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Article

Toward a Proof of the Riemann Hypothesis: Insights from Elliptic Complex Analysis

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Abstract

This study aims to prove the Riemann Hypothesis and the Generalized Riemann Hypothesis by extending the Riemann zeta function and Dirichlet L -functions to the elliptic complex domain, based on a newly constructed system of elliptic complex numbers $\mathbb{C}_\lambda(\lambda < 0)$. The core challenge addressed is the inherent difficulty in resolving these conjectures within the traditional "circular complex domain" framework ($\lambda = -1$); the author posits that a complete proof is unattainable strictly within this conventional setting. The primary innovation of this work lies in the formulation of the theory of elliptic complex numbers, specifically identifying the limiting case as $\lambda \to 0^-$ as the key to the proof. Through rigorous deduction, a bijective correspondence between zeros across different complex planes is established. By employing proof by contradiction and leveraging the correspondence between $\mathbb{C}_\lambda$ (as $\lambda \to 0$) and the circle complex plane $\mathbb{C}$, the Riemann Hypothesis and the Generalized Riemann Hypothesis are ultimately proven. This paper is organized into three parts: (1) Construction and Geometric Properties: The first part details the construction of elliptic complex numbers and their fundamental geometric properties, laying the necessary foundation for subsequent analysis and the proof of the conjectures. (2) Analytic Extension: The second part introduces elliptic complex numbers into mathematical analysis, deriving numerous results analogous to those in classical complex variable function theory. (3) Proof of Conjectures: The final part presents the formal proofs of the Riemann Hypothesis and the Generalized Riemann Hypothesis.

Keywords: the Riemann hypothesis; elliptic complex analysis

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1. Introduction

In the 1740s, Euler first revealed the connection between prime numbers and functions, establishing the famous Euler product formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This formula transformed an additive problem (summation) into a multiplicative one (product over primes), for the first time building a bridge between prime numbers and analytic functions [1].

In 1859, the 32-year-old Bernhard Riemann, in his seminal paper *"On the Number of Primes Less Than a Given Magnitude"* (originally *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*) [2], treated the zeta function as a function of a complex variable. He proposed two revolutionary ideas:

(1) Using analytic continuation, the domain of the zeta function can be extended to the entire complex plane (except for a simple pole at $s = 1$). This continued function is what now known as the Riemann zeta function.
(2) The non-trivial zeros of this analytically continued zeta function determine the precise law governing the distribution of prime numbers [3].

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Riemann's paper, a mere eight pages long, contains profoundly deep insights and opened up entirely new paths for research into the distribution of prime numbers. The function $\zeta(s)$ was formally named the Riemann zeta function in his honor.

1.1. The Formulation of the Riemann Hypothesis

Riemann reached the following conclusion in his 1859 paper: the zeros of the Riemann zeta function $\zeta(s)$ are divided into two parts:

(1) **Real zeros** are all negative even integers, i.e., the trivial zeros [1];

(2) There also exist **complex zeros**, which are distributed in the critical strip $0 < \mathrm{Re}(s) < 1$. Actually, Riemann originally proposed that the non-trivial zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$; later mathematicians refined this conclusion [4].

In his paper, Riemann presented three propositions [2].

Theorem 1.1. *If we define $N(T)$ as the number of non-trivial zeros of $\zeta(s)$ with $0 < Im(s) < T$, then*

$$N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}. \tag{1}$$

Theorem 1.2. *If we define $N_0(T)$ as the number of non-trivial zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$ with $0 < Im(s) < T$, then*

$$N_0(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}.$$

These were Riemann's first and second propositions, which seemed so certain to him. However, the first proposition was only proven 46 years later (in 1905) by the mathematician von Mangoldt [5]. The second proposition was not proven until nearly a century later (in 1942) by the mathematician Atle Selberg [6].

Combining numerous "pieces of evidence," Riemann boldly conjectured: **All non-trivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$** [3].

When the Riemann Hypothesis was first proposed in 1859, it did not immediately create a sensation [1]. On one hand, Riemann's paper was exceedingly brief, with many crucial steps "omitted from the proof," making it difficult for mathematicians of the time to fully grasp its profound implications. On the other hand, its conclusions were so ahead of their time that some mathematicians even viewed it with skepticism [7].

However, there was one mathematician who was full of confidence in it, believing that it would be solved within 50 years. This optimist was **Charles Hermite** [8]. According to some historical sources on mathematics, Hermite expressed this view in letters to friends, showing great optimism about the prospects of the Riemann Hypothesis. History proved him overly optimistic, however, as the conjecture remains unproven to this day.

In 1900, Hilbert delivered his famous lecture at the International Congress of Mathematicians in Paris, presenting the influential "23 Mathematical Problems" that charted the course for 20th-century mathematical research [9]. The Riemann Hypothesis was included as **part of Problem 8**, alongside other number theory challenges such as the Goldbach Conjecture [10]. It was due to Hilbert's enormous influence that the Riemann Hypothesis was elevated to an unprecedented status, becoming a towering peak that mathematicians aspired to conquer [11].

Today, of Hilbert's 23 problems, apart from Problem 8, the remaining 22 problems have either been completely or partially solved, or have been proven to be untenable in certain formulations [10,12]. The Riemann Hypothesis has thus become the most influential and perhaps the most difficult unsolved mystery of all [3,13].

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In May 2000, the Clay Mathematics Institute (CMI) officially announced the seven Millennium Prize Problems at the Collège de France in Paris [14,15]. The Riemann Hypothesis was prominently included in this prestigious list [16–18].

1.2. Important Milestones in the Study

Since Riemann first proposed the hypothesis in 1859, mathematicians have engaged in continuous attempts to prove it [1]. Here we will list only those results that have played a decisive role, hold significant meaning, or even represent the current state-of-the-art conclusions .

1.2.1. The Prime Number Theorem

In 1896, Hadamard and de la Vallée-Poussin independently proved the crucial result that the Riemann zeta function has no zeros on the line $\Re(s) = 1$ [19,20]. From this zero-free region, they were able to prove the Prime Number Theorem, which describes the asymptotic distribution of prime numbers [4]:

$$\pi(x) \sim \frac{x}{\ln x} \sim \mathrm{Li}(x), \quad x \to \infty,$$

where $\pi(x)$ denotes the number of primes less than or equal to x, and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$ is the logarithmic integral function [21].

1.2.2. The Number of Zeros on the Critical Line

In 1905, von Mangoldt proved and obtained the exact formula for the zero-counting function[**?**], namely the Riemann-von Mangoldt Formula [1,4]:

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + O(\ln T).$$

Furthermore, in 1914, Hardy utilized the function $Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$ to obtain a proof of Hardy's Theorem [22,23]:

$$N_0(T) \to \infty, \quad T \to \infty.$$

That is, $\zeta(s)$ has infinitely many zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Of course, at this time, this was still a non-quantitative result. By 1921, Hardy and Littlewood proved that the number of zeros of the Riemann zeta function on the critical line is at least proportional to the height T [24], i.e.,

$$N_0(T) \gg T,$$

laying the foundation for subsequent research on zero-density estimates [6,25–27].

1.2.3. Riemann-Siegel Formula

In 1932, the German mathematician Siegel discovered an astonishing secret from Riemann's manuscripts, which had lain dormant for 73 years [1,28]: Riemann not only proposed the hypothesis based on intuition, but he also personally calculated several zeros, such as $1/2 + i14.134\ldots$ and $1/2 + i21.028\ldots$, using computational methods far ahead of the mathematical community of his time [23]. From these cryptic manuscripts, Siegel reconstructed a highly efficient formula for computing zeros [7]:

$$Z(t) = 2\sum_{n=1}^{N} \frac{\cos[\theta(t) - t\ln n]}{\sqrt{n}} + O(t^{-1/4}),$$

whose computational complexity is only $O(t^{1/2})$, far superior to the Euler-Maclaurin formula's $O(t)$ [29].

Building upon this foundation, mathematicians have employed an efficient "double-counting" strategy [30,31]:

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(1) Zeros on the Critical Line: Using the Riemann-Siegel formula, we can compute the sign changes of the real function $Z(t)$ [6]. Whenever $Z(t)$ changes from positive to negative or from negative to positive, a zero is captured. In this way, we can count that within the range of imaginary parts $0 < t < T$, at least N zeros are found on the critical line.

(2) Total Zeros in the Critical Strip: Using another known mathematical theorem (the argument principle), we can precisely calculate the total number M of all non-trivial zeros within the rectangular region $\{0 < \Re(s) < 1, 0 < \Im(s) < T\}$ [33]. This calculation does not depend on the specific locations of the zeros.

If the number of zeros N found on the critical line in the first step exactly equals the total number M of zeros in the entire region calculated in the second step, then an irrefutable conclusion can be drawn: Within this region, all M non-trivial zeros, without exception, lie on the critical line! [3].

In fact, as of 2004, scientists had verified the first 10 trillion zeros, finding that they all lie on the critical line and are all simple zeros [30,31,34].

1.2.4. Theorem on the Density of Critical Zeros

In 1942, the Norwegian mathematician Atle Selberg achieved a major breakthrough in the study of the Riemann zeta function [6,35]. Using the mollifier method, he carefully constructed the mollifier function

$$M(s) = \sum_{n \le X} \frac{\mu(n)}{n^s} f\left(\frac{\ln n}{\ln X}\right),$$

where $\mu(n)$ is the Möbius function and f is a smooth cutoff function [4]. The parameter X is chosen optimally as a power of T.

Using this mollifier function, Selberg further constructed the core inequality [1,23]:

$$\int_0^T \left| M\left(\frac{1}{2} + it\right) \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \gg T,$$

where the notation $\gg T$ means that the left-hand side is at least a constant multiple of T for sufficiently large T.

From this inequality, Selberg obtained his celebrated theorem [6]: There exists a constant $c > 0$ such that

$$N_0(T) \ge c \cdot N(T),$$

where $N(T)$ is the total number of non-trivial zeros with imaginary part between 0 and T, and $N_0(T)$ is the number of such zeros lying on the critical line $\Re(s) = \frac{1}{2}$.

This result was revolutionary because it was the first time mathematicians could prove that a fixed positive percentage of zeros (not just infinitely many) lie on the critical line [35]. Selberg's original constant c was very small, but subsequent work by Levinson, Conrey, and others has significantly improved this proportion [25–27].

In 1974, Norman Levinson achieved a major breakthrough in the study of the Riemann zeta function [25,36]. Building upon Selberg's mollifier method, he proved that more than one-third of the non-trivial zeros lie on the critical line $\Re(s) = \frac{1}{2}$ [1,23].

Levinson considered a deformation of the ξ function [25]:

$$\eta(s) = \frac{1}{2}\xi(s) - \frac{1}{2}\xi'(s),$$

where $\xi(s)$ is the Riemann xi function, defined as $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ [4].

Using the mollifier function

$$M(s) = \sum_{n \le y} \frac{\mu(n)}{n^s} \left(1 - \frac{\ln n}{\ln y}\right)$$

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with $y = T^{\theta}$ for some $\theta > 0$, he constructed two core integrals [26]

$$I = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)M\left(\frac{1}{2}+it\right)\right|^2 dt,$$

$$J = \int_0^T \zeta\left(\frac{1}{2}+it\right)M\left(\frac{1}{2}+it\right)\zeta'\left(\frac{1}{2}+it\right)M\left(\frac{1}{2}+it\right)dt,$$

and obtained the key inequality

$$\frac{N_0(T)}{N(T)} \geq 1 - \frac{1}{R}\ln\left(\frac{I}{|J|}\right) + o(1),$$

where R is a parameter related to the choice of y [25].

Through precise estimates of the integrals I and J, Levinson proved that

$$\frac{N_0(T)}{N(T)} \geq \frac{1}{3} + o(1),$$

and with more careful calculations, he obtained the numerical value 34.74%.

This result improved significantly upon Selberg's earlier work, which had only established the existence of a positive proportion without giving a specific numerical value [6]. Levinson's work opened the door for subsequent improvements by Conrey and others [26,27].

In 1989, Conrey significantly improved this result to $\liminf_{T\to\infty} \dfrac{N_0(T)}{N(T)} \geq \dfrac{2}{5} = 40\%$; In 2020, Pratt, Robles, Zaharescu, and Zeindler further raised it [37,38] to $\geq \dfrac{5}{12} \approx 41.67\%$.

In summary, this is the Critical Line Density Theorem: At least 41.67% of the zeros lie definitively on the critical line, and this portion of zeros is of the same order as the total zeros.

1.2.5. Ingham Bound

In 1940, Albert Ingham proved the classical result on zero-density estimates, known as Ingham's Theorem [39,40]:

$$N(\sigma,T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}+o(1)}, \quad \frac{1}{2} < \sigma < 1,\ T \to \infty,$$

where $N(\sigma,T) := \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, \beta \geq \sigma, 0 < \gamma \leq T\}$ is the zero-density function [4,23].

For $\sigma = \dfrac{3}{4}$ in particular [40],

$$N\left(\frac{3}{4},T\right) \ll T^{\frac{3}{5}+o(1)}, \quad T \to \infty.$$

The Ingham bound is a foundational result in zero-density estimates, used to control the number of zeros that may deviate from the critical line $\sigma = \dfrac{1}{2}$ [1]. This bound remained essentially unimproved for over 80 years [3,41].

In 2025, MIT mathematician Larry Guth and Oxford mathematician James Maynard (2022 Fields Medalist) published a paper achieving the first substantial improvement to the Ingham bound [42,44], obtaining the Guth-Maynard Theorem:

$$N(\sigma,T) \leq T^{\frac{30(1-\sigma)}{13}+o(1)}.$$

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For $\sigma = \frac{3}{4}$ in particular, this improves the exponent to

$$N\left(\frac{3}{4}, T\right) \ll T^{\frac{15}{26}+o(1)} \approx T^{0.577+o(1)},$$

which is superior to Ingham's $\frac{3}{5} = 0.6$ [42,43].

The Guth-Maynard result represents the first substantial progress on zero-density estimates since Ingham's original work [44,45]. The key innovation lies in deriving new bounds for Dirichlet polynomials, which in turn control the large values that lead to improved zero-density estimates [42].

1.3. Comparative Study of Leading Research Approaches

From the perspective of analytic number theory, mathematicians can directly improve zero-density estimates, gradually pushing the proportion

$$\kappa = \liminf_{T\to\infty} \frac{N_0(T)}{N(T)}$$

towards the ultimate goal of 100% [3,4]. The current best result is $\kappa \geq 0.4173$ obtained by the Levinson-Conrey method [26,27,46].

However, the limit of current methods is approximately 50-60% [3,47], and even advancing by a small fraction may consume decades or even centuries of research effort from the academic community [48]. Even if new methods could be found and eventually push the result to 99%, the final 1% might still be an insurmountable chasm [13].

In fact, even if one could push the result to 100%, i.e., prove that 100% of zeros lie on the critical line, this would not be equivalent to proving the Riemann Hypothesis [1,23].

This is because one would still need to exclude the possibility of exceptional zeros at infinity; such zeros could exist at least one, or even infinitely many [6,25]. Current methods cannot eliminate "sparse but infinite" zeros deviating from the critical line, and therefore entirely new nonlinear or global methods are needed to provide a solution [42,44,45].

The distinction between "100% in density" and "all zeros" is crucial: density results only control the proportion of zeros up to height T, but they do not rule out the existence of a sparse set of zeros off the critical line that grows more slowly than $N(T)$ [40]. Such a set could still contain infinitely many zeros, each of which would be a counterexample to the Riemann Hypothesis [3].

Beyond the classical analytic methods, several profound alternative approaches to the Riemann Hypothesis have emerged, drawing connections to algebraic geometry, noncommutative geometry, random matrix theory, and quantum physics [1,3,13].

1.3.1. The Algebraic Geometry Analogy: Weil Conjectures

Another direction is the algebraic geometry analogy, namely seeking possible proof patterns over number fields through the Weil conjectures over finite fields [49,50]. The Weil conjectures over finite fields were proved by Deligne in 1974 [50], and the Riemann hypothesis part of the Weil conjectures is precisely the Riemann hypothesis over finite fields [51].

However, it should be noted that although Deligne's proof represents the pinnacle of 20th-century mathematics, the essential differences between number fields and finite fields make generalization extremely difficult [52]. This is the fundamental difference between characteristic p and characteristic 0, and the Frobenius endomorphism has no direct analog in number fields [53,54].

1.3.2. Noncommutative Geometry: Connes' Approach

Another promising direction is noncommutative geometry, which transforms number-theoretic problems into space-spectrum problems [55,57]. Connes conjectured a global trace formula, which

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is the sum of contributions from local trace formulas; this conjecture is equivalent to the Riemann hypothesis [55,58].

Currently, the local formula has been rigorously proved, but the convergence of the global summation remains unresolved [54,58]. It can be seen that Connes' method provides the deepest conceptual framework, but the technical details are not yet complete. This may require the development of new tools beyond current noncommutative geometry [54,59].

1.3.3. Random Matrix Theory

In fact, random matrix theory is also a promising approach to studying the Riemann hypothesis [60,61]. Mathematicians use GUE eigenvalue statistics to predict zero distributions and seek deeper structures [62,63].

However, although random matrix theory provides powerful heuristic evidence, it has not yet provided a rigorous path to proving the Riemann Hypothesis [3,13]. This is because the statistical correspondence may be a "coincidence" or "universality" phenomenon, lacking a bridge from statistics to determinism [64].

1.3.4. The Hilbert-Pólya Operator

Another intriguing research direction is the Hilbert-Pólya operator method, whose core strategy is to find a self-adjoint operator H whose eigenvalues correspond to the zeros of the ζ function [65,66]. Scientists hope to construct a concrete quantum Hamiltonian operator and prove a one-to-one correspondence between its spectrum and the zeros; if the operator is self-adjoint, this could further prove the Riemann hypothesis itself [62,66].

Although this method promises that finding the Hilbert-Pólya operator would immediately prove the Riemann hypothesis, no concrete construction has yet been achieved [13]. Moreover, this approach lacks a rigorous correspondence from classical chaos to quantum spectra, requiring an entirely new quantization framework [64,66].

These diverse approaches—algebraic geometry, noncommutative geometry, random matrix theory, and quantum physics—each offer profound insights into the Riemann hypothesis [1,13,54]. While none has yet provided a complete proof, they continue to inspire new mathematical developments and deepen our understanding of the connections between number theory and other areas of mathematics and physics [64,67].

1.4. The Undecidability of the Riemann Hypothesis

In fact, there is a view, which is not mainstream in the mathematical community, that the Riemann Hypothesis (RH) is unprovable. The core idea of this view is that if RH is unprovable, then it must be true. This makes it a potential candidate for the most dramatic instance of Gödel's Incompleteness Theorems in number theory.

Gödel's Incompleteness Theorems state that in any sufficiently powerful axiomatic system that contains basic arithmetic such as ZFC , there exist propositions that can neither be proven nor disproven i.e., "independent" propositions . The history of mathematics indeed contains important number-theoretic statements that have been proven independent . For example , the Continuum Hypothesis is independent of ZFC, but is not an arithmetic statement; while Goodstein's theorem and the Paris-Harrington theorem are arithmetic statements independent of the Peano axioms but provable in ZFC . This invites speculation about whether RH might also be independent of ZFC.

Gregory Chaitin, the founder of algorithmic information theory, is one of the most prominent mathematicians to publicly discuss the idea that the RH might require new axioms.

Chaitin argues that the distribution of prime numbers exhibits a certain "randomness" (pseudo-randomness). In his books and papers, he proposes that the truth of RH may stem from this arithmetic randomness, which, to some extent, resembles the intrinsic randomness of quantum mechanics and cannot be fully captured by a finite, deterministic axiomatic system [32,84]. He speculates that proving

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RH might require introducing "new axioms," analogous to physical laws, or acknowledging that certain mathematical facts are "accidentally true" rather than logical necessities.

Yuri Matiyasevich proved the undecidability of Hilbert's Tenth Problem (the solvability of Diophantine equations). He demonstrated how to transform RH into a question about the existence of solutions to a massive Diophantine equation. Matiyasevich suggests that since the general problem of Diophantine equations is undecidable, a specific, extremely complex equation (such as the one corresponding to RH) might also be undecidable within the current axioms [56,85].

As one of the leading figures in contemporary analytic number theory, Brian Conrey holds a more pragmatic attitude towards RH but also acknowledges this possibility. In his renowned survey article, he points out that while the numerical evidence overwhelmingly supports RH, we cannot rule out the possibility that it is unprovable in ZFC. If RH were undecidable, it would be a groundbreaking discovery because it would mean RH is true (as argued above), but we could never prove it within our existing framework [3].

The idea that RH might be unprovable is a fascinating philosophical possibility and a logical fallback, especially in light of the long-standing failure to find a proof. It serves as a reminder that our system of axioms might be insufficient to capture all mathematical truths. However, in practical research, this is more of a "last resort." The current direction of effort remains focused on finding new mathematical tools (such as noncommutative geometry, random matrix theory, and new developments in algebraic geometry) to prove it, rather than proving its unprovability.

1.5. Research Objectives

Regardless of perspective, we must first reach a consensus: it is impossible to obtain a complete proof of the Riemann Hypothesis using only the basic tool of complex numbers.

In fact, looking back at the history of mathematicians' research on the Riemann Hypothesis, we can observe the following fact: ever since Riemann introduced the ζ function into the complex domain, studied the zero distribution of the function, and proposed formula 1 and the Riemann Hypothesis, mathematicians have become inextricably trapped in the path suggested by Riemann's formula 1 , embarking on the so-called research journey. However, following this direction, our research is doomed to fail. As analyzed in Section 1.3 , a proof of the Riemann Hypothesis itself is still considered 'miles away' even if the results are optimized to their theoretical limit . Although it is undeniable that the new tools developed during this research may advance the field of mathematics, I believe that no matter how much progress we make in this direction, it is ultimately meaningless for the Riemann Hypothesis itself.

Although scientists have attempted to find other possible directions, such as using new methods like random matrix theory to study the Riemann Hypothesis, these efforts are largely approached in a spirit of "verification" rather than proof itself. Such approaches seem more like an act of helplessness when unable to find a genuine proof.

Returning to the very beginning of Riemann's work, i.e., "introducing the ζ function into the complex domain," it might be considered to expand the complex domain, finding algebraic systems parallel to the complex domain, studying the zero distribution of the ζ function in these algebraic systems, and thereby gaining insight into the essence of the Riemann Hypothesis as a whole?

This is the fundamental research approach of this paper. No argument is made regarding whether the Riemann Hypothesis is provable in the complex domain; instead, algebraic systems with properties similar to those of the complex domain—namely, elliptic complex numbers—are directly constructed . Subsequently, the geometric, algebraic, and analytic properties of elliptic complex numbers will be investigated, given their role as essential foundational material for the proof of the Riemann Hypothesis . It should be noted that elliptic complex numbers do not refer to a specific algebraic system, but rather a collection of algebraic systems with properties analogous to complex numbers. In a sense, all elliptic complex numbers, inclusive of ordinary complex numbers, would be regarded as 'variables' within the scope of this study.

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Regarding the proof of the Riemann Hypothesis, this paper will investigate the correspondence of the zero distribution of the Riemann ζ function across the complex planes corresponding to all elliptic complex numbers. From this correspondence, we discover that on one particular complex plane, the Riemann Hypothesis is "self-evident," and this naturally corresponds to all other complex planes, ultimately providing a proof of the Riemann Hypothesis.

Of course, the proof of the Generalized Riemann Hypothesis will also be presented at the end using the same method .

2. Elliptic Complex Numbers and Their Geometric Significance

As a special kind of algebraic system, complex numbers are introduced out of necessity when solving algebriaic questions in the 16th century [68,69].In the mid-16th century, the Italian mathematician Cardan, when solving cubic equations in 1545, first conceived the idea of taking square roots of negative numbers [70,71]. To make square roots of negative numbers meaningful, it was necessary to extend the number system once again, thus introducing the imaginary number $\sqrt{-1}$. Euler systematically established the theory of complex numbers in 1777, using i to replace $\sqrt{-1}$ as the unit of imaginary numbers [72,73].

In fact, in a broad sense, mathematicians have discovered (invented) three types of binary numbers: complex numbers (hereafter referred to as circular complex numbers to distinguish them), hyperbolic complex numbers (hereafter referred to as equilateral hyperbolic complex numbers to distinguish them), and dual numbers [74–76]. Remarkably, all three types of complex numbers satisfy the commutative law and associative law of multiplication; however, only circular complex numbers form a division algebra, while equilateral hyperbolic complex numbers and dual numbers are not divisible [77–79].

This chapter will construct a series of generalized binary complex numbers and study their algebraic properties and underlying geometric properties .

2.1. Construction of Generalized Complex Numbers

Based on the three types of binary numbers given by mathematicians [74–76], we present the construction of generalized binary complex numbers [77,78,80].

Definition 2.1 (Generalized Complex). *For $\forall x, y \in \mathbb{R}$, a number of the form $z = x + iy$ is called a generalized (binary) complex number, where i satisfies*

$$ii = i^2 = \lambda, \quad \lambda \in \mathbb{R}.$$

The set of all generalized complex numbers is denoted by $\mathbb{C}_\lambda$ [79,81]. Here x is called the real part of z, and y is called the imaginary part of z.

In particular, when $\lambda = 1$, this corresponds to equilateral hyperbolic complex numbers; when $\lambda = -1$, this corresponds to circular complex numbers; when $\lambda = 0$, this corresponds to dual numbers [76,77].

The addition and scalar multiplication operations of generalized complex numbers are defined as follows: Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}_\lambda$,

$$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2), \quad kz_1 := kx_1 + i(ky_1), \quad k \in \mathbb{R}.$$

Two generalized complex numbers z_1 and z_2 are equal if and only if their corresponding real and imaginary parts are equal, i.e.,

$$z_1 = z_2 \iff x_1 = x_2,\ y_1 = y_2.$$

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Thus we have the zero element of generalized complex numbers

$$0 := z_1 - z_2 := (x_1 - x_2) + i(y_1 - y_2) = 0 + i0.$$

If the imaginary part of a generalized complex number is 0, i.e., $z = x$, then it is a real number; if the real part of a generalized complex number is 0, i.e., $z = iy$, then it is called a pure generalized complex number, or pure imaginary number [70,72]. These definitions are consistent with those of circular complex numbers [68,69].

2.2. Operational Rules of Generalized Complex Numbers

This section mainly discusses the rules of multiplication. The rules of addition, through the definitions of addition and scalar multiplication given above, can easily be seen to satisfy associativity, commutativity, and distributivity with respect to real numbers; division operations will be discussed subsequently [76,77].

From Definition 2.1, we obtain the multiplication operation of generalized complex numbers [78,80]

$$\begin{aligned} z_1 z_2 &= x_1 x_2 + \lambda y_1 y_2 + i(x_1 y_2 + x_2 y_1) \\ &= \begin{bmatrix} x_1 & \lambda y_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := M_2(z_1) z_2 \\ &= \begin{bmatrix} x_2 & \lambda y_2 \\ y_2 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} := M_2'(z_2) z_1 \end{aligned} \tag{2}$$

where $(x, y)^T$ is the vector form of generalized complex numbers, and

$$M_2(z_1) = \begin{bmatrix} x_1 & \lambda y_1 \\ y_1 & x_1 \end{bmatrix}, \quad M_2'(z_2) = \begin{bmatrix} x_2 & \lambda y_2 \\ y_2 & x_2 \end{bmatrix}$$

are called the multiplier matrix and inverse multiplier matrix with respect to the product $z_1 z_2$, respectively [79,81].

It is easy to see that the multiplication operation of generalized complex numbers satisfies associativity and commutativity [77,78]. This is because

$$z_2 z_1 = M_2'(z_1) z_2 = M_2(z_1) z_2 = z_1 z_2, \quad \forall z_1, z_2 \in \mathbb{C}_\lambda$$

in view of Equation 2, so commutativity holds. On the other hand,

$$M_2(z_1) M_2'(z_2) = \begin{bmatrix} x_1 & \lambda y_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & \lambda y_2 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \lambda y_1 y_2 & \lambda x_1 y_2 + \lambda y_1 x_2 \\ x_1 y_2 + \lambda y_1 x_2 & \lambda y_1 y_2 + x_1 x_2 \end{bmatrix},$$

$$M_2'(z_2) M_2(z_1) = \begin{bmatrix} x_2 & \lambda y_2 \\ y_2 & x_2 \end{bmatrix} \begin{bmatrix} x_1 & \lambda y_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} x_1 x_2 + \lambda y_1 y_2 & \lambda x_1 y_2 + \lambda y_1 x_2 \\ x_1 y_2 + \lambda y_1 x_2 & \lambda y_1 y_2 + x_1 x_2 \end{bmatrix},$$

so $M_2(z_1) M_2'(z_2) = M_2'(z_2) M_2(z_1), \quad \forall z_1, z_2 \in \mathbb{C}_\lambda$, and thus associativity holds [79,80].

Furthermore, it can be determined that generalized complex numbers satisfy the distributive law of multiplication [77,81].

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2.3. Elliptic Complex Numbers

In comparison, we prefer the most familiar real numbers, so no matter how complex the algebraic system we encounter, we tend to find the simplest mapping from that algebraic system to the real numbers [76,77]. Thus,

Definition 2.2 (Conjugate Complex Number)**.** *Let $z = x + iy \in \mathbb{C}_\lambda$, then $z^* = x - iy$ is called the conjugate complex number of z [78,80].*

Obviously,

$$zz^* = x^2 - \lambda y^2.$$

Definition 2.3 (Norm)**.** *Let $z = x + iy \in \mathbb{C}_\lambda$, then $N(z) = zz^*$ is called the norm of the complex number z, i.e.,*

$$N(z) = x^2 - \lambda y^2. \tag{3}$$

When $\lambda < 0$, we always have $N(z) \geq 0$, and in this case $|z| = \sqrt{N(z)}$ is called the modulus of the complex number z [79,81]. In this case, let $\lambda = -p$, from Equation 3 we obtain

$$\frac{x^2}{N(z)} + \frac{y^2}{\frac{N(z)}{p}} = 1, \quad p > 0.$$

This equation represents an ellipse, showing the geometric interpretation of generalized complex numbers with $\lambda < 0$ [76,77]. It can be seen that this is a general equation of an ellipse, whose eccentricity is $e = \begin{cases} \sqrt{1 - \frac{1}{p}}, & p > 1 \\ \sqrt{\frac{1}{p} - 1}, & 0 < p < 1 \end{cases}$. Substituting $p = -\lambda$ gives

$$e = \sqrt{\left|1 + \frac{1}{\lambda}\right|}. \tag{4}$$

When $\lambda = -1$, which corresponds to circular complex numbers $\mathbb{C}$, the eccentricity $e = 0$ [78,80]. Therefore, we have the following definition:

Definition 2.4 (Elliptic Complex Numbers)**.** *In general, generalized complex numbers with $-\infty < \lambda < 0$ are collectively called elliptic complex numbers [79,81].*

This classification unifies the geometric interpretation of generalized complex numbers with negative λ values, with the special case $\lambda = -1$ corresponding to the familiar circular complex numbers (ordinary complex numbers) [76,77].

When $\lambda > 0$, from Equation 3 we see that its norm is not positive definite [76,77], and its norm corresponds to

$$\frac{x^2}{N(z)} - \frac{y^2}{\frac{N(z)}{\lambda}} = 1, \quad \lambda > 0,$$

which is the equation of a hyperbola [78,80], with eccentricity consistent with Equation 4. The equilateral hyperbolic complex numbers $\mathbb{C}_{+1}$ correspond to $\lambda = 1$, with eccentricity $e = \sqrt{2}$, taking the form of an equilateral hyperbola [76,79].

In general, generalized complex numbers with $0 < \lambda < +\infty$ are collectively called hyperbolic complex numbers [77,81]. Obviously, hyperbolic complex numbers cannot represent points on the asymptotes $x = \pm\sqrt{\lambda}y$ in the complex plane, because the norm (or modulus) of points on these asymptotes is zero; this is precisely the geometric reason why hyperbolic complex numbers do not form a division algebra [78,80].

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To distinguish between them, we stipulate that the imaginary unit for elliptic complex numbers is denoted by i, so they take the form

$$z = x + iy \in \mathbb{C}_\lambda \quad (\lambda < 0);$$

and the imaginary unit for hyperbolic complex numbers is denoted by j, so they take the form

$$z = x + jy \in \mathbb{C}_\lambda \quad (\lambda > 0).$$

Subsequent research will mainly focus on the more "perfect" elliptic complex numbers, as they possess positive definite norms and form division algebras, making them more suitable for various applications.

2.4. Euler's Formula for Generalized Complex Numbers

One might doubt whether the generalized complex numbers defined in this way are meaningful [76,77]. Next, we will further explore the algebraic properties of generalized complex numbers and reveal more of their elegant properties [78,80]. Using methods similar to those for complex numbers (i.e., circular complex numbers), we have the following conclusion.

Proposition 2.5 (Euler's Formula)**.** *Let $\varphi \in \mathbb{R}$, and $i^2 = -\lambda = -q^2$, $q \in \mathbb{R}^*$, then we have*

$$e^{i\varphi} = \cos(q\varphi) + \frac{i}{q}\sin(q\varphi),$$

where e is the base of the natural logarithm, and $\sin\theta$, $\cos\theta$ are the sine and cosine functions of θ, respectively [79,81].

Proof. From the Maclaurin series expansions [69,82], we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$$

Thus it follows that

$$\begin{aligned} e^{i\varphi} &= 1 + \frac{i\varphi}{1!} + \frac{(i\varphi)^2}{2!} + \cdots + \frac{(i\varphi)^n}{n!} + o((i\varphi)^n) \\ &= 1 - \frac{(q\varphi)^2}{2!} + \frac{(q\varphi)^4}{4!} - \frac{(q\varphi)^6}{6!} + \cdots + (-1)^n\frac{(q\varphi)^{2n}}{(2n)!} + o((q\varphi)^{2n+1}) \\ &\quad + \frac{i}{q}\left[\frac{(q\varphi)}{1!} - \frac{(q\varphi)^3}{3!} + \frac{(q\varphi)^5}{5!} - \frac{(q\varphi)^7}{7!} + \cdots + (-1)^n\frac{(q\varphi)^{2n+1}}{(2n+1)!} + o((q\varphi)^{2n+2})\right] \\ &= \cos(q\varphi) + \frac{i}{q}\sin(q\varphi), \end{aligned}$$

which proves the proposition [78,80]. □

Similarly, this leads to the following results [76,77].

Proposition 2.6 (Hyperbolic Euler Formula)**.** *Let $\varphi \in \mathbb{R}$, and $j^2 = \lambda = q^2$, $q \in \mathbb{R}^*$, then it gives*

$$e^{j\varphi} = \cosh(q\varphi) + \frac{j}{q}\sinh(q\varphi),$$

where e is the base of the natural logarithm, and $\sinh\theta$, $\cosh\theta$ are the hyperbolic sine and hyperbolic cosine functions of θ, respectively [78,80].

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Corollary 2.7. *From the meaning of scalar multiplication of elliptic complex numbers and Proposition 2.5, we deduce*

$$e^z = e^{x+iy} = e^x\left(\cos(y) + i\frac{1}{q}\sin(y)\right).$$

This is Euler's formula in the generalized sense for elliptic complex numbers [79,81].

Following the same procedure, it follows that

$$e^z = e^{x+jy} = e^x\left(\cosh(qy) + j\frac{1}{q}\sinh(qy)\right),$$

which is Euler's formula in the generalized sense for hyperbolic complex numbers [78,80].

Proposition 2.8 (Exponential Additivity). *Let $z_1, z_2 \in \mathbb{C}_\lambda, \lambda < 0$, then $e^{z_1}e^{z_2} = e^{z_1+z_2}$.*

Proof. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then from Corollary 2.7 [78,80] we conclude that

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1+iy_1}e^{x_2+iy_2} \\
&= e^{x_1}\left(\cos(qy_1) + i\frac{1}{q}\sin(qy_1)\right)e^{x_2}\left(\cos(qy_2) + i\frac{1}{q}\sin(qy_2)\right) \\
&= e^{x_1+x_2}[(\cos(qy_1)\cos(qy_2) - \sin(qy_1)\sin(qy_2)) \\
&\quad + i\frac{1}{q}(\sin(qy_1)\cos(qy_2) + \cos(qy_1)\sin(qy_2))\Big] \\
&= e^{x_1+x_2}\left[\cos(q(y_1+y_2)) + i\frac{1}{q}\sin(q(y_1+y_2))\right] \\
&= e^{x_1+x_2}e^{i(y_1+y_2)} \\
&= e^{x_1+x_2+i(y_1+y_2)} = e^{z_1+z_2}.
\end{aligned}$$

Hence proving the proposition [79,81]. □

In the same way, for $z_1, z_2 \in \mathbb{C}_\lambda, \lambda > 0$, it leads to $e^{z_1}e^{z_2} = e^{z_1+z_2}$ [78,80]. Furthermore,

Corollary 2.9. *From Proposition 2.5 and Proposition 2.8, we can arrive at the multiple-angle form of Euler's formula for elliptic complex numbers [79,81], namely,*

$$e^{i\left(\frac{n}{q}\varphi\right)} = \cos(n\varphi) + \frac{i}{q}\sin(n\varphi), \quad n \in N^*.$$

Analogously, it follows that

$$e^{j\left(\frac{n}{q}\varphi\right)} = \cosh(n\varphi) + \frac{j}{q}\sinh(n\varphi), \quad n \in N^*.$$

From Corollary 2.9, we easily obtain [78,80]

$$e^{\frac{i}{q}\pi} + 1 = 0, \quad i^2 = -q^2, q \in \mathbb{R}^*, \tag{5}$$

where, when $q = 1$, Equation 5 becomes the most beautiful mathematical formula in the eyes of mathematicians [69,83].

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2.5. Division Operations for Generalized Complex Numbers

The definition of division operation is obviously based on the multiplication operation [76,77].

Definition 2.10 (Division for Elliptic Complex Numbers)**.** *Let* $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}_\lambda$, $\lambda < 0$. *Define*

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{N(z_2)} = \frac{z_1 z_2^*}{x_2^2 - \lambda y_2^2}, \quad \lambda = -p$$

as the division operation for elliptic complex numbers [78,80].

Since $N(z) = 0 \Leftrightarrow z = 0$, elliptic complex numbers have no non-zero zero divisors and form a division algebra [77,79]. As the same way,

Definition 2.11 (Division for Hyperbolic Complex Numbers)**.** *Let* $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}_\lambda$, $\lambda > 0$. *Define*

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{N(z_2)} = \frac{z_1 z_2^*}{x_2^2 - \lambda y_2^2}, \quad x_2^2 \neq \lambda y_2^2$$

as the division operation for hyperbolic complex numbers [78,81].

It is worth noting that the division definition is meaningless when ${x_2}^2 = \lambda {y_2}^2$ [76,78]. The reason is, from the Euler formula for hyperbolic complex numbers, that it will lead to

$${x_2}^2 = \lambda {y_2}^2 \Leftrightarrow \sinh(qy_2) = \pm\cosh(qy_2) \Leftrightarrow e^{y_2} = 0,$$

which obviously contradicts the fact that $\forall y_2 \in \mathbb{R}, e^{y_2} > 0$ [80,81].

This explains geometrically why hyperbolic complex numbers do not form a division algebra: there exist non-zero elements with zero norm lying on the asymptotes $x = \pm\sqrt{\lambda}y$, for which division is undefined [76,77].

2.6. Vectors on the Elliptic Complex Plane

Similar to the circular complex plane $\mathbb{C}$, we define the vector corresponding to the complex number $z = x + iy$ in the elliptic complex plane $\mathbb{C}_\lambda$ as $\vec{z} = (x, y)$ [76,77]. For convenience, unless otherwise specified, in the following we always assume $i^2 = \lambda = -q^2$.

This vector representation allows us to study elliptic complex numbers geometrically, with the real part x and imaginary part y serving as coordinates in the elliptic plane [79,81]. The parameter $q = \pm\sqrt{-\lambda}$ determines the geometry of this plane, with the special case $q = 1$ corresponding to the ordinary complex plane $\mathbb{C}$ [76].

Consider the unit complex number. According to Euler's formula $e^{i\alpha} = \cos(q\alpha) + \frac{i}{q}\sin(q\alpha)$, it is easy to see that for a point $z = (x, y)$ in the complex plane $\mathbb{C}_\lambda$ compared to a point on the unit circle in the circular complex plane $\mathbb{C}$, its abscissa remains unchanged while its ordinate becomes $\frac{1}{q}$ times the original [76,77]. As shown in Figure 1, here we take $|\lambda| < 1$ as an example.

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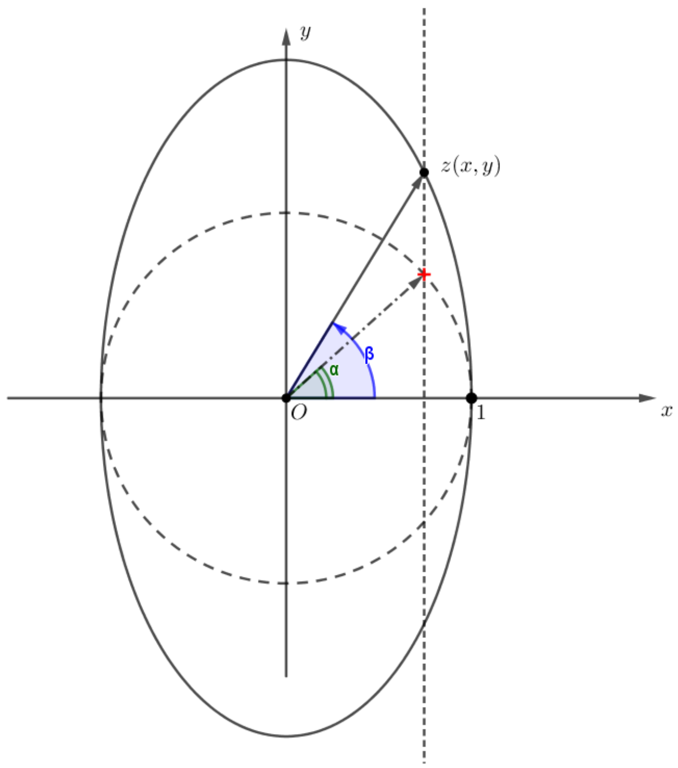

Figure 1. Points on the elliptic complex plane $\mathbb{C}_\lambda$

It is easy to see that $\tan\alpha = \dfrac{\tan\beta}{q}$. From an ordinary geometric perspective, the modulus of the vector z in the figure is greater than 1, which is the modulus of its corresponding point on the unit circle; however, in the complex plane $\mathbb{C}_\lambda$, the modulus of the vector z is exactly 1 . Similarly, from an ordinary geometric perspective, the angle of the vector z should be α, but in $\mathbb{C}_\lambda$, its angle is β .

Corollary 2.12 (the Angle in the Elliptic Complex Plane). *In summary, the angle α in the complex plane $\mathbb{C}_\lambda$, when measured in the ordinary geometric sense, is β, and they satisfy* $\tan\alpha = \dfrac{\tan\beta}{q}$.

Consider two vectors z_1, z_2 in the complex plane $\mathbb{C}_\lambda$. Their angles in the ordinary geometric sense are β_1, β_2 respectively, while their angles in $\mathbb{C}_\lambda$ are α_1, α_2 respectively [81]. Then it follows that

$$\tan\alpha_1 \cdot \tan\alpha_2 = \frac{\tan\beta_1 \cdot \tan\beta_2}{q^2} = -\frac{\tan\beta_1 \cdot \tan\beta_2}{\lambda}.$$

Thus when the angle $\|\beta_1 - \beta_2\|$ is $\pi/2$, it follows that

$$\tan\alpha_1 \cdot \tan\alpha_2 = \frac{1}{\lambda} \tag{6}$$

This relationship shows how orthogonality in the elliptic complex plane $\mathbb{C}_\lambda$ (with angle β_1, β_2 having difference $\pi/2$) translates to a specific product relationship of the ordinary geometric angles α_1, α_2 [76,78].

2.6.1. Length and Angle of Vectors on the Elliptic Complex Plane

In order to correctly describe vectors in the complex plane $\mathbb{C}_\lambda$ corresponding to the definition of elliptic complex multiplication [76,77], we can introduce the following definitions.

Definition 2.13 (Inner Product). *Let $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$ be two vectors in the complex plane $\mathbb{C}_\lambda$. Define*

$$\vec{a} \cdot \vec{b} = a_x b_x + q^2 a_y b_y \tag{7}$$

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as the inner product (or scalar product) of $\vec{a}$ and $\vec{b}$ in $\mathbb{C}_\lambda$ [78,80]. Furthermore, define $|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$ as the modulus (length) of the vector $\vec{a}$ [79,81].

By the same token,

Definition 2.14 (Vector Angle)**.** *Define the angle $\theta = \langle\vec{a},\vec{b}\rangle$ between $\vec{a}$ and $\vec{b}$ in $\mathbb{C}_\lambda$ by*

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta. \tag{8}$$

Now let us analyze the reasonableness of this definition . Let the angles between the vectors $\vec{a},\vec{b}$ and the positive direction of the X-axis in the coordinate system corresponding to the elliptic complex plane $\mathbb{C}_\lambda$ be α,β respectively [78,80]. Then when $|\vec{a}| \neq 0, |\vec{b}| \neq 0$,

$$\cos\alpha = \frac{a_x}{|\vec{a}|}, \quad \sin\alpha = q\frac{a_y}{|\vec{a}|}; \quad \cos\beta = \frac{b_x}{|\vec{b}|}, \quad \sin\beta = q\frac{b_y}{|\vec{b}|}. \tag{9}$$

Obviously, Equation 9 satisfies the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ [79,81]. Furthermore, from Equation 7 it is easy to obtain the unit vectors of $\vec{a}$ and $\vec{b}$ as

$$\vec{e}_a = \left(\cos\alpha, \frac{\sin\alpha}{q}\right), \quad \vec{e}_b = \left(\cos\beta, \frac{\sin\beta}{q}\right). \tag{10}$$

According to Equation 8 combined with Equation 7, we have $\cos\theta = \cos\langle\vec{e}_a,\vec{e}_b\rangle = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, that is

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

This is consistent with the result in trigonometric identities [76]. It can be seen that such a definition is reasonable.

Based on Equation 6, the vectors $\vec{r} = (x,y)$ and $\vec{r}' = (-q^2y, x)$ in the complex plane $\mathbb{C}_\lambda$ are perpendicular to each other [78]. In this case, from Definition 2.13 we know that $\vec{r}\cdot\vec{r}' = 0$, thus we have the conclusion:

Proposition 2.15 (Vector Perpendicularity)**.** *In the elliptic complex plane $\mathbb{C}_\lambda$, two vectors $\vec{a},\vec{b}$ are perpendicular to each other $\Leftrightarrow \vec{a}\cdot\vec{b} = 0$ [80,81].*

This result is the same as the conclusion in the circular complex plane $\mathbb{C}$, which further demonstrates the reasonableness of the above definition [77,79].

2.6.2. Right Angles on the Elliptic Complex Plane

As mentioned above: from an ordinary geometric perspective, the angle of the vector $\vec{a}$ should be α, but in $\mathbb{C}_\lambda$, the magnitude of this angle α is β [76,77]. The angle between two vectors in the complex plane $\mathbb{C}_\lambda$ defined above is precisely the magnitude β of α in $\mathbb{C}_\lambda$ [78,80]. Now we determine the value of the aforementioned α in the ordinary geometric sense, i.e., the value of α in the circular complex plane $\mathbb{C}$, denoted as $\theta_{\mathbb{C}}$ [79,81].

For the unit vectors $\vec{a}_e, \vec{b}_e$ in Equation 10, their angle in $\mathbb{C}$ satisfies

$$\cos\theta_{\mathbb{C}} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta/q^2}{\sqrt{\cos^2\alpha + \sin^2\alpha/q^2}\sqrt{\cos^2\beta + \sin^2\beta/q^2}}. \tag{11}$$

Based on Equation 11, when $\cos\beta = \cos\alpha$, i.e., $\beta = \alpha + 2k\pi$, it can be shown that $\cos\theta_{\mathbb{C}} \equiv 1$. This means

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Corollary 2.16 (Invariance of Straight Lines). *Assuming $\lambda_1 \neq \lambda_2$, a straight line in the complex plane $\mathbb{C}_{\lambda_1}$ remains a straight line in the complex plane $\mathbb{C}_{\lambda_2}$; that is, the geometric property of being a straight line is invariant between complex planes [76,78].*

If the vectors $\vec{a}_e, \vec{b}_e$ are mutually perpendicular in the elliptic complex plane $\mathbb{C}_\lambda$, then from equation (1.18) we know that $\cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$ [80]. Substituting this into Equation 11 yields

$$\cos\theta_{\mathbb{C}} = \frac{1 - 1/q^2}{\sqrt{\frac{1/q^2}{\cos^2\alpha} + 1 - 1/q^2}\sqrt{\frac{1/q^2}{\cos^2\beta} + 1 - 1/q^2}}.$$

It can be seen that even for two vectors $\vec{a}_e, \vec{b}_e$ that are mutually perpendicular in the complex plane $\mathbb{C}_\lambda$, their angle in the circular complex plane will vary with changes in α or β [79,81]. Therefore it leads to

Corollary 2.17 (Non-Congruence of Right Angles). *In the complex plane $\mathbb{C}_\lambda$ with $\lambda \neq -1$, the axiom that "all right angles are congruent" does not hold [77,78].*

This result highlights a fundamental difference between elliptic complex geometry (with $\lambda \neq -1$) and Euclidean geometry: while straight lines remain straight under the transformation between different λ planes, the measure of angles, particularly right angles, is not preserved in a uniform way [76,80].

In fact, when $\cos\beta = \sin\alpha$, i.e., $\beta = \alpha + \pi/2 + 2k\pi$, Equation 11 transforms into [76,78]

$$\cos\theta_c = \frac{\tan\alpha(q^2 - 1)}{q^2 + \tan^2\alpha}. \tag{12}$$

In this case, if $|q| \neq 1$, $\cos\theta_c$ is not identically zero, meaning θ_c is not identically equal to $\pi/2 + k\pi$ [79,80].

It is well known that one of the five postulates of Euclidean geometry states: "All right angles are congruent to one another" [68,69]. Clearly, the geometry corresponding to elliptic complex numbers does not satisfy this postulate [77,81]. This result demonstrates that the geometry induced by the elliptic complex plane $\mathbb{C}_\lambda$ with $\lambda \neq -1$ (i.e., $|q| \neq 1$) is non-Euclidean [76,78].

2.7. Geometric Significance of Elliptic Complex Numbers

For a point $A(x, y)$ in the elliptic complex plane $\mathbb{C}_\lambda$, let the coordinate origin be O [76,77]. Then the corresponding vector is $\overrightarrow{OA} = x + iy$, with $i^2 = \lambda = -p$ [78,80]. Now consider the vector

$$\overrightarrow{OB} = i\overrightarrow{OA} = i(x + iy) = -py + ix,$$

so the coordinates of point B are $B(-py, x)$ [79,81]. As a result,

$$k_{OA} \cdot k_{OB} = (-1) \times \frac{1}{p} = \frac{1}{\lambda},$$

which yields the same result as Equation 6 [78]. In particular, if we set $y = 0$, then A is exactly a point on the positive X-axis, and B is exactly a point on the positive Y-axis. It can be seen that when $\lambda \neq -1$, the angle between the positive X-axis and the positive Y-axis in the complex plane $\mathbb{C}_\lambda$ is no longer a right angle [80]. Consequently,

Corollary 2.18 (Non-Rectangular Coordinate Plane). *When $\lambda \neq -1$, the complex plane $\mathbb{C}_\lambda$ describes points on a non-rectangular coordinate plane [76,81].*

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Furthermore, consider the vector $\overrightarrow{OC} = \frac{i}{q}\overrightarrow{OA} = -qy + \frac{i}{q}x$ [76,77]. Obviously, in the complex plane $\mathbb{C}_\lambda$, we have $OC \perp OA$ and $|\overrightarrow{OC}| = |\overrightarrow{OA}| = \sqrt{x^2 + q^2y^2}$ [78,80]. Consequently,

Corollary 2.19 (Orthogonal Equal-Length Vectors)**.** *Two vectors $\vec{a}, \vec{b}$ in the complex plane $\mathbb{C}_\lambda$ are mutually orthogonal and of equal length if and only if*

$$\vec{a} + \frac{i}{q}\vec{b} = \vec{0}.$$

This is the geometric interpretation of the correspondence between a complex number (or vector) z and the complex number (or vector) $\frac{i}{q}z$ in the complex plane $\mathbb{C}_\lambda$ [79,81].

2.7.1. Normal Ellipse

Just as the definition of a "circle" in the circular complex plane $\mathbb{C}$, we also need to find analogous geometric elements in the elliptic complex plane $\mathbb{C}_\lambda$ [76,77].

From the definitions of norm and modulus for elliptic complex numbers $\mathbb{C}_\lambda, \lambda = -p = -q^2 < 0$, we have [78,80]

$$\frac{x^2}{N(z)} + \frac{y^2}{\frac{N(z)}{q^2}} = 1 \iff \frac{x^2}{|z|^2} + \frac{y^2}{\left(\frac{|z|}{q}\right)^2} = 1,$$

whose geometric interpretation is an ellipse centered at the coordinate origin in the complex plane [79,81]. When $0 < |q| < 1$, the semi-major axis length of this ellipse is $m = \frac{|z|}{|q|}$, and the semi-minor axis length is $n = |z|$; when $|q| = 1$, this ellipse degenerates into a circle with radius $r = |z|$, which is the geometric interpretation of circular complex numbers; when $|q| > 1$, the semi-major axis length is $m = |z|$, and the semi-minor axis length is $n = \frac{|z|}{|q|}$ [76,78].

This type of ellipse, which can define the norm or modulus in the complex plane $\mathbb{C}_\lambda$, possesses very essential characteristics [77,80]. To fully characterize this geometric element, we first give the following definitions:

Definition 2.20. *When $0 < |q| < 1$, the direction of the minor axis of the ellipse is its principal axis direction, and the direction of the major axis is its secondary axis direction; when $|q| > 1$, the direction of the major axis of the ellipse is its principal axis direction, and the direction of the minor axis is its secondary axis direction [78,79].*

Definition 2.21. *The length of the major axis (or minor axis) in the principal axis direction is called the principal axis length of the ellipse. Half of the principal axis length is called the principal radius [81]. An ellipse with principal radius 1 is called the unit ellipse [80].*

Combined with Equation 4, The following definition is established.

Definition 2.22 (Normal Ellipse)**.** *In the complex plane $\mathbb{C}_\lambda$, if an ellipse has eccentricity satisfying $e = \sqrt{\left|1 + \frac{1}{\lambda}\right|}$ and its principal axis direction is parallel to the X-axis, then this ellipse is called a normal ellipse in $\mathbb{C}_\lambda$ [76,78].*

In particular, the normal ellipse of the circular complex plane $\mathbb{C}$ is a circle [77]. Unlike circles in the circular complex plane $\mathbb{C}$, normal ellipses in the complex plane $\mathbb{C}_\lambda$ have directionality [79,81]. When $\lambda \neq -1$, a normal ellipse in $\mathbb{C}_\lambda$, if rotated slightly, is no longer a normal ellipse [80].

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2.7.2. Representation of Normal Ellipses on the Complex Plane

We know that the equation of a straight line in the plane in real coordinates is [76,77]

$$l : ax + by + c = 0, \; a, b, c \in \mathbb{R}.$$

Let $z = x + iy \in \mathbb{C}_\lambda$, $\lambda = -p < 0$ [78,80]. Then $x = \dfrac{z + z^*}{2}$, $y = \dfrac{z - z^*}{2i} = -\dfrac{z - z^*}{2p} \cdot i$. Hence it leads to the complex equation of a straight line [79,81]

$$l : \frac{pa - ib}{2p} z + \frac{pa + ib}{2p} z^* + c = 0,$$

that is

$$l : \beta^* z + \beta z^* + \gamma = 0, \; \gamma \in \mathbb{R},$$

where $\beta = \dfrac{pa + ib}{2p} \in \mathbb{C}_\lambda$. This is the equation representation of a straight line in the elliptic complex plane [76,78].

Now we discuss the representation of a normal ellipse in the complex plane [77,80]. Since

$$\begin{aligned}
|z - z_0|^2 &= (z - z_0)(z^* - z_0^*) = zz^* - z^* z_0 - z z_0^* + z_0 z_0^* \\
&= (x^2 + p^2 y^2) + (x_0^2 + p^2 y_0^2) \\
&\quad - [(x + iy)(x_0 - iy_0) + (x - iy)(x_0 + iy_0)] \\
&= (x - x_0)^2 + p^2 (y - y_0)^2,
\end{aligned}$$

so $|z - z_0|$ represents a normal ellipse centered at $z_0 = (x_0, y_0)^T$ [79,81].

Let a normal ellipse in the complex plane be $|z - z_0| = s$. Squaring both sides and expanding gives

$$-zz^* + z^* z_0 + z_0^* z + (s^2 - z_0 z_0^*) = 0.$$

That is, the equation of a normal ellipse in the complex plane $\mathbb{C}_\lambda$ is

$$C : \alpha z z^* + \beta^* z + \beta z^* + \gamma = 0, \; \alpha, \gamma \in \mathbb{R}, \; \beta \in \mathbb{C}_\lambda,$$

where when $\alpha = 0$, the equation degenerates into the form of a straight line equation [78,80]. Therefore, the normal ellipses and the straight lines in the complex plane are unified, with a straight line being a normal ellipse with infinite principal radius [76,79].

2.7.3. Limiting Cases of Elliptic Complex Numbers

Proposition 2.23 (Conjugate Equality in the Limit)**.** *Let $z \in \mathbb{C}_\lambda$. Then when $\lambda \to 0$, we always have $z^* = z$ [76,77].*

Proof. Since z can be written as $z = Re^{i\theta}$ [78,80]. When $R = 0$, clearly $z^* = z$ holds [79]. When $R \neq 0$, if $\lambda = -q^2 \to 0$, then according to Euler's formula on the elliptic complex domain [81],

$$\frac{z}{z^*} = \lim_{\lambda \to 0^-} e^{2i\theta} = \lim_{q \to 0} [\cos(2q\theta) + i \sin(2q\theta)] = 1 + 2i\theta,$$

where because $|z/z^*| = 1$, it yields $z/z^* = 1$, which proves the proposition [78,80]. □

From this, we can see

Corollary 2.24 (Oblique Coordinate Plane)**.** *When $\lambda \to 0$, the corresponding complex plane represents an oblique coordinate plane where the angle between the positive Y-axis and the positive X-axis tends to 0 [76,79].*

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It should be noted that, as shown in Figure 2 , the angle tending to 0 does not mean that the X-axis and Y-axis coincide [76,77]. The region between the positive X-axis and the positive Y-axis, as shown in (1) of the figure, still contains infinitely many points [78,80], and there is a one-to-one correspondence with the points in the region between the positive X-axis and the positive Y-axis in the orthogonal plane shown in (2) of the figure [79,81].

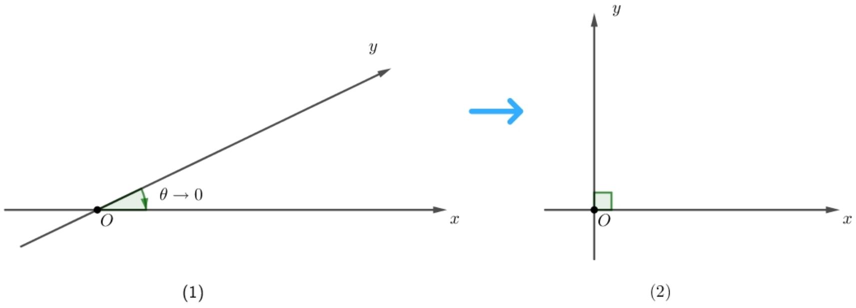

Figure 2. The geometric significance of the complex plane $\mathbb{C}_\lambda$ $(\lambda \to 0)$

Note that if $\alpha = \dfrac{\pi}{2}$, then Equation 12 gives $\cos\theta_c = 0$, i.e., $\theta_c = \dfrac{\pi}{2}$, corresponding to case (2) in Figure 2 [76,77].

Furthermore, the geometric interpretation of the complex plane $\mathbb{C}_\lambda$ as $\lambda \to \infty$ is shown in Figure 3 [78,80]: the corresponding complex plane represents an oblique coordinate plane where the angle between the positive Y-axis and the positive X-axis tends to π [79,81].

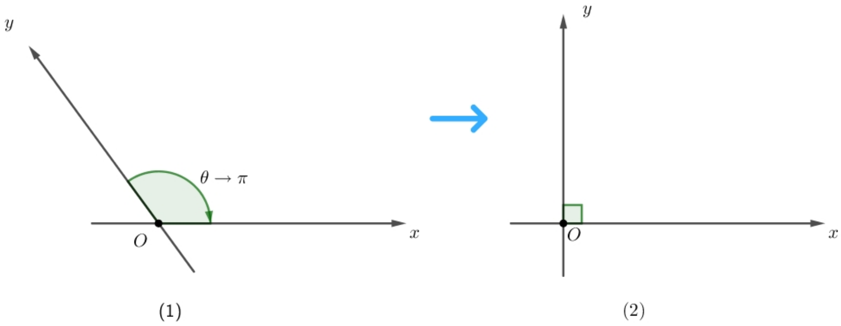

Figure 3. The geometric significance of the complex plane $\mathbb{C}_\lambda$ $(\lambda \to \infty)$

In fact, it is difficult to determine the exact angle between the positive Y-axis and the positive X-axis for the coordinate system corresponding to each complex plane $\mathbb{C}_\lambda$ [76,77]. What we can determine is that as $\lambda \to 0$, the plane $\mathbb{C}_\lambda$ reaches a "sufficient limit" that allows us to glimpse the essential properties within the "black box" of elliptic complex numbers [78,80].

The two limiting cases, $\lambda \to 0$ and $\lambda \to \infty$, represent extreme geometric configurations where the axes become nearly parallel (angle approaching 0) or nearly opposite (angle approaching π) [79,81]. These limits help illuminate the fundamental nature of elliptic complex geometry.

2.8. Conclusion

It should be noted that mathematicians have already given certain special "elliptic complex numbers" in the algebraic sense, such as the integral domains $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[\sqrt{-3}]$, etc., as well as imaginary quadratic fields $k = \mathbb{Q}(\sqrt{d})$, where d is a negative square-free integer. However, mathematicians tend to study the algebraic properties corresponding to these special "elliptic complex numbers," and even more so, to explore the properties of divisibility at present .

In a sense, this behavior of mathematicians is more like studying the discrete mathematical structures of "elliptic complex numbers." But the Riemann Hypothesis itself belongs to the realm of

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analysis. We naturally hope to construct a series of "continuous" algebraic structures, and even treat these algebraic structures as parameters, to study the zero distribution of the Riemann ζ function holistically using the tools of mathematical analysis.

From a purely algebraic perspective, elliptic complex numbers are algebraically isomorphic to circular complex numbers. From a geometric perspective, however , elliptic complex numbers $\mathbb{C}_\lambda(\lambda \neq -1)$ represent geometric elements in a Cartesian coordinate system that have been transformed from an oblique coordinate system.

It is important to note that we require λ to be a constant. The reason is that to solve the proof of the Riemann Hypothesis itself, letting λ be a constant is sufficient. As for further research on the distribution of zeros on the critical line, it may require designing special elliptic complex numbers to resolve. In fact, if λ is a function of one or more variables $t_i(i = 1, 2, \ldots)$, then it corresponds to a curved or even surface coordinate system. This would then correspond to geometry in curved spaces.

On the other hand, we consider the case $\lambda \to 0$, which does not mean that λ is a variable. Its true meaning is that λ equals a constant approaching 0. It can be seen that the elliptic complex numbers $\mathbb{C}_\lambda$ with $\lambda \to 0$ also form a number field, corresponding to an oblique coordinate plane where the angle between the positive Y-axis and the positive X-axis tends to 0. However, it is particularly important to note that the elliptic complex numbers at this point still represent geometric elements in a Cartesian coordinate system.

In fact, using elliptic complex numbers allows us to appreciate the unique charm of observing a geometric object from different geometric perspectives. As is well known, the world in our human eyes is completely different from the world as seen by other creatures such as cats, dogs, etc. Are there civilizations in the universe that are similar to us or even more advanced than us ? What differences would there be between the physical world observed by these civilizations and that observed by us humans? Elliptic complex numbers are precisely what we need to predict our physical world from all these possible perspectives.

As demonstrated, elliptic complex numbers essentially provide an alternative formulation for calculating vector lengths within the defined space. While this definition is fundamentally simple, it possesses the potential to profoundly reshape our conceptual framework for understanding the physical world. The author posits the significant value of this approach and anticipates its validation and further exploration by the broader scientific community.

Part I

Mathematical Analysis on Elliptic Complex Numbers

3. Theory of Elliptic Complex Functions

3.1. Ellipsoidal Representation of Elliptic Complex Numbers and the Extended Complex Plane

In the plane, there is no point corresponding to ∞ [76,77]. Of course, we can introduce an "ideal" point, called the point at infinity [68]. All points in the plane together with the point at infinity form the extended complex plane [69]. We stipulate that every straight line passes through the point at infinity; consequently, there is no half-plane that contains this ideal point [78]. Previously, we unified normal ellipses and straight lines through their equations [80].

In fact, the unification of normal ellipses and straight lines can be achieved by introducing the point at infinity, which leads to the concept of a complex ellipsoid [79,81].

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Definition 3.1 (Ellipsoid and Polar Projection)**.** *Consider in the three-dimensional rectangular coordinate system $O-XYU$ the ellipse $C:\begin{cases} x^2+q^2y^2=1 \\ u=0 \end{cases}$. If we rotate C around the X-axis, we obtain a rotational ellipsoid*

$$S_1: x_1^2+q^2y_1^2+q^2u_1^2=1.$$

The vertex $(0,0,\frac{1}{q})^T$ of S_1 is called the X-system pole of the ellipsoid, denoted by n_X [78,80]. For any point $p'_X \in S_1 \setminus \{n_X\}$ on S_1, draw the line connecting n_X and p'_X and extend it; it necessarily intersects the coordinate plane XOY at a point p_X [79]. We call p_X the X-system ellipsoidal polar projection of p'_X, and the correspondence $p'_X \mapsto p_X$ is called its X-system ellipsoidal polar mapping [81].

Now we find the correspondence relation for $p'_X \mapsto p_X$ [76,77]. Let $p'_X=(x_1,y_1,u_1)$ and $p_X=(x,y,0)$, with its corresponding point in the complex plane being $z=x+iy$ [78]. From geometric relations we obtain

$$\frac{1/q-u_1}{1/q}=\frac{1-q^2u_1^2}{|z|^2}=\frac{x_1}{x}=\frac{y_1}{y}.$$

Thus

$$u_1=\frac{1}{q}\cdot\frac{|z|^2-1}{|z|^2+1},\quad x_1=\frac{z+z^*}{1+|z|^2},\quad y_1=\frac{z-z^*}{i(1+|z|^2)}.$$

Conversely,

$$z=\frac{x_1+iy_1}{1-u_1},$$

and

$$x:y:(-1)=x_1:y_1:(qu_1-1).$$

It can be seen that the points $(x,y,0)$, (x_1,y_1,u_1), and $(0,0,\frac{1}{q})$ are collinear, which again yields the mapping $p'_X \mapsto p_X$ [79,80].

Definition 3.2 (Y-System Ellipsoid and Polar Projection)**.** *If we rotate C around the Y-axis, we obtain a rotational ellipsoid*

$$S_2: x_2^2+q^2y_2^2+u_2^2=1.$$

The vertex $(0,0,1)^T$ of S_2 is called the Y-system pole of the ellipsoid, denoted by n_Y [76,77]. For any point $p'_Y \in S_2 \setminus \{n_Y\}$ on S_2, draw the line connecting n_Y and p'_Y and extend it; it necessarily intersects the coordinate plane XOY at a point p_Y [78,80]. We call p_Y the Y-system ellipsoidal polar projection of p'_Y, and the correspondence $p'_Y \mapsto p_Y$ is called its Y-system ellipsoidal polar mapping [79,81].

Now consider the X-system pole n_X (or the Y-system pole n_Y) [76]. Clearly, n_X (or n_Y) corresponds to the point at infinity in the complex plane [68,69]. Thus we introduce a new elliptic complex number ∞ corresponding to it. The complex plane $\mathbb{C}_\lambda$ together with this point forms the extended complex plane, or complex ellipsoid, denoted by $\overline{\mathbb{C}_\lambda}=\mathbb{C}_\lambda\cup\{\infty\}$ [78,80]. The operations for this new elliptic complex number ∞ are defined as follows:

$$\begin{cases} \infty\pm z=\infty, & \forall z\in\mathbb{C}_\lambda; \\ z\cdot\infty=\infty, & \forall z\in\mathbb{C}_\lambda\setminus\{0\}. \end{cases}$$

In summary, the mapping from points on the ellipsoid to the extended complex plane is a bijection [77,79] — this is precisely the geometric meaning of the divisibility of elliptic complex numbers [76,81].

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3.2. The Topology of the Elliptic Complex Plane

As established in Section 2.7.1, the normal ellipse is the fundamental tool for describing the magnitude and direction of vectors on the elliptic complex plane. Similar to the method for the circular complex plane, we now use normal ellipses to analyze the topological structure within the elliptic complex plane [78,80].

Definition 3.3 (Neighborhood). *The set of points determined by the interior of a normal ellipse centered at z_0 with principal radius $r = \delta > 0$ is called the δ-neighborhood of z_0, denoted by $U(z_0, \delta)$, i.e.,*

$$U(z_0, \delta) = \{z \mid |z - z_0| < \delta, z \in \mathbb{C}_\lambda\},$$

which is an open elliptic disk [79,81]. Furthermore,

- *The set of points determined by the inequality $0 < |z - z_0| < \delta$ is called the punctured δ-neighborhood of z_0, denoted by $\mathring{U}(z_0, \delta)$, i.e., $\mathring{U}(z_0, \delta) = \{z \mid 0 < |z - z_0| < \delta, z \in \mathbb{C}_\lambda\}$;*
- *The set of points determined by the inequality $|z - z_0| \leq \delta$ is called the closed δ-neighborhood of z_0, denoted by $\overline{U}(z_0, \delta)$, i.e., $\overline{U}(z_0, \delta) = \{z \mid |z - z_0| \leq \delta, z \in \mathbb{C}_\lambda\}$, which is a closed elliptic disk [78,80].*

From the properties of neighborhoods around a point, we have:

Definition 3.4 (Limit Point, Interior Point, Boundary Point). *Let $z_0 \in \mathbb{C}_\lambda$ and $E \subseteq \mathbb{C}_\lambda$. Then,*

- *If $\forall r > 0$, we have $\mathring{U}(z_0, r) \cap E \neq \emptyset$, then z_0 is called a limit point (accumulation point) of E [76,77];*
- *If $\forall r > 0$, we have $\mathring{U}(z_0, r) \subset E$, then z_0 is called an interior point of E [79];*
- *If $\forall r > 0$, we have $\mathring{U}(z_0, r) \cap E \neq \emptyset$ and $\mathring{U}(z_0, r) \cap E^c \neq \emptyset$, then z_0 is called a boundary point of E [81].*

If $z_0 \in E$ but z_0 is not a limit point of E, then it is called an isolated point of E [78]. The set of all boundary points of E is called the boundary of E, denoted by ∂E; the set $\overline{E} = E \cup \partial E$ is called the closure of E [80].

Definition 3.5 (Open Set, Closed Set, Bounded Set). *If all points in the set E are interior points, then E is called an open set [76]; conversely, if $E^c = \mathbb{C}_\lambda \setminus E$ is an open set, then E is called a closed set [77]; if $\exists M > 0$ such that $\forall z \in E$, $|z| < M$, then E is called a bounded set; otherwise, E is called an unbounded set [79,81].*

A bounded closed set is called a compact set [78,80].

Definition 3.6 (Region). *Let D be a point set in the complex plane $\mathbb{C}_\lambda$ satisfying*

- *D is an open set [76,77];*
- *Any two finite points in D can be connected by a finite polygonal line, and all points on this finite polygonal line belong entirely to D [78,80],*

Then D is called a region [79,81].

It can be seen that a region is a connected open set [77]. If a region together with its boundary is considered, it is called a closed region [76].

Definition 3.7 (Curves). *For a curve $C : z = z(t), a \leq t \leq b$, i.e., $z(t) = \operatorname{Re} z(t) + i \operatorname{Im} z(t), a \leq t \leq b$ [78,80],*

- *If $\operatorname{Re} z(t)$ and $\operatorname{Im} z(t)$ are continuous on $[a, b]$, then C is called a continuous curve [79];*
- *If $\forall t_1, t_2 \in (a, b), t_1 \neq t_2$, we have $z(t_1) \neq z(t_2)$, then C is called a simple curve [81];*
- *If $\operatorname{Re} z(t)$ and $\operatorname{Im} z(t)$ have continuous derivatives on $[a, b]$, and $z'(t) \neq 0$, then C is called a smooth curve [78].*

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A simple continuous curve is called a Jordan curve [76,77]. A simple continuous closed curve (with its endpoints coinciding) is called a Jordan closed curve (or simple closed curve) [69,78].

Definition 3.8 (Jordan Curve Theorem for $\mathbb{C}_\lambda$). *Any simple closed curve divides the complex plane into two regions with no common points [76,80], one of which is a bounded region (called the interior region) and the other is an unbounded region (called the exterior region) [77,79], and both regions share this closed curve as their common boundary [78,81].*

Definition 3.9 (Simply and Multiply Connected Regions). *If for any simple closed curve within a region $D \subset \mathbb{C}_\lambda$, its interior region is contained in D, then D is called a simply connected region [76,80]; otherwise, D is called a multiply connected region [77,79].*

These definitions establish the fundamental concepts of connectivity in the elliptic complex plane $\mathbb{C}_\lambda$, analogous to those in the standard complex plane [69,81]. The Jordan curve theorem ensures that every simple closed curve has a well-defined interior and exterior, which is essential for understanding the topological structure of regions in $\mathbb{C}_\lambda$ and for the subsequent study of elliptic complex functions, their continuity, and limits [78,80].

3.3. Elliptic Complex Functions: Continuity and Limits

The definition of elliptic complex functions is, in form, actually the same as the definition of functions in mathematical analysis and in (circular) complex function theory studied by mathematicians [76,77].

Definition 3.10 (Elliptic Complex Function). *Let E be a point set in the complex plane $\mathbb{C}_\lambda$ [78,80]. If there exists a rule f such that $\forall z = x + iy \in E$, there corresponds $w = u + iv \in \mathbb{C}_\lambda$, then f is called an elliptic complex function defined on E, referred to simply as a complex function, denoted by*

$$w = f(z),\, z \in E \quad or \quad f : E \to \mathbb{C}_\lambda,\, z \mapsto f(z)$$

Just like circular complex functions, elliptic complex functions also have single-valued and multi-valued complex functions [79,81]. If there is a uniquely determined w corresponding to it, it is a single-valued complex function; if there are multiple or infinitely many w corresponding to it, it is a multi-valued complex function [78]. For now, the complex functions we refer to are all single-valued complex functions.

From the definition, we have $w = f(x + iy) = u(x, y) + iv(x, y)$ [80]. To describe the graph of $w = f(z)$, we would need to use four-dimensional space (u, v, x, y) (which is clearly beyond our imagination) [76]. To avoid this difficulty, we use two complex planes: the z-plane and the w-plane [77]. We understand complex functions as mappings between point sets in these two complex planes [79,81].

Definition 3.11 (Function Limit). *Let $w = f(z)$ be a complex function defined on a point set E, let z_0 be a limit point of E, and let α be a complex constant [78,80]. If $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that whenever $z \in E$ and $0 < |z - z_0| < \delta$,*

$$|f(z) - \alpha| < \varepsilon$$

holds, then as z approaches z_0 in E, $f(z)$ is said to approach the limit α, denoted by

$$\lim_{z \to z_0, z \in E} f(z) = \alpha \quad or \quad f(z) \to \alpha \quad (z \to z_0, z \in E)$$

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Definition 3.12 (Infinite Limit). *Let $w = f(z)$ be a complex function defined on a point set E, and let z_0 be a limit point of E [76,77]. If $\forall A > 0, \exists \delta = \delta(A) > 0$ such that whenever $z \in E$ and $0 < |z - z_0| < \delta$ [78,80],*

$$|f(z)| > A$$

holds, then as z approaches z_0 in E, $f(z)$ is said to approach infinity [79,81], denoted by

$$\lim_{z \to z_0, z \in E} f(z) = \infty \quad \text{or} \quad f(z) \to \infty \quad (z \to z_0, z \in E)$$

As a result, we have the following conclusion.

Proposition 3.13 (Limit in Terms of Real and Imaginary Parts). *Let $w = f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $\alpha = s + it$ be a complex constant [76,77]. Then*

$$\lim_{z \to z_0} f(z) = \alpha \iff \begin{cases} \lim\limits_{(x,y) \to (x_0,y_0)} u(x,y) = s; \\ \lim\limits_{(x,y) \to (x_0,y_0)} v(x,y) = t. \end{cases}$$

Proof. Since $f(z) - \alpha = (u(x,y) - s) + i(v(x,y) - t)$ [78,80], from the inequality

$$\max\{|x|, |q| \cdot |y|\} \leq |x + iy| \leq |x| + |q| \cdot |y| \quad (\lambda = -q^2),$$

it results in [79,81]

$$\max\{|u(x,y) - s|, |q| \cdot |v(x,y) - t|\} \leq |f(z) - \alpha| \leq |u(x,y) - s| + |q| \cdot |v(x,y) - t|.$$

Note that q is a nonzero constant [78]. Hence,

$\Rightarrow$: Since $\lim\limits_{z \to z_0} (f(z) - \alpha) = 0$, the first inequality gives the result;

$\Leftarrow$: Since $\lim\limits_{(x,y) \to (x_0,y_0)} (u(x,y) - s) = 0$ and $\lim\limits_{(x,y) \to (x_0,y_0)} (v(x,y) - t) = 0$, the second inequality gives the result [79,80]. □

Definition 3.14 (Continuity of a Function). *Let $w = f(z)$ be a complex function defined on a point set E, let z_0 be a limit point of E, and $z_0 \in E$ [76]. If*

$$\lim_{z \to z_0} f(z) = f(z_0),$$

then $f(z)$ is said to be continuous at z_0 [77]. If the function $f(z)$ is continuous at every limit point of E, then $f(z)$ is said to be continuous on E [78].

Proposition 3.15. *A function $f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if its real part $u(x,y)$ and imaginary part $v(x,y)$ are continuous at (x_0, y_0) [80,81].*

Analogously, the four arithmetic operations are closed for continuous functions, i.e., if $f(z)$ and $g(z)$ are continuous at z_0, then the functions

$$f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)} \; (g(z_0) \neq 0)$$

are also continuous at z_0 [78,79].

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Definition 3.16 (Uniform Continuity). *Let $w = f(z)$ be a complex function defined on a point set E [76,77]. If for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ independent of z such that for all $z_1, z_2 \in E$ with $|z_1 - z_2| < \delta$,*

$$|f(z_1) - f(z_2)| < \varepsilon$$

holds, then $f(z)$ is said to be uniformly continuous on E [78,80].

It should be noted that continuity is a local property, generally concerning a single point, while uniform continuity is a global property [79,81].

Theorem 3.17 (Uniform Continuity on Compact Sets). *If a function $f(z)$ is continuous on a simple curve or a bounded closed region E, then $f(z)$ is uniformly continuous on E [76,77].*

Theorem 3.18 (Boundedness of Continuous Functions). *Let the function $f(z)$ be continuous on a simple curve or a bounded closed region E [76,77]. Then $f(z)$ is bounded on E, i.e., $\exists M > 0$ such that $|f(z)| \leq M$ for all $z \in E$ [78,80]; and the following equivalent condition holds:*

- *The function $f(z) = u(x,y) + iv(x,y)$ is bounded on E $\Leftrightarrow$ $u(x,y)$ and $v(x,y)$ are bounded on E [79,81].*

Theorem 3.19. *Let the function $f(z)$ be continuous on a simple curve or a bounded closed region E [76,78]. Then $f(z)$ attains its maximum modulus and minimum modulus on E, i.e., $\exists z_1, z_2 \in E$ such that*

$$|f(z_1)| = \max_{z \in E} |f(z)|, \quad |f(z_2)| = \min_{z \in E} |f(z)|$$

Definition 3.20 (Limit at Infinity). *Let $w = f(z)$ be a complex function defined on an unbounded (closed) region E, and let α be a complex constant [77,80]. If $\forall \varepsilon > 0$, $\exists \rho = \rho(\varepsilon) > 0$ such that whenever $z \in E$ and $|z| > \rho$,*

$$|f(z) - \alpha| < \varepsilon$$

holds, then as z approaches ∞ in E, $f(z)$ is said to approach the limit α [79,81], denoted by

$$\lim_{z \to \infty, z \in E} f(z) = \alpha \quad \text{or} \quad f(z) \to \alpha \quad (z \to \infty, z \in E)$$

Therefore far , the fundamental theory of limits and continuity for elliptic complex functions has been established [76,77]. Next comes the corresponding theory of analytic functions [78,80].

3.4. Derivatives of Elliptic Complex Functions and Analytic Functions

Based on the above discussion, we begin to explore the differential structure of elliptic complex functions [76,77].

3.4.1. Basic Concepts of Analytic Functions

Definition 3.21 (Differentiability). *Let $w = f(z)$ be a single-valued complex function defined on a region D, and let $z_0 \in D$ [78,80]. If the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and equals α, then $f(z)$ is said to be differentiable at z_0, and α is called its derivative or differential coefficient, denoted by $f'(z_0)$ or $\left.\frac{d}{dz}f(z)\right|_{z=z_0}$.

The quantity $f'(z)\Delta z$ is called the differential of $w = f(z)$ at z, denoted by dw or $df(z)$, i.e., $dw = f'(z)\Delta z$ [79,81]. In fact, from the properties of differentials, the function $f(z)$ is differentiable at z_0 if and only if

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0)\Delta z + o(|\Delta z|), \quad \Delta z \to 0$$

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Definition 3.22 (Analyticity)**.** *Let $w = f(z)$ be a single-valued complex function defined on a region D, and let $z_0 \in D$ [76,78]. If there exists a neighborhood $U(z_0, r)$ of z_0 such that $f(z)$ is differentiable at every point in this neighborhood, then the function $f(z)$ is said to be analytic at z_0 [77,80]. If the function $f(z)$ is analytic at every point in the region D, then $f(z)$ is said to be analytic in the region D [79,81].*

It can be seen that differentiability is a local concept, while analyticity is a global concept [76,77]. A function may be differentiable at a point without being analytic there, but if it is analytic at a point, it must be differentiable there [78,80]. Analyticity in a region is equivalent to differentiability in that region [79,81].

Similarly, the four arithmetic operations are closed for analytic functions, i.e., if $f(z)$ and $g(z)$ are analytic in a region D, then the functions

$$f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)} \ (g(z) \neq 0)$$

are also analytic in the region D, and it is obtained that

$$\begin{cases} [f(z) \pm g(z)]' = f'(z) \pm g'(z); \\ [f(z)g(z)]' = f'(z)g(z) + f(z)g'(z); \\ \left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2} \quad (g(z) \neq 0). \end{cases}$$

Elliptic complex functions also have a corresponding chain rule [76,78]: If the function $w = f(z)$ is analytic in a region D of the z-plane, and the function $W = F(w)$ is analytic in a region E of the w-plane, with $f(D) \subset E$, then the composite function $s = F(f(z))$ is analytic in the region D, and we have

$$\frac{d}{dz}F(f(z)) = \frac{dF}{dw}(f(z)) \cdot \frac{d}{dz}f(z).$$

These properties demonstrate that the theory of analytic functions in the elliptic complex plane $\mathbb{C}_\lambda$ parallels that of ordinary complex analysis [77,80].

3.4.2. The C.-R. Equations for Analytic Functions

We know that a real-valued function of two variables $u(x, y)$ is differentiable at a point (x, y) if there exist numbers A, B independent of $\Delta x, \Delta y$ such that [76,77]

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y) = A\Delta x + B\Delta y + o(\rho),$$

where $\rho = \sqrt{\Delta x^2 + \Delta y^2} \to 0$ [78,80]. In this case, $A = \frac{\partial u}{\partial x}$, $B = \frac{\partial u}{\partial y}$, and $du = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$ [79,81].

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on a region D [76]. When $u(x, y)$ and $v(x, y)$ are given, the function $f(z)$ is also determined. However, in general, if $u(x, y)$ and $v(x, y)$ are independent of each other, even if the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y exist, the function $f(z)$ may not be differentiable [77,78].

Theorem 3.23 (Cauchy-Riemann Equations for Elliptic Complex Analytic Functions)**.** *Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on a region D in the complex plane $\mathbb{C}_\lambda$ [79,80]. Then $f(z)$ is differentiable at $z = x + iy \in D$ if and only if the functions $u(x, y)$ and $v(x, y)$ are differentiable at (x, y) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -q^2 \frac{\partial v}{\partial x}, \tag{13}$$

where q satisfies $i^2 = -\lambda = -q^2$ $(q \in \mathbb{R}^*)$ [81]. Equation (2.11) is called the Cauchy-Riemann equations (abbreviated as C.-R. equations) for elliptic complex analytic functions [76,78].

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Proof.
- For the necessity : Let $f'(z) = a + ib$ and $\Delta z = \Delta x + i\Delta y$ [77,80]. From the definition of the derivative, we have $f(z+\Delta z) - f(z) = f'(z)\Delta z + o(|\Delta z|)$ as $\Delta z \to 0$, i.e.,

$$\begin{aligned}&[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]\\ =&(a+ib)(\Delta x + i\Delta y) + o(|\Delta z|)\\ =&(a\Delta x - q^2 b\Delta y) + i(b\Delta x + a\Delta y) + o(|\Delta z|).\end{aligned}$$

Comparing the real and imaginary parts on both sides, we obtain [76,77]

$$u(x+\Delta x, y+\Delta y) - u(x,y) = a\Delta x - q^2 b\Delta y + o(\rho), \quad \rho \to 0,$$

$$v(x+\Delta x, y+\Delta y) - v(x,y) = b\Delta x + a\Delta y + o(\rho), \quad \rho \to 0.$$

From the definition of differentiability for real-valued functions of two variables, we know that the functions $u(x,y)$ and $v(x,y)$ are differentiable at (x,y) [78,80], and

$$a = \frac{\partial u}{\partial x}, \quad -q^2 b = \frac{\partial u}{\partial y}, \quad b = \frac{\partial v}{\partial x}, \quad a = \frac{\partial v}{\partial y}.$$

Hence it follows that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -q^2\frac{\partial v}{\partial x}$, which proves the necessity [79,81];
- For the sufficiency : Since the functions $u(x,y)$ and $v(x,y)$ are differentiable at (x,y), we have [77,78]

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + o(\rho), \quad \rho \to 0,$$

$$v(x+\Delta x, y+\Delta y) - v(x,y) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + o(\rho), \quad \rho \to 0.$$

Combining these two equations, we obtain [79,80]

$$\begin{aligned}&[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]\\ =&\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + o(|\Delta z|)\\ &\text{(substituting the C.-R. equations)}\\ =&\left(\frac{\partial u}{\partial x}\Delta x - q^2\frac{\partial v}{\partial x}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + o(|\Delta z|)\\ =&\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + o(|\Delta z|)\\ =&\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + o(|\Delta z|)\end{aligned}$$

Therefore, $f(z)$ is differentiable at $z = x + iy \in D$, and its derivative is [76,81]

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{14}$$

This proves the sufficiency of the proposition [78,80].

□

Substituting Equation 13 into Equation 3.5.1 yields other forms of the derivative $f'(z)$ of the complex function [76,77]. If we set $\lambda = -p$, the C.-R. equations can also be written as [78,80]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -p\frac{\partial v}{\partial x}$$

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Since differentiability in a region is equivalent to analyticity in that region [79,81], we have the following conclusion.

Theorem 3.24 (Differentiability in a Region)**.** *Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on a region D in the complex plane $\mathbb{C}_\lambda$ [76,78]. Then $f(z)$ is differentiable in the region D if and only if the functions $u(x, y)$ and $v(x, y)$ are differentiable at (x, y) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -q^2 \frac{\partial v}{\partial x},$$

where q satisfies $i^2 = \lambda = -q^2$ $(q \in \mathbb{R}^*)$ [77,80].

3.5. Elementary Univalent Elliptic Complex Functions

Next, several fundamental elementary functions in elliptic complex analysis shall be derived based on the Cauchy-Riemann equations.

3.5.1. Exponential Function

Similar to the exponential function in real analysis, the exponential function in elliptic complex function theory should satisfy the following conditions [76,77]:

(1) When restricted to the real numbers, for $z \in \mathbb{R}$, $f(z) = e^z$;
(2) $f(z)$ is analytic in the complex plane $\mathbb{C}_\lambda$ [78,80];
(3) $\forall z_1, z_2 \in \mathbb{C}_\lambda$, $f(z_1 + z_2) = f(z_1)f(z_2)$.

From (1) and (3), $f(z) = f(x + iy) = f(x)f(iy) = e^x f(iy) = e^x(A(y) + iB(y))$ [79,81]. Then from (2), we know that $u(x, y) = e^x A(y)$ and $v(x, y) = e^x B(y)$ must satisfy the Cauchy-Riemann equations [76], giving

$$A(y) = B'(y), \quad A'(y) = -q^2 B(y),$$

with particular solutions $A(y) = \cos(qy)$, $B(y) = \frac{1}{q}\sin(qy)$ [78,80]. Thus we have the definition of the exponential function:

Definition 3.25 (Exponential Function)**.** *Let $z = x + iy \in \mathbb{C}_\lambda$. The exponential function in the elliptic complex function domain is defined as*

$$e^z = \exp(z) := e^x \left[\cos(qy) + \frac{i}{q}\sin(qy)\right],$$

where q satisfies $i^2 = -\lambda = -q^2$ $(q \in \mathbb{R}^*)$ [77,79].

Obviously, the elliptic complex exponential function possesses properties as elegant as those of the (circular) complex exponential function [76,81].

Proposition 3.26. *For $z_1, z_2 \in \mathbb{C}_\lambda$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$.*

Proof. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. From the definition of the exponential function [78,80],

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$$
\begin{aligned}
e^{z_1+z_2} &= e^{x_1+x_2+i(y_1+y_2)} \\
&= e^{x_1}e^{x_2}\left[\cos(qy_1)+i\frac{1}{q}\sin(qy_1)\right]\cdot\left[\cos(qy_2)+i\frac{1}{q}\sin(qy_2)\right] \\
&= e^{x_1+x_2}[\cos(qy_1)\cos(qy_2)-\sin(qy_1)\sin(qy_2)] \\
&\quad + i[\sin(qy_1)\cos(qy_2)+\cos(qy_1)\sin(qy_2)] \\
&= e^{x_1+x_2}\left[\cos(q(y_1+y_2))+i\frac{1}{q}\sin(q(y_1+y_2))\right] \\
&= e^{x_1+x_2}e^{i(y_1+y_2)} = e^{x_1+x_2+i(y_1+y_2)} = e^{z_1+z_2}.
\end{aligned}
$$

Thus the proposition is proved [79,81]. □

Proposition 3.27. $\forall z \in \mathbb{C}_\lambda$, $e^z \neq 0$. *The function* $w = e^z$ *is analytic in the entire complex plane, and* $\dfrac{d}{dz}e^z = e^z$.

Proof. Since [76,78]

$$
|e^z| = |e^{x+iy}| = \left|e^x\left(\cos(qy)+\frac{i}{q}\sin(qy)\right)\right| = |e^x|\sqrt{\cos^2(qy)+q^2\left(\frac{1}{q}\sin(qy)\right)^2} = |e^x|,
$$

it follows that $e^z \neq 0$ [80].

From the definition of the exponential function, let $e^z = u(x,y)+iv(x,y)$ [77]. Then the functions $u(x,y) = e^x\cos(qy)$ and $v(x,y) = \frac{e^x}{q}\sin(qy)$ are differentiable [79], and

$$
\frac{\partial u}{\partial x} = e^x\cos(qy), \quad \frac{\partial u}{\partial y} = -qe^x\sin(qy), \quad \frac{\partial v}{\partial x} = \frac{e^x}{q}\sin(qy), \quad \frac{\partial v}{\partial y} = e^x\cos(qy).
$$

As is evident from the above, the C.-R. equations are satisfied, which implies the analyticity of the function in the entire complex plane, and

$$
\frac{d}{dz}e^z = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = e^x\cos(qy)+i\frac{e^x}{q}\sin(qy) = e^z.
$$

In summary, the proposition is proved . □

Proposition 3.28 (Periodicity)**.** *The function* $w = e^z$ *is a periodic function with fundamental period* $\dfrac{2\pi i}{q}$ *[76,78].*

Proof. For any integer $k \in \mathbb{Z}$, it yields [77,80]

$$
e^{z+\frac{2k\pi i}{q}} = e^z e^{\frac{2k\pi i}{q}} = e^z.
$$

□

3.5.2. Trigonometric Functions

From the Euler formula for elliptic complex numbers [76,77], it is obtained that

$$
e^{i\frac{y}{q}} = \cos y + i\frac{1}{q}\sin y, \quad e^{-i\frac{y}{q}} = \cos y - i\frac{1}{q}\sin y,
$$

giving [78,80]

$$
\cos y = \frac{1}{2}\left(e^{i\frac{y}{q}}+e^{-i\frac{y}{q}}\right), \quad \sin y = \frac{q}{2i}\left(e^{i\frac{y}{q}}-e^{-i\frac{y}{q}}\right).
$$

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Consequently,

Definition 3.29 (Trigonometric Functions). *The cosine and sine functions for elliptic complex numbers are defined as [79,81]*

$$\cos z = \frac{1}{2}\left(e^{i\frac{z}{q}} + e^{-i\frac{z}{q}}\right),$$
$$\sin z = \frac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right),$$

where q satisfies $i^2 = -q^2$ ($q \in \mathbb{R}^*$) [76,77].

Proposition 3.30 (Analyticity and Derivatives). *The functions $\sin z$ and $\cos z$ are analytic in the entire complex plane, and [78,80]*

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z.$$

Proof. Since $w_1 = e^{i\frac{z}{q}}$ and $w_2 = e^{-i\frac{z}{q}}$ are analytic in the entire complex plane $\mathbb{C}_\lambda$ [79,81], and

$$\left(e^{i\frac{z}{q}}\right)' = \frac{i}{q}e^{i\frac{z}{q}}, \quad \left(e^{-i\frac{z}{q}}\right)' = -\frac{i}{q}e^{-i\frac{z}{q}},$$

the functions $\sin z$ and $\cos z$ are analytic in the entire complex plane [76], and

$$(\sin z)' = \left(\frac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right)\right)' = \frac{q}{2i}\left(\frac{i}{q}e^{i\frac{z}{q}} + \frac{i}{q}e^{-i\frac{z}{q}}\right) = \frac{1}{2}\left(e^{i\frac{z}{q}} + e^{-i\frac{z}{q}}\right) = \cos z.$$

Similarly, we obtain $(\cos z)' = -\sin z$ [78,80]. □

Proposition 3.31 (Symmetry and Trigonometric Identities). *The functions $\sin z$ and $\cos z$ are odd and even functions, respectively, and satisfy the trigonometric identities [77,79]*

$$\sin^2 z + \cos^2 z = 1,$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

and other standard trigonometric identities [76,81].

Proposition 3.32 (Periodicity). *The functions $\sin z$ and $\cos z$ are periodic functions with fundamental period 2π [76,77].*

Proof.
$$\sin(z + 2k\pi) = \frac{q}{2i}\left(e^{i\frac{z+2k\pi}{q}} - e^{-i\frac{z+2k\pi}{q}}\right) = \frac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right) = \sin z[78,80].$$

The case for $\cos z$ can be proved similarly [79,81]. □

Proposition 3.33 (Zeros of Sine and Cosine). *The zeros of the function $\sin z$ in the complex plane $\mathbb{C}_\lambda$ are: $k\pi$ ($k \in \mathbb{Z}$); the zeros of the function $\cos z$ in the complex plane $\mathbb{C}_\lambda$ are: $\left(k + \frac{1}{2}\right)\pi$ ($k \in \mathbb{Z}$) [76,77].*

Proof. Setting $\sin z = 0$, i.e., [78,80] $\frac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right) = 0$, hence

$$e^{i\frac{2z}{q}} = 1 = e^{i\frac{2k\pi}{q}} \ (k \in \mathbb{Z}),$$

so the zeros of the function $\sin z$ in the complex plane $\mathbb{C}_\lambda$ are: $k\pi$ ($k \in \mathbb{Z}$) [79,81];
The case for $\cos z$ can be proved by the same token [76,77]. □

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Therefore, when extending the sine and cosine functions to the elliptic complex domain, their zeros do not increase [78,80].

Furthermore, let $k \in \mathbb{Z}$, we have the following formulas [79,81]:

Proposition 3.34. *(1)* $\sin(z+(2k+1)\pi) = -\sin(z), \quad \cos(z+(2k+1)\pi) = -\cos(z);$
(2) $\sin\left(z+\frac{\pi}{2}\right) = \cos(z), \quad \cos\left(z+\frac{\pi}{2}\right) = -\sin(z).$

Proof. Here we only prove $\sin\left(z+\dfrac{\pi}{2}\right)$ in (2) [76,77],

$$\begin{aligned}\sin\left(z+\frac{\pi}{2}\right) &= \frac{q}{2i}\left(e^{i\frac{z}{q}}\cdot e^{i\frac{\pi}{2q}} - e^{-i\frac{z}{q}}\cdot e^{-i\frac{\pi}{2q}}\right)\\ &= \frac{1}{2}\left(e^{i\frac{z}{q}} + e^{-i\frac{z}{q}}\right) = \cos z[78,80].\end{aligned}$$

The other formulas can be verified similarly [79,81]. □

From the definitions of the sine and cosine functions [76,77], we can further define other trigonometric functions.

Definition 3.35 (Trigonometric Tangent, Cotangent, Secant, and Cosecant Functions)**.** *The tangent, cotangent, secant, and cosecant functions for elliptic complex numbers are respectively defined as [78,80]*

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

The functions $\tan z$ and $\cot z$ both have period π [79,81]. Taking $\tan z$ as an example,

$$\tan(z+\pi) = \frac{\sin(z+\pi)}{\cos(z+\pi)} = \frac{-\sin z}{-\cos z} = \tan z.$$

It can be seen that the singularities of the function $\tan z$ are precisely the zeros of $\sin z$, and the singularities of the function $\cot z$ are precisely the zeros of $\cos z$ [76,77]. Furthermore, we have

$$(\tan z)' = \left(\frac{\sin z}{\cos z}\right)' = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z.$$

By symmetry, we conclude [78,80]

$$(\cot z)' = -\csc^2 z.$$

Definition 3.36 (Hyperbolic Functions)**.** *Define* $\sinh z = \dfrac{e^z - e^{-z}}{2}$ *and* $\cosh z = \dfrac{e^z + e^{-z}}{2}$ *as the hyperbolic sine function and hyperbolic cosine function of z, respectively [76,77].*

It is easy to see that the functions $\sinh z$ and $\cosh z$, like e^z, both have period $2\pi i$ [78,80], are analytic on $\mathbb{C}_\lambda$ [79,81], and satisfy

$$(\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z.$$

There is an important conclusion [76,77]:

$$\cosh(iz) = \cos(qz), \quad \cos(iz) = \cosh(qz),$$

$$\sinh(iz) = \frac{i}{q}\sin(qz), \quad \sin(iz) = \frac{i}{q}\sinh(qz).$$

These two formulas have many applications in the derivation of relevant formulas in subsequent series of articles on the Riemann Hypothesis in the elliptic complex domain [78,80]. The proof of the

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formulas can be obtained from the basic definitions; readers are encouraged to attempt it themselves [79,81].

3.6. Elementary Multivalent Elliptic Complex Functions

The preceding discussion has focused exclusively on single-valued complex functions; we now turn our attention to multivalued complex functions.

3.6.1. The Argument and Logarithm Functions

It can be seen that the definition of the argument function in the field of elliptic complex functions is consistent with that in the field of circular complex functions [76,77], namely:

Definition 3.37 (Argument Function). *When $z \in \mathbb{C} \setminus \{0\}$, we call*

$$w = Argz = \arg z + 2k\pi \quad (k \in \mathbb{Z})$$

the argument function of z, where $\arg z$ is called the principal value of $Argz$, generally specified as: $-\pi \leq \arg z \leq \pi$ [78,80].

The above region $D = \mathbb{C} \setminus \{0\}$ can be viewed as a region obtained by cutting the entire complex plane along the negative real axis [79,81], where the negative real axis is called a cut line, serving as the boundary of the region D. In general, we take an unbounded simple continuous curve K connecting the origin O to the point at infinity ∞ as a cut line, obtaining a region $\mathbb{C} \setminus K$, called a slit region, whose boundary is the curve K [76,77].

Let $z_1, z_2 \in E$, with $z_1 \neq z_2$, take $\mathrm{Arg} z_1 = \theta_1$, and consider a simple continuous curve γ in E connecting z_1 and z_2 [78,80]. As z moves continuously from z_1 to z_2 along γ, $\arg z$ also changes continuously from θ_1 to θ_2. Thus, starting from the value θ_1 of $\mathrm{Arg} z$ at z_1, we can determine the values at any other position in E, thereby obtaining a single-valued continuous function in E, i.e., a single-valued continuous branch [79,81].

In fact, for a fixed $k \in \mathbb{Z}$, $w = \arg z + 2k\pi$ is also a single-valued continuous branch of the argument function, showing that the argument function $w = \mathrm{Arg} z$ is a multi-valued function [76,77].

Proposition 3.38 (Addition and Subtraction Properties). *Let $z \in \mathbb{C} \setminus \{0\}$, then*

$$Arg(z_1 z_2) = Argz_1 + Argz_2,$$
$$Arg\frac{z_1}{z_2} = Argz_1 - Argz_2.$$

Now we study the "inverse function" of the exponential function [76,77].

Definition 3.39 (Logarithmic Function). *For a non-zero complex number $z \in \mathbb{C}_\lambda$, if there exists $z = e^w$, then w is called the logarithm of z, denoted by $\log z$ [78,80].*

Now we need to derive the formula for the logarithmic function .

Let $z = |z| e^{\frac{i \arg z}{q}}$ (this definition is chosen for the convenience of later definitions of power functions and root functions) [76,77], and $w = u + iv$. Then from $z = e^w$ it can be shown that

$$|z| e^{\frac{i \arg z}{q}} = e^{u+iv} = e^u e^{iv},$$

so $u = \log |z|$, $v = \frac{\arg z}{q}$, i.e.,

$$w = \log z = \log |z| + i\frac{\arg z}{q}. \tag{15}$$

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Equation 15 is called the logarithmic function of z [78,80]. Due to the (infinite) multi-valuedness of the argument function, the logarithmic function is also multi-valued [79,81]. In fact, the inverse function of the logarithmic function, namely the exponential function, is periodic, so the logarithmic function is multi-valued [76,77].

From $w = \log|z| + i\frac{\arg z}{q} = \log|z| + \frac{i}{q}(\arg z + 2k\pi)$, $k \in \mathbb{Z}$, we call $\log z = \log|z| + \frac{i}{q}\arg z$ the principal value of the logarithm of z [78,80]. Then

$$w = \log z = \text{Log}z + \frac{2k\pi i}{q},\ k \in \mathbb{Z},$$

which shows that non-zero complex numbers have infinitely many logarithms, and any two values differ by an integer multiple of $\frac{2\pi i}{q}$ [79,81].

Proposition 3.40 (Addition and Subtraction Properties)**.** *Let $z_1, z_2 \in \mathbb{C}_\lambda$ be non-zero complex numbers. Then [76,77]*

$$Log(z_1 z_2) = Log(z_1) + Log(z_2),$$
$$Log\left(\frac{z_1}{z_2}\right) = Log(z_1) - Log(z_2).$$

It can be seen that for the logarithmic function $w = \text{Log}z = \log z + \frac{2k\pi i}{q}, k \in \mathbb{Z}$ [78,80], taking each fixed k, the function $f_k(z) = \log z + \frac{2k\pi i}{q}$ is a single-valued continuous branch on the slit region $E = \mathbb{C}_\lambda \setminus \{0, \infty\}$ [79,81]. Thus we have the following property:

Proposition 3.41 (Analyticity and Derivative)**.** *Let $f(z)$ be a single-valued continuous branch of $w = \text{Log}z$ in a region G [76,77]. Then $f(z)$ is analytic in the region G, and*

$$f'(z) = \frac{1}{z}, \quad z \in G.$$

Proof. From the definition of the logarithmic function, when $z, z+h \in G$ [78,80],

$$z = e^{f(z)}, \quad z + h = e^{f(z+h)}.$$

For $h \neq 0$, we have [79,81]

$$\frac{f(z+h) - f(z)}{h} = \frac{f(z+h) - f(z)}{(z+h) - z} = \frac{1}{\frac{e^{f(z+h)} - e^{f(z)}}{f(z+h) - f(z)}}.$$

Therefore,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{\frac{d}{dz}\left(e^{f(z)}\right)} = \frac{1}{e^{f(z)}} = \frac{1}{z}.$$

Thus the proof is complete [76,77]. □

3.6.2. Power and Root Functions

Definition 3.42 (Power Function)**.** *Let α be an arbitrary complex number, and let z be a non-zero complex number. Then [76,77]*

$$w = z^\alpha = e^{\alpha Logz} = e^{\alpha(\log z + \frac{2k\pi i}{q})} = e^{\alpha \log z} e^{\frac{2k\pi\alpha i}{q}}$$

is called the power function of z [78,80]. Here, for a fixed $k \in \mathbb{Z}$, $w = e^{\alpha \log z} e^{\frac{2k\pi\alpha i}{q}}$ is a single-valued branch of $w = z^\alpha$ [79,81].

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For different types of α, it can be shown the following cases [76,77]:

(1) If $\alpha = n \in \mathbb{Z}^+$, then the power function is the usual n-th power [78,80]. In this case, $e^{\frac{2k\pi\alpha i}{q}} = 1$, so $w = z^\alpha$ is a single-valued function [79,81].

(2) If $\alpha = \dfrac{1}{n}, n = 2, 3, 4, \ldots$, then the power function is the n-th root of z, i.e.,

$$z^\alpha = z^{\frac{1}{n}} = e^{\frac{1}{n}\left(\ln|z| + \frac{i}{q}\operatorname{Arg} z\right)} = \sqrt[n]{|z|}\, e^{\frac{i}{nq}(\arg z + 2k\pi)}, \quad k = 0, 1, 2, \ldots, n-1.$$

In this case, $w = z^\alpha$ is a finitely multi-valued function [76,77].

(3) If $\alpha = \dfrac{m}{n}, m \in \mathbb{Z}^+, n = 2, 3, 4, \ldots$, the situation is similar to the case of the n-th root, since $z^\alpha = (z^m)^{\frac{1}{n}}$ [78,80]. Thus $w = z^\alpha$ is also a finitely multi-valued function [79,81].

(4) If α is an irrational number or a complex number, then the function $w = z^\alpha$ is an infinitely multi-valued function [76,77], because

$$w = z^\alpha = e^{\alpha \log z} e^{\frac{2k\pi\alpha i}{q}}.$$

When $k_1 \neq k_2$ (with $k_1, k_2 \in \mathbb{Z}$),

$$e^{\frac{2\alpha k_1 \pi i}{q}} \div e^{\frac{2\alpha k_2 \pi i}{q}} = e^{\frac{2\alpha(k_1 - k_2)\pi i}{q}} \neq 1,$$

so different values of k correspond to different values, making it an infinitely multi-valued function [78,80].

From the single-valuedness of the logarithmic function and the analyticity of rational expressions, it follows that the single-valued transformation branches of the power function are also analytic [79,81].

Proposition 3.43 (Derivative of Power Function Branches). *Let $f(z)$ be a single-valued continuous branch of the power function $w = z^\alpha$ in a region D [76,77]. Then $f(z)$ is analytic in the region D, and*

$$f'(z) = \alpha \cdot \frac{f(z)}{z} \tag{16}$$

(i.e., $f'(z) = \alpha z^{\alpha-1}$) [78,80].

Proof. Since $f(z) = e^{\alpha \log z} e^{\frac{2k\alpha\pi i}{q}}$, where $k \in \mathbb{Z}$ is fixed [79,81], we have

$$f'(z) = e^{\alpha \log z} \cdot \frac{\alpha}{z} \cdot e^{\frac{2\alpha k\pi i}{q}} = \alpha \cdot \frac{f(z)}{z}. \tag{17}$$

The function $w = f(z)$ is called a single-valued analytic branch in the region D [76,77]. □

4. Integral Theory of Analytic Functions

In the previous articles, we studied some fundamental theories of elliptic complex functions, including analytic functions, elementary analytic functions, and the Cauchy–Riemann equations. Building on this foundation, this chapter delves into another very important theory—contour integration.

4.1. Fundamental Theory of Contour Integration

4.1.1. Basic Concepts and Conclusions of Integration

Let C be a simple curve (smooth or piecewise smooth) in the complex plane $\mathbb{C}_\lambda$ connecting z_0 to z, and let the function

$$f(z) = u(x, y) + iv(x, y)$$

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be continuous on C. Insert $n+1$ partition points on the curve C: $z_0, z_1, z_2, \ldots, z_{n-1}, z_n = z$ [79,81]. Let $z_k = x_k + iy_k (k = 0, 1, \ldots, n)$, $\Delta x_k = x_{k+1} - x_k$, $\Delta y_k = y_{k+1} - y_k$. Arbitrarily take

$$\zeta_k = \xi_k + i\psi_k \in z_k z_{k+1} (k = 0, 1, 2, \ldots, n-1),$$

and form the sum:

$$S_n = \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \sum_{k=0}^{n-1} [u(\xi_k, \psi_k) + iv(\xi_k, \psi_k)](\Delta x_k + i\Delta y_k)$$

$$= \sum_{k=0}^{n-1} [u(\xi_k, \psi_k)\Delta x_k - q^2 v(\xi_k, \psi_k)\Delta y_k] + i\sum_{k=0}^{n-1} [u(\xi_k, \psi_k)\Delta y_k + v(\xi_k, \psi_k)\Delta x_k],$$

where q satisfies $i^2 = \lambda = -q^2 (q \in \mathbb{R}^*)$.

Denote $\chi = \max\{|z_k z_{k+1}| : k = 0, 1, 2, \ldots, n-1\}$. Then as $\chi \to 0$,

$$S_n \to \int_C u(x,y)dx - q^2 v(x,y)dy + i\int_C u(x,y)dx + v(x,y)dy.$$

The above limit is defined as the (contour) integral of the function $f(z)$ along the curve C, denoted by $\int_C f(z)dz$ [76,77]. That is,

$$\int_C f(z)\mathrm{d}z = \int_C u(x,y)\mathrm{d}x - q^2 v(x,y)\mathrm{d}y + i\int_C u(x,y)\mathrm{d}x + v(x,y)\mathrm{d}y. \tag{18}$$

Consider expressing the curve in parametric form. Let the simple curve $C : \begin{cases} x = x(t) \\ y = y(t) \end{cases} t \in (t_0, t)$. Then

$$\int_C f(z)\mathrm{d}z = \int_{t_0}^{t} f(z(t))z'(t)\mathrm{d}t \tag{19}$$

In summary, it yields the following conclusion.

Proposition 4.1 (Integrability). *If the function $f(z) = u(x,y) + iv(x,y)$ is continuous along the curve C, then $f(z)$ is integrable along the curve C, and the integral is of the form 18 (with the corresponding parametric form shown in 19) [78,80].*

4.1.2. Basic Properties of Integrals

According to the definition of the integral of elliptic complex functions, its general properties can be easily derived [79,81].

Corollary 4.2 (Properties of Integrals). *Let the functions $f(z), g(z)$ be continuous on the curve C. Then*

(1) $\int_C \alpha f(z)\mathrm{d}z = \alpha \int_C f(z)\mathrm{d}z$, *where α is a complex constant;*

(2) $\int_C [f(z) + g(z)]\mathrm{d}z = \int_C f(z)\mathrm{d}z + \int_C g(z)\mathrm{d}z$;

(3) $\int_C f(z)\mathrm{d}z = \int_{C_1} f(z)\mathrm{d}z + \int_{C_2} f(z)\mathrm{d}z + \cdots + \int_{C_n} f(z)\mathrm{d}z$, *where the curve C is composed of smooth curves $C_1, C_2, \ldots, C_n$ connected together;*

(4) $\int_C f(z)\mathrm{d}z = -\int_{C^-} f(z)\mathrm{d}z$;

(5) *If on the curve C we have $|f(z)| \leq M$, and L is the arc length of the curve C, then we have the integral (estimation) inequality*

$$\left|\int_C f(z)\mathrm{d}z\right| \leq M \cdot L. \tag{20}$$

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Proof. Here we only prove Item 5. From the definition of the integral,

$$S_n = \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) \leq \sum_{k=0}^{n-1} |f(\zeta_k)| \, |(z_{k+1} - z_k)|$$
$$\leq M \sum_{k=0}^{n-1} |(z_{k+1} - z_k)| \leq M \cdot L$$

Let $\chi = \max\{|z_k z_{k+1}| : k = 0, 1, 2, \ldots, n-1\} \to 0$, which proves Equation 20 [76,77]. □

Proposition 4.3 (Fundamental Integral). *Let C be the ellipse $|z - \alpha| = \rho$, oriented counterclockwise. Hence,*

$$\int_C \frac{1}{(z-\alpha)^n} \mathrm{d}z = \begin{cases} \dfrac{2\pi i}{q}, & n = 1 \\ 0, & n \neq 1, n \in \mathbb{Z} \end{cases} \tag{21}$$

Proof. Owing to $C : |z - \alpha| = \rho$, i.e., $C : z = \alpha + \rho e^{i\frac{t}{q}}$ $(0 \leq t \leq 2\pi)$, we have

$$dz = i\frac{\rho}{q} e^{i\frac{t}{q}} dt.$$

When $n = 1$, the integral is

$$\int_C \frac{1}{z-\alpha} \mathrm{d}z = \int_0^{2\pi} \frac{i\frac{\rho}{q} e^{i\frac{t}{q}} \mathrm{d}t}{\rho e^{i\frac{t}{q}}} = \frac{2\pi i}{q};$$

When $n \neq 1$, the integral is

$$\int_C \frac{1}{(z-\alpha)^n} \mathrm{d}z = \int_0^{2\pi} \frac{i\frac{\rho}{q} e^{i\frac{t}{q}} \mathrm{d}t}{\rho^n e^{i\frac{nt}{q}}} = \frac{i}{q\rho^{n-1}} \int_0^{2\pi} e^{-\frac{i}{q}(n-1)t} \mathrm{d}t = 0.$$

Thus the proposition is proved [78,80]. □

Formula 21 is extremely important and serves as the foundation for deriving subsequent integral formulas [79,81].

4.2. Cauchy Integral Theorem

Definition 4.4 (Primitive Function). *Let $f(z)$ and $F(z)$ be functions defined in a region D, and suppose that $F'(z) = f(z)$ in D. Then $F(z)$ is called a primitive function (antiderivative) of $f(z)$ in the region D [76,77].*

If $F_1(z)$ and $F_2(z)$ are both primitive functions of $f(z)$ in the region D, then

$$[F_1(z) - F_2(z)]' = f(z) - f(z) = 0, \quad z \in D.$$

Hence $F_1(z) - F_2(z) = c$ with $c \in \mathbb{C}_\lambda$ and $c \in D$. From this, if a function defined in a region has a primitive function, then it necessarily has infinitely many primitive functions, and any two differ by a constant c [78,80].

Let $F(z)$ be a primitive function of $f(z)$ in the region D. Then its indefinite integral is expressed as

$$\int f(z)\mathrm{d}z = F(z) + c, \quad c \in \mathbb{C}_\lambda.$$

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4.2.1. Three Lemmas

Lemma 4.5 (Polygon Boundary Integral). *Let the function $f(z)$ be analytic in a simply connected region D, and let C be any polygonal boundary in D. Then*

$$\oint_C f(z)\mathrm{d}z = 0.$$

Proof. The proof is similar to the case of circular complex functions, except that the disk is replaced by a normal elliptic disk finally [79,81]. □

Lemma 4.6 (Existence of Primitive Function in Convex Regions). *Let the function $f(z)$ be analytic in a convex region D. Then $f(z)$ necessarily has a primitive function in the region D.*

Proof. Since D is a convex region, for any two points z, z_0 in D, the line segment $\overline{z_0 z}$ connecting them must lie entirely in D. Define

$$F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta.$$

Let $z, z+h \in D$. By Lemma 4.5, if the function $f(z)$ is analytic in a simply connected region D and C is any polygonal boundary in D, then $\oint_C f(z)\mathrm{d}z = 0$. Hence

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\zeta)\mathrm{d}\zeta - \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta = \int_{z}^{z+h} f(\zeta)\mathrm{d}\zeta.$$

The function $f(z)$ is analytic in D, i.e., $f(z)$ is continuous in D. Thus $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\overline{U}(z,\delta) \subset D$, and when $\zeta \in U(z,\delta)$ there exists $|f(\zeta) - f(z)| < \epsilon$ [76,77].

When $0 < |h| < \delta$, it yields $z + h \in U(z,\delta)$. Then

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \left|\frac{F(z+h)-F(z)}{h} - \frac{1}{h}\int_z^{z+h} f(z)\,d\zeta\right| \\ &= \left|\frac{1}{h}\int_z^{z+h} [f(\zeta)-f(z)]\,d\zeta\right| < \frac{1}{|h|}\cdot\varepsilon\cdot|h| = \varepsilon,\end{aligned}$$

Therefore, $F'(z) = \lim_{h\to 0}\dfrac{F(z+h)-F(z)}{h} = f(z), \quad \forall z \in D$. This completes the proof [78,80]. □

Lemma 4.7 (Integral with Primitive Function). *Let $f(z)$ be a continuous function in the region D, and suppose it has a primitive function $F(z)$ in D. If two points $\alpha, \beta \in D$, and C is a simple curve in D connecting α and β, then*

$$\int_C f(z)\mathrm{d}z = F(\alpha) - F(\beta).$$

Proof. Since C is a simple curve $z = z(t), t \in (a,b)$, with $z(a) = \alpha, z(b) = \beta$, it can be seen that

$$\int_C f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t,$$

where $\dfrac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = f(z(t))z'(t)$. Hence $\int_C f(z)\mathrm{d}z = F(z(t))|_a^b = F(\alpha) - F(\beta)$ [79,81]. □

4.2.2. Cauchy Integral Theorem and Related Conclusions

Theorem 4.8 (Cauchy Integral Theorem). *Let $f(z)$ be an analytic function in a simply connected region D in the complex plane $\mathbb{C}_\lambda$.*

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(1) *Let C be any simple closed curve in the region D. Then*

$$\oint_C f(z)\mathrm{d}z = 0.$$

(2) *Let C be any simple curve in the region D connecting z_0 and z. Then the value of the integral from z_0 to z along C does not depend on the path C, but only on z_0 and z themselves.*

This is the Cauchy Integral Theorem. In fact, conclusion (2) is a corollary of conclusion (1); it suffices to prove (1) [76,77].

Proof. For $\zeta_0 \in D$, $\exists \delta_0 > 0$ such that the elliptic disk $K_0 = U(\zeta_0, \delta_0) \subset D$. By Lemma 4.6, $f(z)$ has a primitive function $F_0(z)$ on the convex region K_0, and for any two points z_0, z in K_0, by Lemma 4.7,

$$\int_{z_0 z} f(\zeta)d\zeta = F_0(z) - F_0(z_0).$$

In view of the fact that C is a bounded closed curve (i.e., compact), there exist finitely many open elliptic disks $K_i(i = 1, 2, \ldots, n-1)$ in D that completely cover C [78,80]. Choose points $\zeta_1 \in K_{n-1} \cap K_1 \cap C, \ldots, \zeta_{n-1} \in K_{n-2} \cap K_{n-1} \cap C$. Clearly, $f(z)$ has primitive functions $F_i(z)$ on the convex regions K_i. Let $\overline{z_1 z_2}$ denote the line segment, and let $C(z_1 z_2)$ denote the curve segment along C from z_1 to z_2. Subsequently, by Lemma 4.5,

$$\oint_C f(z)dz = \sum_{i=1}^{n-1} \int_{C(\zeta_i \zeta_{i+1})} f(\zeta)d\zeta = \sum_{i=1}^{n-1} \int_{\overline{\zeta_i \zeta_{i+1}}} f(\zeta)d\zeta = 0.$$

□

Alternatively, Cauchy's theorem can be proved using Green's formula and the C.-R. equations [79,81]:

Proof. Let $z = x + iy \in D$, $f(z) = u(x, y) + iv(x, y)$. Since $f(z)$ is analytic in D, $u(x, y)$ and $v(x, y)$ are differentiable in D. By Green's formula

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

combined with the Cauchy-Riemann equations, it can be shown that

$$\begin{aligned}\oint_C u(x,y)dx - q^2 v(x,y)dy &= \iint_D \left(-q^2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy \\ &= \iint_D \left(-q^2 \frac{\partial v}{\partial x} + q^2 \frac{\partial v}{\partial x} \right) dxdy = 0,\end{aligned}$$

and

$$\int_C v(x,y)dx + u(x,y)dy = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0.$$

Therefore,

$$\oint_C f(z)\mathrm{d}z = \int_C u(x,y)\mathrm{d}x - q^2 v(x,y)\mathrm{d}y + i \int_C u(x,y)\mathrm{d}x + v(x,y)\mathrm{d}y = 0.$$

This proves the proposition. □

Corollary 4.9. *Let C be a simple closed curve, and let the function $f(z)$ be analytic in the bounded region with boundary C. Then $\oint_C f(z)\mathrm{d}z = 0$ holds [76,77].*

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Theorem 4.10 (Existence of Primitive Function)**.** *Let $f(z)$ be an analytic function in a simply connected region D. Then $f(z)$ necessarily has a primitive function in the simply connected region D, i.e., there exists an analytic function $F(z)$ in D such that $F'(z) = f(z)$ for all $z \in D$ [78,80].*

Proof. By Theorem 4.8, the integral $\int_{C(z_0 z)} f(\zeta)\mathrm{d}\zeta$ is path-independent. Define the primitive function as

$$F(z) = \int_{C(z_0 z)} f(\zeta)\mathrm{d}\zeta.$$

Then $F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\mathrm{d}\zeta$. Hence for $\forall \epsilon > 0$,

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h}[f(\zeta)-f(z)]d\zeta\right| < \frac{1}{|h|}\cdot\epsilon\cdot|h| = \epsilon.$$

Therefore, $F'(z) = \lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z)$. This completes the proof [79,81]. □

Now consider multiply connected regions.

Theorem 4.11 (Cauchy's Theorem for Multiply Connected Regions)**.** *Let there be $n+1$ simple closed curves $C_0, C_1, C_2, \ldots, C_n$, where each of the curves $C_1, C_2, \ldots, C_n$ lies in the exterior region of the remaining curves, and all lie in the interior region of C_0. That is, C_0 together with $C_1, C_2, \ldots, C_n$ encloses a multiply connected region D whose closure is $\overline{D}$. If the function $f(z)$ is analytic on the region $\overline{D}$, then*

$$\oint_C f(z)\mathrm{d}z = 0,$$

where $C = C_0 + C_1^- + C_2^- + \cdots + C_n^-$ is the entire boundary of region D. This conclusion is equivalent to

$$\oint_{C_0} f(z)\mathrm{d}z = \oint_{C_1} f(z)\mathrm{d}z + \oint_{C_2} f(z)\mathrm{d}z + \cdots + \oint_{C_n} f(z)\mathrm{d}z. \quad (3.7)$$

Combined with Proposition 4.3, it could be shown as the following conclusion.

Corollary 4.12. *Let C be a simple closed curve containing α. Then*

$$\int_C \frac{1}{(z-\alpha)^n}\mathrm{d}z = \begin{cases} \dfrac{2\pi i}{q}, & n = 1 \\ 0, & n \neq 1, n \in \mathbb{Z} \end{cases}$$

Note that if $f(z)$ is an analytic function in a multiply connected region D, and we define its primitive function as $F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$, then $F(z)$ may be a multi-valued function [76,77]. This is because two simple curves in a multiply connected region D from z_0 to z may not be able to be combined into the same simple curve within D (with the same starting point). Take a simply connected subregion Δ in D, where $z_1 \in \Delta$. Let C_i be a fixed simple curve in D from z_0 to z. Define the function

$$F_i(z) = \int_{C_i} f(\zeta)\mathrm{d}\zeta + \int_{C(z_1 z)} f(\zeta)\mathrm{d}\zeta.$$

Then each determined $F_i(z)$ is a single-valued analytic branch of $F(z)$ in Δ [78,80].

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4.3. Cauchy's Integral Formula

We know that $|z - z_0|$ represents a normal ellipse centered at $z_0 = (x_0, y_0)^T$ in the complex plane $\mathbb{C}_\lambda$ [79,81]. For simplicity, let the normal ellipse be $F : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, i.e., the parametric equation $F : \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, t \in [0, 2\pi]$. Then by the symmetry of the ellipse, its perimeter is

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt.$$

Let $\lambda = -q^2$, and let the principal radius of the ellipse be r. Clearly, $q^2 = \frac{a^2}{b^2}$. Then when $|q| \geq 1$, $r = a$, and the ellipse perimeter is

$$\begin{aligned} L &= 4a\int_0^{\frac{\pi}{2}} \sqrt{(1-\cos^2 t) + \frac{b^2}{a^2}\cos^2 t}\,dt \\ &= 4r\int_0^{\frac{\pi}{2}} \sqrt{1 - \left(1 - \frac{1}{q^2}\right)\cos^2 t}\,dt =: 4rE(q); \end{aligned}$$

When $0 \leq |q| \leq 1$, $r = b$, and the ellipse perimeter is

$$\begin{aligned} L &= 4b\int_0^{\frac{\pi}{2}} \sqrt{\frac{a^2}{b^2}\sin^2 t + (1-\sin^2 t)}\,dt \\ &= 4r\int_0^{\frac{\pi}{2}} \sqrt{1 - (1-q^2)\sin^2 t}\,dt = 4rE'(q). \end{aligned}$$

Here,

$$\begin{aligned} E(q) &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(1 - \frac{1}{q^2}\right)\cos^2 t}\,dt, \\ E'(q) &= \int_0^{\frac{\pi}{2}} \sqrt{1 - (1-q^2)\sin^2 t}\,dt \end{aligned}$$

are called the contour coefficients of the normal ellipse in the complex plane $\mathbb{C}_\lambda$ when $|q| \geq 1$ and when $0 \leq |q| \leq 1$, respectively, collectively referred to as the contour coefficients of the complex plane $\mathbb{C}_\lambda$ [76,77]. Clearly, for a given complex plane $\mathbb{C}_\lambda$, its contour coefficient is a definite constant. When $\lambda = -1$, we have the classical complex plane $\mathbb{C}$, and the contour coefficients $E(q) = E'(q) = \frac{\pi}{2}$ [78,80].

Now we proceed to derive the integral formula from the integral theorem.

Proposition 4.13 (Cauchy Integral Formula)**.** *Let D be a bounded region with boundary consisting of finitely many simple closed curves C, and let the function $f(z)$ be analytic on the closed region $\overline{D}$ consisting of D and C. Then for $\forall z \in D$,*

$$f(z) = \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{22}$$

Proof. For any point z in D, draw an elliptic circle $C_\rho : |\zeta - z| = \rho$ such that the closed elliptic disk enclosed by C_ρ lies entirely within D [79,81].

Since $f(\zeta)$ is continuous at $\zeta = z$, for $\forall \epsilon > 0$, $\exists \delta > 0 (\delta < \rho)$ such that when $\zeta \in U(z, \delta)$,

$$|f(\zeta) - f(z)| < \epsilon.$$

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Therefore, when $r < \delta$, if $|q| > 1$,

$$\left|\frac{1}{4E(q)}\left(\oint_{C_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{2\pi i}{q}f(z)\right)\right| = \left|\frac{1}{4E(q)}\oint_{C_r}\frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta\right| < \frac{1}{4E(q)}\cdot\frac{\varepsilon}{r}\cdot 4rE(q) = \varepsilon;$$

if $0 < |q| < 1$,

$$\left|\frac{1}{4E'(q)}\left(\oint_{C_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{2\pi i}{q}f(z)\right)\right| = \left|\frac{1}{4E'(q)}\oint_{C_r}\frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta\right| < \frac{1}{4E'(q)}\cdot\frac{\varepsilon}{r}\cdot 4rE'(q) = \varepsilon$$

In summary, since for a given $q(q \neq 0)$, $E(q)$ and $E'(q)$ are nonzero constants, it leads to

$$\lim_{r\to 0}\oint_{C_r}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{2\pi i}{q}f(z) \Leftrightarrow f(z) = \frac{q}{2\pi i}\lim_{r\to 0}\oint_{C_r}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

Finally, it is obtained that $f(z) = \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$ [76,77]. □

This is the Cauchy integral formula, which can also be written as

$$\oint_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{2\pi i}{q}f(z). \tag{23}$$

Proposition 4.14 (Mean Value Theorem). *Let $f(z)$ be analytic on the elliptic disk enclosed by $C_\rho : |\zeta - z| = \rho$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i\frac{\theta}{q}}\right)\mathrm{d}\theta.$$

Proof. By the Cauchy integral formula, $f(z_0) = \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\mathrm{d}\zeta$. Let $\zeta = z_0 + \rho e^{i\frac{\theta}{q}}$. Then

$$f(z_0) = \frac{q}{2\pi i}\oint_{C_\rho}\frac{f\left(z_0 + \rho e^{i\frac{\theta}{q}}\right)}{\rho e^{i\frac{\theta}{q}}}\cdot\frac{i}{q}\rho e^{i\frac{\theta}{q}}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i\frac{\theta}{q}}\right)\mathrm{d}\theta.$$

Thus the proposition is proved [78,80]. □

That is, the value of $f(z)$ at the center z_0 of the ellipse equals the arithmetic mean of its values on the ellipse. Theorem 4.14 is also called the Mean Value Theorem for analytic functions [79,81].

4.4. Cauchy's Integral Formula for Derivatives of Arbitrary Order

Now, we need to derive the higher-order Cauchy integral formula based on the above integral formula, as stated in the following proposition:

Proposition 4.15 (Higher-Order Cauchy Integral Formula). *Let D be a bounded region with boundary consisting of finitely many simple closed curves C, and let the function $f(z)$ be analytic on the closed region $\overline{D}$ consisting of D and C. Then $f(z)$ has derivatives of all orders in D, and*

$$f^{(n)}(z) = \frac{q\cdot n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta \quad (n = 1, 2, \ldots). \tag{24}$$

Proof. First consider the case $n = 1$, i.e., prove $f'(z) = \frac{q}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta$ [76,77].

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For any $z \in D$, $\exists d > 0$ such that $U(z, 2d) \subset D$. Let $M = \max_{\zeta \in C}\{|f(\zeta)|\}$, and let L be the arc length of C. Then when $0 < |h| < d$,

$$\begin{aligned}
&\left|\frac{f(z+h)-f(z)}{h} - \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \\
=&\left|\frac{1}{h}\left[\frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z-h}\,d\zeta - \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z}\,d\zeta\right] - \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \\
=&\left|\frac{q\cdot h}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)^2}\,d\zeta\right| \\
\leq&\frac{|q|\cdot|h|}{2\pi}\cdot\frac{M}{d^3}\cdot L \to 0 \quad (h \to 0).
\end{aligned}$$

Therefore,

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{q}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta.$$

Now use mathematical induction to prove the general case [78,80]. Assume the conclusion holds for $n = k$, i.e.,

$$f^{(k)}(z) = \frac{q\cdot k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{k+1}}\mathrm{d}\zeta \quad (k = 1, 2, \ldots).$$

Then for $n = k+1$,

$$\begin{aligned}
&\left|\frac{f^{(k)}(z+h)-f^{(k)}(z)}{h} - \frac{q\cdot(k+1)!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{k+2}}\,d\zeta\right| \\
=&\left|\frac{1}{h}\left[\frac{q\cdot k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z-h)^{k+1}}\,d\zeta - \frac{q\cdot k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta\right] - \frac{q\cdot(k+1)!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{k+2}}\,d\zeta\right| \\
=&\left|\frac{q\cdot(k+1)!}{2\pi i}\oint_C f(\zeta)\left[\frac{1}{(\zeta-z-h)^{k+1}(\zeta-z)} - \frac{1}{(\zeta-z)^{k+2}}\right]d\zeta + hO(1)\right| \to 0 \quad (h \to 0).
\end{aligned}$$

Hence,

$$f^{(k+1)}(z) = \lim_{h\to 0}\frac{f^{(k)}(z+h)-f^{(k)}(z)}{h} = \frac{q\cdot(k+1)!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{k+2}}\mathrm{d}\zeta.$$

The proposition is proved [79,81]. □

Corollary 4.16. *If the function $f(z)$ is analytic in the region D, then $f(z)$ has derivatives of all orders in D [76,77].*

The higher-order Cauchy integral formula can also be rewritten as

$$\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\mathrm{d}\zeta = \frac{2\pi i}{q\cdot n!}f^{(n)}(z) \quad (n = 1, 2, \ldots). \tag{25}$$

We shall invoke equations 25 and 24 in the sequel.

4.5. Applications of Cauchy's Theorem

Proposition 4.17 (Cauchy's Inequality)**.** *Let the function $f(z)$ be analytic on the closed elliptic disk bounded by $C : |z - z_0| = \rho_0$ $(0 < \rho_0 < +\infty)$, and let $M(\rho) = \max\limits_{|z-z_0|=\rho} |f(z)|$ $(0 < \rho < \rho_0)$. Thereafter,*

$$\frac{|f^{(n)}(z_0)|}{n!} \leq \begin{cases} |q|\cdot\dfrac{M(\rho)}{\rho^n}\cdot\dfrac{2E(q)}{\pi}, & |q| > 1 \\ |q|\cdot\dfrac{M(\rho)}{\rho^n}\cdot\dfrac{2E'(q)}{\pi}, & 0 < |q| < 1 \end{cases}. \tag{26}$$

Proof. By the Cauchy integral formula 22 and the higher-order derivative formula 25 [78,80],

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$$f^{(n)}(z_0) = \frac{q \cdot n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \quad (n = 1, 2, \ldots).$$

When $|q| > 1$,

$$|f^{(n)}(z_0)| \leq |q| \cdot \frac{n!}{2\pi} \cdot \frac{M(\rho)}{\rho^{n+1}} \cdot 4\rho E(q) = |q| \cdot \frac{M(\rho)}{\rho^n} \cdot \frac{2E(q)}{\pi} \cdot n!.$$

The case $0 < |q| < 1$ can be shown in the same manner [79,81]. □

In particular, when $|q| = 1$, we have $E(q) = E'(q) = \frac{\pi}{2}$, which is the case of circular complex functions [76,77]. Formula 26 is also called Cauchy's Inequality.

Definition 4.18 (Entire Function). *A function that is analytic in the entire complex plane $\mathbb{C}_\lambda$ is called an entire function (or holomorphic function). A function that is analytic only in some region of the complex plane is called a meromorphic function [78,80].*

Proposition 4.19 (Liouville's Theorem). *A bounded entire function must be constant [79,81].*

Proof. Let $f(z)$ be a bounded function, i.e., $\exists M > 0$ such that $|f(z)| < M$ for all $z \in \mathbb{C}_\lambda$. Then for any $z_0 \in \mathbb{C}_\lambda$ and any $\rho > 0$, when $|q| > 1$,

$$|f'(z_0)| \leq |q| \cdot \frac{M(\rho)}{\rho} \cdot \frac{2E(q)}{\pi} \leq \frac{2E(q) \cdot M \cdot |q|}{\pi} \cdot \frac{1}{\rho} \to 0 \quad (\rho \to +\infty);$$

when $0 < |q| < 1$,

$$|f'(z_0)| \leq |q| \cdot \frac{M(\rho)}{\rho} \cdot \frac{2E'(q)}{\pi} \leq \frac{2E'(q) \cdot M \cdot |q|}{\pi} \cdot \frac{1}{\rho} \to 0 \quad (\rho \to +\infty).$$

As a result, $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}_\lambda$, i.e., $f(z) \equiv C$ is a constant. This theorem, like in the theory of circular complex functions, is also called Liouville's Theorem [76,77]. □

By the same token, in the elliptic complex domain, there is also a corresponding Fundamental Theorem of Algebra [78,80].

Proposition 4.20 (Fundamental Theorem of Algebra). *Any algebraic equation of degree n*

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0 \quad (a_0 \neq 0, \forall z \in \mathbb{C}_\lambda)$$

has at least one root [79,81].

Proof. Assume the conclusion is false. Then the function

$$f(z) = \frac{1}{a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n}$$

is analytic in the entire complex plane, i.e., it is an entire function [76,77].

Clearly, $\lim_{z \to \infty} f(z) = 0$; hence $f(z)$ is bounded in the complex plane $\mathbb{C}_\lambda$. By Liouville's Theorem, $f(z) \equiv C$ is a constant. However, clearly $\exists z_1, z_2 \in \mathbb{C}_\lambda$ with $z_1 \neq z_2$ such that $f(z_1) \neq f(z_2)$, a contradiction. Therefore, the original assumption is false, and the equation has at least one root [78,80]. □

Proposition 4.21 (Morera's Theorem). *If the function $f(z)$ is continuous in the region D, and for every simple closed curve C in D it leads to*

$$\oint_C f(z) \mathrm{d}z = 0,$$

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then $f(z)$ is analytic in D [79,81].

Proof. Let $z, z_0 \in D$. Define $F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$. By the hypothesis, the value of $\int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$ depends only on z and z_0, not on the path of integration. Thus $F(z)$ is a single-valued function [76,77].

It can be easily shown that $F(z)$ is analytic in D and that $F'(z) = f(z)$. Since $F'(z)$ is analytic, $f(z)$ is also analytic in D. □

This theorem, also called Morera's Theorem, is the converse of the Cauchy Integral Theorem [78,80].

5. Series Representations of Analytic Functions

Clearly, the definitions of complex series and the related theory of series convergence in the elliptic complex domain are the same as the corresponding basic theory in the circular complex domain, so we will not repeat them here (interested readers can refer to relevant materials on (circular) complex function theory) [76,77]. We will start directly from the theory of Taylor series [78,80].

5.1. Taylor Expansion of Analytic Functions

Definition 5.1 (Power Series)**.** *Let $\alpha_n (n = 0, 1, 2, \dots)$ and z_0 be complex constants in the complex plane $\mathbb{C}_\lambda$. Then*

$$\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n = \alpha_0 + \alpha_1 (z - z_0) + \cdots + \alpha_n (z - z_0)^n + \dots$$

is called a power series in $(z - z_0)$ [79,81].

For uniformity, unless otherwise specified, "ellipse" below refers to a normal ellipse [76,77].

5.1.1. Basic Theory of Power Series

Proposition 5.2. *(1) If the power series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges at $z_1 (z_1 \neq z_0)$, then for any point z in the elliptic disk region satisfying $|z - z_0| < |z_1 - z_0|$, the series converges absolutely [78,80];*

(2) If the power series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ diverges at z_2, then for any point z in the region outside the elliptic disk satisfying $|z - z_0| > |z_2 - z_0|$, the series diverges [79,81].

Proof. In fact, if (1) is true, then (2) must be true [76,77]. Assume (2) is false, then the power series diverges at z_2, and there exists some point z_2' in the region outside the elliptic disk satisfying $|z - z_0| > |z_2 - z_0|$ such that the series converges. Then by (1), for any point z in the elliptic disk region satisfying $|z - z_0| < |z_2' - z_0|$ (clearly including z_2), the series converges, contradicting the assumption [78,80]. Therefore, we only need to prove (1).

Owing to the series $\sum_{n=0}^{\infty} \alpha_n (z_1 - z_0)^n$ converges, by the necessary condition for convergence: $\lim_{n\to\infty} \alpha_n (z_1 - z_0)^n = 0$, there exists a constant $M > 0$ such that

$$|\alpha_n (z_1 - z_0)^n| < M (n = 0, 1, 2, \dots).$$

Therefore, for any point z in the elliptic disk region satisfying $|z - z_0| < |z_1 - z_0|$, it can be seen that

$$|\alpha_n (z - z_0)^n| = \left|\alpha_n (z_1 - z_0)^n \cdot \left(\frac{z - z_0}{z_1 - z_0}\right)^n\right| \leq M \cdot \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n = Mk^n.$$

Clearly, $0 \leq k < 1$, so the series $\sum_{n=0}^{\infty} Mk^n$ converges, and thus the series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges absolutely [79,81]. □

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Proposition 5.3 (Convergence Principal Radius)**.** *Let the convergence radius of the real power series* $\sum_{n=0}^{\infty} |\alpha_n| x^n$ *be R [76,77].*

(1) *If* $0 < R < \infty$*, then the power series* $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *converges absolutely when* $|z - z_0| < R$*, and diverges when* $|z - z_0| > R$ *[78,80];*

(2) *If* $R = +\infty$*, then the power series* $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *converges absolutely in the entire complex plane* $\mathbb{C}_\lambda$ *[79,81];*

(3) *If* $R = 0$*, then the power series* $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *diverges at every point except* $z = z_0$ *[76,77].*

In this case, R is called the convergence principal radius of the power series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$*. For the case* $0 < R < \infty$*, the region* $|z - z_0| < R$ *is called the convergence elliptic disk of the series [78,80].*

Proof. Here we only prove case (1) [79,81]. Arbitrarily take z_1 satisfying $|z_1 - z_0| < R$. Then there must exist r_1 satisfying

$$|z_1 - z_0| < r_1 < R.$$

By hypothesis, the series $\sum_{n=0}^{\infty} |\alpha_n| r_1^n$ converges, so the series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges absolutely at $r_1 = z - z_0$, i.e., at $z = r_1 + z_0$. Since $|z_1 - z_0| < r_1 = |(z_0 + r_1) - z_0|$, by Proposition 5.2, the series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges absolutely at z_1 [76,77]. Similarly, arbitrarily take z_2 satisfying $|z_2 - z_0| > R$. Then there must exist r_2 satisfying

$$|z_2 - z_0| > r_2 > R.$$

Assume the series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ converges at z_2. Then by Proposition 5.2, the series also converges at $z = r_2 + z_0$, so the corresponding real power series $\sum_{n=0}^{\infty} |\alpha_n| x^n$ has convergence radius $r_2 > R$, a contradiction [78,80]. □

Similar to the case of the circular complex plane, we have the following conclusions [79,81]:

Proposition 5.4 (Convergence Radius Formulas)**.** *If any of the following conditions holds:*

(1) $l = \lim_{n \to \infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right|$;

(2) $l = \lim_{n \to \infty} \sqrt[n]{|\alpha_n|}$;

(3) $l = \lim_{n \to \infty} \sqrt[n]{|\alpha_n|}$,

then the convergence principal radius of the power series $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *is* $R = \frac{1}{l}$ *[76,77].*

Proposition 5.5 (Analyticity of Sum Function)**.** *Let the convergence principal radius of the series* $\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *be* $R (R > 0)$*. Then in the convergence elliptic disk* $|z - z_0| < R$*, the corresponding sum function* $f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ *is analytic [78,80].*

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5.1.2. Basic Theory of Taylor Series

Proposition 5.6 (Taylor Expansion). *Let the function $f(z)$ be analytic in a region D [79,81]. For $z_0 \in D$, as long as the ellipse $K : |z - z_0| < R$ is contained in D, then $f(z)$ can be expanded into a power series in the region K:*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

where the coefficients (let $C : |z - z_0| = \rho, 0 < \rho < R$) are

$$c_n = \frac{q}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta = \frac{f^{(n)}(z_0)}{n!}. \tag{27}$$

Moreover, this expansion is unique [76,77].

Proof. On account of $\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n (|u| < 1)$, because $\left|\dfrac{z - z_0}{\zeta - z_0}\right| < 1$ holds for any $z \in C$, it is obtained that [78,80]

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}, \tag{28}$$

By the Cauchy integral formula $f(z) = \dfrac{q}{2\pi i} \oint_C \dfrac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$, it yields that [79,81]

$$f(z) = \frac{q}{2\pi i} \sum_{n=0}^{\infty} \left(\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \right) (z - z_0)^n; \tag{29}$$

Combined with the Cauchy higher-order derivative formula [76,77]

$$f^{(n)}(z) = \frac{q \cdot n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta \quad (n = 1, 2, \ldots),$$

hence we obtain the conclusion [78,80]. □

Equation 27 gives the Taylor coefficients, and Equation 5.6 is the Taylor series expansion [79,81].

From Proposition 5.6, we can arrive at the Taylor expansions (at $z = 0$) of basic elementary functions [76,77]:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{30}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \tag{31}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \tag{32}$$

and so on.

Proof. For Equation 27 , due to $\dfrac{\mathrm{d}}{\mathrm{d}z} e^z = e^z$, we achieve $(e^z)^{(n)}|_{z=0} = 1$, which gives Equation 30 [78,80];

For Equation 28, given that $\sin z = \dfrac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right)$, substituting Equation 27 gives Equation 31 [79,81];

For Equation 29, since $(\sin z)' = \cos z$, differentiating Equation 28 term by term gives Equation 32 [76,77].

Likewise, we can obtain the Taylor expansion (at $z = 0$) of the logarithmic function $\mathrm{Log}(1 + z)$ [78,80]. Clearly, $\mathrm{Log}(1 + z)$ is an infinitely multi-valued function, with branch points at $z = -1, \infty$ [79,81]. Cutting the z-plane along the negative real axis from -1 to ∞, in the resulting region G,

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the function $\mathrm{Log}(1+z)$ yields infinitely many single-valued analytic branches $f_k(z) = [\log(1+z)]_k$ $(k = 0, \pm 1, \pm 2, \dots)$ [76,77]. Taking the principal value branch $[\log(1+z)]_0$, in view of

$$f_0'(z) = \frac{1}{1+z}, \dots, f_0^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n},$$

the Taylor coefficients are

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n},$$

so it leads to

$$[\log(1+z)]_0 = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{z^n}{n}.$$

Consequently , the Taylor expansion of $\mathrm{Log}(1+z)$ at $z = 0$ is

$$\mathrm{Log}(1+z) = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{z^n}{n} + \frac{2k\pi i}{q}(k = 0, \pm 1, \pm 2, \dots).$$

In the same manner, we can obtain the expansion of the principal value branch of the function $(1+z)^\alpha$ [78,80]:

$$(1+z)^\alpha = 1 + \alpha z + \binom{\alpha}{2}z^2 + \cdots + \binom{\alpha}{n}z^n + \cdots = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n.$$

□

It can be seen that although we are in different complex planes, most of the Taylor expansions obtained for elementary functions are the same, except for some periodic functions whose periods are related to the elliptic coefficient q [79,81].

5.2. Isolation of Zeros and Uniqueness Theorem of Analytic Functions

5.2.1. Zeros of Analytic Functions

In many practical problems, it is often necessary to find points where a function equals zero, i.e., to determine roots [76,77].

Definition 5.7 (Zeros of Analytic Functions). *If the value of a function $f(z)$ at a point z_0 in its analytic region D is 0, then z_0 is called a zero of the analytic function $f(z)$ [78,80]. Suppose the function $f(z)$ has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ in some neighborhood U of the zero z_0 [79,81].*

(1) If $c_n = 0 (n = 0, 1, 2, \dots)$, then $f(z)$ is identically zero in U [76,77];
(2) If $c_n = 0 (n = 0, 1, 2, \dots, m-1)$, and $c_m \neq 0$, then z_0 is called a zero of order m of $f(z)$. When $m = 1$, it is called a simple zero; when $m > 1$, it is called a zero of multiplicity m [78,80].

Proposition 5.8 (Characterization of Zeros). *A necessary and sufficient condition for a non-identically zero analytic function $f(z)$ to have z_0 as a zero of order m is*

$$f(z) = (z - z_0)^m \phi(z)$$

where $\phi(z)$ is analytic at z_0, and $\phi(z_0) \neq 0$ [79,81].

For example, find all zeros of the function $f(z) = \sin z - 1$ [76,77]. Setting $f(z) = 0$, we have

$$\frac{q}{2i}\left(e^{i\frac{z}{q}} - e^{-i\frac{z}{q}}\right) = 1 \Leftrightarrow \left(e^{i\frac{z}{q}} - \frac{i}{q}\right)^2 = 0 \Leftrightarrow e^{i\frac{z}{q}} = \frac{i}{q},$$

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so the zeros are $z = \frac{\pi}{2} + 2k\pi(k = 0, \pm 1, \dots)$. Due to the fact that

$$f'(z)|_{z=\frac{\pi}{2}+2k\pi} = 0, \quad f''(z)|_{z=\frac{\pi}{2}+2k\pi} \neq 0,$$

$z = \frac{\pi}{2} + 2k\pi(k = 0, \pm 1, \dots)$ are zeros of order 2 of $f(z)$ [78,80].

Proposition 5.9 (Isolation of Zeros)**.** *Let the function $f(z)$ be analytic in a neighborhood of z_0, with $f(z_0) = 0$. Then either $f(z)$ is identically zero in some neighborhood of z_0, or there exists a neighborhood of z_0 in which z_0 is the only zero of $f(z)$ [79,81].*

Proof. Let z_0 be a zero of order m of $f(z)$. Then in a neighborhood U of z_0,

$$f(z) = (z - z_0)^m \phi(z), \quad \phi(z_0) \neq 0$$

where $\phi(z)$ is analytic in U [76,77].

Thus, there exists $\epsilon > 0$ such that when $0 < |z - z_0| < \epsilon$, $\phi(z) \neq 0$, i.e., $f(z) \neq 0$ [78,80]. □

Theorem 5.9 states that for a non-identically zero analytic function, its zeros must be isolated [79,81].

5.2.2. Uniqueness Theorem

Lemma 5.10 (Analytic Continuation Along a Curve)**.** *Let the function $f(z)$ be analytic in a region D. If $f(z)$ is identically zero in some neighborhood within D, then $f(z)$ is identically zero in D [76,77].*

Proof. Suppose at some point z_0 in D, there exists a neighborhood K_0 such that $f(z) \equiv 0$. Arbitrarily take $z \in D$, connect z_0 and z with a finite polygonal line L in D, such that the distance between L and the boundary ∂D of D is greater than $\delta > 0$, ensuring that every point on L has some neighborhood contained in D [78,80]. On L, take successively

$$z_0, z_1, z_2, \dots, z_{n-1}, z_n = z,$$

such that $z_1 \in K_0$, and the neighborhood K_1 of z_1 contains z_2, the neighborhood K_2 of z_2 contains z_3, ..., the neighborhood K_{n-1} of z_{n-1} contains z_n. Then $f(z_1) = 0$, so $f^{(n)}(z_1) = 0$. Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n = 0, \quad z \in K_1$$

Thus $f(z_2) = 0$, so $f^{(n)}(z_2) = 0$, further obtaining $f(z_3) = 0$, ..., finally obtaining $f(z_n) = 0$ [79,81]. □

From Proposition 5.9 , for a non-identically zero analytic function $f(z)$, its zeros must be isolated; if the zero is isolated, then the function $f(z) \equiv 0$ [76,77]. Furthermore , the conclusion below follows. [78,80].

Proposition 5.11 (Uniqueness Theorem of Analytic Functions)**.** *Let the functions $f(z)$ and $g(z)$ be analytic in a region D, and let $z_k(k = 1, 2, \dots)$ be distinct points in D such that the sequence $\{z_k\}$ has a limit point in D. If $f(z_k) = g(z_k)(k = 1, 2, \dots)$, then $f(z) \equiv g(z)$ in D [79,81].*

Additionally , we have the following corollaries [76,77].

Corollary 5.12. *Let the functions $f(z)$ and $g(z)$ be analytic in a region D, and let C be a curve segment in D. If for all $z \in C$, $f(z) = g(z)$, then $f(z) \equiv g(z)$ in D [78,80].*

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Corollary 5.13. *Let the functions $f(z)$ and $g(z)$ be analytic in a region D. If $f(z)$ and $g(z)$ are equal in some subregion of D, then $f(z) \equiv g(z)$ in D [79,81].*

5.3. Laurent Expansion of Analytic Functions

Consider the power series in $(z - z_0)$ [76,77]

$$\sum_{n=0}^{\infty} \beta_n (z - z_0)^n = \beta_0 + \beta_1 (z - z_0) + \cdots + \beta_n (z - z_0)^n + \ldots, \tag{33}$$

and the power series in $(z - z_0)^{-1}$ [78,80]

$$\sum_{n=1}^{\infty} \beta_{-n} (z - z_0)^{-n} = \beta_{-1} (z - z_0)^{-1} + \cdots + \beta_{-n} (z - z_0)^{-n} + \ldots. \tag{34}$$

Let the convergence radii of the two power series be R and R' respectively [79,81]. Then series 33 converges in $|z - z_0| < R$, while series 34 converges in $|z - z_0|^{-1} < R'$, i.e., in $|z - z_0| > r := \frac{1}{R'}$ [76,77].

Therefore , when $0 < r < R < +\infty$, the series

$$\sum_{n=-\infty}^{+\infty} \beta_n (z - z_0)^n \tag{35}$$

converges in the elliptic annulus $U : r < |z - z_0| < R$ [78,80]. Series 35 is called a Laurent series (expansion), where series 33 is called the analytic part of the Laurent expansion, and series 34 is called the principal part of the Laurent expansion [79,81].

Proposition 5.14 (Laurent Expansion)**.** *Let the function $f(z)$ be analytic in the elliptic annulus $U : r < |z - z_0| < R$. Then in U, $f(z)$ has the expansion*

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n,$$

and this expansion is unique , where

$$c_n = \frac{q}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{36}$$

and $C : |z - z_0| = \rho$, with ρ any number satisfying $r < \rho < R$ [76,77].

Proof. Arbitrarily take $z \in U$. Then there must exist r_1 and R_1 such that

$$r < r_1 < |z - z_0| < R_1 < R.$$

Let $C_1 : |\zeta - z_0| = r_1$ and $C_2 : |\zeta - z_0| = R_1$. By the Cauchy integral formula [78,80],

$$f(z) = \frac{q}{2\pi i} \oint_{C_2 - C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We now expand $\frac{1}{\zeta - z}$ into series form for $\zeta \in C_2$ and $\zeta \in C_1$ respectively [79,81].

For $\zeta \in C_2$, it is concluded that $|\zeta - z_0| > |z - z_0|$, so

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}};$$

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For $\zeta \in C_1$, we have $|\zeta - z_0| < |z - z_0|$, so

$$\frac{1}{\zeta - z} = \frac{-1}{z - z_0 - (\zeta - z_0)} = \frac{-1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

Therefore,

$$\begin{aligned}
f(z) &= \frac{q}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z_0} d\zeta - \frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z_0} d\zeta \\
&= \frac{q}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \, d\zeta \\
&\quad + \frac{q}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(z - z_0)^{n+1}} (\zeta - z_0)^n \, d\zeta \\
&= \sum_{n=0}^{\infty} \frac{q}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \, d\zeta \\
&\quad + \sum_{n=0}^{\infty} \frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(z - z_0)^{n+1}} (\zeta - z_0)^n \, d\zeta
\end{aligned}$$

For the part $\sum_{n=0}^{\infty} \frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(z - z_0)^{n+1}} (\zeta - z_0)^n \mathrm{d}\zeta$, let $n = -k - 1$. Then when $n = 0, 1, 2, \cdots$, we have $k = -1, -2, -3, \cdots$. Hence,

$$\sum_{n=0}^{\infty} \frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(z - z_0)^{n+1}} (\zeta - z_0)^n \mathrm{d}\zeta = \sum_{n=-\infty}^{-1} \left(\frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \right) (z - z_0)^n.$$

Combined with the Cauchy integral theorem, it follows that

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \left(\frac{q}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\
&\quad + \sum_{n=-1}^{-\infty} \left(\frac{q}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{q}{2\pi i} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\
&\quad + \sum_{n=-1}^{-\infty} \left(\frac{q}{2\pi i} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{q}{2\pi i} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n
\end{aligned}$$

which gives the conclusion [76,77]. Regarding the uniqueness of the expansion, consider another possible Laurent series with coefficients c'_n. By formula 36 [78,80],

$$\begin{aligned}
c'_k &= \frac{q}{2\pi i} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta = \frac{q}{2\pi i} \oint_{C} \frac{\sum_{n=-\infty}^{+\infty} c_n (\zeta - z_0)^n}{(\zeta - z_0)^{k+1}} d\zeta \\
&= \sum_{n=-\infty}^{+\infty} \left(\frac{q}{2\pi i} \oint_{C} \frac{1}{(\zeta - z_0)^{k-n+1}} d\zeta \right) c_n.
\end{aligned}$$

By the basic property of integrals as shown in Proposition 4.3 of Section 4.1.2, $c'_k = c_k$ holds for all k. Hence the Laurent expansion is unique [76,77]. □

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5.4. Isolated Singularities of Analytic Functions

Definition 5.15 (Singular Point)**.** *If the function $f(z)$ is not analytic at z_0, but in every neighborhood of z_0 there exist points where $f(z)$ is analytic, then z_0 is called a singular point of the function $f(z)$ [78,80].*

5.4.1. Basic Theory of Isolated Singularities

Definition 5.16 (Isolated Singularity)**.** *If the function $f(z)$ is not analytic at z_0, and there exists a punctured elliptic disk*

$$D = \{z \mid 0 < |z - z_0| < R\}(0 < R < +\infty),$$

such that $f(z)$ is defined and analytic in this elliptic disk, then z_0 is called an isolated singular point of the function $f(z)$ [79,81].

The difference between a singular point and an isolated singular point is that in any neighborhood of a singular point, there are points where $f(z)$ is analytic, but the function may not be analytic throughout that neighborhood. An isolated singular point requires the existence of such a punctured neighborhood where $f(z)$ is entirely analytic [76,77].

Definition 5.17 (Classification of Isolated Singularities)**.** *Let z_0 be an isolated singular point of the function $f(z)$. Regarding the Laurent coefficients $c_n(n < 0)$ being zero, there are three cases [78,80]:*

(1) *If $c_n = 0$ for all $n < 0$, then z_0 is called a removable singular point of $f(z)$ [79,81];*

(2) *If only finitely many $n < 0$ have $c_n \neq 0$, then z_0 is called a pole of $f(z)$. If there exists a positive integer m such that $c_{-m} \neq 0$ while $c_n = 0$ for all $n < -m$, then z_0 is called a pole of order m of $f(z)$, i.e.,*

$$f(z) = \sum_{n=-m}^{+\infty} c_n(z - z_0)^n.$$

In particular, when $m = 1$, z_0 is called a simple pole of $f(z)$ [76,77];

(3) *If there are infinitely many $n < 0$ with $c_n \neq 0$, then z_0 is called an essential singular point of $f(z)$ [78,80].*

Proposition 5.18 (Characterization of Removable Singularities)**.** *Let z_0 be an isolated singular point of the function $f(z)$. Then z_0 is a removable singular point of $f(z)$ if and only if*

$$\lim_{z \to z_0} f(z) = \alpha(\alpha \neq \infty).$$

Proof. (Necessity) Let z_0 be a removable singular point of $f(z)$. Then $f(z) = \sum_{n=0}^{+\infty} c_n(z - z_0)^n$ for $z \in D = \{z \mid 0 < |z - z_0| < R\}$. Hence $\lim_{z \to z_0} f(z) = c_0$ [79,81].

(Sufficiency) Suppose $\lim_{z \to z_0} f(z)$ exists. Since $f(z)$ is analytic in D, there exist $M > 0, 0 < \delta < R$ such that when $0 < |z - z_0| < \delta$, $|f(z)| \leq M$ [76,77]. Then when $0 < \rho < \delta$, for $|q| > 1$,

$$|c_n| = \left| \frac{q}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \leq \frac{|q|}{2\pi} \cdot \frac{M}{\rho^{n+1}} \cdot 4\rho E(q) = \frac{2|q|E(q)}{\pi} \frac{M}{\rho^n};$$

for $0 < |q| < 1$,

$$|c_n| = \left| \frac{q}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \leq \frac{|q|}{2\pi} \cdot \frac{M}{\rho^{n+1}} \cdot 4\rho E'(q) = \frac{2|q|E'(q)}{\pi} \frac{M}{\rho^n},$$

where $E(q), E'(q)$ are the contour coefficients of the complex plane $\mathbb{C}_\lambda$, satisfying $E(q) \leq \pi, E'(q) \leq \pi$ [78,80]. Hence ,

$$|c_n| \leq \frac{2M|q|}{\rho^n} \to 0 \quad (\rho \to 0)$$

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for all $n = -1, -2, \ldots$. Therefore, $c_n = 0$ for all $n = -1, -2, \ldots$ [79,81]. □

Proposition 5.19 (Characterization of Poles). *Let z_0 be an isolated singular point of the function $f(z)$. Then z_0 is a pole of $f(z)$ if and only if*

$$\lim_{z \to z_0} f(z) = \infty.$$

Proof. (Necessity) Let z_0 be a pole of order m of $f(z)$. Then

$$f(z) = \frac{1}{(z - z_0)^m} \phi(z)$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Thus $\lim\limits_{z \to z_0} f(z) = \infty$ [76,77].

(Sufficiency) Suppose $\lim\limits_{z \to z_0} f(z) = \infty$. Let $F(z) = \dfrac{1}{f(z)}$. Then $\lim_{z \to z_0} F(z) = 0$, so z_0 is a removable singular point of $F(z)$ [78,80]. Thus, in some punctured neighborhood of z_0, $F(z)$ has the Laurent expansion

$$F(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n.$$

Since $F(z_0)$ is not defined at z_0, we can define $F(z_0) = d_0$, making $F(z)$ analytic at z_0. As $d_0 = \lim_{z \to z_0} F(z) = 0$, suppose $d_0 = d_1 = \cdots = d_{m-1} = 0$ and $d_m \neq 0$. Then

$$F(z) = (z - z_0)^m \psi(z),$$

where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Accordingly,

$$f(z) = \frac{1}{(z - z_0)^m} \cdot \frac{1}{\psi(z)}.$$

Due to the fact that $\dfrac{1}{\psi(z)}$ is analytic at z_0 , z_0 is a pole of order m of $f(z)$ [79,81]. □

From Propositions 5.18 and 5.19 and the definition of essential singularities , it is found that essential singularities admit an equivalent characterization [76,77]:

Proposition 5.20 (Characterization of Essential Singularities). *Let z_0 be an isolated singular point of the function $f(z)$. Then z_0 is an essential singular point of $f(z)$ if and only if the limit $\lim\limits_{z \to z_0} f(z)$ does not exist and is not ∞ [78,80].*

Proposition 5.21 (Weierstrass Theorem). *Let z_0 be an isolated singular point of the function $f(z)$, and let $f(z)$ be analytic in $0 < |z - z_0| < R$. Then z_0 is an essential singular point of $f(z)$ if and only if for every finite or infinite complex number γ, there exists a sequence $\{z_n\}$ converging to z_0 in $0 < |z - z_0| < R$ such that $\lim\limits_{n \to \infty} f(z_n) = \gamma$ [79,81].*

Proof. (Sufficiency) Owing to the fact that $\gamma = \lim\limits_{n \to \infty} f(z_n)$ can be any value (including ∞) , the limit $\lim\limits_{z \to z_0} f(z)$ does not exist and is not ∞ . By Proposition 5.20, z_0 is an essential singular point of $f(z)$ [76,77].

(Necessity) Consider the cases $\gamma = \infty$ and $\gamma \neq \infty$ separately [78,80]. □

5.4.2. Properties of Analytic Functions at Infinity

Definition 5.22 (Isolated Singularity at Infinity). *If the function $f(z)$ is analytic in some neighborhood of infinity $R < |z| < +\infty$ ($R \geq 0$), then ∞ is called an isolated singular point of $f(z)$ [79,81].*

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Clearly, $f(z)$ has ∞ as an isolated singular point if and only if the function $f(\frac{1}{w})$ has $w = 0$ as an isolated singular point [76,77]. Thus ,

Definition 5.23 (Laurent Expansion at Infinity). *Let the function $f(z)$ be analytic in $R < |z| < +\infty$ $(R \geq 0)$. Then*

$$f(z) = \sum_{n=-\infty}^{+\infty} c_{-n}z^n = \sum_{n=0}^{+\infty} c_{-n}z^n + \sum_{n=1}^{+\infty} c_{-n}z^n$$

where $s(z) = \sum_{n=0}^{-\infty} c_n z^n$ is called the analytic part of $f(z)$, and $t(z) = \sum_{n=1}^{+\infty} c_n z^n$ is called the principal part of $f(z)$ [78,80].

Proposition 5.24 (Classification of Singularities at Infinity). *Let the function $f(z)$ be analytic in $R < |z| < +\infty$ $(R \geq 0)$. Then $z = \infty$ is a removable singular point, pole, or essential singular point of $f(z)$ if and only if $\lim_{z\to\infty} f(z)$ exists, $\lim_{z\to\infty} f(z) = \infty$, or $\lim_{z\to\infty} f(z)$ does not exist and is not ∞, respectively [79,81].*

Similarly, the Weierstrass theorem also holds for the point at infinity [76,77].

Proposition 5.25 (Weierstrass Theorem at Infinity). *Let z_0 be an isolated singular point of the function $f(z)$, and let $f(z)$ be analytic in $R < |z| < \infty$. Then $z = \infty$ is an essential singular point of $f(z)$ if and only if for every finite or infinite complex number γ, there exists a sequence $\{z_n\}$ converging to ∞ in $R < |z| < \infty$ such that $\lim_{n\to\infty} f(z_n) = \gamma$ [78,80].*

5.5. Entire and Meromorphic Functions

Based on the characteristics of isolated singularities of analytic functions, there exist two classical families of analytic functions that can be obtained [79,81].

Definition 5.26 (Entire and Meromorphic Functions). *A function that is analytic in the entire complex plane $\mathbb{C}_\lambda$ is called an entire function [76,77]. A single-valued analytic function that has no singularities other than poles in the complex plane $\mathbb{C}_\lambda$ is called a meromorphic function [78,80].*

Let $f(z)$ be an entire function. Then $f(z)$ has only $z = \infty$ as an isolated singular point [79,81]. Therefore ,

Proposition 5.27 (Classification of Entire Functions). *Let $f(z)$ be an entire function. As a result [76,77] ,*

(1) *the point ∞ is a removable singular point of $f(z)$ if and only if $f(z)$ is constant [78,80];*
(2) *the point ∞ is a pole of order m of $f(z)$ if and only if $f(z)$ is a polynomial of degree m, i.e.,*

$$f(z) = \sum_{n=0}^{m} c_n z^n \quad (c_m \neq 0, m \geq 1); \tag{37}$$

(3) *the point ∞ is an essential singular point of $f(z)$ if and only if infinitely many c_n are non-zero, i.e., $f(z) = \sum_{n=0}^{\infty} c_n z^n$ [79,81].*

Proof. (1) (Sufficiency) Let ∞ be a removable singular point of $f(z)$, i.e., $\lim_{z\to\infty} f(z) = \alpha$ $(|\alpha| < +\infty)$. Then there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}_\lambda$. By Liouville's theorem as seen in Propsition 4.19 , $f(z)$ is constant [76,77].
(Necessity) When $f(z)$ is constant, clearly $\lim_{z\to\infty} f(z) = \alpha$ $(|\alpha| < +\infty)$, so ∞ is a removable singular point of $f(z)$ [78,80];

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(2) (Sufficiency) Let ∞ be a pole of order m of $f(z)$. Then the principal part of the corresponding Laurent expansion is

$$g(z) = \sum_{n=1}^{m} c_n z^n \quad (c_m \neq 0).$$

Let $F(z) = f(z) - g(z)$. Clearly, $F(z)$ is an entire function, and ∞ is a removable singular point of $F(z)$. Hence

$$F(z) = f(z) - g(z) \equiv C,$$

so Formula 37 holds [79,81]. The necessity is obvious [76,77];

□

From the definition of meromorphic functions, rational functions [78,80]

$$f(z) = \frac{P(z)}{Q(z)} = \frac{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n}{\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_n z^n} \quad (\alpha_n, \beta_n \neq 0)$$

are meromorphic functions [79,81].

Proposition 5.28 (Characterization of Rational Functions)**.** *Let $f(z)$ be a meromorphic function. Then $f(z)$ is a rational function if and only if ∞ is a removable singular point or a pole of $f(z)$ [76,77].*

Proof. (Necessity) Let $f(z)$ be a rational function. Then when $m \geq n$, $\lim_{z\to\infty} f(z)$ exists and is finite, so ∞ is a removable singular point of $f(z)$. When $m \leq n$, $\lim_{z\to\infty} f(z) = \infty$, so ∞ is a pole of $f(z)$ [78,80].

(Sufficiency) Let $f(z)$ be a meromorphic function. Then $f(z)$ has no singularities other than poles in the complex plane $\mathbb{C}_\lambda$, and these poles can only be finitely many. Otherwise, infinitely many poles would necessarily form an accumulation point, contradicting the definition of singularities [79,81]. □

Now suppose the poles of $f(z)$ in the complex plane $\mathbb{C}_\lambda$ are $z_1, z_2, \ldots, z_n$, with orders $s_1, s_2, \ldots, s_n$ respectively. Then the function

$$g(z) = (z - z_1)^{s_1}(z - z_2)^{s_2} \ldots (z - z_n)^{s_n} f(z)$$

has at most $z = \infty$ as a pole, and is analytic in the complex plane $\mathbb{C}_\lambda$. Hence $g(z)$ is a polynomial of finite degree, so $f(z)$ is a rational function [76,77].

6. Theory of Residues

Residue theory is an application of series theory and another powerful tool for studying analytic functions [76,77]. It can be used not only to compute complex integrals but also to determine the zero distribution of functions within a region [78,80].

6.1. Residues and the Residue Theorem

Definition 6.1 (Residue)**.** *Let z_0 be an isolated singular point of the function $f(z)$, and let $f(z)$ be analytic in the punctured elliptic disk $D : 0 < |z - z_0| < R$. Take $C : |z - z_0| = r$ ($0 < r < R$). It could be seen that*

$$\frac{q}{2\pi i} \oint_C f(z) \mathrm{d}z$$

is called the residue of the function $f(z)$ at z_0, denoted by $Res(f, z_0)$ [79,81].

Let the Laurent expansion of $f(z)$ in $D : 0 < |z - z_0| < R$ be

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n.$$

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Thus $\text{Res}(f, z_0) = c_{-1}$, because [76,77]

$$\oint_C f(z)dz = \oint_C \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n dz = \sum_{n=-\infty}^{\infty} c_n \oint_C (z-z_0)^n dz = \frac{2\pi i}{q} c_{-1}.$$

Therefore ,

$$\text{Res}(f, z_0) = \frac{q}{2\pi i} \oint_C f(z)\text{d}z = c_{-1}.$$

It can be shown that c_{-1} is the coefficient corresponding to the $(z-z_0)^{-1}$ term in the Laurent expansion of $f(z)$ in $D : 0 < |z - z_0| < R$. Whence , the residue of a function $f(z)$ at a finite removable singular point (not ∞) is zero [78,80].

Proposition 6.2 (Residue Theorem)**.** *Let D be a bounded region in the complex plane, whose boundary consists of one (or finitely many) simple closed curves. If the function $f(z)$ is analytic in D except for finitely many isolated singular points $z_1, z_2, \ldots, z_n$, and $f(z)$ is also analytic on the boundary $C = \partial D$, then*

$$\oint_C f(z)\text{d}z = \frac{2\pi i}{q} \sum_{k=1}^{n} Res(f, z_k). \tag{38}$$

The integral along C is taken in the positive direction with respect to the region D, i.e., when traversing the boundary, if $q > 0$ the left-hand side is always inside the region, and if $q < 0$ the right-hand side is always inside the region [79,81].

Proof. Construct ellipses $C_1, C_2, \ldots, C_n$ centered at the isolated singular points $z_1, z_2, \ldots, z_n$ within the region D [76,77]. By the Cauchy integral theorem,

$$\oint_{C+C_1^-+C_2^-+\ldots+C_n^-} f(z)\text{d}z = 0,$$

i.e.,

$$\oint_C f(z)\text{d}z = \oint_{C_1+C_2+\ldots+C_n} f(z)\text{d}z = \frac{2\pi i}{q} \sum_{k=1}^{n} \text{Res}(f, z_k).$$

That being said , the proposition is proved [78,80]. □

Proposition 6.2 is also called the Residue Theorem in the elliptic complex domain [79,81].

Proposition 6.3 (Residue at a Pole)**.** *Let z_0 be a pole of order n of the function $f(z)$, and let $f(z) = \dfrac{\phi(z)}{(z-z_0)^n}$, where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Consequently ,*

$$Res(f, z_0) = \frac{\phi^{(n-1)}(z_0)}{(n-1)!}.$$

Note that here $\phi^{(0)}(z_0)$ denotes $\phi(z_0)$, and $\phi^{(n-1)}(z_0) = \lim_{z \to z_0} \phi^{(n-1)}(z)$ [76,77].

Proof. From the definition of residues and the Cauchy higher-order derivative formula [78,80],

$$\text{Res}(f, z_0) = \frac{q}{2\pi i} \oint_C \frac{\phi(z)}{(z-z_0)^n}\text{d}z = \frac{\phi^{(n-1)}(z_0)}{(n-1)!}.$$

This completes the proof [79,81]. □

Specifically , when $n = 1$, i.e., z_0 is a simple pole of $f(z)$,

$$\text{Res}(f, z_0) = \phi(z_0) = \lim_{z \to z_0} (z-z_0) f(z); \tag{39}$$

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when $n = 2$, i.e., z_0 is a pole of order 2 of $f(z)$,

$$\mathrm{Res}(f, z_0) = \phi'(z_0).$$

For example, the function $f(z) = \dfrac{e^{iz}}{1+z^2}$ has two simple poles at $z = \pm\dfrac{i}{q}$ [76,77]. The residues at these singular points are

$$\mathrm{Res}\left(f, \frac{i}{q}\right) = \lim_{z \to \frac{i}{q}} \left(z - \frac{i}{q}\right) f(z) = \lim_{z \to \frac{i}{q}} \frac{e^{iz}}{z + \frac{i}{q}} = \frac{q \cdot e^{-q}}{2i},$$

$$\mathrm{Res}\left(f, -\frac{i}{q}\right) = \lim_{z \to -\frac{i}{q}} \left(z + \frac{i}{q}\right) f(z) = \lim_{z \to -\frac{i}{q}} \frac{e^{iz}}{z - \frac{i}{q}} = -\frac{q \cdot e^{q}}{2i}.$$

Consequently , by the Residue Theorem, for any region in the complex plane $\mathbb{C}_\lambda$ containing these two singular points, it is established that [78,80]

$$\oint_C f(z)\mathrm{d}z = -\frac{q \cdot (e^q - e^{-q})}{2i} = \sin\frac{i}{q}.$$

Proposition 6.4 (Residue for Rational Functions)**.** *Let $f(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are analytic at z_0, and z_0 is a simple zero (zero of order one) of $Q(z)$. Then*

$$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Proof. Owing to the fact that z_0 is a simple zero of $Q(z)$, it is found that $Q(z_0) = 0$. Therefore, z_0 is a simple pole of $f(z)$ [79,81]. By Formula 39 ,

$$\mathrm{Res}(f, z_0) = \phi(z_0) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{P(z)}{\frac{Q(z) - Q(z_0)}{z - z_0}} = \frac{P(z_0)}{Q'(z_0)}.$$

This completes the proof [76,77]. □

6.2. Argument Principle

Lemma 6.5 (Finiteness of Zeros and Poles)**.** *Let $f(z)$ be a meromorphic function in a bounded region D, analytic at every point on the boundary C of D, and having no zeros on C. Then $f(z)$ has at most finitely many zeros and poles in D [78,80].*

Proof. Let D_1 be the region obtained by removing all poles of $f(z)$ in D. Then $f(z)$ is analytic in D_1 and not identically zero. We now prove that $f(z)$ has only finitely many zeros in D_1 [79,81].

Assume that $f(z)$ has infinitely many zeros $\{z_n\} : z_1, z_2, \ldots, z_n, \ldots$ in D_1. Since D_1 is a bounded region, the sequence $\{z_n\}$ must have a limit point z_0. By the Uniqueness Theorem for analytic functions, $z_0 \notin D_1$, because if $z_0 \in D_1$, then $f(z)$ would be identically zero in D_1, a contradiction [76,77]. Thus z_0 may lie on the boundary C or be a pole of $f(z)$.

If $z_0 \in C$, then there exists a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z_0$ as $k \to \infty$. Hence $f(z_0) = \lim_{k\to\infty} f(z_{n_k}) = 0$, contradicting the assumption that $f(z)$ has no zeros on C [78,80]. If z_0 is a pole of $f(z)$, then $\lim_{z \to z_0} f(z) = \infty$, so $\lim_{k \to \infty} f(z_{n_k}) = \infty$, contradicting $f(z_{n_k}) = 0$ [79,81] .

Therefore, $f(z)$ has at most finitely many zeros in D. For poles, consider the function $F(z) = \dfrac{1}{f(z)}$, and define $F(z)$ to be zero at the poles of $f(z)$. Then $F(z)$ has only finitely many zeros in D_1, so $f(z)$ has only finitely many poles in D_1 [76,77]. □

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Lemma 6.6 (Logarithmic Residue)**.** *(1) If z_0 is a zero of order n of $f(z)$, then*

$$Res\left(\frac{f'}{f}, z_0\right) = n.$$

(2) If z_0 is a pole of order m of $f(z)$, then

$$Res\left(\frac{f'}{f}, z_0\right) = -m.$$

Here $Res\left(\frac{f'}{f}, z_0\right)$ is called the logarithmic residue of $f(z)$ at z_0, because $\frac{\mathrm{d}}{\mathrm{d}z}\log f(z) = \frac{f'(z)}{f(z)}$ [78,80].

Proof. (1) If z_0 is a zero of order n of $f(z)$, then in a neighborhood of z_0 ,

$$f(z) = (z - z_0)^n g(z),$$

where $g(z)$ is analytic in a neighborhood of z_0 and $g(z_0) \neq 0$ [79,81]. In conclusion ,

$$\frac{f'(z)}{f(z)} = \frac{n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)}{(z - z_0)^n g(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Clearly, $\frac{g'(z)}{g(z)}$ is analytic in a neighborhood of z_0, so $\frac{g'(z)}{g(z)} = \sum_{n=0}^{+\infty} c_n (z - z_0)^n$. This yields the conclusion [76,77].

(2) If z_0 is a pole of order m of $f(z)$, then in a neighborhood of z_0 ,

$$f(z) = \frac{h(z)}{(z - z_0)^m},$$

where $h(z)$ is analytic in a neighborhood of z_0 and $h(z_0) \neq 0$ [78,80]. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{\frac{(z-z_0)^m h'(z) - m(z-z_0)^{m-1} h(z)}{(z-z_0)^{2m}}}{\frac{h(z)}{(z-z_0)^m}} = -\frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Clearly, $\frac{h'(z)}{h(z)}$ is analytic in a neighborhood of z_0 , so $\frac{h'(z)}{h(z)} = \sum_{n=0}^{+\infty} c_n (z - z_0)^n$. This yields the conclusion [79,81].

□

Proposition 6.7 (Argument Principle)**.** *Let D be a bounded region in the complex plane whose boundary C consists of one or finitely many simple closed curves. Let $f(z)$ be a meromorphic function in D, analytic on C and having no zeros on C. Then*

$$\frac{q}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \mathrm{d}z = N(f, C) - P(f, C),$$

where $N(f, C)$ and $P(f, C)$ are the numbers of zeros and poles, respectively, within the region bounded by C. Here a zero of order n counts as n zeros, and a pole of order m counts as m poles [76,77].

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Proof. By Lemma 6.5 , $f(z)$ has only finitely many zeros and poles in the region [78,80]. Denote them by $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $\beta_1, \beta_2, \ldots, \beta_t$, respectively. Let the residues of $f(z)$ at these zeros and poles be $n_1, n_2, \ldots, n_s$ and $-m_1, -m_2, \ldots, -m_t$, respectively. In view of Lemma 6.6 [79,81],

$$\begin{aligned}\frac{q}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz &= \sum_{p=1}^{s}\operatorname{Res}\left(\frac{f'}{f}, z_p\right) + \sum_{q=1}^{t}\operatorname{Res}\left(\frac{f'}{f}, w_q\right) \\ &= \sum_{p=1}^{s} n_p - \sum_{q=1}^{t} m_q = N(f,C) - P(f,C)\end{aligned}$$

This completes the proof [76,77]. □

Corollary 6.8 (Argument Principle in Terms of Argument Variation)**.** *Under the conditions of Proposition 6.7 , it follows that*

$$N(f,C) - P(f,C) = \frac{1}{2\pi}\Delta_C \arg f(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\Delta_{C_j}\arg f(z),$$

where $C_1, C_2, \ldots, C_n$ are the closed curves that constitute C, each taken in the positive direction with respect to the region D (i.e., when traversing the boundary, if $q > 0$ the left-hand side is always inside the region, and if $q < 0$ the right-hand side is always inside the region). $\Delta_{C_j}\arg f(z)$ denotes the continuous change in $\arg f(z)$ as z traverses C_j once in the positive direction with respect to D [78,80].

Proof. Due to the fact that there are no zeros or poles on C_j, we can find finitely many elliptic disks $U_k (k = 1, 2, \ldots, m)$ that contain no zeros or poles of $f(z)$ and completely cover C_j [79,81]. It is easy to see that $f(z)$ is analytic and non-zero on each elliptic disk U_k, so the function $\log f(z)$ can be decomposed into single-valued analytic branches $\log f(z)$ on U_k, with $\frac{\mathrm{d}}{\mathrm{d}z}\log f(z) = \frac{f'(z)}{f(z)}$ [76,77].

Let points on C_j be chosen as $z_1 \in U_m \cap U_1$, $z_2 \in U_1 \cap U_2$, ..., $z_m \in U_{m-1} \cap U_m$ [78,80].

Fix the value of $\log f(z)$ at z_1 as $\log|f(z_1)| + i\frac{\arg f(z_1)}{q}$. This determines an analytic branch $g_1(z)$ of $\log f(z)$ in U_1. Similarly, we can determine analytic branches $g_i(z)$ of $\log f(z)$ in U_i. Clearly, $g_1(z) = g_2(z)$ for all $z \in U_1 \cap U_2$, and in general,

$$g_{k-1}(z) = g_k(z), \quad \forall z \in U_{k-1} \cap U_k \quad (k = 1, 2, \ldots, m),$$

$$\log f_k(z) = \log|f_k(z)| + i\frac{\arg f_k(z)}{q}.$$

Accordingly,

$$\begin{aligned}\frac{q}{2\pi i}\oint_{C_j}\frac{f'(z)}{f(z)}\,dz &= \frac{q}{2\pi i}\left(\int_{z_1}^{z_2}\frac{f'(z)}{f(z)}\,dz + \int_{z_2}^{z_3}\frac{f'(z)}{f(z)}\,dz + \cdots + \int_{z_m}^{z_1}\frac{f'(z)}{f(z)}\,dz\right) \\ &= \frac{q}{2\pi i}\left[g_1(z_2) - g_1(z_1) + g_2(z_3) - g_2(z_2) + \cdots + g_m(z_1) - g_m(z_m)\right] \\ &= \frac{q}{2\pi i}\left[-g_1(z_1) + g_m(z_1)\right] = \frac{q}{2\pi i}\left[-i\frac{\arg f_1(z_1)}{q} + i\frac{\arg f_m(z_1)}{q}\right]\end{aligned}$$

Thereby , $\frac{q}{2\pi i}\oint_{C_j}\frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi}\Delta_{C_j}\arg f(z)$. Summing over all C_j gives the result [79,81]. □

Corollary 6.8 is also called the Argument Principle in the elliptic complex plane, clearly identical to the Argument Principle in the circular complex plane [76,77].

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6.3. Rouché Theorem

Proposition 6.9 (Rouché Theorem)**.** *Let D be a bounded region in the complex plane whose boundary C consists of one or finitely many simple closed curves. Let the functions $f(z)$ and $g(z)$ be analytic on the closed region $\overline{D}$ consisting of D and C, and suppose*

$$|g(z)| < |f(z)|, \quad \forall z \in C.$$

Whence , the functions $f(z)$ and $f(z) + g(z)$ have the same number of zeros in D [78,80].

Proof. Since $f(z)$ and $g(z)$ are analytic on $\overline{D}$, they have no poles in $\overline{D}$ [79,81]. On the boundary C,

$$|f(z)| > |g(z)| \geq 0, \quad |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0.$$

Consequently , $f(z)$ and $f(z) + g(z)$ have no zeros on C [76,77]. Let N_1 and N_2 be the numbers of zeros of $f(z)$ and $f(z) + g(z)$ in D, respectively. By the Argument Principle,

$$N_1 = \frac{1}{2\pi}\Delta_C \arg f(z),$$

$$\begin{aligned} N_2 &= \frac{1}{2\pi}\Delta_C \arg[f(z) + g(z)] \\ &= \frac{1}{2\pi}\Delta_C \arg f(z) + \frac{1}{2\pi}\Delta_C \arg\left[1 + \frac{g(z)}{f(z)}\right]. \end{aligned}$$

Let $w = 1 + \dfrac{g(z)}{f(z)}$. Owing to $|w - 1| = \left|\dfrac{g(z)}{f(z)}\right| < 1$, the function w lies inside a normal ellipse centered at 1 with principal radius 1, which does not contain the origin [78,80]. Therefore,

$$\Delta_C \arg w = \Delta_C \arg\left[1 + \frac{g(z)}{f(z)}\right] = 0,$$

so $N_1 = N_2$. This completes the proof [79,81]. □

Proposition 6.9 is also called Rouché's Theorem in the elliptic complex plane, which is a corollary of the Argument Principle and can be used to determine the distribution of zeros of a function in a given region [76,77].

Proposition 6.10 (Fundamental Theorem of Algebra)**.** *Rouché's Theorem can also be used to prove the Fundamental Theorem of Algebra: any equation of degree n*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \quad (a_0 \neq 0)$$

has exactly n roots [78,80].

Proposition 6.11 (Zero Distribution of Polynomials)**.** *Let the polynomial of degree n*

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_t z^{n-t} + \cdots + a_n \quad (a_0 \neq 0)$$

satisfy

$$|a_t| > |a_0| + \cdots + |a_{t-1}| + |a_{t+1}| + \cdots + |a_n|. \tag{40}$$

Then $f(z)$ has $n - t$ zeros inside the unit ellipse $|z| < 1$ [79,81].

Proof. Take $g(z) = a_t z^{n-t}$. Clearly, $g(z)$ has a zero of order $n - t$ at $z = 0$ inside $|z| < 1$ [76,77]. Now take

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$$h(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{t-1} z^{n-t+1} + a_{t+1} z^{n-t-1} + \cdots + a_n.$$

From condition 40 , it can be seen that $|g(z)| > |h(z)|$ on $|z| = 1$ [78,80]. By Rouché's Theorem as shown in Proposition 6.9 , the number of zeros of $f(z) = g(z) + h(z)$ inside $|z| < 1$ equals the number of zeros of $g(z)$ inside $|z| < 1$, i.e., $n - t$ zeros [79,81]. □

7. Integral Transforms in the Elliptic Complex Setting

As is well known , a simple periodic motion can be represented as a harmonic function [76,77]

$$y = A\sin(wt + \phi),$$

where A is the amplitude, w is the angular frequency, and ϕ is the initial phase. A complex periodic motion can be represented as a superposition of harmonics [78,80]

$$y = A_0 + \sum_{n=1}^{\infty} A_n \sin(n \cdot wt + \phi_n),$$

where $A_n \sin(n \cdot wt + \phi_n)$ can be expanded as $A_n \sin\phi_n \cos nwt + A_n \cos\phi_n \sin nwt$. Letting $\frac{a_0}{2} = A_0$, $a_n = A_n \sin\phi_n$, $b_n = A_n \cos\phi_n$, and $x = wt$, then it follows that [79,81]

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Theorem 7.1 (Orthogonality of Trigonometric System)**.** *The system of functions that constitute trigonometric series*

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$

are mutually orthogonal on $[-\pi, \pi]$, meaning that the integral over $[-\pi, \pi]$ of the product of any two different functions is zero [76,77].

Proof. Clearly, $\int_{-\pi}^{\pi} 1 \cdot \cos nx\, dx = \int_{-\pi}^{\pi} 1 \cdot \sin nx\, dx = 0$ for $n = 1, 2, \ldots$ [78,80].

If $k \neq n$,

$$\int_{-\pi}^{\pi} \cos kx \cdot \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(k+n)x + \cos(k-n)x]\mathrm{d}x = 0.$$

In the same manner, we arrive at

$$\int_{-\pi}^{\pi} \sin kx \cdot \sin nx\, dx = 0 \quad (k \neq n),$$

$$\int_{-\pi}^{\pi} \sin kx \cdot \cos nx\, dx = \int_{-\pi}^{\pi} \cos kx \cdot \sin nx\, dx = 0.$$

Thus the proposition is proved [79,81]. □

However, the integral of the product of two identical trigonometric functions over $[-\pi, \pi]$ is not zero . For example , [76,77]

$$\begin{cases} \int_{-\pi}^{\pi} 1 \cdot 1\, \mathrm{d}x = 2\pi \\ \int_{-\pi}^{\pi} \sin^2 nx\, \mathrm{d}x = \int_{-\pi}^{\pi} \cos^2 nx\, \mathrm{d}x = \pi \quad (n = 1, 2, \ldots) \end{cases}, \tag{41}$$

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where the second formula in Equation 41 can be obtained using $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$ [78,80]. Now we use these conclusions to derive the Fourier integral formula in the elliptic complex domain $\mathbb{C}_\lambda$ [79,81].

7.1. Trigonometric Series and Fourier Series

Theorem 7.2 (Fourier Coefficients). *Let $f(x)$ be a periodic function with period 2π, and [76,77]*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{42}$$

Then [78,80]

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n = 0,1,2,\dots) \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n = 0,1,2,\dots) \end{cases} \tag{43}$$

Proof. Integrating Equation 42 term by term over $[-\pi,\pi]$ gives [79,81]

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right) \\ &= \frac{a_0}{2}\cdot 2\pi = a_0\pi,\end{aligned}$$

so $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\mathrm{d}x$. Similarly, multiplying both sides by $f(x)\cdot\cos kx$ and integrating term by term yields [76,77]

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos kx\,dx &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos kx\cos nx\,dx \\ &\quad + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\cos kx\sin nx\,dx \\ &= a_k\int_{-\pi}^{\pi}\cos^2 kx\,dx = a_k\pi.\end{aligned}$$

Thus $a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x$. By the same token , we can prove $b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x$ [78,80]. □

Now a_n and b_n as seen in Equation 43 are called the Fourier coefficients of the function $f(x)$, and 42 is the Fourier series (expansion) of $f(x)$ [79,81].

Theorem 7.3 (Fourier Series for Periodic Functions). *Let $f_T(t)$ be a periodic function with period T, satisfying the Dirichlet conditions on $\left[-\dfrac{T}{2},\dfrac{T}{2}\right]$ [76,77] as follows ,*

(1) $f_T(t)$ is continuous or has only finitely many discontinuities of the first kind;
(2) $f_T(t)$ has only finitely many extremal points.

Then $f_T(t)$ can be expanded into a Fourier series, and at points of continuity [78,80],

$$f_T(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t),$$

where $\omega = \frac{2\pi}{T}$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_T(t)\cos n\omega t\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_T(t)\sin n\omega t\,dt$. At points of discontinuity [79,81],

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$$f_T(x) = \frac{f_T(x+0) + f_T(x-0)}{2}.$$

Introducing the forms of trigonometric functions in the elliptic complex domain as given in Definition 3.29 , it is obtained that [78,80]

$$\begin{aligned} f_T(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{i\frac{nwt}{q}} + e^{-i\frac{nwt}{q}}}{2} + b_n \cdot q \frac{e^{i\frac{nwt}{q}} - e^{-i\frac{nwt}{q}}}{2i}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - i\frac{1}{q}b_n}{2} e^{i\frac{nwt}{q}} + \frac{a_n + i\frac{1}{q}b_n}{2} e^{-i\frac{nwt}{q}}\right). \end{aligned}$$

Let $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - i\frac{1}{q}b_n}{2}$, $d_n = \frac{a_n + i\frac{1}{q}b_n}{2}$. Thus [79,81]

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)dt,$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)\left[\cos n\omega t - i\frac{1}{q}\sin n\omega t\right]dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-i\frac{n\pi q}{q}}dt,$$

$$d_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{i\frac{n\pi q}{q}}dt =: c_{-n}.$$

Clearly, the formula for c_n also holds when $n = 0$ [76,77].
In summary, assuming $\omega_n = n\omega$, it leads to [78,80]

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi q}{q}} = \frac{1}{T}\sum_{n=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(s)e^{-i\frac{n\pi q}{q}}ds\right]e^{i\frac{n\pi q}{q}}.$$

This is the Fourier expansion of the periodic function $f_T(t)$ with period T in the elliptic complex domain [79,81]. For a non-periodic function $f(t)$, we can view it as obtained from some periodic function $f_T(t)$ as $T \to \infty$. Therefore [76,77],

$$f(t) = \lim_{T\to\infty}\frac{1}{T}\sum_{n=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(s)e^{-i\frac{n\pi q}{q}}ds\right]e^{i\frac{n\pi q}{q}}. \quad (44)$$

It can be seen that for all integers n, the corresponding ω_n are uniformly distributed on the real line [78,80]. Let $\Delta\omega_n$ denote the distance between two adjacent points, then $\Delta\omega_n = \omega_n - \omega_{n-1} = \frac{2\pi}{T}$, i.e., $\frac{1}{T} = \frac{\Delta\omega_n}{2\pi}$, and as $T \to \infty$, $\Delta\omega_n \to 0$. Hence Formula 44 can be rewritten as [79,81]

$$f(t) = \lim_{\Delta\omega_n\to 0}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_T(s)e^{-i\frac{n\pi q}{q}}ds\right]e^{i\frac{n\pi q}{q}}\Delta\omega_n,$$

that is,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)e^{-i\frac{\omega s}{q}}ds\right]e^{i\frac{\omega t}{q}}d\omega. \quad (45)$$

Formula 45 is also called the elliptic complex form of the Fourier integral formula [76,77]. Furthermore, the following result is established.

Theorem 7.4 (Fourier Integral Theorem). *If the function $f(t)$ satisfies the following conditions on $(-\infty, +\infty)$ [78,80]:*

(1) $f(t)$ satisfies the Dirichlet conditions on any finite interval;

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(2) *$f(t)$ is absolutely integrable on the infinite interval $(-\infty, +\infty)$ (i.e., the integral $\int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t$ converges),*

then Formula 45 holds, and at points of discontinuity, $f(x) = \dfrac{f(x+0)+f(x-0)}{2}$ [79,81].

Theorem 7.4 is also called the Fourier Integral Theorem [76,77].
Formula 45 can also be transformed into trigonometric form [78,80]:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)e^{-i\frac{ws}{q}}\,ds\right]e^{i\frac{wt}{q}}\,dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)e^{i\frac{w(t-s)}{q}}\,ds\right]dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)\cos\frac{w(t-s)}{q}\,ds + \frac{i}{q}\int_{-\infty}^{\infty} f(s)\sin\frac{w(t-s)}{q}\,ds\right]dw.$$

Considering that $\displaystyle\int_{-\infty}^{\infty} f(s)\sin\frac{\omega(t-s)}{q}\mathrm{d}s$ is an odd function of ω [79,81] ,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)\cos\frac{\omega(t-s)}{q}\mathrm{d}s\right]d\omega$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(s)\cos\frac{\omega(t-s)}{q}\mathrm{d}s\right]d\omega.$$

This is the trigonometric form of the Fourier integral formula [76,77]. Besides ,

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(s)\left(\cos\frac{\omega t}{q}\cos\frac{\omega s}{q} + \sin\frac{\omega t}{q}\sin\frac{\omega s}{q}\right)\mathrm{d}s\right]d\omega.$$

When $f(t)$ is an odd function, $f(s)\cos\frac{\omega s}{q}$ is an odd function of s, so [78,80]

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty}\left[\int_{0}^{\infty} f(s)\sin\frac{\omega s}{q}\mathrm{d}s\right]\sin\frac{\omega t}{q}\mathrm{d}\omega. \tag{46}$$

In the same manner , when $f(t)$ is an even function, [79,81]

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty}\left[\int_{0}^{\infty} f(s)\cos\frac{\omega s}{q}\mathrm{d}s\right]\cos\frac{\omega t}{q}\mathrm{d}\omega. \tag{47}$$

Considering the cases where $f(t)$ is odd or even is useful when $f(t)$ is only defined on $(0, +\infty)$ [76,77]. In such cases, we can extend $f(t)$ as an odd or even function (i.e., perform odd or even extension), making it much easier to compute the Fourier integral of $f(t)$ [78,80].

7.2. Fourier Transform

From the above integral formulas, we give the definition of the Fourier integral transform and study its basic properties [76,77].

7.2.1. Basic Theory of Fourier Transform

Definition 7.5 (Fourier Transform). *If the function $f(t)$ satisfies the conditions of the Fourier Integral Theorem on $(-\infty, +\infty)$, the function [78,80]*

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\frac{\omega t}{q}}\mathrm{d}t$$

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is called the Fourier transform of $f(t)$, denoted by $\mathscr{F}(f(t))$, i.e., $\mathscr{F}(f(t)) = F(\omega)$; and the function [79,81]

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\frac{\omega t}{q}}\mathrm{d}\omega$$

is called the inverse Fourier transform of $F(\omega)$, denoted by $\mathscr{F}^{-1}(F(\omega))$, i.e., $f(t) = \mathscr{F}^{-1}(F(\omega))$.

Thus, the functions $F(\omega)$ and $f(t)$ form a Fourier transform pair [76,77]. Clearly, $F(\omega)$ and $f(t)$ have the same parity. When $f(t)$ is an odd function, from formula 46 , the transform function is [78,80]

$$F_s(\omega) = \int_0^{\infty} f(t)\sin\frac{\omega t}{q}\mathrm{d}t$$

called the Fourier sine transform of $f(t)$, and the function [79,81]

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\frac{\omega t}{q}\mathrm{d}\omega$$

is called the inverse Fourier sine transform of $F_s(\omega)$.

Similarly, when $f(t)$ is an even function, from Formula 47 , the function [76,77]

$$F_c(\omega) = \int_0^{\infty} f(t)\cos\frac{\omega t}{q}\mathrm{d}t$$

is called the Fourier cosine transform of $f(t)$, and the function [78,80]

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\frac{\omega t}{q}\mathrm{d}\omega$$

is called the inverse Fourier cosine transform of $F_c(\omega)$.

Similarly, if the function $f(t)$ is only defined on $(0, +\infty)$ and satisfies the Fourier Integral Theorem, we can obtain the sine or cosine transform of $f(t)$ by odd or even extension [79,81].

7.2.2. Unit Impulse Function and Its Fourier Transform

Definition 7.6 (Unit Impulse Function)**.** *For any infinitely differentiable function $f(t)$, if [76,77]*

$$\int_{-\infty}^{+\infty}\delta(t)f(t)\mathrm{d}t = \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\delta_\epsilon(t)f(t)\mathrm{d}t,$$

where $\delta_\epsilon(t) = \begin{cases} 0, & t<0, \\ \frac{1}{\epsilon}, & 0\le t\le\epsilon, \end{cases}$ then [78,80]

$$\delta(t) := \lim_{\epsilon\to 0}\delta_\epsilon(t) \tag{48}$$

is called the unit impulse function (or δ-function) [79,81].

From 48, it can get that [76,77]

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0. \end{cases} \quad (6.21)$$

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From the geometric interpretation of $\delta_\epsilon(t)$, it is easy to see that $\int_{-\infty}^{\infty} \delta_\epsilon(t)\mathrm{d}t = \int_0^{\epsilon} \frac{1}{\epsilon}\mathrm{d}t = 1$. Furthermore, taking $f(t) = 1$, from Definition 7.5 , it follows that [78,80]

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1.$$

Corollary 7.7 (Sifting Property of δ-function)**.** *The δ-function has the following sifting property [79,81].*

$$\int_{-\infty}^{+\infty} \delta(t-t_0)f(t)\mathrm{d}t = f(t_0).$$

Proof. Let $x = t - t_0$ i.e., $t = x + t_0$, $dx = dt$. Hence , [78,80]

$$\begin{aligned}\int_{-\infty}^{+\infty} \delta(t-t_0)f(t)\,dt &= \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} \delta_\varepsilon(t-t_0)f(t)\,dt \\ &= \lim_{\varepsilon\to 0}\delta_\varepsilon(x)\int_{-\infty}^{+\infty} f(x+t_0)\,dx \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-t_0}^{\varepsilon-t_0} f(x+t_0)\,dx.\end{aligned}$$

By the mean value theorem for integrals, $\int_0^\epsilon f(x+t_0)\mathrm{d}x = \epsilon f(\theta)$ for some $\theta \in (t_0, \epsilon + t_0)$. Therefore, [79,81]

$$\int_{-\infty}^{+\infty} \delta(t-t_0)f(t)\mathrm{d}t = \lim_{\epsilon\to 0} f(\epsilon+t_0) = f(t_0).$$

□

From the sifting property of the δ-function, we can find its Fourier transform [76,77]:

$$F(\delta(t)) = F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\frac{\omega t}{q}}\mathrm{d}t = e^{-i\frac{\omega t}{q}}\Big|_{t=0} = 1.$$

The inverse Fourier transform of the δ-function is [78,80]

$$\delta(t) = \mathscr{F}^{-1}(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{\omega t}{q}}\mathrm{d}\omega.$$

Consequently , [79,81]

$$\int_{-\infty}^{\infty} e^{i\frac{\omega t}{q}}\mathrm{d}\omega = 2\pi\delta(t).$$

It can be seen that 1 and $\delta(t)$ form a Fourier transform pair [76,77].

Furthermore, $2\pi\delta(\omega-\omega_0)$ and $e^{i\frac{\omega_0 t}{q}}$ form a Fourier transform pair [78,80].

Proof. Consider the inverse Fourier transform [79,81]:

$$\mathscr{F}^{-1}(2\pi\delta(\omega-\omega_0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{i\frac{\omega t}{q}}\mathrm{d}\omega = e^{i\frac{\omega_0 t}{q}} = f(t).$$

As a result , the conclusion holds [76,77]. □

Setting $\omega_0 = 0$, we have that 1 and $2\pi\delta(\omega)$ form a Fourier transform pair [78,80]. Consequently,

$$\begin{cases} \int_{-\infty}^{\infty} e^{-i\frac{\omega t}{q}}\mathrm{d}t = 2\pi\delta(\omega), \\ \int_{-\infty}^{\infty} e^{-i\frac{(\omega-\omega_0)t}{q}}\mathrm{d}t = 2\pi\delta(\omega-\omega_0). \end{cases} \tag{49}$$

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Using formula 49 , we can find the Fourier transform of the sine function $f(t) = \sin\omega_0 t$ [79,81] as

$$\begin{aligned} F(w) &= \int_{-\infty}^{\infty} \sin w_0 t \cdot e^{-i\frac{wt}{q}}\, dt = \int_{-\infty}^{\infty} \frac{q}{2i}\left(e^{i\frac{w_0 t}{q}} - e^{-i\frac{w_0 t}{q}}\right)e^{-i\frac{wt}{q}}\, dt \\ &= \frac{q}{2i}\int_{-\infty}^{\infty}\left(e^{-i\frac{(w-w_0)t}{q}} - e^{-i\frac{(w+w_0)t}{q}}\right)dt \\ &= \frac{q}{2i}\left[2\pi\delta(w-w_0) - 2\pi\delta(w+w_0)\right] \\ &= \frac{i\pi}{q}\left[\delta(w+w_0) - \delta(w-w_0)\right], \end{aligned}$$

Thus, the Fourier transform of the function $f(t) = \sin\omega_0 t$ also depends on the elliptic complex domain itself [76,77].

Proposition 7.8 (Fourier Transform of Derivatives of δ-function)**.** *If $f(t)$ is an infinitely differentiable function, then [78,80]*

$$\int_{-\infty}^{+\infty} \delta^{(n)}(t-t_0) f(t)\mathrm{d}t = (-1)^n f^{(n)}(t_0). \tag{50}$$

Proof. First, consider the first-order derivative. Using integration by parts [79,81],

$$\int_{-\infty}^{+\infty} \delta'(t-t_0) f(t)\mathrm{d}t = \delta(t-t_0) f(t)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(t-t_0) f'(t)\mathrm{d}t.$$

Since $f(t)$ is infinitely differentiable and as $|t| \to \infty$, $F(t) \to \infty$, it can be seen that [76,77]

$$\int_{-\infty}^{+\infty} \delta'(t-t_0) f(t)\mathrm{d}t = -\int_{-\infty}^{+\infty} \delta(t-t_0) f'(t)\mathrm{d}t = -f'(t_0).$$

Now assume the formula holds for $n = k$, i.e., [78,80]

$$\int_{-\infty}^{+\infty} \delta^{(k)}(t-t_0) f(t)\mathrm{d}t = (-1)^k f^{(k)}(t_0).$$

Then for $n = k+1$, [79,81]

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta^{(k+1)}(t-t_0) f(t)\, dt &= \delta^{(k)}(t-t_0) f(t)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta^{(k)}(t-t_0) f'(t)\, dt \\ &= -\int_{-\infty}^{+\infty} \delta^{(k)}(t-t_0) f'(t)\, dt \\ &= (-1)^{k+1}\left(f^{(k)}(t_0)\right)' = (-1)^{k+1} f^{(k+1)}(t_0). \end{aligned}$$

Whence , the proposition is proved [76,77]. □

In particular, when $t_0 = 0$[78,80] ,

$$\int_{-\infty}^{+\infty} \delta^{(n)}(t) f(t)\mathrm{d}t = (-1)^n f^{(n)}(0).$$

Formula 50 can be used to find the Fourier transform of many functions [79,81]. For example, find the Fourier transform of $\delta'(t-1)$ [76,77]

$$\mathscr{F}(\delta'(t-1)) = \int_{-\infty}^{\infty} \delta'(t-1) e^{-i\frac{\omega t}{q}}\, \mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-i\frac{\omega t}{q}}\right)\big|_{t=1} = i\frac{\omega}{q} e^{-i\frac{\omega}{q}}.$$

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Definition 7.9 (Unit Step Function and Exponential Decay Function)**.** *The function [78,80]* $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$ *is called the unit step function. The function [79,81]*

$$\phi(t) = e^{-\beta t}u(t) \quad (\beta > 0)$$

is called the exponential decay function.

The Fourier transform of the exponential decay function is easily found to be [76,77]

$$\mathscr{F}(\phi(t)) = \frac{1}{\beta + i\frac{\omega}{q}} = \frac{\beta - i\frac{\omega}{q}}{\beta^2 + \left(\frac{\omega}{q}\right)^2}.$$

Proposition 7.10 (Derivative Relationship between δ-function and Step Function)**.** *The δ-function is the derivative of the unit step function* $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$*, i.e., [78,80]*

$$\int_{-\infty}^{t} \delta(s)\mathrm{d}s = u(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}u(t) = \delta(t).$$

Proposition 7.11 (Fourier Transform of Unit Step Function)**.** *The Fourier transform of the unit step function* $u(t)$ *is* $\frac{q}{i\omega} + \pi\delta(\omega)$ *[79,81].*

Proof. This can be proved using the inverse Fourier transform combined with the Dirichlet formula (interested readers may attempt it) [76,77]. □

7.3. Properties of the Fourier Transform

Further, we need to study some operational properties of the Fourier transform [78,80]. For convenience, we assume that all functions satisfy the conditions of the Fourier Integral Theorem [79,81].

Corollary 7.12 (Linearity Property)**.** *Let* $\mathscr{F}(f_1(t)) = F_1(\omega)$, $\mathscr{F}(f_2(t)) = F_2(\omega)$*, and let* α, β *be constants. Then [76,77]*

(1) $\mathscr{F}(\alpha f_1(t) + \beta f_2(t)) = \alpha F_1(\omega) + \beta F_2(\omega)$;
(2) $\mathscr{F}^{-1}(\alpha F_1(\omega) + \beta F_2(\omega)) = \alpha f_1(t) + \beta f_2(t)$.

Corollary 7.13 (Shift Property)**.** *(1)* $\mathscr{F}(f(t \pm t_0)) = e^{\pm i\frac{\omega t_0}{q}}\mathscr{F}(f(t))$;
(2) $\mathscr{F}^{-1}(F(\omega \mp \omega_0)) = e^{\pm i\frac{\omega_0 t}{q}} f(t)$.

Proof. Let $u = t \pm t_0$. Then $t = u \mp t_0$. From the definition of the Fourier transform [78,80],

$$\mathscr{F}(f(t \pm t_0)) = \int_{-\infty}^{+\infty} f(u)e^{-i\frac{\omega(u \mp t_0)}{q}}du = e^{\pm i\frac{\omega t_0}{q}}\mathscr{F}(f(t)).$$

Thus (1) is proved. (2) can be proved similarly [79,81]. □

Corollary 7.14 (Differentiation Property)**.**

$$\mathscr{F}(f'(t)) = i\frac{\omega}{q}\mathscr{F}(f(t)).$$

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Proof. Since $f(t)$ is absolutely integrable, $f(t) \to 0$ as $|t| \to \infty$ [76,77]. Using integration by parts,

$$\begin{aligned}\mathscr{F}(f'(t)) &= \int_{-\infty}^{+\infty} f'(t)e^{-i\frac{wt}{q}}\,dw \\ &= f(t)e^{-i\frac{wt}{q}}\Big|_{-\infty}^{+\infty} + \frac{iw}{q}\int_{-\infty}^{+\infty} f(t)e^{-i\frac{wt}{q}}\,dw = \frac{iw}{q}\mathscr{F}(f(t)).\end{aligned}$$

□

This shows that the Fourier transform of the derivative of a function equals the Fourier transform of the function multiplied by the factor $i\dfrac{\omega}{q}$ [78,80].

Corollary 7.15 (Integration Property). *If the function $g(t) = \int_{-\infty}^{t} f(\tau)d\tau$ satisfies the conditions of the Fourier Integral Theorem, then [79,81]*

$$\mathscr{F}\left(\int_{-\infty}^{t} f(\tau)d\tau\right) = \frac{q}{i\omega}\mathscr{F}(f(t)).$$

Proof. Since $\dfrac{d}{dt}\displaystyle\int_{-\infty}^{t} f(\tau)d\tau = f(t)$, we have $\mathscr{F}\left(\dfrac{d}{dt}\displaystyle\int_{-\infty}^{t} f(\tau)d\tau\right) = \mathscr{F}(f(t))$ [76,77]. By the differentiation property,

$$\mathscr{F}\left(\frac{d}{dt}\int_{-\infty}^{t} f(\tau)d\tau\right) = i\frac{\omega}{q}\mathscr{F}\left(\int_{-\infty}^{t} f(\tau)d\tau\right).$$

Hence, we obtain the conclusion [78,80]. □

This shows that the Fourier transform of the integral of a function equals the Fourier transform of the function divided by the factor $i\frac{\omega}{q}$ [79,81].

Corollary 7.16 (Product Theorem). *Let $\mathscr{F}(f_1(t)) = F_1(\omega)$, $\mathscr{F}(f_2(t)) = F_2(\omega)$. Then [76,77]*

(1) $\displaystyle\int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2(\omega)d\omega;$

(2) $\displaystyle\int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\overline{F_2(\omega)}d\omega,$

where $\overline{f(t)}$ denotes the complex conjugate of $f(t)$ [78,80].

Proof. Assume $F_1(\omega)$ and $F_2(\omega)$ are absolutely integrable on $(-\infty, +\infty)$ [79,81]. Then the order of integration can be interchanged:

$$\begin{aligned}\int_{-\infty}^{\infty} \overline{f_1(t)}f_2(t)\,dt &= \int_{-\infty}^{\infty} \overline{f_1(t)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(w)e^{i\frac{wt}{q}}\,dw\right]dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(w)\left[\int_{-\infty}^{\infty} \overline{f_1(t)e^{-i\frac{wt}{q}}}\,dt\right]dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(w)\overline{\int_{-\infty}^{\infty} f_1(t)e^{-i\frac{wt}{q}}\,dt}\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(w)\overline{F_1(w)}\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F_1(w)}F_2(w)\,dw.\end{aligned}$$

Thus (1) is proved [76,77]. (2) can be proved similarly [78,80]. □

If $f_1(t)$ and $f_2(t)$ are real functions, the product theorem can be written as [79,81]

$$\int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F_1(\omega)}F_2(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\overline{F_2(\omega)}d\omega. \tag{51}$$

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Corollary 7.17 (Energy Integral (Parseval's Identity))**.** *Let $\mathscr{F}(f(t)) = F(\omega)$. Then [76,77]*

$$\int_{-\infty}^{+\infty} [f(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega. \quad (6.28)$$

Formula 51 is also called Parseval's identity, which has important applications in estimating zeros of the Riemann zeta function [78,80].

Proof. In Formula 51 , let $f_1(t) = f_2(t) = f(t)$. Then [79,81]

$$\int_{-\infty}^{+\infty} [f(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\overline{F(\omega)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Therefore , the proposition is proved [76,77]. Here, $S(\omega) = |F(\omega)|^2$ is called the energy density function of $f(t)$, which determines the energy distribution of $f(t)$. Integrating over all frequencies gives the total energy of $f(t)$. Clearly, $S(\omega)$ is an even function [78,80]. □

The theory and results of convolution and the convolution theorem for Fourier transforms are the same in the elliptic complex domain as in the circular complex domain, so we will not repeat them here [79,81].

7.4. Laplace Transform

The Fourier transform requires that the function $f(t)$ satisfy the Dirichlet conditions and be absolutely integrable on $(-\infty, +\infty)$ [76,77]. These conditions are somewhat restrictive. Many classical functions such as the unit step function, sine function, cosine function, and linear functions do not satisfy these conditions. Moreover, many practical problems only require the function to be defined on $(0, +\infty)$ [78,80].

Thus, we wish to modify the Fourier transform. Consider the unit step function introduced in the Fourier transform [79,81]

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

and the exponential decay function [76,77]

$$\phi(t) = e^{-\beta t} u(t) \quad (\beta > 0).$$

Clearly, $u(t)$ can transform the integration interval of a function $g(t)$ from $(-\infty, +\infty)$ to $(0, +\infty)$, while $\phi(t)$ can make it absolutely integrable [78,80]. That is, by modifying $f(t)$ to $f(t)u(t)e^{-\beta t}$ with an appropriate $\beta > 0$, such a Fourier transform will exist [79,81].

7.4.1. Laplace Transform

Taking the Fourier transform of $\phi(t)u(t)e^{-\beta t}$ (with $\beta > 0$), we obtain [76,77]

$$G_\beta(\omega) = \int_{-\infty}^{+\infty} \phi(t)u(t)e^{-\beta t} e^{-i\frac{\omega t}{q}} dt$$
$$= \int_0^{+\infty} f(t) e^{-(\beta + i\frac{\omega}{q})t} dt =: \int_0^{+\infty} f(t)e^{-st} dt,$$

where $s = \beta + i\dfrac{\omega}{q}$ and $f(t) = \phi(t)u(t)$ [78,80]. Letting $L(s) = G_\beta\left(\dfrac{q(s-\beta)}{i}\right)$, it follows that

$$L(s) = \int_0^{+\infty} f(t)e^{-st} dt. \quad (52)$$

Thus, the following definition can be obtained[79,81].

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Definition 7.18 (Laplace Transform)**.** *Let $f(t)$ be a function defined on $(0,+\infty)$, and let the integral $\int_0^{+\infty} f(t)e^{-st}dt$ converge in some region of the complex plane. Then formula 52 is called the Laplace transform of $f(t)$, denoted by $\mathscr{L}(f(t))$ [76,77].*

It can be seen that the result of the Laplace transform is the same in the elliptic complex domain as in the circular complex domain (including the corresponding convolution and convolution theorem) [78,80]. We will not repeat these here. The difference from the theory in the circular complex domain lies in the corresponding inverse Laplace transform [79,81].

7.4.2. Inverse Laplace Transform

From the definition of the Laplace transform, the Laplace transform of $f(t)$ is actually the Fourier transform of $f(t)u(t)e^{-\beta t}$ (with $\beta > 0$) [76,77]. Thus, when $f(t)u(t)e^{-\beta t}$ satisfies the conditions of the Fourier Integral Theorem, at points of continuity of $f(t)$, we have the integral expression [78,80]

$$\begin{aligned} f(t)u(t)e^{-\beta t} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)u(\tau)e^{-\beta\tau}e^{-i\frac{w\tau}{q}}\,d\tau\right]e^{i\frac{wt}{q}}\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{0}^{+\infty} f(\tau)e^{-(\beta+i\frac{w}{q})\tau}\,d\tau\right]e^{i\frac{wt}{q}}\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} L\left(\beta+i\frac{w}{q}\right)e^{i\frac{wt}{q}}\,dw, \quad t>0, \end{aligned}$$

Therefore , [79,81]

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} L\left(\beta+i\frac{\omega}{q}\right)e^{\left(\beta+i\frac{\omega}{q}\right)t}d\omega, \quad t>0.$$

Making the substitution $s = \beta + i\frac{\omega}{q}$, it can be given that $d\omega = \frac{q}{i}ds$, and [76,77]

$$f(t) = \frac{q}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} L(s)e^{st}ds, \quad t>0,$$

which is also called the Laplace inversion formula [78,80]. Here, the integral is the inverse Laplace transform of $L(s) = \mathscr{L}(f(t)) = \int_0^{+\infty} f(t)e^{-st}dt$, denoted by $\mathscr{L}^{-1}(L(s))$. We could say that $L(s)$ and $f(t)$ form a Laplace transform pair [79,81].

Proposition 7.19. *Let $s_1, s_2, \ldots, s_n$ be all the singular points of the function $L(s)$ (choose β appropriately so that these singular points all lie in the region $\mathrm{Re}(s) < \beta$), and assume $L(s) \to 0$ as $s \to \infty$. Then [76,77]*

$$f(t) = \frac{q}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} L(s)e^{st}ds = \sum_{k=1}^{n}\mathrm{Res}\left(L(s)e^{st}, s_k\right), \quad t>0.$$

Proof. As shown in Figure 4 , let $C = L_{AB} + C_{BA}$ be a closed curve, where the line segment L_{AB} is directed from A to B, and the arc C_{BA} is directed from B to A [78,80]. Moreover, C_{BA} lies in the region $\mathrm{Re}(s) < \beta$ and is a (normal) elliptic arc with principal radius r. When r is sufficiently large, all singular points of $L(s)$ are contained within the region enclosed by C [79,81]. It is easy to see that the points are $A = \beta - i\frac{r}{q}$ and $B = \beta + i\frac{r}{q}$.

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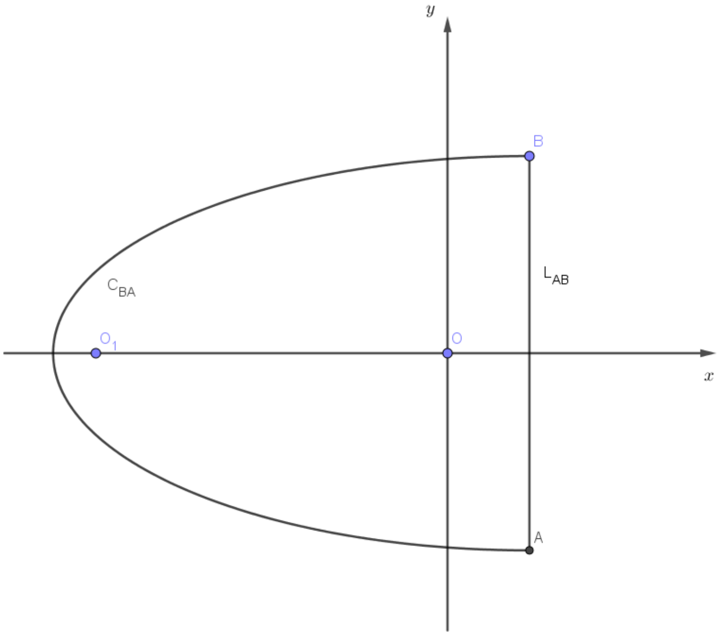

Figure 4. The closed curve $C = L_{AB} + C_{BA}$ in the complex plane $\mathbb{C}_\lambda$ where $\lambda = -q^2$.

Due to the fact that e^{st} is analytic in the entire complex plane $\mathbb{C}_\lambda$, the singular points of $L(s)e^{st}$ are exactly the singular points of $L(s)$ [76,77]. By the residue theorem in the elliptic complex plane, it is obtained that

$$\oint_C L(s)e^{st}ds = \frac{2\pi i}{q}\sum_{k=1}^{n}\text{Res}\big(L(s)e^{st}, s_k\big),$$

i.e., [78,80]

$$\frac{q}{2\pi i}\left(\int_{\beta-i\frac{r}{q}}^{\beta+i\frac{r}{q}} L(s)e^{st}\,ds + \int_{C_{BA}} L(s)e^{st}\,ds\right) = \sum_{k=1}^{n}\text{Res}\big(L(s)e^{st}, s_k\big). \tag{53}$$

It is easy to see that when $t > 0$, [79,81]

$$\lim_{r\to\infty}\int_{C_{BA}} L(s)e^{st}ds = 0.$$

Taking the limit as $r \to \infty$ in Equation 53 yields the proposition [76,77]. □

Using the formula of Proposition 7.19 , it is relatively easy to compute inverse Laplace transforms [78,80].

Part II

The Proof of the Riemann Hypothesis

8. Functional Equation and Zero Distribution of the Riemann ζ - Function

It is well known that Euler gave the definition of the zeta function in the real domain: $\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}$, $s > 1$ [76,77]. Riemann extended it to the circular complex domain $\mathbb{C}$, i.e., the form before analytic continuation [78,80]

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}, \quad \text{Re}(s) > 1, s \in \mathbb{C}$$

called the Riemann zeta function [79,81].

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The purpose of this chapter is to extend the Riemann zeta function to the elliptic complex domain $\mathbb{C}_\lambda$, perform its analytic continuation to the entire elliptic complex plane, and further derive its corresponding functional equation [76,77].

Unless otherwise specified, in this paper, ellipse refers to a normal ellipse in the complex plane $\mathbb{C}_\lambda$ (rather than a general ellipse) [78,80]. In the complex plane $\mathbb{C}_\lambda$, suppose $\lambda = -p = -q^2, q \in \mathbb{R}^*$, where q is called the elliptic coefficient in $\mathbb{C}_\lambda$ [79,81].

8.1. Riemann Zeta Function and Gamma Function

The Riemann zeta function is closely related to the gamma function [76,77]. We first give the integral definition of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

where $z \in \{z \in \mathbb{C}_\lambda \mid \mathrm{Re}(z) > 0\}$ [78,80]. Although the gamma function is introduced into a new algebraic system, namely the elliptic complex domain $\mathbb{C}_\lambda$, it is easy to prove that its basic properties in the elliptic complex domain $\mathbb{C}_\lambda$ are the same as those in the circular complex domain $\mathbb{C}$ [79,81]. Below we will give, without proof, some properties of the gamma function useful for this chapter [76,77].

Corollary 8.1 (Basic Properties of Gamma Function). *(1)* $\Gamma(1) = 1; \Gamma(n+1) = n!,\ n \in \mathbb{N}$;
(2) $\Gamma(z+1) = z\Gamma(z),\ \mathrm{Re}(z) > 0$.

Corollary 8.2 (Limit Definition and Analytic Continuation of Gamma Function). *The gamma function has its limit definition*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\dots(z+n-1)(z+n)},$$

and can be analytically continued to the entire elliptic complex plane $\mathbb{C}_\lambda$ in the form

$$\Gamma(z) = \frac{\Gamma(z+m+1)}{z(z+1)(z+2)\dots(z+m-1)(z+m)}, \tag{54}$$

where $z = -n (n \in \mathbb{N})$ are all simple poles of the function [78,80].

Definition 8.3 (Beta Function). *In the complex plane $\mathbb{C}_\lambda$, define the beta function as*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t$$

where $x, y \in \{z \in \mathbb{C}_\lambda \mid \mathrm{Re}(z) > 0\}$ [79,81].

Corollary 8.4 (Reflection Formula). *In the entire complex plane $\mathbb{C}_\lambda$, we have the relationship between the beta function and the gamma function*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Furthermore, we have the complement formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{55}$$

Similarly, we can use the gamma function to convert the Riemann zeta function into integral form [76,77]:

$$\zeta(s)\Gamma(s) = \sum_{k=1}^\infty \frac{1}{k^s} \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x = \sum_{k=1}^\infty \int_0^\infty \left(\frac{x}{k}\right)^{s-1} e^{-x} \frac{\mathrm{d}x}{k}.$$

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Make the substitution: $t = \frac{x}{k}$, $\mathrm{d}t = \frac{\mathrm{d}x}{k}$. It then follows that [78,80]

$$\zeta(s)\Gamma(s) = \sum_{k=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-kt} \mathrm{d}t = \int_0^{\infty} t^{s-1} e^{-t} \sum_{k=1}^{\infty} e^{-(k-1)t} \mathrm{d}t.$$

According to the property of geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$, we finally obtain the integral form of the Riemann function [79,81]

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1} e^{-t}}{1 - e^{-t}} \mathrm{d}t,$$

i.e.,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \mathrm{d}t. \tag{56}$$

Next, we will perform analytic continuation of the zeta function based on Formula 56 [76,77].

8.2. Analytic Continuation of the Riemann Zeta Function

We know that elliptic complex functions have the corresponding Cauchy integral formula [78,80]

$$f(z) = \frac{q}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

as seen in Proposition 4.13 .

Combined with Formula 56 , we consider the following integral [79,81]:

$$g(s) = \frac{q}{2\pi i} \oint_{\gamma} \frac{z^{s-1}}{e^{-z} - 1} \mathrm{d}z =: \frac{q}{2\pi i} \oint_{\gamma} z^{s-1} f(z) \mathrm{d}z,$$

where $f(z) = \frac{1}{e^{-z} - 1}$, and the contour $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ is shown in Figure 5 , where the normal ellipse corresponding to γ_2 has principal radius $r = \epsilon$ [76,77]

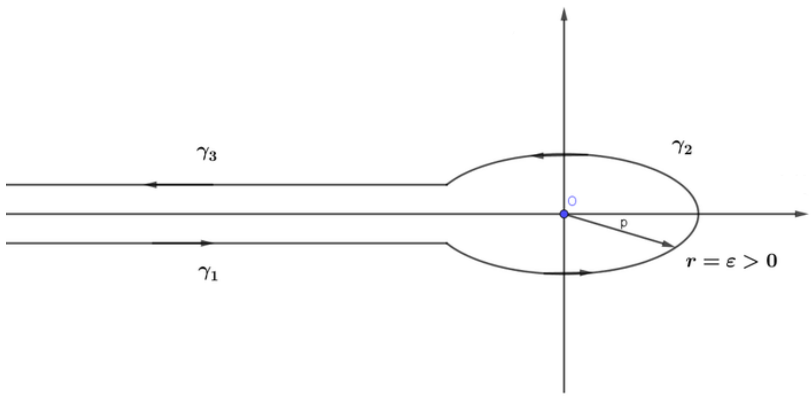

Figure 5. The contour γ employed for the function $g(s)$ in the complex plane $\mathbb{C}_\lambda$.

It is necessary to first show that $g(s)$ is analytic in the entire complex plane, then show that when $\mathrm{Re}(s) > 1$, $\zeta(s)$ can be expressed by $g(s)$, and finally redefine $\zeta(s)$ through $g(s)$, thereby achieving the analytic continuation of $\zeta(s)$ [78,80].

For $g(s)$ to be analytic, each part of the contour integral must be analytic [79,81].

With respect to γ_1 , make the substitution: $z = e^{-i\frac{\pi}{q}} \cdot t = -t$, $\mathrm{d}z = e^{-i\frac{\pi}{q}} \mathrm{d}t$. It is established that [76,77]

$$\int_{\gamma_1} z^{s-1} f(z) \mathrm{d}z = e^{-i\frac{\pi}{q}} \int_{\epsilon}^{\infty} t^{s-1} f(-t) \mathrm{d}t.$$

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Now estimate this integral [78,80] as

$$\left|e^{-i\frac{\pi s}{q}}\int_{\varepsilon}^{\infty} t^{s-1}f(-t)\,dt\right| \leq \int_{\varepsilon}^{\infty}\left|e^{-i\frac{\pi s}{q}}t^{s-1}f(-t)\right|dt$$
$$= e^{q\cdot\pi\,\mathrm{Im}(s)}\int_{\varepsilon}^{\infty} t^{\mathrm{Re}(s)-1}f(-t)\,dt.$$

Regarding $\epsilon > 1$, with $e^t - 1 > e^t/2$, it could be seen that $f(-t) = \dfrac{1}{e^t-1} < 2e^{-t}$ [79,81]. Thus, further estimating this integral by means of that

$$e^{q\cdot\pi\mathrm{Im}(s)}\int_{\epsilon}^{\infty} t^{\mathrm{Re}(s)-1}f(-t)\mathrm{d}t \leq 2e^{q\cdot\pi\mathrm{Im}(s)}\int_{\epsilon}^{\infty} t^{\mathrm{Re}(s)-1}e^{-t}\mathrm{d}t < \infty.$$

Now examine its derivative [76,77]

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\gamma_1} z^{s-1}f(z)\,dz = \int_{\gamma_1}\frac{\partial}{\partial s}z^{s-1}f(z)\,dz = \int_{\gamma_1}\log(z)z^{s-1}f(z)\,dz$$
$$= e^{-i\frac{\pi s}{q}}\int_{\varepsilon}^{\infty}\log\left(e^{-i\frac{\pi}{q}}t\right)t^{s-1}f(-t)\,dt.$$

Through the following estimate [78,80]

$$\left|\frac{d}{ds}\int_{\gamma_1} z^{s-1}f(z)\,dz\right| = \left|e^{-i\frac{\pi s}{q}}\int_{\varepsilon}^{\infty}\log\left(e^{-i\frac{\pi}{q}}t\right)t^{s-1}f(-t)\,dt\right|$$
$$\leq 2e^{q\cdot\pi\,\mathrm{Im}(s)}\int_{\varepsilon}^{\infty}\left|\left(\log(t)-i\frac{\pi}{q}\right)t^{s-1}e^{-t}\right|dt < \infty,$$

it can be shown that the absolute value of the derivative is bounded, thus showing that the integral on γ_1 is analytic [79,81]. Similarly, using analogous methods, we can show that the integral expressions on γ_2 and γ_3 are analytic, finally proving that $g(s)$ is analytic [76,77].

Now let's examine the relationship between $g(s)$ and the Riemann function [78,80].

$$g(s) = \frac{q}{2\pi i}\oint_{\gamma} z^{s-1}f(z)\,dz$$
$$= \frac{q}{2\pi i}\left[e^{-i\frac{\pi s}{q}}\int_{\varepsilon}^{\infty} t^{s-1}f(-t)\,dt \underbrace{-e^{i\frac{\pi s}{q}}\int_{\varepsilon}^{\infty} t^{s-1}f(-t)\,dt}_{=\int_{\gamma_3} z^{s-1}f(z)\,dz} + \underbrace{\int_{\gamma_2} z^{s-1}f(z)\,dz}_{=:M}\right].$$

Clearly, when $r = \epsilon \to 0$, γ_2 becomes a closed normal ellipse [79,81]. According to the Cauchy integral theorem in the elliptic complex domain as given in Proposition 4.8 , the integral over γ_2 tends to 0 [76,77]. Combined with the definition of the sine function in the elliptic complex domain, $g(s)$ further simplifies to

$$\lim_{\varepsilon\to 0^+} g(s) = \frac{q\cdot\left(e^{i\frac{\pi s}{q}} - e^{-i\frac{\pi s}{q}}\right)}{2\pi i}\int_{0}^{\infty} t^{s-1}f(-t)\,\mathrm{d}t = \underbrace{\frac{\sin(\pi s)}{\pi}}_{=:A}\int_{0}^{\infty}\frac{t^{s-1}}{e^t-1}\,\mathrm{d}t.$$

According to Reflection Formula 55 , it follows that $A = \dfrac{1}{\Gamma(s)\Gamma(1-s)}$ [78,80]. Then combined with the previously derived integral expression 56 for the Riemann function, it can be shown that

$$g(s) = \frac{\Gamma(s)\zeta(s)}{\Gamma(s)\Gamma(1-s)} = \frac{\zeta(s)}{\Gamma(1-s)}.$$

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Therefore , we redefine the Riemann function as $\zeta(s) \equiv \Gamma(1-s)g(s)$, i.e., [79,81]

$$\zeta(s) = \frac{q \cdot \Gamma(1-s)}{2\pi i} \oint_\gamma \frac{z^{s-1}}{e^{-z}-1} \mathrm{d}z, \tag{57}$$

where , since $\Gamma(1-s)$ is discontinuous at $s = 1$ while $g(s)$ is analytic for all $s \in \mathbb{C}_\lambda$, the analytically continued Riemann function as established in Equation 57 is everywhere analytic in $\mathbb{C}_\lambda \backslash \{1\}$ [76,77].

8.3. Functional Equation of the Riemann Zeta Function

Now we replace the contour of the analytically continued Riemann zeta function with the one shown in Figure 6 , $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where γ_2 and γ_4 are both normal ellipses in the complex plane $\mathbb{C}_\lambda$, with γ_2 having principal radius $\epsilon > 0$ and γ_4 having principal radius $r = (2k+1)\pi$ [78,80].

Consider the integral

$$g_k(s) = \frac{q}{2\pi i} \oint_{\gamma_k} \frac{z^{s-1}}{e^{-z}-1} \mathrm{d}z.$$

Owing to the fact that the contour is traversed clockwise [1], we need to add a negative sign outside the summation [79,81] when applying the residue theorem . It is easy to obtain that

$$\lim_{k\to\infty} g_k(s) = g(s), \tag{58}$$

which means that it is required to be proven that the integral over γ_4 tends to zero as the principal radius tends to positive infinity [76,77]. Now we can expand this contour integral using the residue theorem [78,80].

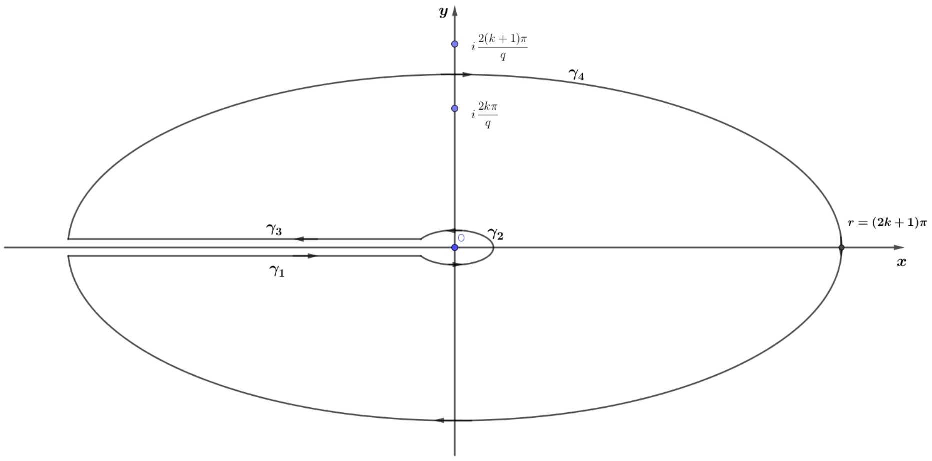

Figure 6. The contour γ employed for the function $g(s)$ in the complex plane $\mathbb{C}_\lambda$.

Clearly, the function

$$g(z) = \frac{z^{s-1}(z-s)}{e^{-z}-1}$$

has poles inside the contour at $z = i\frac{2n\pi}{q}$, $n \in \mathbb{Z}$ [79,81]. It is easy to see that these poles are all simple poles. Therefore, the final integral can be expanded as

$$\begin{aligned} g_k(s) &= (-1) \sum_{-k\le n\le k, n\ne 0} \mathrm{Res}\left(i\frac{2n\pi}{q}\right) \\ &= (-1)\sum_{n=1}^{k}\left[\mathrm{Res}\left(g(s), i\frac{2n\pi}{q}\right) + \mathrm{Res}\left(g(s), -i\frac{2n\pi}{q}\right)\right], \end{aligned} \tag{59}$$

[1] Here we consider the case where $q > 0$, i.e., the positive direction is counterclockwise.

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where [76,77]

$$\operatorname{Res}\left(g(s), i\frac{2n\pi}{q}\right) = \lim_{z\to i\frac{2n\pi}{q}} \frac{\left(z - i\frac{2n\pi}{q}\right)z^{s-1}}{e^{-z}-1} = \lim_{z\to i\frac{2n\pi}{q}} \frac{z^s - i\frac{2n\pi}{q}z^{s-1}}{e^{-z}-1}$$
$$= \lim_{z\to i\frac{2n\pi}{q}} \frac{sz^{s-1} - (s-1)i\frac{2n\pi}{q}z^{s-2}}{-e^{-z}}$$
$$= \frac{s\left(i\frac{2n\pi}{q}\right)^{s-1} - (s-1)\left(i\frac{2n\pi}{q}\right)^{s-1}}{-1}$$
$$= -\left(i\frac{2n\pi}{q}\right)^{s-1} = -2^{s-1}\pi^{s-1}e^{\frac{i\pi(s-1)}{2q}}n^{s-1},$$

i.e.,

$$\operatorname{Res}\left(i\frac{2n\pi}{q}\right) = -2^{s-1}\pi^{s-1}e^{\frac{i\pi(s-1)}{2q}}n^{s-1}. \tag{60}$$

Likewise, we obtain [78,80]

$$\operatorname{Res}\left(-i\frac{2n\pi}{q}\right) = -2^{s-1}\pi^{s-1}e^{-\frac{i\pi(s-1)}{2q}}n^{s-1}. \tag{61}$$

Finally, combining with the definition of the cosine function in the elliptic complex domain, substituting formulas 60 and 61 into 59 yields [79,81]

$$g_k(s) = \sum_{n=1}^{k}\left[2^{s-1}\pi^{s-1}e^{\frac{i\pi(s-1)}{2q}}n^{s-1} + -2^{s-1}\pi^{s-1}e^{-\frac{i\pi(s-1)}{2q}}n^{s-1}\right]$$
$$= 2^s\pi^{s-1}\left[\frac{e^{\frac{i\pi(s-1)}{2q}} + e^{-\frac{i\pi(s-1)}{2q}}}{2}\right]\sum_{n=1}^{k}n^{s-1}$$
$$= 2^s\pi^{s-1}\cos\left(\frac{\pi s}{2} - \frac{\pi}{2}\right)\sum_{n=1}^{k}n^{s-1}$$
$$= 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{k}\frac{1}{n^{1-s}}.$$

Hence according to Formula 58 ,

$$g(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1-s}} = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).$$

Due to $g(s) = \dfrac{\zeta(s)}{\Gamma(1-s)}$, it follows that [79,81]:

$$\frac{\zeta(s)}{\Gamma(1-s)} = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s),$$

that is,

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{62}$$

Note that $\sin\left(\frac{\pi s}{2}\right) = 0$ when s is an even integer. When $s = 2n$ ($n \in \mathbb{Z}^+$), from formula 54 , $\Gamma(1-s)$ has a simple pole, so there are no zeros [76,77]; when $s = 0$, $\zeta(1-s)$ has a pole (non-analytic), so there are also no zeros. Thus, we can find many zeros of the Riemann function [78,80]

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = \zeta(-2k) = 0 \quad (k \in \mathbb{Z}^+).$$

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Being obvious and real numbers , these zeros are called the trivial zeros of the zeta function [79,81]. Zeros other than these are complex numbers and are called non-trivial zeros [76,77].

It can be seen that although the form of the Riemann zeta function in the elliptic complex domain after analytic continuation differs from that in the circular complex domain, its functional equation as seen in Equation 62 is the same as in the circular complex domain [78,80]. This will make our subsequent research very convenient [79,81].

8.4. Symmetric Form of the Functional Equation

Based on the analytic form and functional equation derived above, it is next necessary to derive the symmetric form of the Riemann zeta function's functional equation and further analyze the distribution of its zeros [76,77].

Proposition 8.5. *In the entire complex plane $\mathbb{C}_\lambda$,*

$$\Gamma(2s) = 2^{2s-1}\pi^{-\frac{1}{2}}\Gamma(s)\cdot\Gamma\left(s+\frac{1}{2}\right). \tag{63}$$

Proof. Consider the integral $I = \int_{-1}^{1}(1-x^2)^{s-1}\mathrm{d}x$ [78,80]. Firstly , make the substitution $t = x^2$. It follows that

$$\begin{aligned} I &= 2\int_0^1 (1-t)^{s-1}\frac{1}{2\sqrt{t}}\,dt = \int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}\,dt \\ &= B\left(\frac{1}{2},s\right) = \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}; \end{aligned}$$

Now make another substitution $1+x = 2t$, i.e., $1-x = 2(1-t)$ [79,81] for I. Then

$$\begin{aligned} I &= \int_0^1 [2(1-t)]^{s-1}(2t)^{s-1}(2dt) = 2^{2s-1}\int_0^1 t^{s-1}(1-t)^{s-1}dt \\ &= 2^{2s-1}B(s,s) = 2^{2s-1}\cdot\frac{\Gamma(s)\cdot\Gamma(s)}{\Gamma(2s)}. \end{aligned}$$

Owing to the fact that both expressions equal I, Formula 63 follows by $\Gamma(1/2) = \sqrt{\pi}$ [76,77]. □

Next, substituting Reflection Formula, i.e., Formula 55 , into the functional equation 62 [78,80],

$$\zeta(s) = 2^s\pi^{s-1}\frac{\pi}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}\Gamma(1-s)\zeta(1-s). \tag{64}$$

From formula 63 [79,81],

$$\Gamma(s) = 2^{s-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\cdot\Gamma\left(\frac{1+s}{2}\right). \tag{65}$$

Making the transformation $s \to 1-s$ in 65 gives [76,77]

$$\Gamma(1-s) = 2^{-s}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\cdot\Gamma\left(1-\frac{s}{2}\right). \tag{66}$$

Now substituting 66 into 64 [78,80], it can be seen that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{67}$$

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Equation 67 is just the symmetric form of the Riemann zeta function's functional equation, also called the functional equation of the zeta function [79,81]. Through this transformation, we can study the distribution of zeros of the zeta function more intuitively [76,77].

8.5. Distribution of Zeros of the Riemann Zeta Function

The reason mathematicians are passionate about studying the Riemann zeta function originates from Euler's product formula, which reveals the direct connection between the zeta function and the distribution of all prime numbers [78,80].

8.5.1. Euler Product Formula

Proposition 8.6 (Euler Product Formula)**.** *For* $s = \sigma + i\frac{t}{q} \in \mathbb{C}_\lambda$*,*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{68}$$

where $\Re(s) = \sigma > 1$*. This formula is also called the Euler product formula [79,81].*

Proof. It is easy to see that when $\text{Re}(s) = \sigma > 1$, the series

$$\sum_{n=0}^{\infty} (p^{-s})^n = 1 + \frac{1}{p^s} + \frac{1}{(p^s)^2} + \cdots + \frac{1}{(p^s)^n} + \ldots$$

converges [76,77]. Note that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Thus, for $X \geq 1$, by the fundamental theorem of arithmetic [78,80],

$$\prod_{p \leq X, p \in \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \leq X, p \in \text{prime}} \left(\sum_{n=0}^{\infty} (p^{-s})^n\right),$$

that is ,

$$\prod_{p \leq X, p \in \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \leq X, p \in \text{prime}} \left(1 + \frac{1}{p^s} + \frac{1}{(p^s)^2} + \cdots + \frac{1}{(p^s)^n} + \ldots\right) = \sum_{n \leq X} \frac{1}{n^s} + R(s; X),$$

where $|R(s;X)| \leq \sum_{n>X} \left|\frac{1}{n^s}\right| = \sum_{n>X} \left|\frac{1}{n^\sigma}\right| \leq \frac{X}{\sigma X^\sigma - X^\sigma} = \frac{1}{\sigma - 1} X^{1-\sigma}$ [79,81]. Thus, as $X \to +\infty$, $R(s;X) \to 0$, proving formula 68 [76,77]. □

Furthermore, formula 68 can be rewritten as

$$\prod_{p \in \text{prime}} \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, \quad \text{Re}(s) > 1.$$

According to the convergence of infinite products (an infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely) [78,80], since when $\text{Re}(s) > 1$, the series $\sum_p \frac{1}{p^s}$ converges absolutely, $\prod_{p \in \text{prime}} \left(1 - \frac{1}{p^s}\right)$ converges, so $\frac{1}{\zeta(s)} < \infty$, i.e.,

Corollary 8.7. *The function* $\zeta(s)$ *has no zeros when* $\text{Re}(s) > 1$ *[79,81].*

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8.5.2. Distribution of Zeros

From the symmetric form of the Riemann zeta function's functional equation, i.e., Equation 67 , it can be defined that [76,77]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{69}$$

which means that

$$\xi(1-s) = \xi(s). \tag{70}$$

Here note that $\Gamma(\frac{s}{2})$ has poles but no zeros , and $s(s-1) = 0 \Leftrightarrow s = 0,1$ [79,81]. However, $\xi(1) = \xi(0) = -\zeta(0) = \frac{1}{2}$. Therefore, from Equation 69 , the zeros of $\zeta(s)$ are precisely the zeros of $\xi(s)$ [76,77].

Furthermore, the trivial zeros of $\zeta(s)$ at $s = -2n$ $(n \in \mathbb{Z}^+)$ are exactly the poles of $\Gamma(\frac{s}{2}+1)$. Thus, $s = -2n$ $(n \in \mathbb{Z}^+)$ are not zeros of $\xi(s)$ either [78,80]. In summary,

Corollary 8.8 (Non-Trivial Zeros)**.** *All non-trivial zeros of the function $\zeta(s)$ are precisely all the zeros of the function $\xi(s)$ [79,81].*

Now let's examine whether $\zeta(s)$ has zeros when $\Re(s) = 1$ [76,77].

Proposition 8.9 (No Zeros on $\mathrm{Re}(s) = 1$)**.** *For all $t \in \mathbb{R}$, $\zeta(1+i\frac{t}{q}) \neq 0$ [78,80].*

Proof. When $\Re(s) = \sigma > 1$, taking logarithms on both sides of the Euler product formula 68 gives [79,81]

$$\log\zeta(s) = -\sum_{p\in\text{prime}} \log\left(1-\frac{1}{p^s}\right), \quad s = \sigma + i\frac{t}{q}.$$

Using the expansion of the logarithm $-\log(1-z) = \sum_{k=1}^{\infty}\frac{z^k}{k}$ [76,77] ,

$$\log\zeta(s) = \sum_{p\in\text{prime}}\sum_{m=1}^{\infty}\frac{p^{-sm}}{m} = \sum_{p\in\text{prime}}\sum_{m=1}^{\infty}\frac{p^{-sm}}{m}e^{-imt\log p}.$$

Therefore, the real part of $\log\zeta(s)$ is [78,80]

$$\Re[\log\zeta(s)] = \sum_{p\in\text{prime}}\sum_{m=1}^{\infty}\frac{p^{-sm}}{m}\cos(mt\log p).$$

Thus, [79,81]

$$3\Re[\log\zeta(\sigma)] + 4\Re\left[\log\zeta\left(\sigma+i\frac{t}{q}\right)\right] + \Re\left[\log\zeta\left(\sigma+2i\frac{t}{q}\right)\right]$$
$$= \sum_{p\in\text{prime}}\sum_{m=1}^{\infty}\frac{p^{-sm}}{m}[3+4\cos(mt\log p)+\cos(2mt\log p)].$$

Since [76,77]

$$2(1+\cos\theta)^2 = 3+4\cos\theta+\cos(2\theta) \geq 0, \quad \theta\in\mathbb{R},$$

it follows that [78,80]

$$|\zeta(\sigma)|^3\left|\zeta\left(\sigma+i\frac{t}{q}\right)\right|^4\left|\zeta\left(\sigma+2i\frac{t}{q}\right)\right| \geq 1. \tag{71}$$

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Now assume there exists $m = \frac{t}{q} \in \mathbb{R}$ such that $\zeta\left(1 + i\frac{t}{q}\right) = 0$ [79,81]. Then by the mean value theorem ,

$$\begin{aligned}\left|\zeta\left(\sigma + \frac{i}{q}t\right)\right| &= \left|\zeta\left(\sigma + \frac{i}{q}t\right) - \zeta\left(1 + \frac{i}{q}t\right)\right| \\ &= |\sigma - 1|\left|\zeta'\left(\sigma_0 + \frac{i}{q}t\right)\right|, \quad 1 < \sigma_0 < \sigma \\ &\leq A_1(\sigma - 1),\end{aligned}$$

where A_1 is a constant depending only on t [76,77]. Similarly, $\left|\zeta\left(\sigma + 2i\frac{t}{q}\right)\right| \leq A_2$, where A_2 is a constant depending only on t [78,80]. On the other hand, we know that $\zeta(s)$ has a simple pole at $\sigma = 1$, and its Laurent expansion around $\sigma = 1$ is [79,81]

$$\zeta(s) = \frac{1}{\sigma - 1} + c_0 + c_1(\sigma - 1) + \cdots = \frac{1}{\sigma - 1} + g(\sigma),$$

where $g(\sigma)$ is analytic at $\sigma = 1$ [76,77]. Hence, in some neighborhood of $\sigma = 1$, by the maximum modulus principle in elliptic complex function theory, $|g(s)| \leq A_0$ [78,80]. Thus, $|\zeta(\sigma)| \leq \frac{1}{\sigma - 1} + A_0$.

Therefore, [79,81]

$$\begin{aligned}&\lim_{\sigma \to 1^+} |\zeta(\sigma)|^3 \left|\zeta\left(\sigma + i\frac{t}{q}\right)\right|^4 \left|\zeta\left(\sigma + 2i\frac{t}{q}\right)\right| \\ \leq &\lim_{\sigma \to 1^+} \left(\frac{1}{\sigma - 1} + A_0\right)^3 (A_1(\sigma - 1))^4 A_2 = 0,\end{aligned}$$

which contradicts formula 71 [76,77]. Hence, for all $t \in \mathbb{R}$, $\zeta\left(1 + i\frac{t}{q}\right) \neq 0$ [78,80]. □

From Proposition 8.9 , combined with formula 70 [79,81] , the following conclusion holds.

Corollary 8.10. *All non-trivial zeros of the function $\zeta(s)$, i.e., all zeros of the function $\xi(s)$, lie in the critical strip $0 < \Re(s) < 1$, and the zeros are symmetrically distributed about the point $s = \frac{1}{2}$ [76,77].*

Moreover , when $\Re(s) > 1$, it can be found that the series [78,80]

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^s} = (1 - 2^{1-s})\zeta(s)$$

converges absolutely [79,81]. Therefore,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} = \frac{1}{1 - 2^{1-s}} \eta(s). \tag{72}$$

On the other hand, by the Dirichlet test, $\eta(s)$ converges when $\Re(s) = \sigma > 0$ [76,77]. Thus, an expression for $\zeta(s)$ on $0 < \Re(s) < 1$, i.e., Formula 72 could be obtained[78,80].

9. Proof of the Riemann Hypothesis

In his 1859 paper "On the Number of Primes Less Than a Given Magnitude" , Riemann proposed the Riemann Hypothesis in $\mathbb{C}$: all non-trivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$ [76,77]. Naturally, it could be conjectured that this proposition also holds in the elliptic complex domain $\mathbb{C}_\lambda$ [78,80].

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According to the reflection principle for elliptic complex functions that $f(z^*) = (f(z))^*$, it can be shown that $\zeta(s^*) = (\zeta(s))^*$ [79,81]. Combined with the definition of the gamma function,

$$\xi(s^*) = (\xi(s))^*. \tag{73}$$

Thus, the following conclusion would be given [76,77].

Proposition 9.1 (Imaginary Part on Critical Line). *For all $t \in \mathbb{R}$, $\Im\left(\xi\left(\frac{1}{2} + i\frac{t}{q}\right)\right) = 0$ [78,80].*

Proof. Combining formulas 73 and 70 [79,81],

$$\begin{aligned}
\Im\left[\xi\left(\frac{1}{2} + it\right)\right] &= q\frac{\xi\left(\frac{1}{2} + it\right) - \left(\xi\left(\frac{1}{2} + it\right)\right)^*}{2i} \\
&= q\frac{\xi\left(\frac{1}{2} + it\right) - \xi\left[\left(\frac{1}{2} + it\right)^*\right]}{2i} \\
&= q\frac{\xi\left(\frac{1}{2} + it\right) - \xi\left(\frac{1}{2} - it\right)}{2i} \\
&= q\frac{\xi\left(\frac{1}{2} + it\right) - \xi\left(\frac{1}{2} + it\right)}{2i} = 0.
\end{aligned}$$

Therefore the proposition is proved [76,77]. □

It is precisely because of Proposition 9.1 that we can determine the values of t by the sign changes of $\xi\left(\frac{1}{2} + i\frac{t}{q}\right)$, i.e., determine the positions of non-trivial zeros on $\Re(s) = \frac{1}{2}$ [78,80]. Furthermore, if we define $\Xi(t) = \xi\left(\frac{1}{2} + i\frac{t}{q}\right)$ [79,81] ,

$$\Xi(-t) = \xi\left(\frac{1}{2} - i\frac{t}{q}\right) = \xi\left(1 - \left(\frac{1}{2} - i\frac{t}{q}\right)\right) = \xi\left(\frac{1}{2} + i\frac{t}{q}\right) = \Xi(t).$$

Thus, $\Xi(t) = \xi\left(\frac{1}{2} + i\frac{t}{q}\right)$ is an even function of t [76,77]. Therefore, the zeros of the Riemann zeta function on the critical line $\Re(s) = \frac{1}{2}$ are symmetric about the X-axis. Generally, we only need to study the upper half-plane $s = \frac{1}{2} + i\frac{t}{q}$, with $t > 0$ [78,80].

Having laid the groundwork, we proceed in this chapter to demonstrate the Riemann Hypothesis by exploiting the inherent structure of the elliptic complex domain.

9.1. Mellin Transform on the Elliptic Complex Plane

The proof in this chapter requires the use of the Fourier transform and its "variant," the Mellin transform [76,77].

From the Fourier series, we can easily obtain the Fourier integral formula in the elliptic complex domain $\mathbb{C}_\lambda$ [78,80]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)e^{-i\frac{\omega s}{q}}\,\mathrm{d}s\right]e^{i\frac{\omega t}{q}}\,\mathrm{d}\omega.$$

Thus, the Fourier transform in the elliptic complex domain is obtained [79,81] .

Definition 9.2 (Fourier Transform). *If the function f satisfies the conditions of the Fourier integral theorem on $(-\infty, +\infty)$, then the function*

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\frac{\omega t}{q}}\,\mathrm{d}t \tag{74}$$

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is called the Fourier transform of $f(t)$, denoted by $\mathscr{F}(f(t))$; and the function

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\frac{\omega t}{q}} \mathrm{d}\omega \tag{75}$$

is called the inverse Fourier transform of $F(\omega)$, denoted by $\mathscr{F}^{-1}(F(\omega))$, i.e., $f(t) = \mathscr{F}^{-1}(F(\omega))$ [76,77].

Supposing $s = c - \frac{i}{q}\omega$, $\omega = \frac{i}{q}(s-c)$, $\mathrm{d}\omega = \frac{i}{q}\mathrm{d}s$ [78,80]. Consequently ,

$$F[i(s-c)] = \int_{-\infty}^{\infty} e^{(s-c)t} f(t)\mathrm{d}t = \int_{-\infty}^{\infty} e^{st} \cdot [e^{-ct} f(t)]\mathrm{d}t,$$

$$f(t) = \frac{i}{2\pi q} \int_{c-i\infty}^{c+i\infty} e^{(c-s)t} F[i(s-c)]\mathrm{d}s = \frac{q \cdot e^{ct}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-st} F[i(s-c)]\mathrm{d}s.$$

Next, set $t = \ln x$, $\mathrm{d}t = x^{-1}\mathrm{d}x$ [79,81]. It follows that

$$F[i(s-c)] = \int_0^{\infty} x^{s-1}[x^{-c} f(\ln x)]\mathrm{d}x,$$

$$f(\ln x) = \frac{q \cdot x^c}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F[i(s-c)]\mathrm{d}s.$$

Now let $g(x) = x^{-c} f(\ln x)$ and $G(s) = F[i(s-c)]$. We obtain the formulas for the Mellin transform and its inverse [76,77].

Definition 9.3 (Mellin Transform). *Let $s \in \mathbb{C}_\lambda$. The function*

$$G(s) = \int_0^{\infty} x^{s-1} g(x)\mathrm{d}x$$

is called the Mellin transform of the function $g(x)$, denoted by $\{\mathcal{M}g\}(s)$. Correspondingly, the function

$$g(x) = \frac{q}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} G(s)\mathrm{d}s$$

is called the inverse Mellin transform of the function $G(s)$, denoted by $\{\mathcal{M}^{-1}G\}(x)$, i.e., $g(x) = \{\mathcal{M}^{-1}G\}(x)$ [78,80]. Note that the real number c appears in the inverse transform formula; appropriate choice of c can avoid poles in the integration path [79,81].

9.2. An Equivalent Proposition of the Riemann Hypothesis

Define the function $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ [76,77]. It is easy to obtain that

$$2\psi(x) - x^{-\frac{1}{2}} = \frac{q}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-s} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)\mathrm{d}s.$$

Differentiating both sides with respect to x on it[78,80] ,

$$2\psi'(x) + \frac{1}{2}x^{-\frac{3}{2}} = -\frac{q}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-s-1} \frac{1}{2} s \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)\mathrm{d}s.$$

Multiplying both sides by $x^{3/2}$ and setting $x = e^{2u}$ [79,81] ,

$$2\psi'(e^{2u})e^{3u} + \frac{1}{2} = -\frac{q}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{-u(s-1)} \frac{1}{2} s \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)\mathrm{d}s.$$

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Now differentiating both sides again with regard to u [76,77] ,

$$2\frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u})e^{3u}] = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{-u(s-1)}\xi(s)\mathrm{d}s. \tag{76}$$

And let $s = \frac{1}{2} + \frac{i}{q}t$. Then Equation 76 can be transformed into [78,80]

$$\underbrace{2\,\frac{\mathrm{d}}{\mathrm{d}u}\left[\psi'(e^{2u})e^{3u}\right]\,e^{-u/2}}_{\Psi(u)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{i}{q}ut}\xi\left(\frac{1}{2}+\frac{i}{q}t\right)\mathrm{d}t$$

According to the previous content, we know that $\xi\left(\frac{1}{2}+\frac{i}{q}t\right)$ must be a real-valued even function [79,81]. As a result , in the above formula,

$$\Psi(u) = \frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u})e^{3u}]e^{-u/2} \tag{77}$$

must also be an even function [76,77].

Now let $\Phi(u) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{2u}}$. Expanding formula 77 yields [78,80]

$$\begin{aligned}\Psi(u) &= \frac{\mathrm{d}}{\mathrm{d}u}\left[\psi'(e^{2u})e^{3u}\right]e^{-u/2} = (-2n^2\pi)\sum_{n\geq 1}\frac{\mathrm{d}}{\mathrm{d}u}\left[e^{-n^2\pi e^{2u}+3u}\right]e^{-u/2} \\ &= (-2n^2\pi)\sum_{n\geq 1}(3-2n^2\pi e^{2u})e^{-n^2\pi e^{2u}+3u}e^{-u/2} \\ &= 2\sum_{n\geq 1}(2\pi^2 n^4 e^{9u/2} - 3n^2\pi e^{5u/2})e^{-n^2\pi e^{2u}},\end{aligned}$$

that is , $\Psi(u) = 2\Phi\left(\frac{u}{2}\right)$, which shows that $\Phi(u)$ is also an even function [79,81]. Therefore, combined with the inverse Fourier transform formula 74 in the elliptic complex domain [76,77] ,

$$\xi\left(\frac{1}{2}+\frac{i}{q}t\right) = 4\int_{-\infty}^{\infty}\Phi\left(\frac{u}{2}\right)e^{\frac{i}{q}ut}\mathrm{d}u = 8\int_{0}^{\infty}\Phi\left(\frac{u}{2}\right)\cos(ut)\mathrm{d}u = 16\int_{0}^{\infty}\Phi(x)\cos(2xt)\mathrm{d}x.$$

Now define the function $H(\varpi, z) = \int_{0}^{\infty} e^{\varpi u^2}\Phi(u)\cos(zu)\mathrm{d}u$ [78,80]. Then it follows that

$$H(0,z) = \frac{1}{16}\xi\left(\frac{1}{2}+\frac{iz}{2q}\right). \tag{78}$$

Combining this with the properties of $\xi(s)$, we obtain an equivalent proposition of the Riemann Hypothesis [79,81]:

Proposition 9.4 (Equivalent Form of the Riemann Hypothesis)**.** *The Riemann Hypothesis in the elliptic complex domain $\mathbb{C}_\lambda$ holds if and only if all zeros of $H(0,z)$ are real [76,77].*

9.3. Correspondence of Zeros Between Complex Planes

Proposition 9.4 above suggests that we can discuss the real and imaginary parts of the zeros s of $\xi(s)$ separately [78,80].

Consider the region of the critical strip $\Re(s) \in (0,1)$. Let $\Delta \in (0,1)$ be a given constant [79,81]. Then formula 76 can be directly transformed into

$$2\frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u})e^{3u}] = \frac{q}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty} e^{-u(s-1)}\xi(s)\mathrm{d}s.$$

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Substituting $s = \Delta + \frac{i}{q}t$ into the above formula [76,77] ,

$$\underbrace{2\frac{\mathrm{d}}{\mathrm{d}u}\left[\psi'(e^{2u})e^{3u}\right]}_{\Psi^{\Delta}(u)} e^{u(\Delta-1)} = \frac{1}{2\pi}\int_{\Delta-i\infty}^{\Delta+i\infty} e^{-\frac{i}{q}ut}\xi\left(\Delta+\frac{i}{q}t\right)\mathrm{d}t. \tag{79}$$

where , clearly, $\Psi^{\Delta}(u)$ in this formula is no longer necessarily an even function [78,80]. From equation 79 combined with the inverse Fourier transform formula as seen in 75 ,

$$\xi\left(\Delta+\frac{i}{q}t\right) = 2\int_{-\infty}^{\infty}\Psi^{\Delta}(u)e^{i\frac{ut}{q}}\mathrm{d}u. \tag{80}$$

Define the functions [79,81]

$$R^{\Delta}(\varpi,z) = \int_{-\infty}^{\infty} e^{\varpi u^2}\Psi^{\Delta}(u)\cos(zu)\mathrm{d}u$$

and

$$I^{\Delta}(\varpi,z) = \int_{-\infty}^{\infty} e^{\varpi u^2}\Psi^{\Delta}(u)\sin(zu)\mathrm{d}u.$$

According to equation 80 [76,77] ,

$$\begin{aligned} R^{\Delta}(0,z) &= \frac{1}{2}\xi\left(\Delta+\frac{iz}{q}\right)+\frac{1}{2}\xi\left(\Delta-\frac{iz}{q}\right) = \Re\left[\xi\left(\Delta+\frac{iz}{q}\right)\right], \\ I^{\Delta}(0,z) &= \frac{1}{2}\xi\left(\Delta+\frac{iz}{q}\right)-\frac{1}{2}\xi\left(\Delta-\frac{iz}{q}\right) = q\Im\left[\xi\left(\Delta+\frac{iz}{q}\right)\right]. \end{aligned}$$

Thus, [78,80]

$$\xi\left(\Delta+\frac{i}{q}z\right) = R^{\Delta}(0,z)+\frac{i}{q}\cdot I^{\Delta}(0,z). \tag{81}$$

When z is a fixed real number, the functions $R^{\Delta}(0,z)$ and $I^{\Delta}(0,z)$ have the same values on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$, because at this point $R^{\Delta}(0,z)$ and $I^{\Delta}(0,z)$ are both real functions [79,81].

On the other hand, assuming z is real, $\xi\left(\Delta+\frac{i}{q}z\right)$ already accounts for all cases in the region $\Re(s)\in(0,1)$ of the complex plane [76,77]. Therefore, we have the following conclusion.

Proposition 9.5 (Correspondence of Zeros). *(1) The zeros of the function $\xi(s)$ on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$ are in one-to-one correspondence [78,80];*

(2) Let $\lambda_1 = -q_1^2$ and $\lambda_2 = -q_2^2$. If $s = \Delta+\frac{i}{q_1}z$ (with z real) is a zero of the function $\xi(s)$ on the complex plane $\mathbb{C}_{\lambda_1}$, then $s' = \Delta+\frac{i}{q_2}z$ must be a zero of the function $\xi(s)$ on the complex plane $\mathbb{C}_{\lambda_2}$ [79,81].

Based on Proposition 9.5 , if we set $\Delta = \frac{1}{2}+R\cos\theta$ and $z = R\sin\theta$, where $R\geq 0$, then $s = \frac{1}{2}+Re^{i\frac{\theta}{q_1}}$ and $s' = \frac{1}{2}+Re^{i\frac{\theta}{q_2}}$ [76,77]. According to the definition of a normal ellipse, we have the following conclusion.

Corollary 9.6 (Elliptic Correspondence). *If a zero of the function $\xi(s)$ on the complex plane $\mathbb{C}_{\lambda_1}$ lies on a normal ellipse centered at $s = \frac{1}{2}$ with principal radius $R\geq 0$, then its corresponding zero on the complex plane $\mathbb{C}_{\lambda_2}$ also lies on a normal ellipse centered at $s = \frac{1}{2}$ with principal radius R [78,80].*

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9.4. Final Proof

In what follows, recourse is had to proof by contradiction, a method readily accessible to the general reader [79,81]. Assume the Riemann Hypothesis is false, i.e., on the circular complex plane $\mathbb{C}$, the zeta function has a non-trivial zero not on the critical line $\Re(s) = \frac{1}{2}$ [76,77].

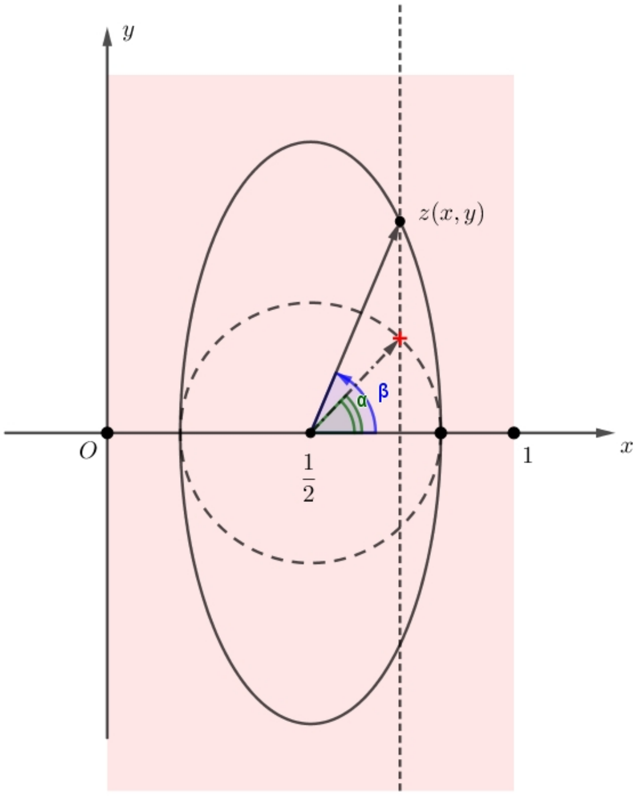

Figure 7. The correspondence of zeros of $\zeta(s)$ between elliptic complex planes.

According to Lemma 8.10 , assume this zero is $s = \frac{1}{2} + Re^{i\alpha}$. Then, by Proposition 9.5 and Corollary 9.6 , the zeta function has a corresponding zero $s' = \frac{1}{2} + Re^{i\alpha/q}$ on the complex plane $\mathbb{C}_\lambda$, where $\lambda = -q^2$ [78,80].

Let the angle corresponding to "α" on the complex plane $\mathbb{C}_\lambda$ in the ordinary geometric sense be β [79,81]. Returning to Proposition 2.12 , it is easy to see that $\tan\alpha = \dfrac{\tan\beta}{q}$ as shown in Figure 7 . Therefore, as $\lambda \to 0^-$,

$$\tan\alpha = \lim_{q\to 0}\frac{\tan\beta}{q} = \infty. \tag{82}$$

It should be noted that as $\lambda \to 0^-$, the corresponding zeros tend to infinity away from the X -axis by Proposition 9.5 , i.e., $\beta \to \pi/2$, which implies $\tan\beta \to \infty$.

Equation 82 means that the angle α corresponding to the zero s on the complex plane $\mathbb{C}$ should be $\pm\frac{\pi}{2}$, that is, this zero should lie on the critical line $\Re(s) = \frac{1}{2}$ [78,80]. This contradicts the initial assumption.

Thus, the Riemann Hypothesis is correct [79,81]. In fact, according to the above derivation, the Riemann Hypothesis holds on all complex planes $\mathbb{C}_\lambda$ [76,77].

To summarize, in conjunction with Theorem 1.1 , the following result is established.

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Theorem 9.7. *On the complex planes $\mathbb{C}_\lambda$, if we define $N(T)$ as the number of non-trivial zeros of $\zeta(s)$ with $0 < Im(s) < T$, and $N_0(T)$ as the number of non-trivial zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$ with $0 < Im(s) < T$, then*

$$N_0(T) = N(T) \sim \frac{qT}{2\pi}\log\frac{T}{2\pi} - \frac{qT}{2\pi}, \tag{83}$$

in which q is subject to the condition $\lambda = -q^2$.

The result regarding the order appearing in Formula 83 may also be established via alternative methods of greater rigor.

10. Proof of the Generalized Riemann Hypothesis

Dirichlet functions are generalizations of the Riemann zeta function, and the distribution of zeros of Dirichlet functions is a generalization of the Riemann Hypothesis, which we call the Generalized Riemann Hypothesis [76,77].

Around 1735, Euler discovered the formula [78,80]

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}; \tag{84}$$

In 1673, Leibniz obtained [79,81]

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots = \frac{\pi}{4}. \tag{85}$$

Subsequently, mathematicians successively discovered [76,77]

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots + \frac{1}{n^4} + \cdots = \frac{\pi^4}{90}, \tag{86}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots + \frac{(-1)^{n+1}}{(2n+1)^3} + \cdots = \frac{\pi^3}{32}, \tag{87}$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \cdots = \frac{\pi}{3\sqrt{3}} \tag{88}$$

and the Dirichlet formula [78,80]

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + \cdots = \frac{\log(1+\sqrt{2})}{2}. \tag{89}$$

In fact, according to the definition of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, equations 84 and 86 correspond to $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ respectively, while the other series belong to special Dirichlet series [79,81].

10.1. Dirichlet Functions and Dirichlet Characters

To consider other series, we introduce Dirichlet characters [76,77]. Noting that the residue classes modulo m

$$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} = \{\bar{1}, \bar{2}, \bar{3}, \ldots, \overline{m-1}\}$$

form a ring, and the set of multiplicative invertible elements [78,80]

$$\mathbb{Z}_m^* = (\mathbb{Z}/m\mathbb{Z})^* = \{\bar{a} \mid (a,m) = 1, 1 \le a \le m-1\}$$

forms a multiplicative group of order $\phi(m)$, the following definition could be established [79,81].

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Definition 10.1 (Dirichlet Character)**.** *Let* $\chi : \mathbb{Z}_m^* \to \mathbb{C}_\lambda^*$ *be a group homomorphism satisfying the multiplicative property*

$$\chi(\bar{a}\bar{b}) = \chi(\bar{a})\chi(\bar{b}), \quad \forall \bar{a}, \bar{b} \in \mathbb{Z}_m^*.$$

Then χ *is called a Dirichlet character modulo* m *[76,77]. For convenience,* χ *can also be denoted as* $\chi(n;m)$*. Furthermore, define*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \chi(n) = \chi(n;m) \tag{90}$$

as the Dirichlet function associated with the character χ *[78,80].*

Since $\mathbb{Z}_m^*$ does not contain elements not coprime to m , it always holds that $\chi(n) = \chi(n;m) = 0$ while $(n,m) > 1$ [79,81].

On the other hand, because $\chi(1) = \chi(1 \times 1) = \chi(1)\chi(1)$ and $\chi(1) \neq 0$, there invariably holds the identity $\chi(1) = 1$.

Let χ be a Dirichlet character modulo m. When $(n,m) = 1,$, $n^{\phi(m)} \equiv 1 \pmod{m}$ due to Euler's theorem [76,77]. Owing to periodicity and complete multiplicativity,

$$1 = \chi(n^{\phi(m)}) = [\chi(n)]^{\phi(m)}, \quad (n,m) = 1.$$

Thus $|\chi(n)| = 1$ when $(n,m) = 1$, which means that the values of the character χ are all $\phi(m)$-th roots of unity [78,80]. Consequently, $\chi(-1) = \pm 1$. If $\chi(-1) = 1$, then χ is called an even character; if $\chi(-1) = -1$, then χ is called an odd character [79,81].

Now re-examining the previous series, we find that the Dirichlet functions corresponding to formulas 85 and 87 are associated with the Dirichlet character modulo 4 [76,77] as

$$\chi : \mathbb{Z}_4^* = \{\bar{1}, \bar{3}\} \to \mathbb{C}_\lambda^*,$$

$$\chi(\bar{1},4) = 1, \quad \chi(\bar{3},4) = -1.$$

Therefore , it implies that $L(1,\chi) = \dfrac{\pi}{4}$ for $\chi = \chi(n;4)$ and $L(3,\chi) = \dfrac{\pi^3}{32}$ for $\chi = \chi(n;4)$ [78,80]. Here we can seen that $\chi(\bar{3},4) = \chi(-\bar{1},4) = -1$, so χ is an odd character [79,81].

The Dirichlet function corresponding to formula 88 is associated with the Dirichlet character modulo 3 [76,77] like

$$\chi : \mathbb{Z}_3^* = \{\bar{1}, \bar{2}\} \to \mathbb{C}_\lambda^*,$$

$$\chi(\bar{1},3) = 1, \quad \chi(\bar{2},3) = -1.$$

Consequently , we have $L(1,\chi) = \dfrac{\pi}{3\sqrt{3}}$ with $\chi = \chi(n;3)$ [78,80]. And χ is also an odd character in view of that $\chi(\bar{2},3) = \chi(-\bar{1},3) = -1$ [79,81].

The Dirichlet function corresponding to formula 89 is associated with the Dirichlet character modulo 8 [76,77] in accordance with

$$\chi : \mathbb{Z}_8^* = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\} \to \mathbb{C}_\lambda^*,$$

$$\chi(\bar{1},8) = 1, \quad \chi(\bar{3},8) = -1, \quad \chi(\bar{5},8) = -1, \quad \chi(\bar{7},8) = 1.$$

Accordingly, it leads to that $L(1,\chi) = \dfrac{\log(1+\sqrt{2})}{2}$ for $\chi = \chi(n;8)$ [78,80]. In light of that $\chi(\bar{7},8) = \chi(-\bar{1},8) = 1$, χ is an even character [79,81].

Summarizing the above analysis , the following conclusion is reached [76,77] .

Corollary 10.2 (Properties of Dirichlet Characters)**.** *Let* $t \geq 1$ *be a positive integer. The arithmetic function* $\chi(n) = \chi(n;t)$ *satisfies the following properties [78,80]:*

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(1) *When* $(n,t) > 1$, $\chi(n) = 0$;
(2) *Periodicity: For any integer* n, $\chi(n+t) = \chi(n)$;
(3) *Complete multiplicativity: For any integers* m,n, $\chi(mn) = \chi(m)\chi(n)$.

Given that $\chi(n)$ is a root of unity, the inverse of the character χ is given by $\chi^{-1}(n) = [\chi(n)]^{-1} = \overline{\chi(n)}$ [79,81]. As a result , the inverse character χ^{-1} can usually be written directly as $\overline{\chi}$ and is called the conjugate character of χ [76,77]. What is more ,

Definition 10.3 (Principal Character)**.** *If* $\chi(n) = 1$ *for all* n *with* $(n,t) = 1$, *then* χ *is called the principal character (or trivial character), denoted by* $\chi_0(n)$. *That is,*

$$\chi_0(n) = \chi_0(n;t) = \begin{cases} 1, & (n,t) = 1 \\ 0, & (n,t) > 1. \end{cases}$$

All other characters are called non-trivial characters. If the character values are real, it is called a real character; otherwise, it is called a complex character [78,80].

Obviously, when $t = 1$, there is only one character, namely the principal character: $\chi_0(n;1) = 1$ [79,81].

When $t = 2$, since $\mathbb{Z}_2^* = \{\overline{1}\}$, there is also only one character, namely the principal character $\chi_0(n;2) = \begin{cases} 1, & 2 \nmid n \\ 0, & 2 \mid n \end{cases}$ i.e., $\chi_0(\overline{1};2) = 1$ [76,77].

When $t = 3$, since $\mathbb{Z}_3^* = \{\overline{1},\overline{2}\} = \{\pm\overline{1}\}$, and $\chi(-1) = \pm 1$, we have the principal character $\chi_0(n;3) = \chi_0(\overline{1};3) = 1$ and the real character $\chi_1(n;3) = \chi_0(-\overline{1};3) = -1$ [78,80].

When $t \geq 4$, complex characters may appear [79,81]. For example, it could be shown that $\chi(\overline{2};5) = \frac{i}{q}$, $\chi(\overline{3};5) = -\frac{i}{q}$ [76,77]. Specifically, $\chi(n;5)$ has the following characters.

(1) Principal (trivial) character: $\chi_0(\overline{1}) = 1$, $\chi_0(\overline{2}) = 1$, $\chi_0(\overline{3}) = 1$, $\chi_0(\overline{4}) = 1$;
(2) Complex character (also odd character): $\chi_1(\overline{1}) = 1$, $\chi_1(\overline{2}) = \frac{i}{q}$, $\chi_1(\overline{3}) = -\frac{i}{q}$, $\chi_1(\overline{4}) = -1$;
(3) Real character (also even character): $\chi_2(\overline{1}) = 1$, $\chi_2(\overline{2}) = -\frac{1}{q}$, $\chi_2(\overline{3}) = -1$, $\chi_2(\overline{4}) = 1$;
(4) Complex character (also odd character): $\chi_3(\overline{1}) = 1$, $\chi_3(\overline{2}) = -\frac{i}{q}$, $\chi_3(\overline{3}) = \frac{i}{q}$, $\chi_3(\overline{4}) = -1$.

Several other properties of Dirichlet characters are also very important [78,80].

Corollary 10.4 (Additional Properties)**.** *(1)* *From* $\chi(0) = \chi(tm) = \chi(t)\chi(m) = \chi(0)\chi(m)$, *it is clear that* $\chi(0)(1-\chi(m)) = 0$ *, which shows that* $\chi(0) = 0$ *if* χ *is not identically 1[79,81] .*
(2) *If* $\gcd(k,t) = 1$, *then* $\chi(k;t) \neq 0$. *This stems from the fact that there exist integers* x,y *such that* $xk + yt = 1$ *if* $\gcd(k,t) = 1$ *, which means that* $1 = \chi(1) = \chi(xk+yt) = \chi(x)\chi(k)$. *By the divisibility property of elliptic complex numbers , it follows that* $\chi(k) \neq 0$ *[76,77].*

10.2. Analytic Continuation of Dirichlet Functions

10.2.1. Gauss Sums on the Elliptic Complex Plane

It is easy to see that the n-th roots of unity on the complex plane $\mathbb{C}_\lambda$ are [78,80]

$$w_k = e^{\frac{i2k\pi}{qn}} = \left(\cos\frac{2k\pi}{n} + \frac{i}{q}\sin\frac{2k\pi}{n}\right), \quad k = 0,1,\ldots,n-1,$$

and the conjugate of w_k is $w_k^* = w_{n-k}$ [79,81].

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Let $e(x) := \exp\left(\frac{2\pi i}{q}x\right)$ be a function on the complex plane $\mathbb{C}_\lambda$. From the properties of roots of unity on $\mathbb{C}_\lambda$ [76,77] ,

$$\sum_{k\in\mathbb{Z}_h} e\left(\frac{kn}{h}\right) = \begin{cases} h, & h \mid n \\ 0, & \text{otherwise} \end{cases}. \tag{91}$$

Furthermore, define the Gauss sum for a character $\chi(\text{mod } h)$ on the complex plane $\mathbb{C}_\lambda$ as [78,80]

$$G(n;\chi) = \sum_{k\in\mathbb{Z}_h} \chi(k) e\left(\frac{kn}{h}\right), \tag{92}$$

which means that multiplying χ by the Gauss sum yields [79,81]

$$\chi(n)G(n;\chi) = \sum_{k\in\mathbb{Z}_h} \chi(nk) e\left(\frac{kn}{h}\right) = \sum_{m\in\mathbb{Z}_h} \chi(m) e\left(\frac{m}{h}\right).$$

If $\gcd(n,h) = 1$, then $n\mathbb{Z}_h = \mathbb{Z}_h$, so [76,77]

$$G(n;\chi) = \overline{\chi}(n)G(1;\chi). \tag{93}$$

Therefore, combining formulas 91 and 93 , it holds that [78,80]

$$\begin{aligned} |G(1;\chi)|^2 = \overline{G(1;\chi)}G(1;\chi) &= \sum_{k\in\mathbb{Z}_h} \overline{\chi}(k) G(1;\chi) e\left(\frac{-k}{h}\right) \\ &= \sum_{k\in\mathbb{Z}_h} G(k;\chi) e\left(\frac{-k}{h}\right) = \sum_{k,m\in\mathbb{Z}_h} \chi(m) e\left(\frac{km}{h}\right) e\left(\frac{-k}{h}\right) \\ &= \sum_{m\in\mathbb{Z}_h} \chi(m) \sum_{k\in\mathbb{Z}_h} e\left(\frac{k(m-1)}{h}\right) = \chi(1) \sum_{k\in\mathbb{Z}_h} 1 = h, \end{aligned}$$

i.e.,

$$G(1;\chi) = \sqrt{h}. \tag{94}$$

Now we can use Gauss sums to perform analytic continuation of Dirichlet functions [79,81].

10.2.2. Two Lemmas

Let $\chi(\text{mod} h)$ be a character on the complex plane $\mathbb{C}_\lambda$. Corresponding to the parity of the character, define [76,77]

$$\epsilon(\chi) = \begin{cases} 0, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1 \end{cases}, \tag{95}$$

Let $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$, and define the completed Dirichlet function [78,80]

$$\hat{L}(s,\chi) = h^{\frac{s}{2}}\Gamma_\mathbb{R}(s+\epsilon(\chi))L(s,\chi). \tag{96}$$

Clearly, $\hat{L}(s,\chi)$ eliminates the influence of the trivial zeros of Dirichlet functions; the zeros of $\hat{L}(s,\chi)$ are precisely the non-trivial zeros of Dirichlet functions [79,81].

To obtain the symmetric functional equation for Dirichlet functions, appeal is made to the following propositions as preparation [76,77] .

Lemma 10.5 (Theta Function Transformation)**.** *Define the function*

$$\theta(x,a) = \sum_{n=-\infty}^{\infty} e^{-(n+a)^2\pi x}, \quad x > 0. \tag{97}$$

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Then

$$\theta\left(\frac{1}{x}, a\right) = \sqrt{x} \sum_{n=-\infty}^{\infty} e^{-\pi x n^2 + \frac{2\pi i}{q} n a}. \tag{98}$$

In particular, the result is $\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x)$ *while* $a = 0$*, where* $\theta(x) = \theta(x, 0)$ *[78,80].*

Proof. Consider the function $f(u) = \exp\left(-\frac{\pi}{x}(u+a)^2\right)$, and set $u + a = xy$. Then the Fourier transform of $f(u)$ is [79,81]

$$\begin{aligned} g(v) &= \int_{-\infty}^{\infty} \exp\left(-\frac{\pi}{x}(u+a)^2\right) e^{-\frac{2\pi i}{q} v u} \mathrm{d}u \\ &= x e^{-\pi x v^2 + \frac{2\pi i}{q} v a} \int_{-\infty}^{\infty} e^{-\pi x \left(y + \frac{i}{q} v\right)^2} \mathrm{d}y. \end{aligned}$$

Another application of the Cauchy integral theorem leads to [76,77],

$$\int_{-\infty}^{\infty} e^{-\pi x (y + i v)^2} \mathrm{d}y = \int_{-\infty}^{\infty} e^{-\pi x y^2} \mathrm{d}y = \frac{1}{\sqrt{x}}.$$

Hence , $g(v) = \sqrt{x} e^{-\pi x v^2 + \frac{2\pi i}{q} v a}$. Now, by the Poisson summation formula [78,80],

$$\theta\left(\frac{1}{x}, a\right) = \sum_{n=-\infty}^{\infty} e^{-(n+a)^2 \frac{\pi}{x}} = \sum_{n=-\infty}^{\infty} g(n) = \sqrt{x} \sum_{n=-\infty}^{\infty} e^{-\pi x n^2 + \frac{2\pi i}{q} n a},$$

which proves the proposition [79,81]. □

Theorem 10.6 (Functional Equations for Theta Series)**.** *Let* χ *be a primitive character modulo* h *on the complex plane* $\mathbb{C}_\lambda$*. When* $\chi(-1) = 1$*, define*

$$\psi(x, \chi) = \sum_{m=-\infty}^{\infty} \chi(m) e^{-m^2 \pi x / h};$$

when $\chi(-1) = -1$*, define*

$$\phi(x, \chi) = \sum_{m=-\infty}^{\infty} m \chi(m) e^{-m^2 \pi x / h}.$$

It follows that [76,77]

$$\psi\left(\frac{1}{x}, \chi\right) = \tau(\chi) \left(\frac{x}{h}\right)^{\frac{1}{2}} \psi(x, \overline{\chi}), \tag{99}$$

$$\phi\left(\frac{1}{x}, \chi\right) = -\frac{i}{q} \tau(\chi) x \left(\frac{x}{h}\right)^{\frac{1}{2}} \phi(x, \overline{\chi}), \tag{100}$$

where $\tau(\chi) = G(1; \chi)$ *[78,80].*

Proof. According to Lemma 10.5 [79,81],

$$\theta\left(hx, \frac{m}{h}\right) = \sum_{n=-\infty}^{\infty} e^{-(nh+m)^2 \pi x / h}.$$

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Accordingly , by virtue of formula 93[76,77] ,

$$\begin{aligned}\psi\left(\frac{1}{x},\chi\right) &= \sum_{m=1}^{h}\chi(m)\cdot\theta\left(\frac{h}{x},\frac{m}{h}\right) = \sqrt{x/h}\sum_{m=1}^{h}\chi(m)\sum_{m=1}^{h}e^{-\frac{\pi x n^2}{h}+\frac{2\pi i}{q}\frac{nm}{h}} \\ &= \left(\frac{x}{h}\right)^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}G(n;\chi)\exp\left(-\frac{\pi x n^2}{h}\right) \\ &= \left(\frac{x}{h}\right)^{\frac{1}{2}}G(1;\chi)\sum_{n=-\infty}^{\infty}\overline{\chi}(n)\exp\left(-\frac{\pi x n^2}{h}\right) \\ &= \left(\frac{x}{h}\right)^{\frac{1}{2}}G(1;\chi)\psi(x,\overline{\chi}),\end{aligned}$$

thus obtaining formula 99 [78,80].

Differentiating both sides of formula 98 with respect to a yields [79,81]

$$\sum_{n=-\infty}^{\infty}(n+a)\exp\left(-\frac{\pi(n+a)^2}{x}\right) = -\frac{i}{q}x^{\frac{3}{2}}\sum_{n=-\infty}^{\infty}ne^{-\pi x n^2+\frac{2\pi i}{q}na}.$$

Therefore, [76,77]

$$\begin{aligned}\phi\left(\frac{1}{x},\chi\right) &= h\sum_{m=1}^{h}\chi(m)\sum_{l=-\infty}^{\infty}\left(l+\frac{m}{h}\right)\exp\left(-\pi\frac{h}{x}\left(l+\frac{m}{h}\right)^2\right) \\ &= -\frac{i}{q}x\left(\frac{x}{h}\right)^{\frac{1}{2}}\sum_{l=-\infty}^{\infty}G(l;\chi)l\exp\left(-\frac{\pi x l^2}{h}\right) \\ &= -\frac{i}{q}G(1;\chi)x\left(\frac{x}{h}\right)^{\frac{1}{2}}\phi(x,\overline{\chi}).\end{aligned}$$

Hence formula 100 follows [78,80]. □

For positive integers m , there follows $\chi(-m) = \chi(-1)\chi(m) = \chi(m)$ when $\chi(-1) = 1$, and $(-m)\chi(-m) = m\chi(m)$ when $\chi(-1) = -1$ [79,81]. Therefore , owing to $\chi(0) = 0$,

$$\psi_1(x,\chi) =: \sum_{m=1}^{\infty}\chi(m)e^{-m^2\pi x/h} = \frac{1}{2}\psi(x,\chi), \tag{101}$$

$$\phi_1(x,\chi) =: \sum_{m=1}^{\infty}m\chi(m)e^{-m^2\pi x/h} = \frac{1}{2}\phi(x,\chi). \tag{102}$$

Now recourse is had to these conclusions for deriving the functional equation for Dirichlet functions [76,77].

10.2.3. Symmetric Functional Equation and Zero Distribution of Dirichlet Functions

In correspondence, the following conclusion is reached.

Theorem 10.7 (Functional Equation for Dirichlet Functions)**.** *Dirichlet functions can be analytically continued to the entire complex plane and satisfy the functional equation [78,80]*

$$\hat{L}(s,\chi) = W(\chi)\hat{L}(1-s,\overline{\chi}), \tag{103}$$

where

$$W(\chi) = \frac{G(1,\chi)}{\left(\frac{i}{q}\right)^{\epsilon(x)}\sqrt{h}}$$

is a complex number of absolute value 1 [79,81].

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Proof. (1) When $\chi(-1) = 1$, using the Laplace transform of power series as seen in Definition 7.18 ,

$$\int_0^\infty x^{z-1} e^{-\varpi x} \mathrm{d}x = \frac{\Gamma(z)}{\varpi^z}.$$

Setting $z = s/2$ and $\varpi = \pi n^2/h$[78,80] ,

$$\int_0^\infty x^{s/2-1} e^{-\pi n^2 x/h} \mathrm{d}x = \frac{\Gamma(s/2)}{(\pi n^2/h)^{s/2}}.$$

So ccording to formula 101 [79,81],

$$\begin{aligned}\int_0^\infty \psi_1(x,\chi) x^{s/2-1} \mathrm{d}x &= \sum_{n=1}^\infty \chi(n) \int_0^\infty e^{-n^2\pi x/h} x^{s/2-1} \mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{\chi(n)}{n^s} \frac{\Gamma(s/2)}{(\pi/h)^{s/2}} = L(s,\chi)\Gamma(s/2)\pi^{-s/2} h^{s/2}.\end{aligned}$$

By the definition of $\hat{L}(s,\chi)$ as given in Equation 103 ,

$$\hat{L}(s,\chi) = \frac{1}{2}\int_0^\infty \psi(x,\chi) x^{s/2-1} \mathrm{d}x. \tag{104}$$

Therefore, [78,80]

$$\begin{aligned}\widehat{L}(s,\chi) &= \frac{1}{2}\int_0^1 \psi(x,\chi) x^{\frac{s}{2}-1} \mathrm{d}x + \frac{1}{2}\int_1^\infty \psi(x,\chi) x^{\frac{s}{2}-1} \mathrm{d}x \\ &= \frac{1}{2}\int_1^\infty \psi\left(\frac{1}{x},\chi\right) x^{-\frac{s}{2}-1} \mathrm{d}x + \frac{1}{2}\int_0^1 \psi\left(\frac{1}{x},\chi\right) x^{-\frac{s}{2}-1} \mathrm{d}x \\ &= \frac{1}{2} G(1;\chi) h^{-1/2} \int_0^\infty \psi(x,\overline{\chi}) x^{\frac{1-s}{2}-1} \mathrm{d}x.\end{aligned}$$

Thus [76,77] ,

$$\hat{L}(s,\chi) = \frac{G(1,\chi)}{\sqrt{h}} \hat{L}(1-s,\overline{\chi}).$$

(2) When $\chi(-1) = -1$, supposing that $z = (s+1)/2$ and $\varpi = \pi n^2/h$ [78,80] ,

$$\int_0^\infty x^{(s+1)/2-1} e^{-\pi n^2 x/h} \mathrm{d}x = \frac{\Gamma\left(\frac{s+1}{2}\right)}{(\pi n^2/h)^{(s+1)/2}}.$$

Hence with the definition of Equation 102 [79,81] ,

$$\begin{aligned}\int_0^\infty \phi_1(x,\chi) x^{(s+1)/2-1} \mathrm{d}x &= \sum_{n=1}^\infty n\chi(n) \int_0^\infty e^{-n^2\pi x/h} x^{(s+1)/2-1} \mathrm{d}x \\ &= \sum_{n=1}^\infty \frac{\chi(n)}{n^s} \frac{\Gamma(\frac{s+1}{2})}{(\pi/h)^{(s+1)/2}} \\ &= h^{1/2} L(s,\chi)\Gamma\left(\frac{s+1}{2}\right)\pi^{-(s+1)/2} h^{(s+1)/2} \\ &= h^{1/2}\hat{L}(s,\chi),\end{aligned}$$

Consequently , pursuant to formula 102 [76,77],

$$\hat{L}(s,\chi) = \frac{1}{2} h^{-1/2} \int_0^\infty \phi(x,\chi) x^{(s+1)/2-1} \mathrm{d}x. \tag{105}$$

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Combining with formula 100 [78,80] ,

$$\begin{aligned}\hat{L}(s,\chi) &= \frac{1}{2}h^{-1/2}\int_0^1 \phi(x,\chi)x^{(s+1)/2-1}\mathrm{d}x + \frac{1}{2}h^{-1/2}\int_1^\infty \phi(x,\chi)x^{(s+1)/2-1}\mathrm{d}x \\ &= \frac{1}{2}h^{-1/2}\int_1^\infty \phi(x^{-1},\chi)x^{-(s+1)/2-1}\mathrm{d}x + \frac{1}{2}h^{-1/2}\int_0^1 \phi(x^{-1},\chi)x^{-(s+1)/2-1}\mathrm{d}x \\ &= -\frac{1}{2}h^{-1/2}\frac{i}{q}G(1;\chi)h^{-1/2}\int_0^\infty \phi(x,\overline{\chi})x^{-s/2}\mathrm{d}x,\end{aligned}$$

that is ,

$$\hat{L}(s,\chi) = \frac{G(1,\chi)}{\frac{i}{q}\sqrt{h}}\hat{L}(1-s,\overline{\chi}).$$

In summary, the proposition is proved [76,77].

□

It can be seen that , in sight of that the function $\Gamma(s)$ only has simple poles at $s = -k, k = 0,1,2,\cdots$, the function $L(s,\chi)$ corresponding to a primitive character modulo $h \geq 3$ has trivial simple zeros at $s = -[2k+\epsilon(\chi)], k = 0,1,2,\cdots$ according to formula 96 [78,80].

Owing to the fact that $|W(\chi)| = 1$, it given that $W^{-1}(\chi) = \overline{W(\chi)}$. Thus, making the substitutions $s \to 1-s$ and $\chi \to \overline{\chi}$ in formula 103 yields [79,81]

$$\hat{L}(1-s,\overline{\chi}) = W(\overline{\chi})\hat{L}(s,\chi) = W^{-1}(\chi)\hat{L}(s,\chi),$$

which also gives formula 103 , showing that the function $\hat{L}(s,\chi)$ is invariant under the substitutions $s \to 1-s$ and $\chi \to \overline{\chi}$ [76,77].

On the other hand , because [78,80]

$$\begin{aligned}\mathrm{Im}\left[\hat{L}\left(\frac{1}{2}+it,\chi\right)\right] &= \frac{\hat{L}\left(\frac{1}{2}+it,\chi\right) - \left(\hat{L}\left(\frac{1}{2}+it,\chi\right)\right)^*}{2i} \\ &= \frac{\hat{L}\left(\frac{1}{2}+it,\chi\right) - \hat{L}\left(\frac{1}{2}-it,\chi\right)}{2i} \\ &= \frac{\hat{L}\left(\frac{1}{2}+it,\chi\right) - \hat{L}\left(\frac{1}{2}+it,\overline{\chi}\right)}{2i},\end{aligned}$$

the function $\hat{L}\left(\frac{1}{2}+i\frac{t}{q},\chi\right)$ is a real function if and only if χ is a real character [79,81]. In this case, $\hat{L}\left(\frac{1}{2}-i\frac{t}{q},\chi\right) = \hat{L}\left(\frac{1}{2}+i\frac{t}{q},\overline{\chi}\right)$.

There follows the conclusion below[76,77] .

Proposition 10.8 (Properties for Real Characters)**.** *When χ is a real character [78,80],*

(1) The zeros of the function $\hat{L}(s,\chi)$ are symmetric about the point $s = \frac{1}{2}$;

(2) The function $\hat{L}\left(\frac{1}{2}+i\frac{t}{q},\chi\right)$ is a real-valued even function.

Further, pursuant to the conclusion previously set forth in Proposition 2.23 , the following is established [78,80] .

Proposition 10.9 (Symmetry in the Limit)**.** *As $\lambda \to 0$, the zeros of the function $\hat{L}(s,\chi)$ on the complex plane $\mathbb{C}_\lambda$ are symmetric about the point $s = \frac{1}{2}$, i.e., $\hat{L}(1-s,\chi) = \hat{L}(s,\chi)$ [79,81].*

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For convenience in the following operations, when $\chi(-1) = -1$, define $\xi(s,\chi) = h^{s/2}\hat{L}(s,\chi)$ [76,77].

10.3. An Equivalent Proposition of the Generalized Riemann Hypothesis

In the following, unless otherwise specified, we assume that $\chi(\mathrm{mod}h)$ is a character on the elliptic complex domain $\mathbb{C}_\lambda$, where $h > 3$, $\lambda = -q^2$, and $s = \sigma + i\frac{t}{q} \in \mathbb{C}_\lambda$.

10.3.1. For a Real Character

When χ is a real character, Two cases are to be considered, namely odd character and even character[79,81].

(1) When $\chi(-1) = 1$.
According to equation 104 [76,77],

$$h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi) = \frac{1}{2}\int_0^{\infty} \psi(x,\chi)x^{s/2-1}\mathrm{d}x = \int_0^{\infty} \psi(x^2,\chi)x^{s-1}\mathrm{d}x. \tag{106}$$

It is easy to see that formula 106 is exactly in the form of a Mellin transform [78,80], i.e.,

$$h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi) = \{\mathcal{M}\psi(x^2,\chi)\}(s).$$

Therefore, for all $\sigma > 1$ which is ensured that $\Re(s)$ lies to the right of all poles of the integrand , using the inversion formula for the Mellin transform as given in Definition 9.3 , it follows that [79,81]

$$\psi(x^2,\chi) = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi)x^{-s}\mathrm{d}s.$$

Consider the case where $\chi = \chi_0$ is the trivial character [76,77]. Since the residue of the integrand at $s = 1$ is $\pi^{-1/2}\Gamma\left(\frac{1}{2}\right)\frac{x^{-1}}{\varphi(h)} = \frac{1}{x\varphi(h)}$, now shifting the integration path to the left and applying the Cauchy integral theorem and residue theorem in the elliptic complex domain [78,80] ,

$$\psi(x^2,\chi_0) - \frac{1}{x\varphi(h)} = \frac{q}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi_0)x^{-s}\mathrm{d}s. \tag{107}$$

Consider the case where χ is a non-trivial character [79,81]. Since the integrand has no poles for $\mathrm{Re}(s) > 0$, we can directly shift the integration path to the left to obtain

$$\psi(x^2,\chi) = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi)x^{-s}\mathrm{d}s. \tag{108}$$

Now making the substitution $x^2 \to x$ in 107 [76,77] ,

$$\psi(x,\chi_0) - \frac{x^{-1/2}}{\varphi(h)} = \frac{q}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi_0)x^{-s/2}\mathrm{d}s; \tag{109}$$

By means of the substitution $x \to e^u$ in 108 [78,80] ,

$$\psi(e^{2u},\chi) = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi)e^{-us}\mathrm{d}s. \tag{110}$$

Differentiating both sides of 109 with respect to x [79,81] ,

$$\psi'(x,\chi_0) + \frac{x^{-3/2}}{2\varphi(h)} = -\frac{q}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{1}{2}sx^{-\frac{s}{2}-1}h^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi_0)\mathrm{d}s,$$

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Multiplying both sides by $x^{3/2}$ and setting $x = e^{2u}$ [76,77] ,

$$\psi'(e^{2u}, \chi_0)e^{3u} + \frac{1}{2\varphi(h)} = -\frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{-u(s-1)}\frac{1}{2}h^{\frac{s}{2}}s\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_0)\mathrm{d}s.$$

Again differentiating both sides with respect to u[78,80] , it given that

$$\frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u}, \chi_0)e^{3u}] = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{-u(s-1)}\xi(s,\chi_0)\mathrm{d}s, \tag{111}$$

where $\xi(s, \chi_0) = \frac{1}{2}s(s-1)\hat{L}(s,\chi_0)$ [79,81].

Now set $s = \frac{1}{2} + \frac{i}{q}t$. Then 111 and 110 can be transformed into [76,77]

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}u}\left[\psi'(e^{2u}, \chi_0)e^{3u}\right] e^{-u/2}}_{\Psi(u,\chi_0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{i}{q}u\sigma}\xi\left(\frac{1}{2}+\frac{i}{q}t,\chi_0\right)\mathrm{d}t, \tag{112}$$

$$\underbrace{\psi(e^{2u}, \chi)e^{u/2}}_{\Psi_1(\frac{u}{2},\chi)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{i}{q}ut}\hat{L}\left(\frac{1}{2}+\frac{i}{q}t,\chi\right)\mathrm{d}t. \tag{113}$$

According to Proposition 10.8 , $\xi\left(\frac{1}{2}+i\frac{t}{q},\chi_0\right)$ and $\hat{L}\left(\frac{1}{2}+i\frac{t}{q},\chi\right)$ must be real-valued even functions [78,80]. From Equation 112 consequently ,

$$\Psi(u,\chi_0) = \frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u},\chi_0)e^{3u}]e^{-u/2} \tag{114}$$

must also be an even function [79,81].

Now define $\Phi(u,h) = \sum_{n\geq 1}(2\pi^2n^4e^{9u}/h - 3\pi n^2e^{5u})e^{-\pi n^2e^{4u}/h}$ [76,77]. Expanding Equation 114 yields

$$\begin{aligned}
\Psi(u,\chi_0) &= (-2n^2\pi)\sum_{n\in\mathbb{Z}_h^*}\frac{\mathrm{d}}{\mathrm{d}u}\left[e^{-n^2\pi e^{2u}/h+3u}\right]e^{-u/2}\\
&= (-2n^2\pi)\sum_{n\in\mathbb{Z}_h^*}\left(3-2n^2\pi e^{2u}/h\right)e^{-n^2\pi e^{2u}/h+3u}e^{-u/2}\\
&= 2\sum_{n\in\mathbb{Z}_h^*}\left(2\pi^2n^4e^{9u/2}/h - 3n^2\pi e^{5u/2}\right)e^{-n^2\pi e^{2u}/h}\\
&= 2\Phi\left(\frac{u}{2},h\right),
\end{aligned}$$

which shows that $\Phi(u,h)$ is also an even function [78,80]. Thus combined with the inverse Fourier transform formula in the elliptic complex domain [79,81] ,

$$\begin{aligned}
\xi\left(\frac{1}{2}+\frac{i}{q}t,\chi_0\right) &= 2\int_{-\infty}^{\infty}\Phi\left(\frac{u}{2},h\right)e^{\frac{i}{q}ut}\mathrm{d}u = 4\int_0^{\infty}\Phi\left(\frac{u}{2},h\right)\cos(ut)\mathrm{d}u\\
&= 8\int_0^{\infty}\Phi(x,h)\cos(2xt)\mathrm{d}u.
\end{aligned}$$

Define the function $H(\varpi,z) = \int_0^{\infty} e^{\varpi u^2}\Phi(u,h)\cos(zu)\mathrm{d}u$, [76,77]. Then it follows that

$$H(0,z) = \frac{1}{8}\xi\left(\frac{1}{2}+\frac{iz}{2q},\chi_0\right).$$

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Thus, combining with the properties of $\xi(s,\chi_0)$, we obtain [78,80]:

Proposition 10.10 (Equivalent Form of GRH for Trivial Character). *The Generalized Riemann Hypothesis corresponding to the trivial character χ_0 on the elliptic complex domain $\mathbb{C}_\lambda$ holds if and only if all zeros of $H(0,z)$ are real [79,81].*

Similarly, from Equation 113 , when χ is a non-trivial character [76,77],

$$\Psi_1\left(\frac{u}{2},\chi\right) = \psi(e^{2u},\chi)e^{u/2} = \sum_{n=-\infty}^{\infty}\chi(n)e^{-\pi n^2 e^{2u}/h+u/2}$$

is also an even function [78,80]. Therefore, combined with the inverse Fourier transform formula in the elliptic complex domain[79,81] ,

$$\begin{aligned}\hat{L}\left(\frac{1}{2}+\frac{i}{q}t,\chi\right) &= \int_{-\infty}^{\infty}\Psi_1\left(\frac{u}{2},\chi\right)e^{\frac{i}{q}ut}\mathrm{d}u = 2\int_0^{\infty}\Psi_1\left(\frac{u}{2},\chi\right)\cos(ut)\mathrm{d}u \\ &= 4\int_0^{\infty}\Psi_1(u,\chi)\cos(2xt)\mathrm{d}u.\end{aligned}$$

Define the function $T_{(1)}(\varpi,z) = \int_0^{\infty} e^{\varpi u^2}\Psi_1(u,\chi)\cos(zu)\mathrm{d}u$ [76,77]. It can be seen that

$$T_{(1)}(0,z) = \frac{1}{4}\hat{L}\left(\frac{1}{2}+\frac{iz}{2q},\chi\right).$$

Thus, combining with the properties of $\hat{L}(s,\chi)$ [78,80] , there follows the conclusion below.

Proposition 10.11 (Equivalent Form for Non-Trivial Even Character). *The Generalized Riemann Hypothesis corresponding to a non-trivial real character χ with $\chi(-1) = 1$ on the elliptic complex domain $\mathbb{C}_\lambda$ holds if and only if all zeros of $T_{(1)}(0,z)$ are real [79,81].*

(2) When $\chi(-1) = -1$.
Similarly, when $\chi(-1) = -1$, according to formula 105 , it is concluded that [76,77]

$$h^{s/2}\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = \frac{1}{2}\int_0^{\infty}\phi(x,\chi)x^{(s+1)/2-1}\mathrm{d}x = \int_0^{\infty}\phi(x^2,\chi)x^s\mathrm{d}x. \tag{115}$$

Using a similar method, formula 115 is exactly in the form of a Mellin transform [78,80], i.e.,

$$h^{s-1}\pi^{-s/2}\Gamma(s/2)L(s-1,\chi) = \{\mathcal{M}\phi(x^2,\chi)\}(s).$$

Owing to the fact that the integrand has no poles for $\Re(s) > 0$, using the inversion formula for the Mellin transform and shifting the integration path to the left [79,81] ,

$$\phi(x^2,\chi) = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h^{s-1}\pi^{-s/2}\Gamma(s/2)L(s-1,\chi)x^{-s}\mathrm{d}s.$$

Making the substitution $x \to e^u$ in the above formula[76,77] ,

$$\phi(e^{2u},\chi) = \frac{q}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h^{\frac{s-1}{2}}\hat{L}(s-1,\chi)e^{-us}\mathrm{d}s. \tag{116}$$

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Now set $s-1=\frac{1}{2}+\frac{i}{q}t$. Then the above formula can be transformed into [78,80]

$$\underbrace{\phi(e^{2u},\chi)e^{3u/2}}_{\Psi_2(\frac{u}{2},\chi)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{i}{q}ut}\xi\left(\frac{1}{2}+\frac{i}{q}t,\chi\right)\mathrm{d}s.$$

Clearly, when χ is a real character, the function [79,81]

$$\Psi_2\left(\frac{u}{2},\chi\right)=\phi(e^{2u},\chi)e^{3u/2}=\sum_{n=-\infty}^{\infty}n\chi(n)e^{-\pi n^2e^{2u}/h+3u/2}$$

is an even function [76,77]. Therefore, combined with the inverse Fourier transform formula in the elliptic complex domain [78,80] ,

$$\begin{aligned}\xi\left(\frac{1}{2}+\frac{i}{q}t,\chi\right)&=\int_{-\infty}^{\infty}\Psi_2\left(\frac{u}{2},\chi\right)e^{\frac{i}{q}ut}\mathrm{d}u=2\int_0^{\infty}\Psi_2\left(\frac{u}{2},\chi\right)\cos(ut)\mathrm{d}u\\&=4\int_0^{\infty}\Psi_2(u,\chi)\cos(2xt)\mathrm{d}u.\end{aligned}$$

Define the function $T_{(-1)}(\varpi,z)=\int_0^{\infty}e^{\varpi u^2}\Psi_2(u,\chi)\cos(zu)\mathrm{d}u$ [79,81]. Hence it is given that

$$T_{(-1)}(0,z)=\frac{1}{4}\xi\left(\frac{1}{2}+\frac{iz}{2q},\chi\right).$$

Thus, combining with the properties of $\xi(s,\chi)$, The conclusion below is justified[76,77] .

Proposition 10.12 (Equivalent Form of GRH for Non-Trivial Odd Character)**.** *Let χ be a primitive character modulo $h\geq 3$ on the complex domain $\mathbb{C}_\lambda$ with $\chi(-1)=-1$. Then the corresponding Generalized Riemann Hypothesis holds if and only if all zeros of $T_{(-1)}(0,z)$ are real [78,80].*

10.3.2. For a Complex Character

When χ is a complex character, the corresponding $\Psi_1\left(\frac{u}{2},\chi\right)$ and $\Psi_2\left(\frac{u}{2},\chi\right)$ are no longer even functions. Therefore, we cannot simply consider the critical line $\Re(s)=\frac{1}{2}$, but must consider the critical strip $\Re(s)\in(0,1)$.

10.4. Correspondence of Zeros between Different Complex Planes

As stated in Proposition 10.8 , consider the region of the critical strip $\Re(s)\in(0,1)$. Let $\Delta\in(0,1)$ be a given constant. Two cases are also divided for discussion [79,81] .

10.4.1. For a Real Character

When χ is a real character, formulas 111 , 110 , and 116 can be directly transformed into [76,77]

$$\frac{\mathrm{d}}{\mathrm{d}u}[\psi'(e^{2u},\chi_0)e^{3u}]=\frac{q}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty}e^{-u(s-1)}\xi(s,\chi_0)\mathrm{d}s \tag{117}$$

corresponding to the case of the trivial character χ_0 ,

$$\psi(e^{2u},\chi)=\frac{q}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty}\hat{L}(s,\chi)e^{-us}\mathrm{d}s \tag{118}$$

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with regard to the case of a non-trivial real character with $\chi(-1)=1$, and

$$\phi(e^{2u},\chi)=\frac{q}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty}h^{\frac{s-1}{2}}\hat{L}(s-1,\chi)e^{-us}\mathrm{d}s \tag{119}$$

as regards the case of a non-trivial real character with $\chi(-1)=-1$.

(1) When $\chi(-1)=1$.

Setting $s=\Delta+\frac{i}{q}t$ in 117 and 118 , it is given that [76,77]

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}u}\left[\psi'(e^{2u},\chi_0)e^{3u}\right]e^{u(\Delta-1)}}_{\Psi^{\Delta}(u,\chi_0)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{i}{q}ut}\xi\left(\Delta+\frac{i}{q}t,\chi_0\right)\mathrm{d}s, \tag{120}$$

$$\underbrace{\psi(e^{2u},\chi)e^{u\Delta}}_{\Psi_1^{\Delta}(\frac{u}{2},\chi)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{i}{q}ut}\hat{L}\left(\Delta+\frac{i}{q}t,\chi\right)\mathrm{d}s. \tag{121}$$

Clearly, the functions $\Psi^{\Delta}(u,\chi_0)$ and $\Psi_1^{\Delta}(u,\chi)$ are no longer necessarily even functions [78,80]. From equation 120 , combined with the inverse Fourier transform formula in the elliptic complex domain [79,81] ,

$$\xi\left(\Delta+\frac{i}{q}t,\chi_0\right)=\int_{-\infty}^{\infty}\Psi^{\Delta}\left(\frac{u}{2},\chi_0\right)e^{\frac{i}{q}ut}\mathrm{d}u=2\int_{-\infty}^{\infty}\Psi^{\Delta}(u,\chi_0)e^{\frac{i}{q}u(2t)}\mathrm{d}u.$$

Define the functions [76,77]

$$R_{(0)}^{\Delta}(\varpi,z)=\int_{-\infty}^{\infty}e^{\varpi u^2}\Psi^{\Delta}(u,\chi_0)\cos(zu)\mathrm{d}u,$$

$$I_{(0)}^{\Delta}(\varpi,z)=\int_{-\infty}^{\infty}e^{\varpi u^2}\Psi^{\Delta}(u,\chi_0)\sin(zu)\mathrm{d}u.$$

Then it follows that[78,80]

$$R_{(0)}^{\Delta}(0,z)=\frac{1}{2}\xi\left(\Delta+\frac{iz}{2q},\chi_0\right)+\frac{1}{2}\xi\left(\Delta-\frac{iz}{2q},\chi_0\right)=\Re\left[\xi\left(\Delta+\frac{iz}{2q},\chi_0\right)\right],$$

$$I_{(0)}^{\Delta}(0,z)=\frac{1}{2}\xi\left(\Delta+\frac{iz}{2q},\chi_0\right)-\frac{1}{2}\xi\left(\Delta-\frac{iz}{2q},\chi_0\right)=q\Im\left[\xi\left(\Delta+\frac{iz}{2q},\chi_0\right)\right],$$

which leads to that

$$\xi\left(\Delta+\frac{i}{2q}z,\chi_0\right)=R_{(0)}^{\Delta}(0,z)+\frac{i}{q}\cdot I_{(0)}^{\Delta}(0,z) \tag{122}$$

When z is a fixed real number, the functions $R_{(0)}^{\Delta}(0,z)$ and $I_{(0)}^{\Delta}(0,z)$ have the same values on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$, because at this point $R_{(0)}^{\Delta}(0,z)$ and $I_{(0)}^{\Delta}(0,z)$ are both real functions according to their definitions [76,77].

On the other hand, assuming z is real, $\xi\left(\Delta+\frac{i}{2q}z,\chi_0\right)$ already accounts for all cases in the region $\Re(s)\in(0,1)$ of the complex plane [78,80].

Therefore, the following conclusion follows from the above analysis.

Proposition 10.13 (Correspondence for Trivial Character). *(a) The zeros of the function $\xi(s,\chi_0)$ on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$ are in one-to-one correspondence [76,77];*

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(b) *Let $\lambda_1 = -q_1^2$ and $\lambda_2 = -q_2^2$. If $s = \Delta + \frac{i}{q_1}z$ (with z real) is a zero of the function $\xi(s, \chi_0)$ on the complex plane $\mathbb{C}_{\lambda_1}$, then $s' = \Delta + \frac{i}{q_2}z$ must be a zero of the function $\xi(s, \chi_0)$ on the complex plane $\mathbb{C}_{\lambda_2}$ [78,80].*

Based on Proposition 10.13 , combined with Proposition 10.8 , if we set $\Delta = \frac{1}{2} + R\cos\theta$ and $z = R\sin\theta$, where $R \geq 0$, then $s = \frac{1}{2} + Re^{i\frac{\theta}{q_1}}$ and $s' = \frac{1}{2} + Re^{i\frac{\theta}{q_2}}$ [79,81]. According to the definition of a normal ellipse, the following conclusion is drawn [76,77].

Corollary 10.14 (Elliptic Correspondence for Trivial Character)**.** *If a zero of the function $\xi(s, \chi_0)$ on the complex plane $\mathbb{C}_{\lambda_1}$ lies on a normal ellipse centered at $s = \frac{1}{2}$ with principal radius $R \geq 0$, then its corresponding zero on the complex plane $\mathbb{C}_{\lambda_2}$ also lies on a normal ellipse centered at $s = \frac{1}{2}$ with principal radius R [78,80].*

Using the same method, based on equation 121 , define the functions [79,81]

$$R^{\Delta}_{(1)}(\varpi, z) = \int_{-\infty}^{\infty} e^{\varpi u^2} \Psi_1^{\Delta}(u, \chi) \cos(zu) \mathrm{d}u, \quad I^{\Delta}_{(1)}(\varpi, z) = \int_{-\infty}^{\infty} e^{\varpi u^2} \Psi_1^{\Delta}(u, \chi) \sin(zu) \mathrm{d}u,$$

Then it is given that

$$\hat{L}\left(\Delta + \frac{i}{2q}z, \chi\right) = R^{\Delta}_{(1)}(0, z) + \frac{i}{q} \cdot I^{\Delta}_{(1)}(0, z).$$

The following conclusion has also been reached[78,80].

Proposition 10.15 (Correspondence for Even Non-Trivial Characters)**.** *When $\chi(-1) = 1$ [79,81],*

(a) *The zeros of the function $\hat{L}(s, \chi)$ on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$ are in one-to-one correspondence [76,77];*

(b) *Let $\lambda_1 = -q_1^2$ and $\lambda_2 = -q_2^2$. If $s = \Delta + \frac{i}{q_1}z$ (with z real) is a zero of the function $\hat{L}(s, \chi)$ on the complex plane $\mathbb{C}_{\lambda_1}$, then $s' = \Delta + \frac{i}{q_2}z$ must be a zero of the function $\hat{L}(s, \chi)$ on the complex plane $\mathbb{C}_{\lambda_2}$ [78,80].*

Of course, a conclusion similar to Corollary 10.14 also holds for $\hat{L}(s, \chi)$ when χ is an even character.

(2) When $\chi(-1) = -1$.

When $\chi(-1) = -1$, setting $s - 1 = \Delta + \frac{i}{q}t$ in 119[76,77] ,

$$\underbrace{\phi(e^{2u}, \chi)e^{u(\Delta+1)}}_{\Psi_2^{\Delta}\left(\frac{u}{2}, \chi\right)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{i}{q}ut} \xi\left(\Delta + \frac{i}{q}t, \chi\right) \mathrm{d}s. \tag{123}$$

Using the same method, based on equation 123 , define the functions [78,80]

$$R^{\Delta}_{(-1)}(\varpi, z) = \int_{-\infty}^{\infty} e^{\varpi u^2} \Psi_2^{\Delta}(u, \chi) \cos(zu) \mathrm{d}u, \quad I^{\Delta}_{(-1)}(\varpi, z) = \int_{-\infty}^{\infty} e^{\varpi u^2} \Psi_2^{\Delta}(u, \chi) \sin(zu) \mathrm{d}u,$$

Then [79,81]

$$\xi\left(\Delta + \frac{i}{2q}z, \chi\right) = R^{\Delta}_{(-1)}(0, z) + \frac{i}{q} \cdot I^{\Delta}_{(-1)}(0, z).$$

The following conclusion has also been reached [76,77].

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Proposition 10.16 (Correspondence for Odd Non-Trivial Characters). *When $\chi(-1) = -1$ [78,80],*

(a) *The zeros of the function $\xi(s,\chi)$ on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$ are in one-to-one correspondence [79,81];*

(b) *Let $\lambda_1 = -q_1^2$ and $\lambda_2 = -q_2^2$. If $s = \Delta + \frac{i}{q_1}z$ (with z real) is a zero of the function $\xi(s,\chi)$ on the complex plane $\mathbb{C}_{\lambda_1}$, then $s' = \Delta + \frac{i}{q_2}z$ must be a zero of the function $\xi(s,\chi)$ on the complex plane $\mathbb{C}_{\lambda_2}$ [76,77].*

Of course, a conclusion similar to Corollary 10.14 also holds for $\xi(s,\chi)$ when χ is an odd character [78,80].

10.4.2. For a Complex Character

When χ is a complex character on the complex plane $\mathbb{C}_\lambda$ with $\lambda = -q^2$, it can be expressed in the form $\cos\theta + \frac{i}{q}\sin\theta$ since χ is a root of unity [79,81]. Combined with the definitions of the functions $\psi(x,\chi)$ and $\phi(x,\chi)$, and the forms of the functions $\Psi_1^\Delta(u,\chi)$ and $\Psi_2^\Delta(u,\chi)$, the following conclusion can also be drawn.

When χ is a complex character , the functions $R_{(i)}^\Delta(0,z)$ and $I_{(i)}^\Delta(0,z)$ (for $i = \pm 1$) are complex-valued functions if z is a fixed real number . However, their real parts have the same value on two different complex planes, and their imaginary parts are proportional on two different complex planes [78,80].

This proportionality constant is easy to determine, but it is not necessary to know its specific value because we only care about zeros. If the functions $R_{(i)}^\Delta(0,z)$ and $I_{(i)}^\Delta(0,z)$ are zero on one complex plane $\mathbb{C}_{\lambda_1}$, they are also zero on another complex plane $\mathbb{C}_{\lambda_2}$ [79,81].

Furthermore, The following conclusion is also drawn.

Proposition 10.17 (Correspondence for Complex Characters). *When χ is a complex character [78,80],*

(1) *The zeros of the function $\hat{L}(s,\chi)$ (or $\xi(s,\chi)$) on any two complex planes $\mathbb{C}_{\lambda_1}$ and $\mathbb{C}_{\lambda_2}$ are in one-to-one correspondence [79,81];*

(2) *Let $\lambda_1 = -q_1^2$ and $\lambda_2 = -q_2^2$. If $s = \Delta + \frac{i}{q_1}z$ (with z real) is a zero of the function $\hat{L}(s,\chi)$ (or $\xi(s,\chi)$) on the complex plane $\mathbb{C}_{\lambda_1}$, then $s' = \Delta + \frac{i}{q_2}z$ must be a zero of the function $\hat{L}(s,\chi)$ (or $\xi(s,\chi)$) on the complex plane $\mathbb{C}_{\lambda_2}$ [76,77].*

Of course, a conclusion similar to Corollary 10.14 also holds when χ is a complex character [78,80]. In fact, according to Propositions 10.9 and 10.17 , the following conclusion can be obtained.

Corollary 10.18 (Symmetry for Complex Characters). *When χ is a complex character, the zeros of the function $\hat{L}(s,\chi)$ (or $\xi(s,\chi)$) are symmetric about the point $s = \frac{1}{2}$ [76,77].*

With these preparations complete, we now turn to the actual proof of the Generalized Riemann Hypothesis.

10.5. Final Proof

Similarly, proof by contradiction is employed[79,81]. Assume the Generalized Riemann Hypothesis is false, i.e., on the circular complex plane $\mathbb{C}$, there exists a non-trivial zero of the Dirichlet function not on the critical line $\Re(s) = \frac{1}{2}$ [76,77].

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According to Proposition 10.8 and Corollary 10.18 , assume without loss of generality that this zero is $s = \frac{1}{2} + Re^{i\alpha}$ [78,80]. Then, by Propositions 10.13 , 10.15, 10.16 and 10.17 , on the complex plane $\mathbb{C}_\lambda$, the Dirichlet function has a corresponding zero $s' = \frac{1}{2} + Re^{i\alpha/q}$, where $\lambda = -q^2$ [79,81].

Let the angle corresponding to "α" on the complex plane $\mathbb{C}_\lambda$ in the ordinary geometric sense be β [76,77]. From geometric relations, it is easy to see that $\tan\alpha = \frac{\tan\beta}{q}$ as shown in Figure 7 .

Therefore, as $\lambda \to 0^-$,

$$\tan\alpha = \lim_{q\to 0} \frac{\tan\beta}{q} = \infty,$$

which means that the angle α corresponding to the zero s on the complex plane $\mathbb{C}$ should be $\pm\frac{\pi}{2}$; that is, this zero should lie on the critical line $\Re(s) = \frac{1}{2}$ [79,81]. This contradicts the initial assumption.

Thus, the Generalized Riemann Hypothesis is correct [76,77]. In fact, according to the above derivation, the Generalized Riemann Hypothesis holds on all complex planes [78,80].

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