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Posted Date: 9 June 2025

doi: 10.20944/preprints202506.0668.v1

Keywords: Euler-Mascheroni constant; convergence; inequality

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Another New Sequence Which Converges Faster Towards to the Euler-Mascheroni Constant

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Abstract: In this paper we propose another new sequence approximating the Euler-Mascheroni constant which converges faster towards its limit and we establish new inequalities for this constant.

Keywords: MSC2010; 40A05; 11Y60; 33B15

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1. Introduction

It is well known (see [1–6]) that the sequence

$$\gamma_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n,\ n \geq 1,$$

is convergent to a limit denoted $\gamma = 0,5772...$ now known as Euler-Mascheroni constant. Many authors have obtained different estimations for $\gamma_n - \gamma$; for example, the following inequalities increase better [1–6] :

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)}, n \geq 2,$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, n \geq 1,$$

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, n \geq 1,$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, n \geq 1,$$

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, n \geq 1,$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, n \geq 1.$$

The convergence of the sequence γ_n to γ is very slow.

DeTemple [7] modified the logarithmic term of γ_n and showed that the sequence

$$R_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2})$$

converges to γ with rate of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2},\ n \geq 1.$$

Chen [8] proved that for all integers $n \geq 1$,

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2},$$

 doi:10.20944/preprints202506.0668.v1

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(\frac{3}{2})]}} - 1 = 0,55106... \text{ and } b = \frac{1}{2}.$$

Vernescu [9] provided the sequence

$$V_n = 1 + \frac{1}{2} + ... + \frac{1}{n-1} + \frac{1}{2n} - \ln n \; (= \gamma_n - \frac{1}{2n}),$$

which also converges to γ with rate of convergence n^{-2}, since

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}.$$

A similar convergence result to γ with rate of convergence n^{-2} was obtained by Ivan [10]:

$$lim_{n\to\infty} n^2(c_n - \gamma) = \frac{1}{6},$$

where $c_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln \sqrt{n(n+1)}$.

Cristea and Mortici [11] introduced the family of sequences

$$v_n(a,b) = 1 + \frac{1}{2} + ... + \frac{1}{n-2} + \frac{an+b}{n(n-1)} - \ln n,$$

where a, b are real parameters.

Furthermore, they proved that, among the sequences $(v_n(a,b))_{n\geq 1}$, the privileged one $(v_n(\frac{3}{2}, -\frac{5}{12}))_{n\geq 1}$ offers the best approximation to γ, since it has the rate of convergence n^{-3}. More precisely, they obtained the bounds

$$\frac{1}{12n^3} + \frac{11}{120n^4} < v_n(\frac{3}{2}, -\frac{5}{12}) - \gamma < \frac{1}{12n^3} + \frac{13}{120n^4} \quad (n \geq 9),$$

where

$$v_n(\frac{3}{2}, -\frac{5}{12}) = 1 + \frac{1}{2} + ... + \frac{1}{n-2} + \frac{13}{12(n-1)} + \frac{5}{12n} - \ln n.$$

Lu [12] used continued fraction approximation to obtain the following faster sequences converging to the Euler–Mascheroni constant:

$$r_n^{(2)} = 1 + \frac{1}{2} + ... + \frac{1}{n} - \frac{3}{6n+1} - \ln n,$$
$$r_n^{(3)} = 1 + \frac{1}{2} + ... + \frac{1}{n} - \frac{6n-1}{12n^2} - \ln n,$$

which satisfie

$$\frac{1}{72(n+1)^3} < \gamma - r_n^{(2)} < \frac{1}{72n^3},$$
$$\frac{1}{120(n+1)^4} < r_n^{(3)} - \gamma < \frac{1}{120(n-1)^4}.$$

Negoi [13] modified the logarithmic term of γ_n and showed that the sequence

$$T_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n}),$$

 doi:10.20944/preprints202506.0668.v1

is strictly increasing and convergent to γ with rate of convergence n^{-3}. Moreover, he proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3},\ n \geq 1.$$

Chen and Mortici [14] proved that for all integers $n \geq 1$,

$$\frac{1}{48(n+a)^3} \leq \gamma - T_n < \frac{1}{48(n+b)^3},$$

with the best possible constants

$$a = \frac{1}{\sqrt[3]{48[1-\gamma+\ln(\frac{37}{24})]}} - 1 = 0,27380525... \text{ and } b = \frac{83}{360} = 0,23055555....$$

Recently You and Chen [15] modified the logarithmic term of γ_n and they showed that the sequences

$$r_2(n) = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} + \frac{1}{48n^2}),$$
$$r_3(n) = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} - \frac{1}{48n^3} - \frac{1}{576n^4} - \frac{1}{1152n^5}),$$

converge to γ with rates of convergence n^{-3}, respectively n^{-4}, since

$$\frac{1}{24(n+1)^3} < \gamma - r_2(n) < \frac{1}{24n^3},$$

$$\frac{143}{5760(n+1)^4} < r_3(n) - \gamma < \frac{143}{5760n^4}.$$

By modifying the logarithmic term of γ_n, Cringanu [16] defined a new sequence

$$S_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2})$$

and showed as he converges to γ with rate of convergence n^{-4}, since

$$\frac{23}{5760(n+a)^4} \leq S_n - \gamma < \frac{23}{5760(n+b)^4},$$

with the best possible constants $a = \sqrt[4]{\frac{23}{5760[1-\gamma-\ln(\frac{73}{48})]}} - 1 = 0,0315..., b = -\frac{7}{46} = -0,1521....$

Now we define the sequence $L_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3})$, for $n \geq 1$, and we prove that for all integers $n \geq 1$,

$$\frac{17}{3840(n+\alpha)^5} \leq L_n - \gamma < \frac{17}{3840(n+\beta)^5},$$ with the best possible constants

$$\alpha = \sqrt[5]{\frac{17}{3840[1-\gamma-\ln(\frac{8783}{5760})]}} - 1 = 0,37407..., \beta = \frac{3305}{12852} = 0,25715....$$

2. The Main Result

Starting from the sequences R_n, T_n and S_n we consider the family of sequences

$$L_n(a,b,c,d) = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}),$$

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for $a, b, c, d \in R, n \geq 1$, and

$$K_n(a,b,c,d) = L_n - \gamma = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}) - \gamma,$$

which converges to zero.

Using a Maclaurin growth series we get

$$K_{n+1}(a,b,c,d) - K_n(a,b,c,d) = \frac{1}{n^2}(a - \frac{1}{2}) + \frac{1}{n^3}(-a^2 - a + 2b + \frac{2}{3}) + \frac{1}{n^4}(a^3 - 3ab + \frac{3a^2}{2} + a - 3b + 3c - \frac{3}{4}) +$$

$$+ \frac{1}{n^5}(-a^4 - 2a^3 + 4a^2b - 2a^2 - 2b^2 + 6ab - 4ac - a + 4b - 6c + 4d + \frac{4}{5}) +$$

$$+ \frac{1}{n^6}(a^5 + \frac{5a^4}{2} - 5a^3b - 10a^2b + 5ab^2 + 5a^2c + \frac{10a^3}{3} +$$

$$\frac{5a^2}{2} + 5b^2 - 10ab + 10ac - 5ad - 5bc + a - 5b + 10c - 10d - \frac{5}{6}) + O(\frac{1}{n^7}).$$

If $a - \frac{1}{2} = 0, -a^2 - a + 2b + \frac{2}{3} = 0, a^3 - 3ab + \frac{3a^2}{2} + a - 3b + 3c - \frac{3}{4} = 0,$
$-a^4 - 2a^3 + 4a^2b - 2a^2 - 2b^2 + 6ab - 4ac - a + 4b - 6c + 4d + \frac{4}{5} = 0$ then
$a = \frac{1}{2}, b = \frac{1}{24}, c = -\frac{1}{48}, d = \frac{23}{5760}$ and so

$$a^5 + \frac{5a^4}{2} - 5a^3b - 10a^2b + 5ab^2 + 5a^2c + \frac{10a^3}{3} + \frac{5a^2}{2} + 5b^2 - 10ab + 10ac - 5ad - 5bc + a - 5b +$$

$$10c - 10d - \frac{5}{6} = -\frac{17}{768}.$$

It results that

$$L_n = L_n(\frac{1}{2}, \frac{1}{24}, -\frac{1}{48}, \frac{23}{5760}) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3})$$

and

$$K_n = K_n(\frac{1}{2}, \frac{1}{24}, -\frac{1}{48}, \frac{23}{5760}) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}) - \gamma.$$

Thus, we get

$$K_{n+1} - K_n = -\frac{17}{768n^6} + O(\frac{1}{n^7}).$$

By a standard result, if a sequence x_n converges to zero and there exists the $\lim_{n\to\infty} n^k(x_n - x_{n+1}) = l$, then $\lim_{n\to\infty} n^{k-1}x_n = \frac{l}{k-1}$ (see e.g., [17]).

In our case of K_n, we have $\lim_{n\to\infty} n^6(K_n - K_{n+1}) = \frac{17}{768}$ and so

$$\lim_{n\to\infty} n^5 K_n = \frac{l}{k-1} = \frac{17}{3840}.$$

Starting from this result and using an elementary sequence method and MATLAB software for computation, we obtain the following:

Theorem 1. *For every integer $n \geq 1$ we have*

$$\frac{17}{3840(n+\alpha)^5} \leq L_n - \gamma < \frac{17}{3840(n+\beta)^5}, \text{with the best possible constants}$$

$$\alpha = \sqrt[5]{\frac{17}{3840[1 - \gamma - \ln(\frac{8783}{5760})]}} - 1 = 0,37407..., \beta = \frac{3305}{12852} = 0,25715....$$

 doi:10.20944/preprints202506.0668.v1

Proof. We define the sequence

$$a_n = L_n - \gamma - \frac{17}{3840(n+a)^5} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}) - \gamma - \frac{17}{3840(n+a)^5},$$

for $a > 0$ and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \frac{1}{n+1} - \ln(n + \frac{3}{2} + \frac{1}{24(n+1)} - \frac{1}{48(n+1)^2} + \frac{23}{5760(n+1)^3}) + \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}) -$$

$$- \frac{17}{3840(n+a+1)^5} + \frac{17}{3840(n+a)^5}.$$

The derivative of function f is equal to

$$f'(n) = -\frac{1}{(n+1)^2} - \frac{3(1920n^4 + 7680n^3 + 11440n^2 + 7600n + 1897)}{(n+1)(5760n^4 + 25920n^3 + 43440n^2 + 32040n + 8783)} +$$

$$+ \frac{3(1920n^4 - 80n^2 + 80n - 23)}{n(5760n^4 + 2880n^3 + 240n^2 - 120n + 23)} - \frac{17}{768} \frac{(n+a+1)^6 - (n+a)^6}{(n+a)^6(n+a+1)^6} = \frac{P(n)}{Q(n)},$$

where

$$P(n) = 768(4406400n^4 + 7490400n^3 + 2117280n^2 - 1647796n - 606027)(n+a)^6(n+a+1)^6 -$$

$$-17n(n+1)^2(5760n^4 + 2880n^3 + 240n^2 - 120n + 23)(5760n^4 + 25920n^3 + 43440n^2 + 32040n + 8783) \cdot$$

$$\cdot [(n+a+1)^6 - (n+a)^6],$$

and

$$Q(n) = 768n(n+1)^2(5760n^4 + 2880n^3 + 240n^2 - 120n + 23) \cdot$$

$$\cdot (5760n^4 + 25920n^3 + 43440n^2 + 32040n + 8783)(n+a)^6(n+a+1)^6.$$

By using MATLAB software, we obtain that

$$P(n) = (23688806400a - 6091776000)n^{15} +$$

$$+(189510451200a^2 + 140097945600a - 54101237760)n^{14} +$$

$$+(710664192000a^3 + 1208781619200a^2 + 283562311680a - 216840281088)n^{13} +$$

$$+(1658216448000a^4 + 4345122816000a^3 + 3195768176640a^2 + 27217096704a - 517434349824)n^{12} +$$

$$+(2676835123200a^5 + 9421194240000a^4 + 11191051468800a^3 +$$

$$+4313542551552a^2 - 951258977280a - 816556190976)n^{11} +$$

$$+(3126922444800a^6 + 13913159270400a^5 + 22443335884800a^4 +$$

$$+15431215165440a^3 + 2584804713984a^2 - 2085448478976a - 895178414400)n^{10} +$$

$$+(2680219238400a^7 + 14696194867200a^6 + 29955583918080a^5 + 28487564835840a^4 +$$

$$+11558431964160a^3 - 677704596480a^2 - 2380540040640a - 696766207680)n^9 +$$

$$+(1675137024000a^8 + 11256628838400a^7 + 28108735119360a^6 + 33662464303104a^5 +$$

$$+19582377488640a^4 + 3364432450560a^3 - 2380953893760a^2 - 1694043099360a - 386195711526)n^8 +$$

$$+(744505344000a^9 + 6197824512000a^8 + 18769457479680a^7 + 27144229367808a^6 + 19496312727552a^5 +$$

$$+5329924312320a^4 - 1566049991040a^3 - 1857543344640a^2 - 776337703566a - 149937998355)n^7 +$$

$$+(223351603200a^{10}+2382336000000a^{9}+87842838528008a^{8}+15008630759424a^{7}+$$
$$+12269823390720a^{6}+3216811753728a^{5}-1958475850560a^{4}-1844984923200a^{3}-$$
$$-744713732028a^{2}-224024317080a-38882029144)n^{6}+$$
$$+(40609382400a^{11}+603024998400a^{10}+2763719884800a^{9}+5475548298240a^{8}+$$
$$+4600589211648a^{7}+13519070208a^{6}-2858407439040a^{5}-2140061613120a^{4}-$$
$$-720254785500a^{3}-155641811586a^{2}-36388303746a-6000763190)n^{5}+$$
$$+(3384115200a^{12}+89336217600a^{11}+537755811840a^{10}+1188768215040a^{9}+$$
$$+700593096960a^{8}-1318070983680a^{7}-2664208535040a^{6}-1999216550880a^{5}-$$
$$-736245769470a^{4}-1130981320660a^{3}-12904066140a^{2}-2408847810a-423322168)n^{4}+$$
$$+(5752627200a^{12}+54028615680a^{11}+110086612992a^{10}-161048540160a^{9}-$$
$$-936009699840a^{8}-1519292712960a^{7}-1239385743360a^{6}-540538312566a^{5}-$$
$$-116468696355a^{4}-9205598920a^{3}-99484425a^{2}-131085198a-40496924)n^{3}+$$
$$+(1626071040a^{12}-5429661696a^{11}-89850714624a^{10}-310896161280a^{9}-$$
$$-517564650240a^{8}-477213143040a^{7}-247005305856a^{6}-66245936460a^{5}-$$
$$-7106635020a^{4}-235621700a^{3}-228228570a^{2}-111896346a-22083544)n^{2}-$$
$$-(1265507328a^{12}+13178188800a^{11}+49700906496a^{10}+95124456960a^{9}+$$
$$+102759782400a^{8}+63444492288a^{7}+20813514240a^{6}+2813177334a^{5}+$$
$$+51512295a^{4}+68683060a^{3}+51512295a^{2}+20604918a+3434153)n-$$
$$-465428736a^{12}-2792572416a^{11}-6981431040a^{10}-9308574720a^{9}-$$
$$-6981431040a^{8}-2792572416a^{7}-465428736a^{6}.$$

If $23688806400a-6091776000=0$, so that $a=\frac{3305}{12852}$, then

$$P(n)=-\frac{1978301112320}{357}n^{14}-\frac{6614222397555712}{127449}n^{13}-\frac{26775073877883399270 4}{1228480911}n^{12}-$$
$$-\frac{6410863378097015826387904}{11841327501129}n^{11}-\frac{112034623964251324641178745 60}{12682061753709159}n^{10}-$$
$$-\frac{283656463491678607733321331433820}{285232250902672695069}n^{9}-\frac{9458940152242502727508996226916173}{119797545379122531928 98}n^{8}-$$
$$-\frac{3143249208914985444050893551827075913725 5}{706693866424529596181224190 64}n^{7}-$$
$$-\frac{80478660328007330472740977151857448170459391}{4541214785644027185060546649052 64}n^{6}-$$
$$-\frac{16214760589885408521174449082511424923648820905}{3242427356949835410133230307423584 96}n^{5}-$$
$$-\frac{156623253881180975307929252680508229943201835250784 1}{15001803500946942488771619327962849006131 2}n^{4}-$$
$$-\frac{5080478942246096058576980892862211265795283987629213 83}{2754331122773858640938469308613979077525688 32}n^{3}-$$
$$-\frac{20244841378617164823764709930798913021994554417695397}{688582780693464660234617327153494769381422 08}n^{2}-$$

$$-\frac{1978798466755139648164437374814331556679548768427578889}{6610394694657260738252326340673549786061651968}n-$$
$$-\frac{4683428037913070737204329712156882479692563209390625}{881385959287634765100310178756473304808220262 4} < 0,$$

for all $n \geq 1$, and then f is strictly decreasing.

We have $lim_{n\to\infty} f(n) = 0$ and then it follows $f(n) > 0$ for all $n \geq 1$, such that $(a_n)_{n\geq 1}$ is strictly increasing. Since (a_n) converges to zero it results that $a_n < 0$ for all $n \geq 1$, such that

$$L_n - \gamma < \frac{17}{3840(n+\frac{3305}{12852})^5}, \text{ for all } n \geq 1.$$

If $a = \sqrt[5]{\frac{17}{3840[1-\gamma-\ln(\frac{8783}{5760})]}} - 1 = 0,37407...$, then $a_1 = L_1 - \gamma - \frac{17}{3840(1+a)^5} = 0$,

$P(n) > 0$ for all $n \geq 2$ and then f is strictly increasing on $[2, \infty)$.

Since $lim_{n\to\infty} f(n) = 0$ it results that $f(n) < 0$ for all $n \geq 2$, such that $(a_n)_{n\geq 2}$ is strictly decreasing.

The sequence (a_n) converges to zero and then it results that $a_n > 0$ for all $n \geq 2$, such that

$$\frac{17}{3840(n+a)^5} \leq L_n - \gamma, \text{ for all } n \geq 1.$$

□

Remark 1. Let us remark that, if $a > \frac{3305}{12852}$, then $23688806400a - 6091776000 > 0$ and then there exists $n_a \geq 1$ such that $P(n) > 0$ for all $n \geq n_a$ and then

$$\frac{17}{3840(n+a)^5} < L_n - \gamma < \frac{17}{3840(n+\frac{3305}{12852})^5},$$

for all $n \geq n_a$.

Remark 2. Returning to the sequences

$$R_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2})$$

$$T_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n})$$

$$S_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}),$$

$$L_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3})$$

with rates of convergence n^{-2}, n^{-3}, n^{-4} and respectively n^{-5} , after a few iterations we can observe the faster convergence of the sequence L_n to $\gamma = 0,577215664901...$ compared to R_n, T_n and S_n:

n	R_n	T_n	S_n	L_n
10	0.5775929968...	0.5771962501...	0.5772160837...	0.5772157035...
11	0.5775303095...	0.5772009829...	0.5772159500...	0.5772156892...
12	0.5774820339...	0.5772042946...	0.5772158656...	0.5772156808...
13	0.5774440696...	0.5772066809...	0.5772158102...	0.5772156756...
14	0.5774136771...	0.5772084436...	0.5772157727...	0.5772156723...

 doi:10.20944/preprints202506.0668.v1

15	0.5773889693...	0.5772097738...	0.5772157465...	0.5772156702...
16	0.5773686123...	0.5772107964...	0.5772157278...	0.5772156687...
17	0.5773516417...	0.5772115954...	0.5772157142...	0.5772156677...
18	0.5773373461...	0.5772122288...	0.5772157040...	0.5772156670...
19	0.5773251915...	0.5772127372...	0.5772156964...	0.5772156665...
20	0.5773147709...	0.5772131501...	0.5772156905...	0.5772156661...
21	0.5773057696...	0.5772134889...	0.5772156859...	0.5772156659...
22	0.5772979410...	0.5772137694...	0.5772156823...	0.5772156657...
23	0.5772910899...	0.5772140037...	0.5772156795...	0.5772156655...
24	0.5772850602...	0.5772142010...	0.5772156772...	0.5772156654...
25	0.5772797255...	0.5772143682...	0.5772156753...	0.5772156653...
26	0.5772749832...	0.5772145109...	0.5772156738...	0.5772156652...
27	0.5772707485...	0.5772146334...	0.5772156725...	0.5772156651...
28	0.5772669516...	0.5772147391...	0.5772156715...	0.5772156651...
29	0.5772635342...	0.5772148309...	0.5772156706...	0.5772156651...
30	0.5772604473...	0.5772149110...	0.5772156699...	0.5772156650...
40	0.5772410648...	0.5772153449...	0.5772156664...	0.5772156649...
50	0.5772320020...	0.5772155005...	0.5772156654...	0.5772156649...

3. Conclusion

By modifying the logarithmic term of γ_n we have constructed another new sequence that converges faster to the Euler-Mascheroni constant, with the convergence rate n^{-5}, compared to the sequences in [7,8] with convergence rates n^{-2} or those in [13–15] with convergence rates n^{-3}, or those in [15,16] with convergence rates n^{-4}. Also the idea of constructing the sequence L_n allows the construction of a new sequence with the convergence rate n^{-6}, starting from the family of sequences

$$L_n^{(5)}(a,b,c,d,e) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3} + \frac{e}{n^4}),$$

for $a,b,c,d,e \in \mathbf{R}$, $n \geq 1$.

More generally, starting from the family of sequences

$$L_n^{(k)}(a_1,a_2,\ldots,a_k) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + a_1 + \frac{a_2}{n} + \ldots + \frac{a_k}{n^{k-1}}),$$

for $a_1,a_2,\ldots,a_k \in \mathbf{R}$, $k \geq 5, n \geq 1$, we find a sequence with a convergence rate n^{-k-1}.

Funding: The APC is funded by “Dunarea de Jos” University of Galati, Romania.

Data Availability Statement: The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest: The author declares no conflict of interest.

doi:10.20944/preprints202506.0668.v1

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