Article

Not peer-reviewed version

Momentum Conservation in a Charged Retarded Field Engine

Prachi Sharma and Asher Yahalom *

Posted Date: 29 October 2024

doi: 10.20944/preprints202410.2252.v1

Keywords: Newton's third law; Electromagnetism; Relativity

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

Article

Momentum Conservation in a Charged Retarded Field Engine

Prachi Sharma [1] and Asher Yahalom [1,2,*]

[1] Ariel University, Faulty of Engineering, Department of Electrical & Electronic Engineering, Ariel 40700, Israel
[2] Center for Astrophysics, Geophysics, and Space Sciences (AGASS), Ariel University, Ariel 40700, Israel
* Correspondence: asya@ariel.ac.il; Tel.: +972-54-7740294

Abstract: Addressing the potential conflicts between momentum conservation laws and retardation requires a comprehensive understanding of both principles within the context of specific physical systems and theoretical frameworks. Indeed it was claimed that a relativistic engine is impossible due to the lack of linear momentum conservation. The purpose of this work is to settle this issue by demonstrating that linear momentum is indeed conserved for a charged retarded field engine. Earlier research has already introduced the concept of a charged retarded engine. The system gains mechanical momentum and energy when it is subjected to a total force for a certain amount of time. If retardation is taken into account a total force is present even if no external force affects the system, thus momentum seems to appear from nowhere violating the law of momentum conservation. However, momentum is indeed conserved as the present work demonstrates. We present a mathematical proof that the momentum is conserved in a time dependent electromagnetic retarded motor. The expression of field momentum comes out to be equal and opposite to the mechanical momentum gained by the material system as obtain in [1]. This calculation of charged relativistic engine is an extension work of papers published earlier [2,3] which explored a non-charged relativistic engine. We prove that the total momentum in a relativistic motor including both material and electromagnetic linear momentum is indeed conserved as dictated by Noether's theorem.

Keywords: Newton's third law; electromagnetism; relativity

---

1. Introduction

The foundation of modern science was established by Newton's laws of motion. These rules explain how motion reacts to acting forces and how those forces relate to one another. In his 1687 publication Philosophiae Naturalis Principia Mathematica, Isaac Newton formulated three fundamental rules [4–6]. Here we deal with the third law only which state: When a body applies force to another, the second body reciprocally applies a force of equal magnitude but opposite direction onto the first body. As per Newton's third law, in a system unaffected by external forces, the total force sum is zero. This principle has undergone numerous experimental validations and stands as a fundamental pillar of physics. However, due to the finite speed of signal propagation, it is evident from the theory of relativity that an action and its reaction cannot be formed simultaneously. In accordance with the theory of relativity, it is postulated that no object, message, signal (regardless of its nature, even if non-electromagnetic), or field can exceed the speed of light in a vacuum [7–9]. As a result, the forces cannot add up to zero. However, as Griffiths & Heald [10] pointed out, retardation can be disregarded in the quasi-static approximation.

The majority of contemporary engines operate on the principle that two material components acquire momentum, one of which is equal to and opposite from the momentum acquired by the other (for example a rocket that propels itself by ejecting matter). Nevertheless, relativistic effects indicate a novel motor design where the system doesn't consist of two material bodies, but rather a material body in conjunction with a field. In [1] we thoroughly elucidated the concept of a relativistic engine, delving into its significance for space exploration [11,12]. A retarded engine is a type of propulsion system where the motion of the center of mass is achieved through the interaction between

 doi:10.20944/preprints202410.2252.v1

its internal components, which may either move relative to one another or remain fixed within a rigid structure. The focus is primarily on the movement of the center of mass, which can occur in all directions, including vertical motion. Unlike traditional engines, a retarded engine doesn't have moving mechanical parts or rely on conventional fuel, thereby eliminating the need for fuel combustion and reducing carbon emissions. It operates by harnessing electromagnetic energy, such as from solar panels, making it especially advantageous for space travel, where large amounts of fuel storage are typically required. This approach offers a cleaner, more efficient way to power spacecraft.

Yet some scholars claim that such device cannot exist as it does not respect conservation of linear momentum, it is especially those scholars for which the current paper is written. In a nutshell, linear momentum is conserved in such a device if one takes both mechanical linear momentum and electromagnetic linear momentum into account.

Griffiths and Heald [13] noted that the laws of Coulomb and Biot-Savart govern the configurations of electric and magnetic fields exclusively for stationary sources. Time-dependent extensions of these laws, as outlined by [14], were employed to explore the validity of Coulomb and Biot-Savart formulas beyond static conditions.

In an earlier work, the author addressed the force between two current-carrying coils [2] using Jefimenko's [8,14] equation. Later on, this was extended to encompass the relationship between a permanent magnet and a current-carrying loop [15]. The earlier calculations have showed the expressions for mechanical momentum and field momentum when dealing with macroscopically neutral bodies and verified the conservation of momentum [3]. However, in the case of charged relativistic engine, momentum conservation was still a question. Energy conservation in an uncharged relativistic motor was also discussed [16].

The case of a charged relativistic motor was discussed in [1] but without discussing in detail the problem of momentum conservation.

In the present work, we have calculated the field's momentum when electromagnetic fields are time dependent and compared with the mechanical momentum of the same system which comprises of two arbitrary charged bodies. The main result of this work is that the field momentum for a two charged body system is equal and opposite to the mechanical momentum gained by the material components. This result is not new, as it was already pointed out by a few authors [17–21]. In particular Feynman [9] describes two orthogonally moving charges, apparently contradicting Newton's third law, as the forces that the charges induce do not cancel (last part of 26-2); this is resolved in 27-6 in which it is noticed that the momentum gained by the two-charge system is lost to the field. However, here we derive a general expression for field momentum that is applicable to any charge density and current density and not just point particles and prove mathematically that the total linear momentum of matter and field is indeed conserved. This result can be applied also to quantum systems [22].

2. Electromagnetic Field Momentum

According to Noether's theorem, momentum conservation is inherent in any system exhibiting translational symmetry. When considering a system involving charge and current densities, the expression of momentum conservation law manifests in the following manner.

$$\frac{d\vec{P}_{mechi}}{dt} + \frac{d\vec{P}_{fieldi}}{dt} = \oint_S T_{ij} n_j da \tag{1}$$

In the above, $\vec{P}_{mechi}$ represents the i-th component of the mechanical momentum within the system, while $\vec{P}_{fieldi}$ denotes the i-th component of the field momentum. T_{ij} stands for the Maxwell stress tensor. S denotes a closed surface that encloses the volume where the system is situated, with $\hat{n}$ representing a

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

unit vector perpendicular to the surface. Additionally, the notation assumes Einstein's summation convention. T_{ij}, Maxwell stress tensor can be calculated from the electromagnetic fields as follows:

$$T_{ij} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \left(\vec{E}^2 + c^2 \vec{B}^2 \right) \delta_{ij} \right]. \tag{2}$$

Here, $\epsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1} is the vacuum permittivity, c is the velocity of light and δ_{ij} is Kronecker's delta.

The field linear momentum is defined to be:

$$\vec{P}_{field} \equiv \epsilon_0 \int \vec{E} \times \vec{B} d^3 x. \tag{3}$$

Let us suppose we have two subsystems labeled by 1 and 2, positioned far apart from each other to the extent that their interaction can be neglected. Under this condition, Equation (1) holds true for each subsystem individually, indicating:

$$\frac{dP_{mech\ 1\ i}}{dt} + \frac{dP_{field\ 1\ i}}{dt} = \oint_S T_{1\ ij} n_j da \tag{4}$$

$$\frac{dP_{mech\ 2\ i}}{dt} + \frac{dP_{field\ 2\ i}}{dt} = \oint_S T_{2\ ij} n_j da. \tag{5}$$

We will assume that the mechanical momentum of each subsystem is insignificant.

Next we will explore the possibility of interaction between them while ensuring that it doesn't alter the charge and current densities of either subsystem. The total fields of the combined system are:

$$\vec{E} = \vec{E}_1 + \vec{E}_2, \qquad \vec{B} = \vec{B}_1 + \vec{B}_2 \tag{6}$$

As both Equation (3) governing field momenta and Equation (2) describing the Maxwell stress tensor are quadratic in the fields, the following outcome is attained.

$$\begin{aligned}
\vec{P}_{field} &= \epsilon_0 \int \vec{E} \times \vec{B} d^3 x = \vec{P}_{field\ 1} + \vec{P}_{field\ 2} + \vec{P}_{field\ 12} + \vec{P}_{field\ 21} \\
\vec{P}_{field\ 1} &\equiv \epsilon_0 \int \vec{E}_1 \times \vec{B}_1 d^3 x \\
\vec{P}_{field\ 2} &\equiv \epsilon_0 \int \vec{E}_2 \times \vec{B}_2 d^3 x \\
\vec{P}_{field\ 12} &\equiv \epsilon_0 \int \vec{E}_1 \times \vec{B}_2 d^3 x \\
\vec{P}_{field\ 21} &\equiv \epsilon_0 \int \vec{E}_2 \times \vec{B}_1 d^3 x
\end{aligned} \tag{7}$$

$$\begin{aligned}
T_{ij} &\equiv \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} (\vec{E}^2 + c^2 \vec{B}^2) \delta_{ij} \right] \\
&= T_{1\ ij} + T_{2\ ij} + 2 T_{12\ ij} \\
T_{1\ ij} &\equiv \epsilon_0 \left[E_{1\ i} E_{1\ j} + c^2 B_{1\ i} B_{1\ j} - \frac{1}{2} (\vec{E}_1^2 + c^2 \vec{B}_1^2) \delta_{ij} \right] \\
T_{2\ ij} &\equiv \epsilon_0 \left[E_{2\ i} E_{2\ j} + c^2 B_{2\ i} B_{2\ j} - \frac{1}{2} (\vec{E}_2^2 + c^2 \vec{B}_2^2) \delta_{ij} \right] \\
T_{12\ ij} &\equiv \epsilon_0 \left[E_{1\ i} E_{2\ j} + c^2 B_{1\ i} B_{2\ j} - \frac{1}{2} (\vec{E}_1 \cdot \vec{E}_2 + c^2 \vec{B}_1 \cdot \vec{B}_2) \delta_{ij} \right]
\end{aligned} \tag{8}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

Subtracting the expressions given in Equation (4) and Equation (5) from Equation (1):

$$\frac{d(P_{mech\ i} - P_{mech\ 1\ i} - P_{mech\ 2\ i})}{dt} + \frac{d(P_{field\ i} - P_{field\ 1\ i} - P_{field\ 2\ i})}{dt}$$
$$= \oint_S (T_{ij} - T_{1\ ij} - T_{2\ ij}) n_j da, \tag{9}$$

now with the help of Equation (7) and Equation (8) we obtain:

$$\frac{d(P_{mech\ i} - P_{mech\ 1\ i} - P_{mech\ 2\ i})}{dt} + \frac{d(P_{field\ 12\ i} + P_{field\ 21\ i})}{dt}$$
$$= 2 \oint_S T_{12\ ij} n_j da \tag{10}$$

Next, we presume that the mechanical momentum produced within each subsystem is insignificant compared to the mechanical momentum generated in one subsystem as a result of the fields produced in the other subsystem, and vice versa. Consequently, the self-generated mechanical momenta are negligible, yielding:

$$\frac{dP_{mech\ i}}{dt} + \frac{d(P_{field\ 12\ i} + P_{field\ 21\ i})}{dt} \simeq 2 \oint_S T_{12\ ij} n_j da \tag{11}$$

It is well known that the exact expressions for both the electric and magnetic fields [8,14] (see also Equation (15) and Equation (16) below) contain terms which reduce as $\frac{1}{R^2}$ and terms which reduce as $\frac{1}{R}$, the later are proportion to at least $\frac{1}{c}$ (and higher powers of $\frac{1}{c}$). Now the surface area grows as R^2, thus the surface integral will converge at large distances to $R^2 \frac{1}{R^4} = \frac{1}{R^2} \to 0$ for any square of $\frac{1}{R^2}$ terms. For mixed terms of the $\frac{1}{R^2}$ and $\frac{1}{R}$ type it will converge at large distances to $R^2 \frac{1}{R^3} = \frac{1}{R} \to 0$. Thus at infinity we will be left with a multiplication of $\frac{1}{R}$ terms which must be proportion to $\frac{1}{c^2}$ and thus will be neglected in the current approximation. Therefore:

$$\frac{d(\vec{P}_{mech} + \vec{P}_{field\ 12} + \vec{P}_{field\ 21})}{dt} \simeq 0 \tag{12}$$

Hence provided that there is no field or mechanical momenta at $t = 0$ we arrive at the result:

$$\vec{P}_{mech} \simeq -\vec{P}_{field\ 12} - \vec{P}_{field\ 21}. \tag{13}$$

3. Retarded Field Momentum of Two Subsystems

Consider two bodies having volume elements d^3x_1, d^3x_2 located at $\vec{x}_1, \vec{x}_2$, having charge densities ρ_1, ρ_2 and current densities $\vec{J}_1, \vec{J}_2$ respectively. Now, We will calculate the integral:

$$\vec{P}_{field12} \equiv \epsilon_0 \int \vec{E}_1 \times \vec{B}_2 d^3x. \tag{14}$$

The expressions for electric field and magnetic field in the case of charged relativistic motor is given by [8,14]:

$$\vec{E}_1(\vec{x}, t) = -k \int d^3x_1 \frac{1}{R_1^2} \left[\left(\rho_1(\vec{x_1}, t_{ret}) + \left(\frac{R_1}{c} \right) \partial_t \rho_1(\vec{x_1}, t_{ret}) \right) \hat{R}_1 \right.$$
$$\left. + \left(\frac{R_1}{c} \right)^2 \frac{\partial_t \vec{J}_1(\vec{x_1}, t_{ret})}{R_1} \right], \tag{15}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

And

$$\vec{B}_2(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x_2 \left[\frac{\hat{R}_2}{R_2^2} \times \left(\vec{J}_2(\vec{x}_2,t_{ret}) + \left(\frac{R_2}{c}\right)\partial_t \vec{J}_2(\vec{x}_2,t_{ret})\right)\right]. \tag{16}$$

Here $k = \frac{1}{4\pi\epsilon_0}$, $\mu_0 = 4\pi \times 10^{-7}$ is the vacuum magnetic permeability, $\vec{R}_1 = \vec{x} - \vec{x}_1$ and $\vec{R}_2 = \vec{x} - \vec{x}_2$. So, substituting Equation (15) and Equation (16) in Equation (14), we obtain:

$$\begin{aligned}\vec{P}_{field12} = -\frac{\mu_0}{16\pi^2}\int \Bigg[&\Bigg(\int d^3x_1 \frac{1}{R_1^2}\Bigg(\left(\rho_1(\vec{x}_1,t_{ret}) + \left(\frac{R_1}{c}\right)\partial_t\rho_1(\vec{x}_1,t_{ret})\right)\hat{R}_1 \\ &+ \left(\frac{R_1}{c}\right)^2 \frac{\partial_t \vec{J}_1(\vec{x}_1,t_{ret})}{R_1}\Bigg)\Bigg) \\ &\times \left(\int d^3x_2 \frac{\hat{R}_2}{R_2^2} \times \left(\vec{J}_2(\vec{x}_2,t_{ret}) + \left(\frac{R_2}{c}\right)\partial_t \vec{J}_2(\vec{x}_2,t_{ret})\right)\right)\Bigg] d^3x. \end{aligned}\tag{17}$$

Neglecting all terms proportional to c^{-1} and consequently approximating any retarded quantity X such that $X(\vec{x},t_{ret}) \simeq X(\vec{x},t)$ we obtain:

$$\vec{P}_{field12} \simeq -\frac{\mu_0}{16\pi^2}\int\left[\int\int d^3x_1 d^3x_2 \left(\rho_1 \frac{\hat{R}_1}{R_1^2} \times \left(\frac{\hat{R}_2}{R_2^2} \times \vec{J}_2\right)\right)\right] d^3x, \tag{18}$$

Using the vector identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - (\vec{A}.\vec{B})\vec{C}, \tag{19}$$

we may write:

$$\vec{P}_{field12} \simeq -\frac{\mu_0}{16\pi^2}\int\left[\int\int d^3x_1 d^3x_2 \frac{\hat{R}_2}{R_2^2}\left(\rho_1 \frac{\hat{R}_1}{R_1^2}\cdot \vec{J}_2\right) - \frac{\rho_1}{R_1^2 R_2^2}\left(\hat{R}_1 \cdot \hat{R}_2\right)\vec{J}_2\right] d^3x. \tag{20}$$

Thus we may write the l component of the electromagnetic momentum in the Einstein summation convention in the form:

$$\begin{aligned} P_{field12\ l} \quad \simeq \quad & -\frac{\mu_0}{16\pi^2}\int\int d^3x_1 d^3x_2 \rho_1 \left[\left(\int d^3x \frac{\hat{R}_{2\ l}}{R_2^2}\frac{\hat{R}_{1\ k}}{R_1^2}\right) J_{2\ k} \right. \\ & \left. - \left(\int d^3x \frac{\hat{R}_1 \cdot \hat{R}_2}{R_1^2 R_2^2}\right) J_{2\ l}\right]. \end{aligned}\tag{21}$$

Now using the identities:

$$\frac{\hat{R}_1}{R_1^2} = \vec{\nabla}\left(\frac{1}{R_1}\right), \qquad \frac{\hat{R}_1}{R_1^2} = -\vec{\nabla}_{\vec{x}_1}\left(\frac{1}{R_1}\right) \tag{22}$$

$$\frac{\hat{R}_2}{R_2^2} = \vec{\nabla}\left(\frac{1}{R_2}\right), \qquad \frac{\hat{R}_2}{R_2^2} = -\vec{\nabla}_{\vec{x}_2}\left(\frac{1}{R_2}\right) \tag{23}$$

We may write:

$$I_{kl} \equiv \int d^3x \frac{\hat{R}_{2\ l}}{R_2^2}\frac{\hat{R}_{1\ k}}{R_1^2} = -\partial_{2\ l}\left(\int d^3x \frac{1}{R_2}\partial_k \frac{1}{R_1}\right). \tag{24}$$

In which we use the shortened notation:

$$\partial_k \equiv \frac{\partial}{\partial_{x_k}} = \vec{\nabla}_k, \qquad \partial_{1\ k} \equiv \frac{\partial}{\partial_{x_{1\ k}}} = \vec{\nabla}_{\vec{x}_{1\ k}}, \qquad \partial_{2\ k} \equiv \frac{\partial}{\partial_{x_{2\ k}}} = \vec{\nabla}_{\vec{x}_{2\ k}} \tag{25}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

Next we introduce the identity (the derivation is not given here but can be found in [3])

$$\vec{G}(\vec{x}_1, \vec{x}_2) \equiv \int \frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} d^3x = -2\pi \hat{R}, \tag{26}$$

in which:

$$\vec{R} \equiv \vec{R}_{12} \equiv \vec{x}_1 - \vec{x}_2, \quad R \equiv |\vec{R}|, \quad \hat{R} \equiv \frac{\vec{R}}{R}. \tag{27}$$

Interchanging the labels 1 and 2 and using index notation we obtain:

$$\int d^3x \frac{1}{R_2} \partial_k \frac{1}{R_1} = +2\pi \hat{R}_k \tag{28}$$

Inserting Equation (28) into Equation (24) we may write I_{kl} as an explicit expression rather than an integral:

$$I_{kl} = -2\pi \partial_{2\ l} \hat{R}_k. \tag{29}$$

By explicit substitution we show below that I_{kl} is symmetric with respect to the interchange of k and l:

$$\partial_{2\ l} \hat{R}_k = \partial_{2\ l} \left(\frac{\vec{R}_k}{R} \right) = -\frac{\delta_{lk}}{R} + \vec{R}_k \partial_{2\ l} \left(\frac{1}{R} \right) = -\frac{\delta_{lk}}{R} + \vec{R}_k \left(-\frac{\hat{R}_l}{R^2} \right) = -\frac{\delta_{lk}}{R} - \frac{\hat{R}_k \hat{R}_l}{R}, \tag{30}$$

in the above δ_{lk} is Kronecker's delta. Thus:

$$I_{kl} = -2\pi \partial_{2\ l} \hat{R}_k = -2\pi \partial_{2\ k} \hat{R}_l. \tag{31}$$

We now turn our attention to the second integral of the type $\int d^3x$ appearing in Equation (21):

$$II = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3x = \int \left(\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} \right) d^3x. \tag{32}$$

Using the identity:

$$\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} = \vec{\nabla} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) - \frac{1}{R_1} \vec{\nabla}^2 \frac{1}{R_2}, \tag{33}$$

and taking into account that [8]:

$$\vec{\nabla}^2 \frac{1}{R_2} = -4\pi \delta(\vec{R}_2), \tag{34}$$

in which $\delta(\vec{R}_2)$ is a three dimensional delta function. We obtain:

$$\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} = \vec{\nabla} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) + \frac{4\pi}{R_1} \delta(\vec{R}_2) \tag{35}$$

The first term on the right is a divergence. Thus, using Gauss theorem its volume integral will become a surface integral, the second term contains a delta function. This means that there is no contribution to the volume integral from the delta term unless $\vec{x} = \vec{x}_2$. It follow that Equation (32) becomes:

$$II = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3x = \int d\vec{S} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) + \frac{4\pi}{R} \tag{36}$$

Let us look at the surface integral and assume that the system is contained inside an infinite sphere, that is such that the surface integral is taken over a spherical surface of radius $r = |\vec{x}| \to \infty$:

$$\lim_{r \to \infty} \int d\vec{S} \cdot \frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} = \lim_{r \to \infty} \int d\vec{S} \cdot \frac{1}{R_1} \frac{\hat{R}_2}{R_2^2}. \tag{37}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 October 2024 doi:10.20944/preprints202410.2252.v1

in this case $d\vec{S} = r^2 d\Omega\, \hat{r}$, in which Ω is the solid angle. Also both $R_1 \to r \to \infty$, $R_2 \to r \to \infty$ and $\hat{R}_2 \to \hat{r}$. It follows that:

$$\lim_{r\to\infty} \int d\vec{S} \cdot \frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} = \lim_{r\to\infty} \int r^2 d\Omega \frac{1}{r^3} = \lim_{r\to\infty} \int d\Omega \frac{1}{r} = 0. \tag{38}$$

(see also Appendix A.1.2. of [16] for complete explanation of Equation (37)). We conclude that simply:

$$II = \frac{4\pi}{R} \tag{39}$$

Inserting I from Equation (31) and II from Equation (39) into Equation (21) and using again vector notation we obtain:

$$\begin{aligned} P_{field12\,l} &\simeq -\frac{\mu_0}{16\pi^2} \int\int d^3x_1 d^3x_2 \rho_1 \left[-2\pi \vec{J}_2 \cdot \vec{\nabla}_{\vec{x}_2} \hat{R}_l - \frac{4\pi}{R} J_{2\,l} \right] \\ &= \frac{\mu_0}{8\pi} \int\int d^3x_1 d^3x_2 \rho_1 \left[\vec{J}_2 \cdot \vec{\nabla}_{\vec{x}_2} \hat{R}_l + \frac{2}{R} J_{2\,l} \right]. \end{aligned} \tag{40}$$

Now we may write:

$$\vec{J}_2 \cdot \vec{\nabla}_{\vec{x}_2} \hat{R}_l = \vec{\nabla}_{\vec{x}_2} \cdot \left(\vec{J}_2 \hat{R}_l \right) - \hat{R}_l \vec{\nabla}_{\vec{x}_2} \cdot \vec{J}_2 = \vec{\nabla}_{\vec{x}_2} \cdot \left(\vec{J}_2 \hat{R}_l \right) + \hat{R}_l \partial_t \rho_2, \tag{41}$$

in which we take into account the charge continuity equation:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0, \qquad \partial_t \rho \equiv \frac{\partial \rho}{\partial t}. \tag{42}$$

Taking the volume integral of Equation (41) and using again Gauss theorem it follows that:

$$\int d^3x_2 \left(\vec{J}_2 \cdot \vec{\nabla}_{\vec{x}_2} \hat{R}_l \right) = \int d\vec{S}_2 \cdot \left(\vec{J}_2 \hat{R}_l \right) + \int d^3x_2 \hat{R}_l \partial_t \rho_2, \tag{43}$$

taking the integration over the entire space volume, and assuming null current density at infinity it follows that the surface integral vanishes and thus:

$$\int d^3x_2 \left(\vec{J}_2 \cdot \vec{\nabla}_{\vec{x}_2} \hat{R}_l \right) = \int d^3x_2 \hat{R}_l \partial_t \rho_2. \tag{44}$$

Inserting Equation (44) into Equation (40) and using vector notation we obtain the simplified expression:

$$\begin{aligned} \vec{P}_{field12} &\simeq \frac{\mu_0}{8\pi} \int\int d^3x_1 d^3x_2 \rho_1 \left[\partial_t \rho_2 \hat{R} + \frac{2}{R} \vec{J}_2 \right] \\ &= \frac{\mu_0}{4\pi} \int\int d^3x_1 d^3x_2 \left[\frac{1}{2} \rho_1 \partial_t \rho_2 \hat{R} + \frac{\rho_1}{R} \vec{J}_2 \right]. \end{aligned} \tag{45}$$

From the above expressions it is easy to calculate $\vec{P}_{field21}$ by exchanging the indices 1 and 2:

$$\vec{P}_{field21} \simeq \frac{\mu_0}{4\pi} \int\int d^3x_1 d^3x_2 \left[-\frac{1}{2} \rho_2 \partial_t \rho_1 \hat{R} + \frac{\rho_2}{R} \vec{J}_1 \right]. \tag{46}$$

Please note that:

$$\vec{R}_{21} = \vec{x}_2 - \vec{x}_1 = -\vec{R}_{12} = -\vec{R} \Rightarrow R_{21} = |\vec{R}_{21}| = |\vec{R}_{12}| = R_{12} = R \tag{47}$$

 doi:10.20944/preprints202410.2252.v1

Combining Equation (45) and Equation (46) and taking the Equation (47) into account, it follows that the total interaction field momentum is:

$$\begin{aligned}\vec{P}_{field} &= \vec{P}_{field12} + \vec{P}_{field21} \\ &= -\frac{\mu_0}{4\pi}\int\int d^3x_1 d^3x_2 \left[\frac{1}{2}\left(\rho_2\partial_t\rho_1 - \rho_1\partial_t\rho_2\right)\hat{R} - \left(\rho_1\vec{J}_2 + \rho_2\vec{J}_1\right)R^{-1}\right] \end{aligned} \tag{48}$$

Which is exactly equal and opposite to the mechanical momentum calculated in [1] in the case of charged relativistic motor, given by:

$$\vec{P}_{mech} = \frac{\mu_0}{4\pi}\int\int d^3x_1 d^3x_2 \left[\frac{1}{2}\left(\rho_2\partial_t\rho_1 - \rho_1\partial_t\rho_2\right)\hat{R} - \left(\rho_1\vec{J}_2 + \rho_2\vec{J}_1\right)R^{-1}\right] \tag{49}$$

Thus, the total interaction momentum of the system remains null:

$$\vec{P}_T = \vec{P}_{field} + \vec{P}_{mech} = 0 \tag{50}$$

as it was in time $t = 0$, the **total** momentum is a conserved quantity as expected from Noether's theorem. On the other hand neither the mechanical momentum nor the electromagnetic momentum are constant in time, but still: $\vec{P}_{mech} = -\vec{P}_{field}$ for any future time t.

4. Conclusions

In conclusion, we investigate the legitimacy of momentum conservation concerning a charged relativistic engine in this paper. We make use of Jefimenko's field expression for the electric and magnetic fields in which the field sources are time dependent. This is of course a common situation in nature and it entails taking into account retardation phenomena. With these formulae, we determine the momentum acquired by the electromagnetic fields. When accounting for both the momentum of the field and the mechanical momentum of the device's material component, conservation of momentum is upheld. Therefore, the charged relativistic motor is capable of generating forward momentum independently of any external forces, solely through interaction with its own electromagnetic field. Thus, this paper has explored the principles of linear momentum conservation in charged relativistic systems. This system offers a groundbreaking method for space propulsion, aiming to address the major drawbacks of conventional rocket fuel. By harnessing the power of electromagnetic fields, spacecraft fitted with these engines could achieve greater speeds and distances while using significantly less fuel, thereby making interplanetary and interstellar travel more achievable.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rajput, S.; Yahalom, A. Newton's Third Law in the Framework of Special Relativity for Charged Bodies. *Symmetry* **2021**, *13*, 1250.
2. Yahalom, M.T..A. Newton's Third Law in the Framework of Special Relativity. *The European Physical Journal Plus* **2014**, *129*, 240.
3. Tuval, M.; Yahalom, A. Momentum conservation in a relativistic engine. *The European Physical Journal Plus* **2016**, *131*, 1–12.
4. Newton, I. *Philosophiae naturalis principia mathematica*; Vol. 1, S. PEPYS, Reg. Soc. PRÆSES: London, England, 1687.
5. Goldstein P, P.C.J.; JL, S. *Classical Mechanics*; Pearson; 3 edition: London, England, 2001.
6. Arya, A. *Introduction to Classical Mechanics*; Allyn and Bacon: Boston, MA, USA, 1990.

 doi:10.20944/preprints202410.2252.v1

7. Einstein, A. On the electrodynamics of moving bodies **1905**.
8. Jackson, J.D. *Classical electrodynamics*, 3rd ed. ed.; Wiley: New York, NY, 1999.
9. Feynman, R.P.; Leighton, R.B.; Sands, M. *The Feynman lectures on physics, Vol. I: The new millennium edition: mainly mechanics, radiation, and heat*; Vol. 1, Basic books: New York,New York,USA, 2011.
10. D'Abramo, G. On apparent faster-than-light behavior of moving electric fields. *The European Physical Journal Plus* **2021**, *136*, 301.
11. Yahalom, A. Newton's Third Law in the Framework of Special Relativity for Charged Bodies Part 2: Preliminary Analysis of a Nano Relativistic Motor. *Symmetry* **2022**, *14*, 94. doi:https://doi.org/10.3390/sym14010094.
12. Yahalom, A. Implementing a Relativistic Motor over Atomic Scales. *Symmetry* **2023**, *15*, 1613. doi:https://doi.org/10.3390/sym15081613.
13. Griffiths, D.J.; Heald, M.A. Time-dependent generalizations of the Biot–Savart and Coulomb laws. *American Journal of Physics* **1991**, *59*, 111–117.
14. Jefimenko, O. *Electricity and Magnetism 2nd edn*; Electret Scientific: Star City, WV:, 1989.
15. Yahalom, A. Retardation in Special Relativity and the Design of a Relativistic Engine. *Acta Physica Polonica A* **2017**, *131*, 1285–1288. doi:DOI: 10.12693/APhysPolA.131.1285.
16. Rajput, S.; Yahalom, A.; Qin, H. Lorentz symmetry group, retardation and energy transformations in a relativistic engine. *Symmetry* **2021**, *13*, 420.
17. Breitenberger, E. Magnetic interaction between charged particles. *American Journal of Physics* **1968**, *36*, 505–515.
18. Scanio, J. Conservation of momentum in electrodynamics - an example. *American Journal of Physics* **1975**, *43*, 258–260.
19. Portis, A. *Electromagnetic Fields, Source and Media*; John Wiley & Sons Inc: New-York, NY, USA, 1978.
20. Jefimenko, O. A Relativistic Paradox Seemingly Violating Conservation of Momentum in Electromagnetic Systems. *Eur. J. Phys.* **1999**, *20*, 39.
21. Jefimenko, O. *Cuasality, Electromagnetic Induction and Gravitation 2nd edn*; Electret Scientific: Star City, WV:, 2000.
22. Yahalom, A. Quantum Retarded Field Engine. *Symmetry* **2024**, *16*, 1109.