for arbitrary curvilinear coordinates. This formulation provides a rigorous theoretical basis for stress evaluation in curved structures and lays the groundwork for future experimental validation and engineering implementation.

2. Kinematics

In the context of acoustoelastic theory, a solid body may be subject to a certain level of pre-existing stress—referred to as the initial stress—before the propagation of any small perturbation or wave. This initial stress is defined as the stress associated with a finite deformation that brings the material from its natural state to a pre-deformed or initial state. The presence of this initial stress fundamentally alters the material's response to subsequent incremental motions and must be appropriately accounted for in both the governing equations and constitutive modeling.

To accurately describe this layered deformation process—first from the natural to the initial configuration (finite deformation), and then from the initial to the final configuration (small perturbation)—a rigorous kinematic framework is required. This necessitates the introduction of multiple configurations (natural, initial, and final), as well as corresponding deformation gradients that map between them. Such a framework allows us to capture not only the geometric changes associated with finite pre-deformation but also the superimposed infinitesimal wave-induced motion.

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In this section, we systematically define the relevant configurations, deformation gradients, and associated kinematic measures such as the Green–Lagrange strain tensor, setting the foundation for the nonlinear acoustoelastic formulation developed in subsequent sections.

2.1. Definition of Configurations, Position Vectors and Coordinate Systems

Let $\mathscr{B}$ be a **continuum body** composed of a set of material points $\mathscr{P}$ (for precise definitions see [32]). A configuration of $\mathscr{B}$ is a one-to-one mapping that assigns each material point to a unique position in Euclidean space. As the body deforms over time, it moves through a continuous sequence of spatial regions denoted by Ω_t, for $t \in (0, \infty]$. The region Ω_t represents the **configuration** of $\mathscr{B}$ at time t.

A continuum body can undergo infinitely many configurations. In this study, we focus on three particular configurations [7], (see Figure 1):

- **Natural configuration** $\mathscr{R}$: The stress-free, undeformed state.
- **Initial configuration** $\mathscr{F}$: A statically deformed state with finite pre-existing stress or strain, which is the reference for incremental analysis.
- **Final configuration** $\mathscr{f}$: The result of applying a dynamic perturbation, which is infinitesimal, to the pre-deformed body. It is time-dependent by definition.

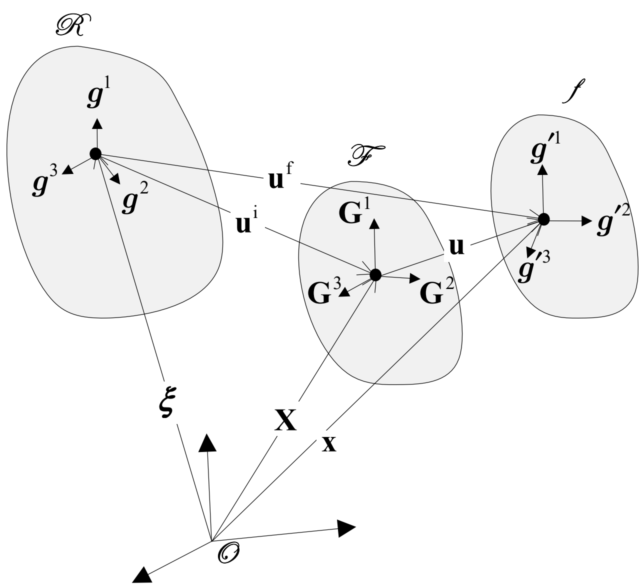

Figure 1. Natural, initial, and final configurations of a predeformed body.

The position vector of a material point in each configuration is denoted by:

- ξ in the natural configuration
- $\mathbf{X}$ in the initial configuration
- $\mathbf{x}$ in the final configuration

All position vectors are measured from the same origin in a common coordinate system. Component notation is as follows:

Greek subscripts (e.g., ξ_α) refer to the natural configuration; uppercase Roman subscripts (e.g., X_I) refer to the initial configuration; lowercase Roman subscripts (e.g., x_i) refer to the final configuration.

Thus, the configurations are described via the following mappings:

$$\xi = \mathscr{R}(\mathscr{P}) \tag{2.1a}$$

$$\mathbf{X} = \mathscr{F}(\mathscr{P}) \tag{2.1b}$$

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$$\mathbf{x} = \mathscr{f}(\mathscr{P}, t) \tag{2.1c}$$

Since the material point $\mathscr{P}$ can be equivalently represented by $\mathbf{X}$ or ξ, the following functional dependencies arise:

$$\mathbf{X} = \hat{\mathbf{X}}(\xi) \tag{2.2a}$$

$$\mathbf{x} = \hat{\chi}(\xi, t) \tag{2.2b}$$

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t) \tag{2.2c}$$

2.2. Definitions of Displacements

Displacement vectors quantify the relative motion of material points between configurations. The incremental displacement vector, referred to the initial configuration, is defined as:

$$\mathbf{U} = \hat{\mathbf{x}}(\mathbf{X}, t) - \mathbf{X} = U^I \mathbf{G}_I = U_I \mathbf{G}^I \tag{2.3a}$$

The initial displacement vector and final displacement vector, both referred to the natural configuration, are:

$$\mathbf{u}^{\mathrm{i}} = \hat{\mathbf{X}}(\xi) - \xi = u^{\mathrm{i}\alpha} \boldsymbol{g}_\alpha = u^{\mathrm{i}}{}_\alpha \boldsymbol{g}^\alpha \tag{2.3b}$$

$$\mathbf{u}^{\mathrm{f}} = \hat{\chi}(\xi, t) - \xi = u^{\mathrm{f}\alpha} \boldsymbol{g}_\alpha = u^{\mathrm{f}}{}_\alpha \boldsymbol{g}^\alpha \tag{2.3c}$$

Hereafter, superscripts "i" and "f" refer to variables in the initial and final configurations, respectively.

In Eq. (2.3a-c), $\mathbf{G}_I$ is the set of covariant bases in initial system and $\mathbf{G}^I$ is the set of contravariant bases in initial system. Their definition and properties can be found in any textbook of continuum mechanics such as [13] [33]. And $\boldsymbol{g}_\alpha$ and $\boldsymbol{g}^\alpha$ are the corresponding set of covariant and contravariant basis of the natural system respectively.

Another essential quantity is the incremental displacement vector in the natural frame, defined as:

$$\mathbf{u} = \mathbf{u}^{\mathrm{f}} - \mathbf{u}^{\mathrm{i}} = u^\alpha \boldsymbol{g}_\alpha = u_\alpha \boldsymbol{g}^\alpha \tag{2.3d}$$

It should be emphasized that lowercase letter $\mathbf{u}$ is used to denote the incremental displacement expressed in the Lagrangian description with the natural configuration as the reference, whereas uppercase letter $\mathbf{U}$ denote, strictly speaking, the incremental displacement formulated in the Lagrangian description with the initial configuration as the reference. However, since the incremental deformation is assumed to be infinitesimal, the difference between the initial and final configurations can be neglected at this level of approximation. Under this assumption, the incremental displacement $\mathbf{U}$ may equivalently be interpreted in an Eulerian description with the initial configuration serving as the current configuration.

With this simplification, the natural configuration is consistently taken as the reference configuration for the Lagrangian description, while the initial configuration is treated as the current configuration for the Eulerian description. Similar situations arise later in the paper, where different symbols are used to represent the same physical quantity expressed in different descriptions (Lagrangian or Eulerian). Such distinctions will be explicitly indicated whenever necessary.

2.3. Deformation Gradients and Displacement Gradients

Deformation gradients describe local changes in configuration. Three relevant deformation gradients are:

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$$\mathbf{F}^{\mathrm{f}}(\mathbf{x},\xi)=\hat{\boldsymbol{\chi}}(\xi,t)\nabla_{\xi}=\chi^{i}{}_{,\alpha}\boldsymbol{g}_{i}'\otimes\boldsymbol{g}^{\alpha} \tag{2.4a}$$

$$\mathbf{F}^{\mathrm{i}}(\mathbf{X},\xi)=\hat{\mathbf{X}}(\xi)\nabla_{\xi}=X^{I}{}_{\alpha}\mathbf{G}_{I}\otimes\boldsymbol{g}^{\alpha} \tag{2.4b}$$

$$\mathbf{F}(\mathbf{x},\mathbf{X})=\hat{\mathbf{x}}(\mathbf{X},t)\nabla_{\mathbf{X}}=x^{i}{}_{,I}\boldsymbol{g}_{i}'\otimes\mathbf{G}^{I} \tag{2.4c}$$

where $\mathbf{F}^{\mathrm{f}}$ is the final deformation gradient, $\mathbf{F}^{\mathrm{i}}$ the initial deformation gradient and $\mathbf{F}$ the incremental deformation gradient. These are two-point tensors [34] that map between different configurations. The operator "$\otimes$" denotes the dyadic product [13].

Throughout this paper, the gradient of a field is consistently defined using the right-gradient convention, i.e. the Nabla operator ∇ acts on a field by right multiplication. Specifically, for a vector field $\mathbf{v}$, its gradient is defined as the second-order tensor

$$\operatorname{grad}(\mathbf{v})=\mathbf{v}\nabla$$

With this convention, all deformation gradients and displacement gradients appearing in the sequel follow the same right-multiplication rule, and transposition is explicitly indicated whenever required.

Using the identity (obtained from Eq. (2.2b) and (2.2c)):

$$\hat{\boldsymbol{\chi}}(\xi,t)=\hat{\mathbf{x}}\left(\hat{\mathbf{X}}(\xi),t\right)$$

and differentiating it with respect to ξ, there is the multiplicative decomposition of the final deformation gradient:

$$\mathbf{F}\mathbf{F}^{\mathrm{i}}=\left(x^{i}{}_{,I}\boldsymbol{g}_{i}'\mathbf{G}^{I}\right)\cdot\left(X^{I}{}_{\alpha}\mathbf{G}_{I}\boldsymbol{g}^{\alpha}\right)=\chi^{i}{}_{,\alpha}\boldsymbol{g}_{i}'\otimes\boldsymbol{g}^{\alpha}=\mathbf{F}^{\mathrm{f}}$$

i.e.

$$\mathbf{F}^{\mathrm{f}}=\mathbf{F}\mathbf{F}^{\mathrm{i}} \tag{2.5}$$

The displacement gradients can be also defined as:

$$\mathbf{H}^{\mathrm{f}}=\mathbf{u}^{\mathrm{f}}\nabla_{\xi}=\nabla_{\beta}u^{\mathrm{f}}{}_{\alpha}\boldsymbol{g}^{\alpha}\otimes\boldsymbol{g}^{\beta} \tag{2.6a}$$

$$\mathbf{H}^{\mathrm{i}}=\mathbf{u}^{\mathrm{i}}\nabla_{\xi}=\nabla_{\beta}u^{\mathrm{i}}{}_{\alpha}\boldsymbol{g}^{\alpha}\otimes\boldsymbol{g}^{\beta} \tag{2.6b}$$

$$\mathbf{H}=\mathbf{U}\nabla_{\mathbf{X}}=\nabla_{J}U_{I}\mathbf{G}^{I}\otimes\mathbf{G}^{J} \tag{2.6c}$$

where $\mathbf{H}^{\mathrm{f}}$ is the final displacement gradient and $\mathbf{H}^{\mathrm{i}}$ is the initial displacement gradient, they are variables of Lagrangian description with natural configuration as reference. And $\mathbf{H}$ denotes the incremental displacement gradient. For convenience in the subsequent analysis, it is defined in the Eulerian description, with the initial configuration treated as the current configuration.

From the definition of $\mathbf{U}$ (Eq.(2.3a)) and $\mathbf{F}$ (Eq. (2.4c)), the identity below can then be deduced [13]:

$$\mathbf{F}=\mathbf{H}+\mathbf{I}=\mathbf{U}\nabla_{\mathbf{X}}+\mathbf{I} \tag{2.7a}$$

Similarly, there are:

$$\mathbf{F}^{\mathrm{i}}=\mathbf{u}^{\mathrm{i}}\nabla_{\xi}+\mathbf{I} \tag{2.7b}$$

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$$\begin{aligned}\mathbf{F}^{\mathrm{f}} &= \mathbf{u}^{\mathrm{f}}\nabla_{\xi} + \mathbf{I} \\ &= \mathbf{u}^{\mathrm{i}}\nabla_{\xi} + \mathbf{u}\nabla_{\xi} + \mathbf{I}\end{aligned} \tag{2.7c}$$

From the relations above, it can be seen that:

$$\begin{aligned}\mathbf{H}\mathbf{F}^{\mathrm{i}} &= (\mathbf{F}-\mathbf{I})\mathbf{F}^{\mathrm{i}} = \mathbf{F}^{\mathrm{f}} - \mathbf{F}^{\mathrm{i}} \\ &= \mathbf{H}^{\mathrm{f}} - \mathbf{H}^{\mathrm{i}} = \nabla_{\alpha} u_{\beta} \boldsymbol{g}^{\beta} \otimes \boldsymbol{g}^{\alpha} \\ &= \mathbf{u}\nabla_{\xi}\end{aligned}$$

Taking the transpose on both sides of the equation above,

$$\mathbf{F}^{\mathrm{iT}}\nabla_{\mathbf{X}}\mathbf{U} = \nabla_{\xi}\mathbf{u} \tag{2.8}$$

where the additional superscript "T" on a tensor denotes the transpose of the (second-order) tensor.

Thus, with Eq. (2.7b), Eq. (2.8) is equivalent to

$$\begin{aligned}\nabla_{\xi}\mathbf{u} &= \left(\nabla_{\xi}\mathbf{u}^{\mathrm{i}} + \mathbf{I}\right)\nabla_{\mathbf{X}}\mathbf{U} \\ &= \nabla_{\mathbf{X}}\mathbf{U} + \nabla_{\xi}\mathbf{u}^{\mathrm{i}}\nabla_{\mathbf{X}}\mathbf{U}\end{aligned}$$

In component form, it can be expressed as:

$$\nabla_{\alpha} u_{\beta} = \nabla_{K} U_{I} g^{K}{}_{\alpha} g^{I}{}_{\beta} + \nabla_{\alpha} u^{\mathrm{i}}{}_{\lambda} g^{\lambda K} \nabla_{K} U_{I} g^{I}{}_{\beta} \tag{2.9}$$

We may also define the initial displacement gradient in the initial configuration:

$$.\ \mathbf{U}^{\mathrm{i}}\nabla_{\mathbf{X}} = \mathbf{u}^{\mathrm{i}}\nabla_{\xi}\left(\mathbf{F}^{\mathrm{i}}\right)^{-1} = \nabla_{J} U^{\mathrm{i}}{}_{I} \mathbf{G}^{I} \otimes \mathbf{G}^{J}\ . \tag{2.10}$$

where $\left(\mathbf{F}^{\mathrm{i}}\right)^{-1}$ is the inverse tensor of $\mathbf{F}^{\mathrm{i}}$. Following a similar reasoning path to arrive at Eq. (2.8), it can be obtained that:

$$\nabla_{\alpha} u^{\mathrm{i}}{}_{\beta} = \nabla_{K} U^{\mathrm{i}}{}_{I} g^{K}{}_{\alpha} g^{I}{}_{\beta} + \nabla_{\alpha} u^{\mathrm{i}}{}_{\lambda} g^{\lambda K} \nabla_{K} U^{\mathrm{i}}{}_{I} g^{I}{}_{\beta} \tag{2.11}$$

2.4. Strain Tensors

In order to analyze the effect of initial strain on the propagation of small perturbations, it is essential to adopt a strain measure that can accurately capture finite background deformations. The Green-Lagrange strain tensor (Green strain tensor) is introduced in this context, as it inherently includes nonlinear terms of the displacement gradient, making it suitable for describing finite initial deformations.

While the final formulation in this study focuses on the linear response of small disturbances superimposed on a pre-strained body, the use of a nonlinear strain measure is indispensable. It ensures that the linearized equations retain the correct dependence on the initial strain, which is critical for capturing the first-order acoustoelastic effects.

For the final configuration, the Green strain tensor is defined as:

$$\mathbf{E}^{\mathrm{f}} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{fT}}\mathbf{F}^{\mathrm{f}} - \mathbf{I}\right) \tag{2.12a}$$

For the initial configuration:

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$$\mathbf{E}^{\mathrm{i}}=\frac{1}{2}\left(\mathbf{F}^{\mathrm{iT}}\mathbf{F}^{\mathrm{i}}-\mathbf{I}\right) \tag{2.12b}$$

Substituting Eq. (2.7b) and (2.7c) into the definition of strain leads to a decomposition involving the displacement gradients:

$$\mathbf{E}^{\mathrm{f}}=\frac{1}{2}\left(\mathbf{H}^{\mathrm{f}}+\mathbf{H}^{\mathrm{fT}}+\mathbf{H}^{\mathrm{f}}\mathbf{H}^{\mathrm{fT}}\right) \tag{2.12c}$$

$$\mathbf{E}^{\mathrm{i}}=\frac{1}{2}\left(\mathbf{H}^{\mathrm{i}}+\mathbf{H}^{\mathrm{iT}}+\mathbf{H}^{\mathrm{i}}\mathbf{H}^{\mathrm{iT}}\right) \tag{2.12d}$$

The incremental strain tensor represented in the initial system (Eulerian description) is defined as:

$$\begin{aligned}\mathbf{E}&=\frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}\right)\\&=\frac{1}{2}\left(\mathbf{H}+\mathbf{H}^{\mathrm{T}}+\mathbf{H}\mathbf{H}^{\mathrm{T}}\right)\end{aligned} \tag{2.13a}$$

Whereas the corresponding incremental strain tensor represented in the natural system (Lagrangian description) is:

$$\bar{\mathbf{E}}=\mathbf{E}^{\mathrm{f}}-\mathbf{E}^{\mathrm{i}} \tag{2.13b}$$

According to Eq. (2.8), they are related by:

$$\bar{\mathbf{E}}=\mathbf{F}^{\mathrm{iT}}\mathbf{E}\mathbf{F}^{\mathrm{i}} \tag{2.14}$$

All relations presented in this section are exact, as no assumptions regarding the magnitude of the incremental deformation have been made thus far. It is important to emphasize that quantities such $\mathbf{E}$ and $\bar{\mathbf{E}}$ (for example) represent the same physical entity, merely expressed in different configurations. In other words, while their mathematical representations differ due to the coordinate systems associated with each configuration, their physical significance remains identical.

3. Deduction of Acoustoelastic Theory in General Form

Following the formulation in [31], the equation of motion for the final configuration expressed in the natural coordinate system (i.e. Lagrangian description of final motion with the natural configuration as reference configuration) is given by:

$$\rho^{0}\frac{\partial^{2}\mathbf{u}}{\partial t^{2}}=\nabla_{\boldsymbol{\xi}}\cdot\mathbf{P}^{\mathrm{f}} \tag{3.1a}$$

where ρ^0 denotes the mass density in the natural configuration, $\mathbf{u}$ is the displacement field relative to the natural state, $\boldsymbol{\xi}$ is the material coordinate, and $\mathbf{P}^{\mathrm{f}}$ is the first Piola–Kirchhoff stress tensor corresponding to the final state.

Alternatively, when expressed by the initial coordinate system (i.e. Lagrangian description of final motion with the initial configuration as reference configuration), the corresponding equation of motion reads:

$$\rho^{\mathrm{i}}\frac{\partial^{2}\mathbf{U}}{\partial t^{2}}=\nabla_{\mathbf{X}}\cdot\mathbf{p}^{\mathrm{f}} \tag{3.1b}$$

where ρ^{i} is the density in the initial configuration, $\mathbf{U}$ is the displacement relative to the initial state, $\mathbf{X}$ is the spatial coordinate in the initial frame, and $\mathbf{p}^{\mathrm{f}}$ is the corresponding Piola–Kirchhoff stress tensor defined with respect to the initial configuration.

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In parallel, the static equilibrium (or quasi-static approximation) of the initial deformation state can be described by the following equations:

$$\nabla_{\xi} \cdot \mathbf{P}^{\mathrm{i}} = 0 \tag{3.2a}$$

$$\nabla_{\mathbf{X}} \cdot \boldsymbol{\sigma}^{\mathrm{i}} = 0 \tag{3.2b}$$

where $\boldsymbol{\sigma}^{\mathrm{i}}$ denotes the Cauchy stress tensor in the initial configuration, and $\mathbf{P}^{\mathrm{i}}$ is the associated first Piola–Kirchhoff stress tensor in the natural frame. In (3.2a) Eulerian description is adopted with the initial configuration as current configuration, whereas in (3.2b) Lagrangian description is adopted with the natural configuration as reference configuration.

3.1. Analysis of Stress Tensors and Constitutive Law

3.1.1. Substitution of Stress Tensors

In Eq. (3.2a), the stress tensor $\boldsymbol{\sigma}^{\mathrm{i}}$ represents the conventional Cauchy stress in the initial configuration. In contrast, the tensors $\mathbf{P}^{\mathrm{f}}$, $\mathbf{P}^{\mathrm{i}}$, $\mathbf{p}^{\mathrm{f}}$ in Eqs. (3.1) and (3.2b) are nominal (Piola-Kirchhoff) stress tensors, whose definitions and properties are detailed in [35]. These are two-point tensors, as their definitions rely on area vectors and force vectors referred to different configurations.

These nominal stress tensors are related to the Kirchhoff stress tensors $\mathbf{T}^{\mathrm{f}}$ $\mathbf{T}^{\mathrm{i}}$ and $\boldsymbol{\tau}^{\mathrm{f}}$, which are defined with respect to a single configuration, via the following relations [31]:

$$\mathbf{P}^{\mathrm{f}}(\xi, \mathbf{x}) = \mathbf{T}^{\mathrm{f}}\mathbf{F}^{\mathrm{fT}} \tag{3.3a}$$

$$\mathbf{P}^{\mathrm{i}}(\xi, \mathbf{X}) = \mathbf{T}^{\mathrm{i}}\mathbf{F}^{\mathrm{iT}} \tag{3.3b}$$

$$\mathbf{p}^{\mathrm{f}}(\mathbf{X}, \mathbf{x}) = \boldsymbol{\tau}^{\mathrm{f}}\mathbf{F}^{\mathrm{T}} \tag{3.3c}$$

Here, the notation $\mathbf{P}^{\mathrm{f}}(\xi, \mathbf{x})$ indicates that the surface area vector is measured in the natural configuration $\mathscr{R}$, while the force vector is evaluated in the final configuration $\mathscr{f}$. Similar interpretations hold for $\mathbf{P}^{\mathrm{i}}(\xi, \mathbf{X})$ and $\mathbf{p}^{\mathrm{f}}(\mathbf{X}, \mathbf{x})$. Accordingly, $\mathbf{T}^{\mathrm{f}}$ and $\mathbf{T}^{\mathrm{i}}$ are represented in the natural configuration, while $\boldsymbol{\tau}^{\mathrm{f}}$ is represented in the initial configuration.

Eq. (3.3a) can be rewritten as:

$$\begin{aligned}\mathbf{P}^{\mathrm{f}} &= \mathbf{T}^{\mathrm{f}}\mathbf{F}^{\mathrm{fT}} = \left(\mathbf{F}^{\mathrm{f}}\mathbf{T}^{\mathrm{f}}\right)^{\mathrm{T}} \\ &= \left[\mathbf{F}^{\mathrm{f}}\left(\mathbf{T}^{\mathrm{f}} - \mathbf{T}^{\mathrm{i}}\right) + \mathbf{F}^{\mathrm{f}}\mathbf{T}^{\mathrm{i}}\right]^{\mathrm{T}}\end{aligned}$$

Using Eq. (2.5) and Eq. (3.3b), this becomes:

$$\begin{aligned}\mathbf{P}^{\mathrm{f}} &= \left[\mathbf{F}^{\mathrm{f}}\left(\mathbf{T}^{\mathrm{f}} - \mathbf{T}^{\mathrm{i}}\right)\right]^{\mathrm{T}} + \mathbf{P}^{\mathrm{i}}\mathbf{F}^{\mathrm{T}} \\ &= \mathbf{T}\mathbf{F}^{\mathrm{fT}} + \mathbf{P}^{\mathrm{i}} + \mathbf{P}^{\mathrm{i}}\mathbf{H}^{\mathrm{T}}\end{aligned} \tag{3.4}$$

where $\mathbf{T}$ is the incremental Kirchhoff stress tensor in the natural configuration, defined by:

$$\mathbf{T} = \mathbf{T}^{\mathrm{f}} - \mathbf{T}^{\mathrm{i}} \tag{3.5}$$

Similarly, the incremental Kirchhoff stress in the initial configuration is given by:

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{f}} - \boldsymbol{\sigma}^{\mathrm{i}} \tag{3.6}$$

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Since both $\mathbf{T}$ and $\boldsymbol{\tau}$ represent the same physical quantity expressed in different configurations, i.e. $\mathbf{T}$ is of Lagrangian description with natural configuration as reference and $\boldsymbol{\tau}$ is of Lagrangian description with initial configuration as reference, they are related via:

$$\boldsymbol{\tau} = \frac{1}{\det \mathbf{F}^{\mathrm{i}}} \mathbf{F}^{\mathrm{i}} \mathbf{T} \mathbf{F}^{\mathrm{iT}} \tag{3.7}$$

Subtracting Eq. (3.2a) from Eq. (3.1a), and substituting Eq.(3.3b) and Eq. (3.4), yields the incremental motion equation in the natural system:

$$\rho^0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla_{\xi} \cdot \left(\mathbf{T}\mathbf{F}^{\mathrm{fT}} + \mathbf{T}^{\mathrm{i}} \mathbf{F}^{\mathrm{iT}} \mathbf{H}^{\mathrm{T}} \right) \tag{3.8}$$

Likewise, subtracting Eq. (3.2b) from Eq. (3.1b) and substituting Eqs. (3.6) and (3.7), we obtain the incremental motion equation in the initial system:

$$\rho^{\mathrm{i}} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla_{\mathbf{x}} \cdot \left(\frac{1}{J^{\mathrm{i}}} \mathbf{F}^{\mathrm{i}} \mathbf{T} \mathbf{F}^{\mathrm{iT}} + \boldsymbol{\tau}^{\mathrm{f}} \mathbf{H}^{\mathrm{T}} \right) \tag{3.9}$$

where $J^{\mathrm{i}} \equiv \det \mathbf{F}^{\mathrm{i}}$. From their deductive process it is reasonable to view Eq. (3.8) as the equation of incremental motion with Lagrangian description of initial deformation and Eq. (3.9) as the equation of incremental motion with Eulerian description of initial deformation.

In contrast to the classical elasticity equation Eq. (3.10), which is formulated in a single reference configuration and assumes small deformations throughout, both Eqs. (3.8) and (3.9) are derived under a multi-configuration framework that accounts for a pre-existing finite deformation. Specifically, Eq. (3.10) describes the motion of a body in its natural (unstressed) configuration as:

$$\rho^0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla_{\xi} \cdot \mathbf{T} \tag{3.10}$$

In this formulation, no effect of the initial stress or strain is accounted for, and thus the influence of pre-deformation on the incremental displacement $\mathbf{u}$ is entirely neglected.

By contrast, Eqs. (3.8) and (3.9) incorporate additional terms arising from the mapping between multiple configurations (natural, initial, and final), reflecting the geometrical nonlinearity induced by large initial deformations. These equations describe the evolution of small perturbations superimposed on a finitely deformed body, with the deformation gradient tensors $\mathbf{F}^{\mathrm{i}}$, $\mathbf{F}^{\mathrm{if}}$, and the perturbation gradient $\mathbf{H}$ capturing the sequence of deformations.

The emergence of the initial stress tensors $\mathbf{T}^{\mathrm{i}}$ and $\boldsymbol{\tau}^{\mathrm{f}}$, as well as the composite deformation gradients, clearly distinguishes these equations from the classical form (Eq. (3.10)). These additional terms are not introduced by material nonlinearity but are purely a consequence of the geometric nonlinearity associated with configuration transformation. The dependence on higher-order gradients and configuration-specific quantities captures how pre-existing stresses and finite deformations influence wave propagation.

3.1.2. Deduction of Constitutive Relations

It is important to note that only the constitutive relation for $\mathbf{T}$ is needed, as the natural configuration is the only objective reference suitable for defining material response functions [13]. The constitutive relation for $\boldsymbol{\tau}$ can be derived directly from that of $\mathbf{T}$.

This study is restricted to elastic materials, which are defined as those for which the Kirchhoff stress tensor depends solely on the Green strain tensor [36]:

$$\mathbf{T}^{\mathrm{f}} = \mathbf{g}\left(\mathbf{E}^{\mathrm{f}}\right)$$

where $\mathbf{g}$ is a tensor-valued function satisfying $\mathbf{g}(\mathbf{0}) = \mathbf{0}$.

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To appropriately describe the influence of initial stress or strain on the propagation of small disturbances, it is necessary to expand the constitutive relation beyond the linear regime. Specifically, expanding $\mathbf{g}$ in a Taylor series about the natural state ($\mathbf{E}^{\mathrm{f}}=0$), we obtain:

$$\begin{aligned}\mathbf{g}\left(\mathbf{E}^{\mathrm{f}}\right)&=\mathbf{g}(\mathbf{0})+\frac{\mathrm{d}\mathbf{g}(\mathbf{0})}{\mathrm{d}\mathbf{E}^{\mathrm{f}}}:\mathbf{E}^{\mathrm{f}}+\frac{1}{2!}\frac{\mathrm{d}^2\mathbf{g}(\mathbf{0})}{\mathrm{d}\left(\mathbf{E}^{\mathrm{f}}\right)^2}::\left(\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}\right)\\&+\frac{1}{3!}\frac{\mathrm{d}^3\mathbf{g}(\mathbf{0})}{\mathrm{d}\left(\mathbf{E}^{\mathrm{f}}\right)^3}:::\left(\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}\right)+O\left(\left\|\mathbf{E}^{\mathrm{f}}\right\|^3\right)\end{aligned} \tag{3.11}$$

here, the operator "$:$" denotes double contraction, and "$::$" denotes fourfold contraction over appropriate indices.

Despite the nonlinearity introduced, a second-order expansion is sufficient for the purpose of acoustoelastic theory [31], which concerns only the first-order variation of wave velocity due to the presence of initial stress or strain. Higher-order terms would complicate the formulation without offering significant additional accuracy for the linearized perturbation analysis. This approach is consistent with the assumption that the incremental response remains within the linear regime, even though it is influenced by a finite initial deformation.

In accordance with the retaining rule (i.e., discarding terms with strain tensor of order higher than two under the assumption of small initial deformation [6]), Eq. (3.11) simplifies to:

$$\mathbf{T}^{\mathrm{f}}=\mathbf{C}:\mathbf{E}^{\mathrm{f}}+\frac{1}{2}\mathbf{D}::\left(\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}\right) \tag{3.12}$$

where, $\mathbf{C}$ second-order elasticity tensor, a fourth-rank tensor that reflects the material's linear elastic behavior in the natural configuration. Its symmetry properties are:

$$C_{\alpha\beta\gamma\delta}=C_{\beta\alpha\gamma\delta}=C_{\alpha\beta\delta\gamma}=C_{\gamma\delta\alpha\beta}$$

Similarly, $\mathbf{D}$ is the third-order elasticity tensor, a sixth-rank tensor describing the second-order (nonlinear) elastic response, with symmetries:

$$\begin{aligned}D_{\alpha\beta\gamma\delta\mu\eta}&=D_{\beta\alpha\gamma\delta\mu\eta}=\cdots=D_{\beta\alpha\gamma\delta\eta\mu}\\&=D_{\gamma\delta\alpha\beta\mu\eta}=D_{\gamma\delta\mu\eta\alpha\beta}\end{aligned}$$

Likewise, the Kirchhoff stress in the initial configuration $\mathbf{T}^{\mathrm{i}}$ is given by:

$$\mathbf{T}^{\mathrm{i}}=\mathbf{C}:\mathbf{E}^{\mathrm{i}}+\frac{1}{2}\mathbf{D}::\left(\mathbf{E}^{\mathrm{i}}\otimes\mathbf{E}^{\mathrm{i}}\right) \tag{3.13}$$

Taking the difference between Eqs. (3.12) and (3.13), the incremental Kirchhoff stress becomes:

$$\begin{aligned}\mathbf{T}&=\mathbf{T}^{\mathrm{f}}-\mathbf{T}^{\mathrm{i}}\\&=\mathbf{C}:\left(\mathbf{E}^{\mathrm{f}}-\mathbf{E}^{\mathrm{i}}\right)+\frac{1}{2}\mathbf{D}::\left(\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}-\mathbf{E}^{\mathrm{i}}\otimes\mathbf{E}^{\mathrm{i}}\right)\end{aligned}$$

The first term simplifies to $\mathbf{C}:\bar{\mathbf{E}}$, via Eq. (2.13b). The second term involves a quadratic strain difference, which can be expressed as:

$$\mathbf{E}^{\mathrm{f}}\otimes\mathbf{E}^{\mathrm{f}}-\mathbf{E}^{\mathrm{i}}\otimes\mathbf{E}^{\mathrm{i}}=2\mathbf{E}^{\mathrm{i}}\otimes\bar{\mathbf{E}}+\bar{\mathbf{E}}\otimes\bar{\mathbf{E}} \tag{3.14}$$

Under the retaining rule, the second term $\bar{\mathbf{E}}\otimes\bar{\mathbf{E}}$ can be neglected. However, the interproduct term $2\mathbf{E}^{\mathrm{i}}\otimes\bar{\mathbf{E}}$ is preserved to fully capture the influence of initial stress on incremental motion [37]. It should be emphasized that the linearization is performed with respect to the incremental strain $\bar{\mathbf{E}}$, rather than the initial strain $\mathbf{E}^{\mathrm{i}}$. In the present small-on-large framework, the initial deformation and the associated strain $\mathbf{E}^{\mathrm{i}}$ may be finite, whereas the incremental strain $\bar{\mathbf{E}}$ is assumed infinitesimal. Accordingly, terms are retained or discarded based on their order in $\bar{\mathbf{E}}$. All terms linear

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in $\bar{\mathbf{E}}$ are preserved, including mixed terms involving $\mathbf{E}^{\mathrm{i}} \otimes \bar{\mathbf{E}}$, regardless of the magnitude of $\mathbf{E}^{\mathrm{i}}$. In contrast, terms of order $O\left(\bar{\mathbf{E}}^2\right)$, such as $\bar{\mathbf{E}} \otimes \bar{\mathbf{E}}$, are neglected. This retaining rule is consistent with the standard linearization procedure in acoustoelastic theory, where small perturbations are superimposed on a finitely deformed or pre-stressed configuration.

Based on this reasoning, a preliminary form of the linearized constitutive relation for the incremental stress, expressed in terms of the Green strain tensor, takes the form:

$$\mathbf{T} = \mathbf{C} : \bar{\mathbf{E}} + \mathbf{D} :: \left(\mathbf{E}^{\mathrm{i}} \otimes \bar{\mathbf{E}}\right) \tag{3.15}$$

To simplify subsequent formulations and emphasize the leading-order contribution relevant in acoustoelasticity theory, it is useful to introduce the concept of infinitesimal strain. According to Eqs. (2.12c) and (2.12d), the incremental Green strain tensor can be expanded as:

$$\bar{\mathbf{E}} = \mathbf{e} + \mathbf{R} + O\left(\varepsilon^2\right)$$

where $\mathbf{e}$ denotes the infinitesimal strain tensor, given by the symmetric part of the gradient of the incremental displacement field:

$$\mathbf{e} = \frac{1}{2}\left(\nabla_\xi \mathbf{u} + \mathbf{u}\nabla_\xi\right)$$

and $\mathbf{R}$ represents the interproduct term, involving the gradient of the initial and incremental displacements:

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{u}\nabla_\xi\right)\cdot\left(\nabla_\xi \mathbf{u}^{\mathrm{i}}\right) + \frac{1}{2}\left(\mathbf{u}^{\mathrm{i}}\nabla_\xi\right)\cdot\left(\nabla_\xi \mathbf{u}\right)$$

here $\varepsilon = \|\mathbf{H}\|$, denotes the norm (an indicator of magnitude) of the incremental displacement gradient tensor.

Substituting this expansion into Eq. (3.15), the linearized constitutive relation becomes:

$$\mathbf{T} = \mathbf{C} : \left(\mathbf{e} + \mathbf{R}\right) + \mathbf{D} :: \left(\mathbf{e}^{\mathrm{i}} \otimes \mathbf{e}\right) \tag{3.16a}$$

Given the symmetry properties of the material tensors—namely, that $\mathbf{C}$ is symmetric with respect to its third and fourth indices, and the same holds for $\mathbf{D}$, the component-wise expression of Eq. (3.16a) is written as:

$$T^{\alpha\beta} = C^{\alpha\beta\gamma\delta}\nabla_\gamma u_\delta + C^{\alpha\beta\rho\delta} g^{\sigma\gamma}\nabla_\sigma u^{\mathrm{i}}{}_\rho \nabla_\gamma u_\delta + D^{\alpha\beta\gamma\delta\varepsilon\eta}\nabla_\gamma u^{\mathrm{i}}{}_\delta \nabla_\varepsilon u_\eta \tag{3.16b}$$

3.2. General Acoustoelasticity Formulation in the Natural Configuration

By substituting Eqs. (3.16a), (2.7c), and (2.8) into Eq. (3.8), we obtain the incremental motion equation fully expressed in the natural coordinate system:

$$\rho^0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla_\xi \cdot \left\{\left[\mathbf{C} : \left(\mathbf{e} + \mathbf{R}\right) + \mathbf{D} :: \left(\mathbf{e}^{\mathrm{i}} \otimes \mathbf{e}\right)\right]\left(\nabla_\xi \mathbf{u}^{\mathrm{i}} + \nabla_\xi \mathbf{u} + \mathbf{I}\right) + \mathbf{T}^{\mathrm{i}}\nabla_\xi \mathbf{u}\right\}$$

Applying the simplification rule that neglects higher-order infinitesimals of strain tensors, the above equation reduces to:

$$\rho^0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla_\xi \cdot \left[\mathbf{C} : \mathbf{e} + \mathbf{C} : \mathbf{R} + \left(\mathbf{C} : \mathbf{e}\right)\nabla_\xi \mathbf{u}^{\mathrm{i}} + \mathbf{D} :: \left(\mathbf{e}^{\mathrm{i}} \otimes \mathbf{e}\right) + \mathbf{T}^{\mathrm{i}}\nabla_\xi \mathbf{u}\right] \tag{3.17a}$$

This reduction is based on the following order-of-magnitude approximations:

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$$\begin{aligned}&\left[\mathbf{C}:(\mathbf{e}+\mathbf{R})\right]\nabla_{\xi}\mathbf{u}\rightarrow O\left(\varepsilon^{2}\right)\\&(\mathbf{C}:\mathbf{R})\nabla_{\xi}\mathbf{u}^{\mathrm{i}}\rightarrow O\left(\varepsilon^{\mathrm{i}2}\right)\\&\left[\mathbf{D}::\left(\mathbf{e}^{\mathrm{i}}\otimes\mathbf{e}\right)\right]\left(\nabla_{\xi}\mathbf{u}^{\mathrm{i}}+\nabla_{\xi}\mathbf{u}\right)\rightarrow O\left(\varepsilon^{2}\right)+O\left(\varepsilon^{\mathrm{i}2}\right)\end{aligned}$$

here, the symbol " $\rightarrow$ " " indicates "on the order of", and the notation $\varepsilon^{\mathrm{i}}=\left\|\mathbf{H}^{\mathrm{i}}\right\|$ is implied.

The component form of Eq. (3.17a) reads:

$$\rho^{0}\frac{\partial^{2}u_{\beta}}{\partial t^{2}}=\nabla_{\alpha}\left(\Lambda^{\alpha\beta\gamma\delta}\nabla_{\gamma}u_{\delta}+g^{\alpha\mu}g^{\gamma\eta}g^{\beta\delta}T^{\mathrm{i}}{}_{\mu\eta}\nabla_{\gamma}u_{\delta}\right) \tag{3.17b}$$

The **equivalent elasticity tensor** $\Lambda^{\alpha\beta\gamma\delta}$ is defined by:

$$\Lambda^{\alpha\beta\gamma\delta}=C^{\alpha\beta\gamma\delta}+\underbrace{C^{\alpha\beta\rho\delta}g^{\sigma\gamma}\nabla_{\sigma}u^{\mathrm{i}}{}_{\rho}}_{1243}+\underbrace{C^{\alpha\sigma\gamma\delta}g^{\rho\beta}\nabla_{\sigma}u^{\mathrm{i}}{}_{\rho}}_{1342}+D^{\alpha\beta\gamma\delta\varepsilon\eta}\nabla_{\varepsilon}u^{\mathrm{i}}{}_{\eta} \tag{3.18a}$$

Here $\Lambda^{\alpha\beta\gamma\delta}$ is the initial-deformation-dependent elasticity tensor governing the incremental motion in the natural configuration. It reduces to the classical elastic tensor $C^{\alpha\beta\gamma\delta}$in the absence of initial deformation, while the additional terms account for the influence of the initial strain field and the third-order elastic moduli on wave propagation. The index-permutation notation $\underbrace{X^{\alpha\beta\gamma\delta}}_{1243}$ indicates a rearrangement of free indices, introduced only to compactly express the required tensor symmetries.

Together, Eqs. (3.17) and (3.18) constitute the general acoustoelastic formulation in the natural system (Lagrangian description of initial deformation) under an arbitrary curvilinear coordinate basis. This formulation governs wave motion in a solid medium subjected to an initial stress field. The additional three terms in Eq. (3.18a), relative to classical linear elasticity, stem from nonlinear constitutive contributions.

When specialized to a Cartesian coordinate system, the formulation simplifies to:

$$\rho^{0}\frac{\partial^{2}u_{\beta}}{\partial t^{2}}=\frac{\partial}{\partial\xi_{\alpha}}\left(\Lambda^{\alpha\beta\gamma\delta}\frac{\partial u_{\gamma}}{\partial\xi_{\delta}}+T^{\mathrm{i}}{}_{\gamma\alpha}\frac{\partial u_{\beta}}{\partial\xi_{\gamma}}\right) \tag{3.17c}$$

$$\Lambda^{\alpha\beta\gamma\delta}=C^{\alpha\beta\gamma\delta}+\underbrace{C^{\alpha\beta\rho\delta}\frac{\partial u^{\mathrm{i}}{}_{\gamma}}{\partial\xi_{\rho}}}_{1243}+\underbrace{C^{\rho\beta\gamma\delta}\frac{\partial u^{\mathrm{i}}{}_{\alpha}}{\partial\xi_{\rho}}}_{2341}+D^{\alpha\beta\gamma\delta\varepsilon\eta}\frac{\partial u^{\mathrm{i}}{}_{\eta}}{\partial\xi_{\varepsilon}} \tag{3.18b}$$

It is evident that Eq. (3.17c) corresponds exactly to Eq. (23) in [9], and Eq. (3.18b) corresponds to Eq. (24), both of which are widely referenced in acoustoelastic literature [16].

For a homogeneously predeformed medium, where the initial stress tensor $T^{\mathrm{i}}{}_{\mu\eta}$ and the elastic coefficients are spatially constant, Eq. (3.17b) simplifies further to:

$$\rho^{0}\frac{\partial^{2}u_{\beta}}{\partial t^{2}}=A^{\alpha\beta\gamma\delta}\nabla_{\alpha}\nabla_{\gamma}u_{\delta} \tag{3.17d}$$

Here, the natural acoustoelastic coefficient tensor is defined as:

$$A^{\alpha\beta\gamma\delta}=\Lambda^{\alpha\beta\gamma\delta}+g^{\alpha\mu}g^{\gamma\eta}g^{\beta\delta}T^{\mathrm{i}}{}_{\mu\eta}$$

which encapsulates the combined effect of the elastic constants and the initial stress components, expressed in the natural coordinate system.

To expand the double gradient $\nabla_{\alpha}\nabla_{\gamma}u_{\delta}$ in a general basis, we first express the single gradient of the incremental displacement as:

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$$\nabla_\gamma u_\delta = \frac{\partial u_\delta}{\partial \xi^\gamma} - \Gamma^\eta_{\gamma\delta} u_\eta$$

where $\Gamma^\eta_{\gamma\delta}$ denotes the Christoffel symbol. For its definition and properties, refer to [38]. Taking the gradient again yields:

$$\nabla_\alpha \nabla_\gamma u_\delta = \frac{\partial u_\delta}{\partial \xi^\alpha \partial \xi^\gamma} - u_\eta \frac{\partial \Gamma^\eta_{\gamma\delta}}{\partial \xi_\alpha} - \left(\frac{\partial u_\mu}{\partial \xi^\gamma} - u_\eta \Gamma^\eta_{\gamma\mu}\right)\Gamma^\mu_{\alpha\delta} - \left(\frac{\partial u_\delta}{\partial \xi^\rho} - u_\eta \Gamma^\eta_{\rho\delta}\right)\Gamma^\rho_{\alpha\gamma}$$

Substituting this into Eq. (3.17d), we obtain the fully expanded motion equation:

$$\rho^0 \frac{\partial^2 u_\beta}{\partial t^2} = A^{\alpha\beta\gamma\delta}\left[\frac{\partial u_\delta}{\partial \xi^\alpha \partial \xi^\gamma} - u_\eta \frac{\partial \Gamma^\eta_{\gamma\delta}}{\partial \xi_\alpha} - \left(\frac{\partial u_\mu}{\partial \xi^\gamma} - u_\eta \Gamma^\eta_{\gamma\mu}\right)\Gamma^\mu_{\alpha\delta} - \left(\frac{\partial u_\delta}{\partial \xi^\rho} - u_\eta \Gamma^\eta_{\rho\delta}\right)\Gamma^\rho_{\alpha\gamma}\right] \quad (3.17e)$$

When restricted to the Cartesian case, where Christoffel symbols vanish, Eq. (3.17e) reduces exactly to Eq. (32) in [9].

3.3 General acoustoelasticity formulation in the initial configuration

Given that the difference between the final stress tensor $\boldsymbol{\tau}^{\mathrm{f}}$ and the initial stress $\boldsymbol{\sigma}^{\mathrm{i}}$ is of order $O(\varepsilon)$, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{f}} - \boldsymbol{\sigma}^{\mathrm{i}} = O(\varepsilon)$$

the difference between $\boldsymbol{\tau}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}$ and $\boldsymbol{\sigma}^{\mathrm{i}}\mathbf{H}^{\mathrm{T}}$ is then of order $O(\varepsilon^2)$. According to the perturbation retention rules outlined previously, Eq. (3.9) can therefore be reformulated as:

$$\rho^{\mathrm{i}} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla_{\mathbf{X}} \cdot \left(\frac{1}{J^{\mathrm{i}}}\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} + \boldsymbol{\sigma}^{\mathrm{i}}\mathbf{H}^{\mathrm{T}}\right) \quad (3.19)$$

This substitution removes any explicit dependence on the final configuration from the incremental motion equation expressed in the initial configuration.

To explicitly demonstrate that Eq. (3.19) is indeed formulated entirely in the initial frame, we substitute relations Eqs. (2.6c) and (3.16a), yielding:

$$\rho^{\mathrm{i}} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla_{\mathbf{X}} \cdot \left(\frac{1}{J^{\mathrm{i}}}\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} + \boldsymbol{\sigma}^{\mathrm{i}}\nabla_{\mathbf{X}}\mathbf{U}\right) \quad (3.20)$$

Among the terms on the right-hand side of Eq. (3.20), the most analytically intricate is the first term within the parentheses. To analyze its structure, we momentarily disregard the prefactor $1/J^i$, and focus on $\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}}$, which expands as:

$$\begin{aligned}\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} &= \left(\nabla_\xi u^{\mathrm{i}\sigma} g^I_\sigma + g^I_\xi\right) T^{\gamma\delta}\left(\nabla_\lambda u^{\mathrm{i}\sigma} g^J_\sigma + g^J_\lambda\right)\mathbf{G}_I \otimes \boldsymbol{g}^\xi \cdot \boldsymbol{g}_\gamma \otimes \boldsymbol{g}_\delta \cdot \boldsymbol{g}^\lambda \otimes \mathbf{G}_J \\ &= T^{\xi\lambda}\left(\nabla_\xi u^{\mathrm{i}\sigma} g^I_\sigma + g^I_\xi\right)\left(\nabla_\lambda u^{\mathrm{i}\sigma} g^J_\sigma + g^J_\lambda\right)\mathbf{G}_I \otimes \mathbf{G}_J\end{aligned}$$

Retaining only first-order terms in the incremental displacement gradient yields:

$$\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} \approx T^{\xi\lambda}\left(\nabla_\xi u^{\mathrm{i}\sigma} g^I_\sigma g^J_\lambda + g^I_\xi \nabla_\lambda u^{\mathrm{i}\sigma} g^J_\sigma + g^I_\xi g^J_\lambda\right)\mathbf{G}_I \otimes \mathbf{G}_J \quad (3.21)$$

Substituting Eq. (3.16b) for $T^{\xi\lambda}$ into this expression, we obtain:

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$$\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} = g^{\alpha\xi} g^{\beta\lambda} \left(C_{\alpha\beta}{}^{\gamma\delta} \nabla_\gamma u_\delta + C_{\alpha\beta}{}^{\gamma\delta} \nabla_\gamma u^{\mathrm{i}\rho} \nabla_\delta u_\rho + D_{\alpha\beta}{}^{\gamma\delta\varepsilon\eta} \nabla_\gamma u^{\mathrm{i}}{}_\delta \nabla_\varepsilon u_\eta \right)$$
$$\times \left(\nabla_\xi u^{\mathrm{i}\sigma} g_\sigma^I g_\lambda^J + g_\xi^I \nabla_\lambda u^{\mathrm{i}\sigma} g_\sigma^J + g_\xi^I g_\lambda^J \right)$$

After collecting terms and retaining only linear orders in either initial or incremental displacement gradients, the expression simplifies to:

$$\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} = C^{IJ\gamma\delta} \nabla_\gamma u_\delta + C^{IJ\gamma\delta} \nabla_\gamma u^{\mathrm{i}\rho} \nabla_\delta u_\rho + D^{IJ\gamma\delta\varepsilon\eta} \nabla_\gamma u^{\mathrm{i}}{}_\delta \nabla_\varepsilon u_\eta$$
$$+ g_\sigma^I C^{\xi J\gamma\delta} \nabla_\xi u^{\mathrm{i}\sigma} \nabla_\gamma u_\delta + C^{I\lambda\gamma\delta} \nabla_\lambda u^{\mathrm{i}\sigma} g_\sigma^J \nabla_\gamma u_\delta$$

In this expression, elastic moduli tensors $\mathbf{C}$ and $\mathbf{D}$ refer to the natural configuration. To rewrite all quantities in the initial configuration, the two-point transformation tensor g_I^α can be applied. Hence, we obtain:

$$\begin{aligned}
\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} = {} & C^{IJKL} g_K^\gamma g_L^\delta \nabla_\gamma u_\delta \\
& + C^{IJML} g_M^\gamma g_L^\delta g^{\sigma\rho} \nabla_\gamma u^{\mathrm{i}}{}_\sigma \nabla_\delta u_\rho \\
& + g^{I\rho} C^{MJKL} g_M^\xi g_K^\gamma g_L^\delta \nabla_\xi u^{\mathrm{i}}{}_\rho \nabla_\gamma u_\delta \\
& + C^{IMKL} g_K^\gamma g_L^\delta g^{\rho J} g_M^\lambda \nabla_\lambda u^{\mathrm{i}}{}_\rho \nabla_\gamma u_\delta \\
& + D^{IJKLMN} g_K^\gamma g_L^\delta g_M^\varepsilon g_N^\eta \nabla_\varepsilon u^{\mathrm{i}}{}_\eta \nabla_\gamma u_\delta
\end{aligned} \tag{3.22}$$

To complete the transition to the initial frame, the displacement gradients can be converted as:

$$\nabla_\gamma u_\delta = X^P{}_\gamma \nabla_P U_Q g^Q{}_\delta, \; \nabla_\gamma u^{\mathrm{i}}{}_\delta = X^P{}_\gamma \nabla_P U^{\mathrm{i}}{}_Q g^Q{}_\delta \tag{3.23}$$

according to Eq. (2.9) and (2.11), where

$$X^P{}_\gamma = \nabla_\gamma u^{\mathrm{i}}{}_\sigma g^{\sigma P} + g_\gamma^P \tag{3.24}$$

Substituting Eq. (3.23) into (3.22), it can be observed that:

$$\begin{aligned}
\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} = {} & C^{IJKL} g_K^\gamma X^P{}_\gamma \nabla_P U_L \\
& + C^{IJML} g_M^\gamma g_L^\delta g^{QS} X^P{}_\gamma X^R{}_\delta \nabla_P U^{\mathrm{i}}{}_Q \nabla_R U_S \\
& + g^{IQ} C^{MJKL} g_M^\xi g_K^\gamma X^P{}_\xi \nabla_P U^{\mathrm{i}}{}_Q X^R{}_\gamma \nabla_R U_L \\
& + C^{IMKL} g^{QJ} g_K^\gamma g_M^\lambda X^P{}_\lambda X^R{}_\gamma \nabla_P U^{\mathrm{i}}{}_Q \nabla_R U_L \\
& + D^{IJKLMN} g_K^\gamma g_M^\varepsilon X^P{}_\varepsilon \nabla_P U^{\mathrm{i}}{}_N X^R{}_\gamma \nabla_R U_L
\end{aligned} \tag{3.25}$$

Note that in all terms except the first, $X^P{}_\gamma$ appears contracted with $\nabla_P U^{\mathrm{i}}{}_Q$, which is itself of $O\left(\varepsilon^{\mathrm{i}}\right)$. Therefore, since terms of $O\left(\varepsilon^{\mathrm{i}2}\right)$ are consistently neglected throughout the formulation, we may replace $X^P{}_\gamma$ by g_γ^P in the last four terms. However, the first term involves no such coupling and must be expanded:

$$\begin{aligned}
g_K^\gamma X^P{}_\gamma \nabla_P U_L &= g_K^\gamma \left(\nabla_\gamma u^{\mathrm{i}}{}_\sigma g^{\sigma P} + g_\gamma^P \right) \nabla_P U_L \\
&= \left(g_K^\gamma \nabla_\gamma u^{\mathrm{i}}{}_\sigma g^{\sigma P} + g_K^P \right) \nabla_P U_L
\end{aligned} \tag{3.26}$$

As $\nabla_\gamma u^{\mathrm{i}}{}_\sigma$ is of the same order as $\nabla_K U^{\mathrm{i}}{}_Q$, we finally approximate:

$$g_K^\gamma X^P{}_\gamma \nabla_P U_L = g^{QP} \nabla_K U^{\mathrm{i}}{}_Q \nabla_P U_L + \nabla_K U_L \tag{3.27}$$

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Substituting Eq. (3.27) into Eq. (3.25) and replacing $X^{P}{}_{\gamma}$ with g^{P}_{γ} in the last four terms on the right-hand side yields the complete expression of $\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}}$:

$$\begin{aligned}\mathbf{F}^{\mathrm{i}}\mathbf{T}\mathbf{F}^{\mathrm{iT}} &= C^{IJKL}\nabla_K U_L + C^{IJML} g^{PK}\nabla_M U^{\mathrm{i}}{}_P \nabla_K U_L + C^{IJKM} g^{PL}\nabla_M U^{\mathrm{i}}{}_P \nabla_K U_L \\ &+ g^{IP} C^{MJKL}\nabla_M U^{\mathrm{i}}{}_P \nabla_K U_L + C^{IMKL} g^{JP}\nabla_M U^{\mathrm{i}}{}_P \nabla_K U_L \\ &+ D^{IJKLMN}\nabla_M U^{\mathrm{i}}{}_N \nabla_K U_L\end{aligned} \tag{3.28}$$

It is evident that all quantities and derivatives in Eq. (3.28) are expressed with respect to the initial configuration. Notably, the careful treatment of $X^{P}{}_{\gamma}$ —especially its retention in the first term—introduces an additional contribution that would otherwise be lost under naïve approximation. The presence of this term plays a critical role, as it ensures that the resulting stressed (or effective) elastic coefficient tensor retains the same symmetry properties as the original material elasticity tensor, as will be demonstrated subsequently.

Another important quantity yet to be addressed is the determinant of the deformation gradient tensor in the initial configuration, denoted as J^{i}, according to its definition:

$$\begin{aligned}J^{\mathrm{i}} &= \det\left(\mathbf{F}^{\mathrm{i}}\right) \\ &= \det\left(\mathbf{H}^{\mathrm{i}} + \mathbf{I}\right)\end{aligned}$$

where $\mathbf{H}^{\mathrm{i}}$ denotes the displacement gradient.

Using the standard expansion of the determinant (see, e.g., Chp 5 of [38]) for small but finite deformation gradients, one obtains

$$\begin{aligned}\det(\mathbf{I} + \mathbf{H}^{\mathrm{i}}) = 1 + \mathrm{tr}(\mathbf{H}^{\mathrm{i}}) &+ \frac{1}{2}\left[(\mathrm{tr}\mathbf{H}^{\mathrm{i}})^2 - \mathrm{tr}\left(\mathbf{H}^{\mathrm{i}2}\right)\right] \\ &+ \frac{1}{6}\left[(\mathrm{tr}\mathbf{H}^{\mathrm{i}})^3 - 3\mathrm{tr}\mathbf{H}^{\mathrm{i}}\mathrm{tr}\left(\mathbf{H}^{\mathrm{i}2}\right) + 2\mathrm{tr}\left(\mathbf{H}^{\mathrm{i}3}\right)\right] \\ &+ \cdots\end{aligned}$$

In the present work, only the first-order approximation of initial deformation is required, yielding

$$\det(\mathbf{I} + \mathbf{H}^{\mathrm{i}}) \approx 1 + \mathrm{tr}(\mathbf{H}^{\mathrm{i}})$$

The displacement gradient $\mathbf{H}^{\mathrm{i}}$ can be decomposed into its symmetric and skew-symmetric parts:

$$\mathbf{H}^{\mathrm{i}} = \mathbf{e}^{\mathrm{i}} + \boldsymbol{\omega}^{\mathrm{i}}$$

where $\mathbf{e}^{\mathrm{i}} = \left(\mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{iT}}\right)/2$ is the symmetric part (linear strain) and $\boldsymbol{\omega}^{\mathrm{i}} = \left(\mathbf{H}^{\mathrm{i}} - \mathbf{H}^{\mathrm{iT}}\right)/2$ is the skew-symmetric part. It follows immediately that

$$\mathrm{tr}(\mathbf{H}^{\mathrm{i}}) = \mathrm{tr}(\mathbf{e}^{\mathrm{i}})$$

since the skew-symmetric tensor $\boldsymbol{\omega}^{\mathrm{i}}$ has zero trace.

Therefore, we obtain the linear approximation:

$$J^{\mathrm{i}} = \mathrm{tr}\mathbf{e}^{\mathrm{i}} + 1 \tag{3.29}$$

where $\mathrm{tr}\mathbf{e}^{\mathrm{i}}$ is the trace of $\mathbf{e}^{\mathrm{i}}$.

Since both J^{i} and $\mathrm{tr}\mathbf{e}^{\mathrm{i}}$ are scalar invariants, Eq. (3.29) holds in any coordinate system. Moreover, recalling that $\mathbf{e}^{\mathrm{i}}$ is the symmetric part of the displacement gradient $\nabla_{\boldsymbol{\xi}}\mathbf{u}^{\mathrm{i}}$, we have:

$$\mathrm{tr}\mathbf{e}^{\mathrm{i}} = \mathrm{tr}\left(\nabla_{\boldsymbol{\xi}}\mathbf{u}^{\mathrm{i}}\right)$$

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From Eq. (2.9), the difference between $\nabla_{\boldsymbol{\xi}}\mathbf{u}^{\mathrm{i}}$ and $\nabla_{\mathbf{X}}\mathbf{U}^{\mathrm{i}}$ is of second order, i.e., $O\left(\varepsilon^{\mathrm{i}2}\right)$. Hence, Eq. (3.29) can be equivalently written as:

$$J^{\mathrm{i}} = \mathrm{tr}\left(\nabla_{\mathbf{X}}\mathbf{U}^{\mathrm{i}}\right)+1$$

Taking the reciprocal of this expression and neglecting higher-order terms of the trace leads to:

$$\frac{1}{J^{\mathrm{i}}} = 1-\mathrm{tr}\left(\nabla_{\mathbf{X}}\mathbf{U}^{\mathrm{i}}\right) \tag{3.30a}$$

or in index notation:

$$\frac{1}{J^{\mathrm{i}}} = 1-\nabla_{P}U^{\mathrm{i}P} \tag{3.30b}$$

This relation (Eq. 3.30b) should be substituted into Eq. (3.20) to fully express the constitutive relation for the Kirchhoff stress tensor $\boldsymbol{\tau}$ in the initial configuration. However, as previously noted, the factor $1/J^{\mathrm{i}}$ only affects the first term on the right-hand side of Eq. (3.28), based on a similar analysis involving $X^{P}{}_{\gamma}$.

We now arrive at the final form of the incremental displacement equation in the initial frame:

$$\rho^{\mathrm{i}}\frac{\partial^2 U_J}{\partial t^2} = \nabla_I\left[\left(\Sigma^{IJKL}+\underbrace{g^{IM}g^{KN}g^{JL}\sigma^{\mathrm{i}}{}_{MN}}_{1324}\right)\nabla_K U_L\right] \tag{3.31a}$$

where the effective elasticity tensor Σ^{IJKL} is given by:

$$\begin{aligned}\Sigma^{IJKL} = {} & C^{IJKL}\left(1-g^{PQ}\nabla_P U^{\mathrm{i}}{}_Q\right) \\ & +\underbrace{C^{IJML}g^{PK}\nabla_M U^{\mathrm{i}}{}_P}_{1243}+C^{IJKM}g^{PL}\nabla_M U^{\mathrm{i}}{}_P \\ & +g^{IP}C^{MJKL}\nabla_M U^{\mathrm{i}}{}_P+\underbrace{C^{IMKL}g^{JP}\nabla_M U^{\mathrm{i}}{}_P}_{1342} \\ & +D^{IJKLMN}\nabla_M U^{\mathrm{i}}{}_N\end{aligned} \tag{3.32a}$$

In Eq. (3.31a), the underbrace notation (e.g., 1324) indicates the required permutation of free indices in order to preserve the symmetry structure. Specifically, the permutation denotes a tensor transpose operation rearranging the indices as $IJKL \to IKJL$. Similarly, in Eq. (3.32a), the annotated permutations such as 1243 and 1342 denote transpositions that maintain the symmetries of Σ^{IJKL} consistent with those of the fourth-order elasticity tensor C^{IJKL}.

Equation (3.31a) is the incremental equation of motion written in the initial (pre-deformed) configuration, which is particularly convenient for wave-propagation analysis in stressed components because the geometry and boundary conditions are posed on the current body shape. The term Σ^{IJKL} represents an effective acoustoelastic stiffness incorporating material nonlinearity and initial deformation, while the $\sigma^{\mathrm{i}}{}_{MN}$-dependent term arises from the geometric contribution of the initial stress state to incremental equilibrium. Unlike classical elasticity, where stiffness depends only on material properties, here the coefficients depend explicitly on the initial state of the body. In ultrasonic NDT applications, this governing equation leads directly to stress-dependent wave speeds (and hence acoustoelastic coefficients) that are experimentally measurable.

The tensor Σ^{IJKL} in Eq. (3.32a) is the effective (incremental) elasticity tensor in the initial configuration. For clarity, its terms can be interpreted as follows: the prefactor $C^{IJKL}\left(1-g^{PQ}\nabla_P U^{\mathrm{i}}{}_Q\right)$ originates from the first-order linearization of the volumetric change of the initial deformation; the terms proportional to $C^{\cdots\cdot}\nabla U^{\mathrm{i}}$ arise from the geometric mapping between configurations (push-forward of the incremental stress measure); and the last term involving D^{IJKLMN} represents the intrinsic material nonlinearity through the third-order elastic constants.

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Eqs. (3.31a) and (3.32a) together constitute the general formulation of acoustoelasticity in the initial configuration (Eulerian description of initial deformation) for arbitrary curvilinear coordinates. When restricted to a Cartesian coordinate system, they reduce to:

$$\rho^{\mathrm{i}}\frac{\partial^2 U_J}{\partial t^2}=\frac{\partial}{\partial X_I}\left[\left(\Sigma^{IJKL}+\underbrace{\delta_{IK}\sigma^{\mathrm{i}}_{JL}}_{1324}\right)\frac{\partial U_K}{\partial X_L}\right] \tag{3.31b}$$

$$\begin{aligned}\Sigma^{IJKL}&=C^{IJKL}\left(1-\frac{\partial U^{\mathrm{i}}_{Q}}{\partial X_Q}\right)+\underbrace{C^{IJML}\frac{U^{\mathrm{i}}_{K}}{\partial X_M}}_{1243}+C^{IJKM}\frac{U^{\mathrm{i}}_{L}}{\partial X_M}\\&+\frac{U^{\mathrm{i}}_{I}}{\partial X_M}C^{MJKL}+\underbrace{C^{IMKL}\frac{U^{\mathrm{i}}_{J}}{\partial X_M}}_{1342}+D^{IJKLMN}\frac{U^{\mathrm{i}}_{N}}{\partial X_M}\end{aligned} \tag{3.32b}$$

here, Eq. (3.31b) corresponds to Eq. (29), and Eq. (3.32b) to Eq. (27), in the well-known reference [9].

The Cartesian forms (3.31b)–(3.32b) are provided for two reasons: (i) they serve as a consistency check by recovering the classical acoustoelastic equations reported in [9]; and (ii) they offer a familiar representation for readers to connect the present tensorial formulation with standard Cartesian-based NDT implementations.

For a homogeneously pre-deformed medium, the initial stress σ^{i}_{MN} and elasticity tensor Σ^{IJKL} are spatially uniform. In this case, the governing equation simplifies to:

$$\rho^{\mathrm{i}}\frac{\partial^2 U_J}{\partial t^2}=B^{IJKL}\nabla_I\nabla_K U_L \tag{3.31c}$$

where the acoustoelastic coefficient tensor is defined by:

$$B^{IJKL}=\Sigma^{IJKL}+g^{IM}g^{KN}g^{JL}\sigma^{\mathrm{i}}_{MN}$$

To further expand the second-order derivative term in general curvilinear coordinates, recall that for a second-order tensor H_{KL}, the covariant derivative is given by [38]:

$$\nabla_I H_{KL}=\frac{\partial H_{KL}}{\partial X^I}-\Gamma^{Q}_{IL}H_{KQ}-\Gamma^{R}_{IK}H_{RL}$$

where Γ^{Q}_{IL} and Γ^{Q}_{IL} are the Christoffel symbols in the initial configuration. Since

$$H_{KL}=\nabla_K U_L=\frac{\partial U_L}{\partial X^K}-\Gamma^{P}_{KL}U_P$$

the full expansion of $\nabla_I\nabla_K U_L$ becomes:

$$\begin{aligned}\nabla_I\nabla_K U_L=&\frac{\partial U_L}{\partial X^I\partial X^K}-U_P\frac{\partial\Gamma^{P}_{KL}}{\partial X^I}\\&-\left(\frac{\partial U_Q}{\partial X^K}-U_P\Gamma^{P}_{KQ}\right)\Gamma^{Q}_{IL}-\left(\frac{\partial U_L}{\partial X^R}-U_P\Gamma^{P}_{RL}\right)\Gamma^{R}_{IK}\end{aligned}$$

Therefore, the equation of motion (Eq. 3.31c) in its fully expanded form reads:

$$\rho^{\mathrm{i}}\frac{\partial^2 U_J}{\partial t^2}=B^{IJKL}\left[\begin{aligned}&\frac{\partial U_L}{\partial X^I\partial X^K}-U_P\frac{\partial\Gamma^{P}_{KL}}{\partial X^I}-\left(\frac{\partial U_Q}{\partial X^K}-U_P\Gamma^{P}_{KQ}\right)\Gamma^{Q}_{IL}\\&-\left(\frac{\partial U_L}{\partial X^R}-U_P\Gamma^{P}_{RL}\right)\Gamma^{R}_{IK}\end{aligned}\right] \tag{3.31d}$$

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In Cartesian coordinates, all Christoffel symbols vanish, and Eq. (3.31d) reduces to Eq. (38) in [9].

Although no higher-order terms of the displacement field explicitly appear in Eqs. (3.17) and (3.31), the acoustoelastic formulation remains inherently nonlinear. This nonlinearity originates from three essential aspects:

Geometric nonlinearity introduced by multiple configurations: The adoption of different configurations to describe various deformation states and their associated stress measures results in nonlinear terms in the governing equations—particularly in Eqs. (3.8) and (3.9). These terms reflect the geometric nonlinearity due to the mapping between configurations.

Nonlinear strain measure via the Green strain tensor: The use of the Green strain tensor in the constitutive relations introduces nonlinear terms of the displacement gradient. As a second-order measure of strain, it naturally accounts for finite (non-infinitesimal) deformations, embedding geometric nonlinearity into the material response.

Material nonlinearity via second-order constitutive expansion: The constitutive relation is expanded to second order in strain, resulting in a nonlinear stress-strain relationship and the emergence of third-order elastic moduli. These moduli describe the intrinsic nonlinear elastic behavior of the material.

It should be emphasized that not all nonlinear effects are retained in the final formulation. A fully nonlinear treatment would make the analytical process excessively complex and is generally unnecessary for the objective of this study. Since the primary focus is on the linear influence of the initial stress or strain on the wave propagation, a linearized treatment suffices for stress measurement purposes. This is consistent with the small-on-large framework commonly adopted in acoustoelastic theory, wherein small dynamic perturbations are superimposed on a finitely deformed or pre-stressed state. Such a linearized approach effectively captures the essential physical effects and is widely employed in existing research.

In conclusion, this section presents the general tensorial formulation of acoustoelasticity theory in both the natural and initial configurations. The derived equations are valid for arbitrary curvilinear coordinate systems because they are expressed entirely in coordinate-independent tensor form. As demonstrated in Eqs. (3.17e) and (3.31d), the only essential distinction between the two formulations lies in the definitions of the acoustoelastic coefficient tensors, which represent stress- and deformation-dependent effective elastic moduli governing incremental wave motion.

The formulation in the natural configuration provides a conceptually transparent reference description and serves as the theoretical foundation of the nonlinear framework, whereas the formulation in the initial configuration is directly suited for practical wave-propagation analysis because boundary conditions and geometry are naturally specified on the pre-stressed body. In applications such as ultrasonic nondestructive testing, these governing equations form the basis for deriving stress-dependent wave speeds and acoustoelastic coefficients that can be measured experimentally. Each formulation therefore offers distinct advantages depending on the analysis objective and implementation context [39].

4. Acoustoelasticity Theory in the Cylindrical System

The acoustoelastic equations derived in the previous sections are formulated in a general curvilinear coordinate system and therefore remain valid for any curvilinear coordinates. In practical applications, however, orthogonal coordinate systems are most commonly used—such as cylindrical, spherical, or natural coordinates. Cylindrical coordinates are particularly suitable for analyzing wave propagation in cylindrical components or geological structures like boreholes. Spherical coordinates are more appropriate for components with spherical surfaces or for modeling wave behavior on the Earth's surface. For curved structures such as turbine blades or shell-like components, natural coordinates offer a more convenient framework for describing ultrasonic wave propagation. While the governing acoustoelastic equations retain their general form, their specific expressions differ among coordinate systems due to variations in the Christoffel symbols Γ^k_{ij} associated with the

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underlying coordinate curves. In this section, we present the explicit formulation of the acoustoelastic equations in cylindrical coordinates as a representative example.

4.1. Cylindrical Coordinate System

In an orthonormal curvilinear coordinate system, the components of the metric tensor g_{ij} satisfy the following properties [40]:

$$\begin{aligned} &g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = 0, \quad \text{for} \quad i \neq j; \\ &g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j = 0, \quad \text{for} \quad i \neq j; \\ &g^{\underline{ii}} = 1/g_{\underline{ii}}; \\ &g_{11}g_{22}g_{33} = g. \end{aligned} \tag{4.1}$$

Here, an underline beneath a repeated index (e.g., $\underline{ii}$) indicates no summation over that index. To obtain explicit partial-differential forms of the governing equations in cylindrical coordinates, we expand the covariant derivatives in terms of Christoffel symbols. The Christoffel symbols Γ_{ij}^k are related to the metric tensor g_{ij} by [38]:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{4.2}$$

In an orthonormal curvilinear coordinate system (as assumed here), the metric tensor is diagonal and the Christoffel symbols simplify accordingly,

$$\begin{aligned} &\Gamma_{ij}^k = 0, \quad (i \neq j \neq k \neq i) \\ &\Gamma_{\underline{i}j}^{\underline{i}} = \Gamma_{j\underline{i}}^{\underline{i}} = \frac{1}{2g_{\underline{ii}}} \frac{\partial g_{\underline{ii}}}{\partial x_j} \\ &\Gamma_{\underline{ii}}^j = -\frac{1}{2} g^{\underline{jj}} \frac{\partial g_{\underline{ii}}}{\partial x^j} \end{aligned} \tag{4.3}$$

More specifically, for cylindrical system [40], there is:

$$\begin{aligned} &\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}; \\ &\Gamma_{22}^1 = -r, \\ &\Gamma_{jk}^i = 0 \qquad (\text{other cases}) \end{aligned} \tag{4.4}$$

4.2. Material Symmetricity

In this section, we develop the acoustoelastic formulation for orthotropic materials, which are commonly encountered in industrial applications. For such materials, the normal stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ depend only on the normal strains $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ and are independent of shear strains $\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}$. Each shear stress depends solely on its corresponding shear strain [41].

For orthotropic solids, it is convenient to express the fourth-order elasticity tensor $\mathbf{C}$ in Voigt notation for engineering implementation. This matrix form is merely a contracted representation of the tensorial constitutive relation and does not alter the coordinate-independent formulation developed in Sections 3.1–3.3. The second-order elasticity tensor $\mathbf{C}$ then takes the form:

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$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{4.5}$$

This is expressed in Voigt notation using the mapping: $1 \to 11$, $2 \to 22$, $3 \to 33$, $4 \to 23$, $5 \to 13$, $6 \to 12$.

In general, the third-order elasticity tensor $\mathbf{D}$ has 56 independent components, but only 20 are non-zero for orthotropic materials [42]. These are:

$$\begin{array}{llllll} D_{111}, & D_{222}, & D_{333} & & & \\ D_{144}, & D_{255}, & D_{366} & & & \\ D_{112}, & D_{223}, & D_{133}, & D_{113}, & D_{122}, & D_{233}, \\ D_{155}, & D_{244}, & D_{344}, & D_{166}, & D_{266}, & D_{355}, \\ D_{123}, & D_{456}. & & & & \end{array} \tag{4.6}$$

The orthotropic assumption is adopted here because it covers a wide range of engineering materials (e.g., rolled metals and fiber-reinforced composites) while keeping the cylindrical expansion tractable. If the material symmetry axes are rotated relative to the cylindrical axes, the formulation remains applicable by transforming $\mathbf{C}$and $\mathbf{D}$using the standard direction-cosine (Bond-type) transformations in Eq. (4.7).

4.3. Acoustoelasticity Formulation in Cylindrical System

In the present cylindrical formulation, it is assumed that the material coordinate system coincides with both the principal directions of orthotropy and the principal directions of the initial stress. Under this assumption, the material symmetrical axes are aligned with the cylindrical coordinate directions, and the initial shear stress components vanish. This configuration is representative of many practical cases, such as rolled or filament-wound cylindrical structures.

If the material coordinate system does not coincide with the principal stress directions or the geometric (cylindrical) coordinate system, the formulation remains applicable through appropriate tensor transformations. In such cases, the elastic constants and higher-order elastic moduli defined in the material coordinate system can be transformed into the working coordinate system using standard tensor rotation rules. Specifically, the second- and third-order elastic tensors transform as

$$\begin{aligned} C_{mnop} &= \beta_{mi}\beta_{nj}\beta_{ok}\beta_{pl}C'_{ijkl} \\ D_{opqrst} &= \beta_{oi}\beta_{pj}\beta_{qk}\beta_{rl}\beta_{sm}\beta_{tn}D'_{ijklmn} \end{aligned} \tag{4.7}$$

where β_{ij} denotes the direction cosine matrix relating the material coordinate system to the principal stress or geometric coordinate system.

This transformation framework allows the present cylindrical acoustoelastic equations to be applied to more general situations in which the orthotropic material axes are rotated with respect to the geometric axes, without altering the governing equations themselves. The restriction to aligned coordinate systems is therefore adopted solely for clarity and notational simplicity and does not constitute a limitation of the underlying tensorial formulation. For orthotropic materials, this tensor transformation is equivalent to the well-known Bond matrix transformation commonly used in anisotropic elasticity. Detailed matrix representations can be found, for example, in Section 2.2 of [41].

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Under this symmetry and configuration, for the natural configuration, only 21 independent components of the coefficient tensor $A_{\alpha\beta\gamma\delta}$ remain which are listed in the Appendix. Substituting into Eq. (3.17e), the wave equations in cylindrical coordinates become:

$$\begin{aligned}\rho^0 \frac{\partial^2 u_r}{\partial t^2} &= A_{1111}\frac{\partial^2 u_r}{\partial r^2} + \frac{A_{1212}}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + A_{1313}\frac{\partial^2 u_r}{\partial z^2} \\ &+ \frac{A_{1122}+A_{1221}}{r}\frac{\partial^2 u_\theta}{\partial r\,\partial\theta} + \left(A_{1331}+A_{1133}\right)\frac{\partial^2 u_z}{\partial r\,\partial z} \\ &+ \frac{A_{1111}+A_{1122}-A_{2211}}{r}\frac{\partial u_r}{\partial r} - \frac{A_{1212}+A_{2222}}{r^2}\frac{\partial u_\theta}{\partial\theta} \\ &+ \frac{A_{1133}-A_{2233}}{r}\frac{\partial u_z}{\partial z} - \frac{A_{2222}}{r^2}u_r\end{aligned} \tag{4.8a}$$

$$\begin{aligned}\rho^0 \frac{\partial^2 u_\theta}{\partial t^2} &= A_{2121}\frac{\partial^2 u_\theta}{\partial r^2} + \frac{A_{2222}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + A_{2323}\frac{\partial^2 u_\theta}{\partial z^2} \\ &+ \frac{A_{2112}+A_{2211}}{r}\frac{\partial^2 u_r}{\partial r\,\partial\theta} + \frac{A_{2233}+A_{2332}}{r}\frac{\partial^2 u_z}{\partial\theta\,\partial z} \\ &+ \frac{A_{2222}+A_{1212}}{r^2}\frac{\partial u_r}{\partial\theta} \\ &+ \frac{A_{2121}+A_{1221}-A_{2112}}{r}\frac{\partial u_\theta}{\partial r} - \frac{A_{1212}}{r^2}u_\theta\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\rho^0 \frac{\partial^2 u_z}{\partial t^2} &= A_{3131}\frac{\partial^2 u_z}{\partial r^2} + \frac{A_{3232}}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + A_{3333}\frac{\partial^2 u_z}{\partial z^2} \\ &+ \left(A_{3113}+A_{3311}\right)\frac{\partial^2 u_r}{\partial r\,\partial z} + \frac{A_{3223}+A_{3322}}{r}\frac{\partial^2 u_\theta}{\partial\theta\,\partial z} \\ &+ \frac{A_{3322}+A_{3113}}{r}\frac{\partial u_r}{\partial z} + \frac{A_{3131}}{r}\frac{\partial u_z}{\partial r}\end{aligned} \tag{4.8c}$$

For the initial configuration, the coefficient tensor B_{IJKL} is major-symmetric and only 15 independent components remain which are listed in the Appendix. Substituting into Eq. (3.31d), we obtain:

$$\begin{aligned}\rho^{\mathrm{i}} \frac{\partial^2 U_r}{\partial t^2} &= B_{1111}\frac{\partial^2 U_r}{\partial r^2} + \frac{B_{1212}}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} + B_{1313}\frac{\partial^2 U_r}{\partial z^2} \\ &+ \frac{B_{1122}+B_{1221}}{r}\frac{\partial^2 U_\theta}{\partial r\,\partial\theta} + \left(B_{1331}+B_{1133}\right)\frac{\partial^2 U_z}{\partial r\,\partial z} \\ &+ \frac{B_{1111}}{r}\frac{\partial U_r}{\partial r} - \frac{B_{1212}+B_{2222}}{r^2}\frac{\partial U_\theta}{\partial\theta} \\ &+ \frac{B_{1133}-B_{2233}}{r}\frac{\partial U_z}{\partial z} - \frac{B_{2222}}{r^2}U_r\end{aligned} \tag{4.9a}$$

$$\begin{aligned}\rho^{\mathrm{i}} \frac{\partial^2 U_\theta}{\partial t^2} &= B_{2121}\frac{\partial^2 U_\theta}{\partial r^2} + \frac{B_{2222}}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + B_{2323}\frac{\partial^2 U_\theta}{\partial z^2} \\ &+ \frac{B_{1122}+B_{1221}}{r}\frac{\partial^2 U_r}{\partial r\,\partial\theta} + \frac{B_{2233}+B_{2332}}{r}\frac{\partial^2 U_z}{\partial\theta\,\partial z} \\ &+ \frac{B_{2222}+B_{1212}}{r^2}\frac{\partial U_r}{\partial\theta} + \frac{B_{2121}}{r}\frac{\partial U_\theta}{\partial r} - \frac{B_{1212}}{r^2}U_\theta\end{aligned} \tag{4.9b}$$

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$$\begin{aligned}\rho^{\mathrm{i}}\frac{\partial^2 U_z}{\partial t^2} &= B_{3131}\frac{\partial^2 U_z}{\partial r^2}+\frac{B_{3232}}{r^2}\frac{\partial^2 U_z}{\partial \theta^2}+B_{3333}\frac{\partial^2 U_z}{\partial z^2}\\ &+\left(B_{1133}+B_{1331}\right)\frac{\partial^2 U_r}{\partial r\,\partial z}+\frac{B_{2233}+B_{2332}}{r}\frac{\partial^2 U_\theta}{\partial \theta\,\partial z}\\ &+\frac{B_{1331}+B_{2233}}{r}\frac{\partial U_r}{\partial z}+\frac{B_{3131}}{r}\frac{\partial U_z}{\partial r}\end{aligned} \quad (4.9c)$$

Eqs (4.8)-(4.9) constitute the acoustoelastic formulations in cylindrical coordinates attached to natural and initial configuration, respectively.

Although the acoustoelastic equations expressed in cylindrical coordinates—specifically Eqs. (4.8) and (4.9)—do not contain higher-order terms of the displacement field, they are nevertheless derived from a fundamentally nonlinear analysis. As previously emphasized, this nonlinearity is not manifested in the form of the equations, but rather embedded in their coefficients, which explicitly depend on both the initial stress components and the third-order elastic constants, as detailed in the Appendix. These coefficients reflect the geometric and material nonlinearities retained in the small-on-large formulation and are essential for capturing the first-order influence of initial stress on wave propagation.

The acoustoelastic equations derived in this chapter, particularly Eqs. (4.8) and (4.9), are not merely of theoretical interest—they are also experimentally verifiable. A particularly suitable application for validation is the use of Rayleigh surface waves for stress evaluation on cylindrical surfaces. In traditional approaches, acoustoelastic coefficients and wave speeds are derived under the assumption of planar geometry within the Cartesian coordinate system, and are then directly applied to cylindrical components such as pipes or shafts. This simplification neglects the geometric curvature, which can introduce errors, especially in cases involving small radii or significant stress gradients near the surface.

By contrast, our formulation explicitly accounts for the curvature effects through the use of cylindrical coordinates. The resulting acoustoelastic equations predict modified dispersion relations, phase velocities, and stress sensitivity coefficients of surface waves, which differ from those predicted under planar assumptions. These differences, although subtle, are physically meaningful and can be quantified experimentally.

5. Conclusions

In this work, a general acoustoelastic formulation has been developed within the framework of small-on-large theory. The governing equations are derived in a fully tensorial form, making the formulation independent of any specific coordinate system or material symmetry assumptions. This coordinate-free representation clarifies the intrinsic structure of acoustoelasticity and allows the governing equations to be consistently specialized to arbitrary curvilinear coordinate systems through standard tensor transformations.

To facilitate practical applications, the general equations were explicitly expanded in cylindrical coordinates for orthotropic materials, which are representative of many engineering components such as rods, pipes, pressure vessels, and boreholes. Unlike conventional approaches that directly apply Cartesian-based acoustoelastic results to curved geometries, the present formulation explicitly accounts for geometric curvature through the metric and connection coefficients, leading to potentially more accurate predictions of wave velocity and acoustoelastic coefficients for surface and guided waves on curved structures.

The proposed theory also provides a clear framework for experimental and numerical validation, by comparing predicted stress-dependent wave speeds with experimental measurements on prestressed specimens. In particular, surface wave propagation on cylindrical specimens subjected to known initial stresses offers a direct means of assessing the influence of curvature on acoustoelastic response. By comparing experimentally measured wave velocities with predictions from both the

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present cylindrical formulation and classical Cartesian-based models, the importance of geometric consistency and tensor-based descriptions can be quantitatively evaluated. Such studies will form the basis of future work.

From a practical perspective, the main advantage of the proposed model lies in its generality and extensibility: it accommodates anisotropic materials, arbitrary initial stress states, and different coordinate systems within a unified framework. The primary limitation is the increased mathematical complexity associated with curvilinear coordinates and higher-order tensor transformations, which may require additional computational effort in implementation. Nevertheless, this trade-off is justified in applications where curvature, anisotropy, or complex stress states play a critical role.

Overall, this study establishes a rigorous and flexible theoretical foundation for acoustoelastic analysis beyond the Cartesian setting, and it opens new possibilities for stress evaluation and material characterization in curved and anisotropic structures.

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Appendix A

1. Expressions of $A_{\alpha\beta\gamma\delta}$ components for Eq. (4.8).

$$A_{1111} = C_{11} + 2C_{11}e^{i}_{rr} + D_{111}e^{i}_{rr} + D_{112}e^{i}_{\theta\theta} + D_{113}e^{i}_{zz} + \sigma^{i}_{rr}$$
$$A_{2222} = C_{22} + 2C_{22}e^{i}_{\theta\theta} + D_{122}e^{i}_{rr} + D_{222}e^{i}_{\theta\theta} + D_{223}e^{i}_{zz} + \sigma^{i}_{\theta\theta}$$
$$A_{3333} = C_{33} + 2C_{33}e^{i}_{zz} + D_{133}e^{i}_{rr} + D_{233}e^{i}_{\theta\theta} + D_{333}e^{i}_{zz} + \sigma^{i}_{zz}$$

$$A_{1122} = C_{12} + C_{12}e^{i}_{\theta\theta} + C_{66}e^{i}_{\theta\theta} + D_{112}e^{i}_{rr} + D_{122}e^{i}_{\theta\theta} + D_{123}e^{i}_{zz}$$
$$A_{2233} = C_{23} + C_{23}e^{i}_{zz} + C_{44}e^{i}_{zz} + D_{123}e^{i}_{rr} + D_{223}e^{i}_{\theta\theta} + D_{233}e^{i}_{zz}$$
$$A_{3311} = C_{31} + C_{31}e^{i}_{rr} + C_{55}e^{i}_{rr} + D_{113}e^{i}_{rr} + D_{123}e^{i}_{\theta\theta} + D_{133}e^{i}_{zz}$$

$$A_{2211} = C_{12} + C_{12}e^{i}_{rr} + C_{66}e^{i}_{rr} + D_{112}e^{i}_{rr} + D_{122}e^{i}_{\theta\theta} + D_{123}e^{i}_{zz}$$
$$A_{3322} = C_{23} + C_{23}e^{i}_{\theta\theta} + C_{44}e^{i}_{\theta\theta} + D_{123}e^{i}_{rr} + D_{223}e^{i}_{\theta\theta} + D_{233}e^{i}_{zz}$$
$$A_{1133} = C_{31} + C_{31}e^{i}_{zz} + C_{55}e^{i}_{zz} + D_{113}e^{i}_{rr} + D_{123}e^{i}_{\theta\theta} + D_{133}e^{i}_{zz}$$

$$A_{2323} = C_{44} + C_{44}e^{i}_{zz} + C_{44}e^{i}_{\theta\theta} + D_{144}e^{i}_{rr} + D_{244}e^{i}_{\theta\theta} + D_{344}e^{i}_{zz} + \sigma^{i}_{zz}$$
$$A_{3131} = C_{55} + C_{55}e^{i}_{rr} + C_{55}e^{i}_{zz} + D_{155}e^{i}_{rr} + D_{255}e^{i}_{\theta\theta} + D_{355}e^{i}_{zz} + \sigma^{i}_{rr}$$
$$A_{1212} = C_{66} + C_{66}e^{i}_{rr} + C_{66}e^{i}_{\theta\theta} + D_{166}e^{i}_{rr} + D_{266}e^{i}_{\theta\theta} + D_{366}e^{i}_{zz} + \sigma^{i}_{\theta\theta}$$

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$$A_{2332} = C_{44} + C_{44}e^{i}_{zz} + C_{23}e^{i}_{\theta\theta} + D_{144}e^{i}_{rr} + D_{244}e^{i}_{\theta\theta} + D_{344}e^{i}_{zz}$$
$$A_{1331} = C_{55} + C_{55}e^{i}_{zz} + C_{31}e^{i}_{rr} + D_{155}e^{i}_{rr} + D_{255}e^{i}_{\theta\theta} + D_{355}e^{i}_{zz}$$
$$A_{1221} = C_{66} + C_{66}e^{i}_{\theta\theta} + C_{12}e^{i}_{rr} + D_{166}e^{i}_{rr} + D_{266}e^{i}_{\theta\theta} + D_{366}e^{i}_{zz}$$

$$A_{3232} = C_{44} + C_{44}e^{i}_{\theta\theta} + C_{44}e^{i}_{zz} + D_{144}e^{i}_{rr} + D_{244}e^{i}_{\theta\theta} + D_{344}e^{i}_{zz} + \sigma^{i}_{\theta\theta}$$
$$A_{1313} = C_{55} + C_{55}e^{i}_{rr} + C_{55}e^{i}_{zz} + D_{155}e^{i}_{rr} + D_{255}e^{i}_{\theta\theta} + D_{355}e^{i}_{zz} + \sigma^{i}_{zz}$$
$$A_{2121} = C_{66} + C_{66}e^{i}_{rr} + C_{66}e^{i}_{\theta\theta} + D_{166}e^{i}_{rr} + D_{266}e^{i}_{\theta\theta} + D_{366}e^{i}_{zz} + \sigma^{i}_{rr}$$

$$A_{3223} = C_{44} + C_{44}e^{i}_{\theta\theta} + C_{23}e^{i}_{zz} + D_{144}e^{i}_{rr} + D_{244}e^{i}_{\theta\theta} + D_{344}e^{i}_{zz}$$
$$A_{3113} = C_{55} + C_{55}e^{i}_{rr} + C_{31}e^{i}_{zz} + D_{155}e^{i}_{rr} + D_{255}e^{i}_{\theta\theta} + D_{355}e^{i}_{zz}$$
$$A_{2112} = C_{66} + C_{66}e^{i}_{rr} + C_{12}e^{i}_{\theta\theta} + D_{166}e^{i}_{rr} + D_{266}e^{i}_{\theta\theta} + D_{366}e^{i}_{zz}$$

$$e^{i}_{rr} = \frac{\partial u^{i}_{r}}{\partial r}, \quad e^{i}_{\theta\theta} = \frac{\partial u^{i}_{\theta}}{r\partial\theta} + \frac{u^{i}_{r}}{r}, \quad e^{i}_{zz} = \frac{\partial u^{i}_{z}}{\partial z}$$

2. Expressions of B_{IJKL} components for Eq. (4.9).

$$B_{1111} = C_{11} - C_{11}e^{i}_{\theta\theta} + 3C_{11}e^{i}_{rr} - C_{11}e^{i}_{zz} + D_{112}e^{i}_{\theta\theta} + D_{111}e^{i}_{rr} + D_{113}e^{i}_{zz} + \sigma^{i}_{rr}$$

$$B_{2222} = C_{22} + 3C_{22}e^{i}_{\theta\theta} - C_{22}e^{i}_{rr} - C_{22}e^{i}_{zz} + D_{222}e^{i}_{\theta\theta} + D_{122}e^{i}_{rr} + D_{223}e^{i}_{zz} + \sigma^{i}_{\theta\theta}$$

$$B_{3333} = C_{33} - C_{33}e^{i}_{\theta\theta} - C_{33}e^{i}_{rr} + 3C_{33}e^{i}_{zz} + D_{233}e^{i}_{\theta\theta} + D_{133}e^{i}_{rr} + D_{333}e^{i}_{zz} + \sigma^{i}_{zz}$$

$$B_{1122} = C_{12} + C_{12}e^{i}_{\theta\theta} + C_{12}e^{i}_{rr} - C_{12}e^{i}_{zz} + D_{122}e^{i}_{\theta\theta} + D_{112}e^{i}_{rr} + D_{123}e^{i}_{zz}$$

$$B_{2233} = C_{23} + C_{23}e^{i}_{\theta\theta} - C_{23}e^{i}_{rr} + C_{23}e^{i}_{zz} + D_{223}e^{i}_{\theta\theta} + D_{123}e^{i}_{rr} + D_{233}e^{i}_{zz}$$

$$B_{3311} = C_{31} - C_{31}e^{i}_{\theta\theta} + C_{31}e^{i}_{rr} + C_{31}e^{i}_{zz} + D_{123}e^{i}_{\theta\theta} + D_{113}e^{i}_{rr} + D_{133}e^{i}_{zz}$$

$$B_{2323} = C_{44} + C_{44}e^{i}_{\theta\theta} - C_{44}e^{i}_{rr} + C_{44}e^{i}_{zz} + D_{244}e^{i}_{\theta\theta} + D_{144}e^{i}_{rr} + D_{344}e^{i}_{zz} + \sigma^{i}_{zz}$$

$$B_{1313} = C_{55} - C_{55}e^{i}_{\theta\theta} + C_{55}e^{i}_{rr} + C_{55}e^{i}_{zz} + D_{255}e^{i}_{\theta\theta} + D_{155}e^{i}_{rr} + D_{355}e^{i}_{zz} + \sigma^{i}_{zz}$$

$$B_{1212} = C_{66} + C_{66}e^{i}_{\theta\theta} + C_{66}e^{i}_{rr} - C_{66}e^{i}_{zz} + D_{266}e^{i}_{\theta\theta} + D_{166}e^{i}_{rr} + D_{366}e_{zz} + \sigma^{i}_{\theta\theta}$$

$$B_{2332} = B_{2323} - \sigma^{i}_{zz}; B_{1331} = B_{1313} - \sigma^{i}_{zz}; B_{1221} = B_{1212} - \sigma^{i}_{\theta\theta}$$

$$e^{i}_{rr} = \frac{\partial U^{i}_{r}}{\partial r}, \quad e^{i}_{\theta\theta} = \frac{\partial U^{i}_{\theta}}{r\partial\theta} + \frac{U^{i}_{r}}{r}, \quad e^{i}_{zz} = \frac{\partial U^{i}_{z}}{\partial z}$$

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