

Article

Not peer-reviewed version

Weighted Optimal Quadrature Formulas in Sobolev Space and Their Applications

[Kholmat Shadimetov](#) and [Khojjakbar Usmanov](#) *

Posted Date: 30 April 2025

doi: 10.20944/preprints202504.2518.v1

Keywords: optimization of computational algorithms; quadrature formula with derivatives; Sobolev space; error functional; optimal coefficients



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a Creative Commons CC BY 4.0 license, which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Article

Weighted optimal quadrature formulas in Sobolev space and their applications

Kholmat Shadimetov^{1,2} , Khojiakbar Usmanov²

¹ Tashkent State Transport University, 1, Odilkhodjayev str., Tashkent 100167, Uzbekistan

² Computational Mathematics Laboratory, V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 4b University str., Tashkent 100174, Uzbekistan

* Correspondence: aziz_boltayev@mail.ru;

Abstract: Optimization of computational algorithms is one of the main problems of computational mathematics. This optimization is well demonstrated by the example of the theory of quadrature and cubature formulas. It is known that numerical integration of definite integrals is of great importance in basic and applied sciences. In this paper we consider the optimization problem of weighted quadrature formulas with derivatives in Sobolev space. Using the extremal function, the square of the norm of the error functional of the considered quadrature formula is calculated. Then, minimizing this norm by coefficients, we obtain a system to find the optimal coefficients of this quadrature formula. The uniqueness of solutions of this system is proved and an algorithm for solving this system is given. The proposed algorithm is used to obtain the optimal coefficients of the derivative weight quadrature formulas. It should be noted that the optimal weighted quadrature formulas constructed in this work are optimal for the approximate calculation of regular, singular, fractional and strongly oscillating integrals. The constructed optimal quadrature formulas are applied to the approximate solution of linear Fredholm integral equations of the second kind. Finally, the numerical results are compared with the known results of other authors.

Keywords: optimization of computational algorithms; quadrature formula with derivatives; Sobolev space; error functional; optimal coefficients

1. Introduction

An important practical goal of computational mathematics is to create the best, i.e., the fastest and cheapest ways of solving mathematical problems. In short, optimization of computational algorithms. Optimization of computational algorithms is well demonstrated by examples of the construction of cubature and quadrature formulas on the functional formulation. In this formulation, we consider functions $\varphi(x)$ belonging to some Banach space B . It is assumed that this space is nested in the space of continuous functions defined in the domain Ω . The integral of the function $\varphi(x)$ with the weight function $p(x)$ over the region Ω

$$\int_{\Omega} p(x) \varphi(x) dx$$

is a linear functional in B . Its approximate expression is.

$$\sum_{k=1}^N C_k \varphi(x^{(k)})$$

will be another linear functional. Then the error functional [1] and [2] of the cubature formula will also be linear

$$(\ell, \varphi) = \int_{\Omega} p(x) \varphi(x) dx - \sum_{k=1}^N C_k \varphi(x^{(k)}) =$$

$$= \int_{\Omega} \varepsilon_{\omega}(x) p(x) - \sum_{k=1}^N C_k \delta(x - x^{(k)})] \varphi(x) dx \quad (1)$$

The problem of constructing a cubature formula

$$\int_{\Omega} p(x) \Omega(x) dx \cong \sum_{k=1}^N C_k \varphi(x^{(k)}) \quad (2)$$

in the functional formulation consists in finding such a functional (1) whose norm in the space B^* is minimal.

Studies of optimal and asymptotic cubature formulas are found in [1] - [11]. Optimization studies of quadrature formulas are presented in [12] - [14].

Currently, there are various methods for constructing optimal approximate integration formulas: the spline method, the φ function method, and the Sobolev method.

In recent years, a number of new results have been obtained on the construction and their error estimates of optimal quadrature formulas for approximate computation of regular, singular, and integrals from rapidly oscillating functions using the Sobolev method. These results can be found, for example, in [15] - [27].

In this paper, the construction of composite optimal quadrature formulas with weight in Sobolev space is studied by the variational method. Here, the square of the norm of the error functional of composite quadrature formulas with weight function is computed using the extremal function. Minimizing this norm by coefficients, the system of algebraic equations is obtained. The uniqueness of the solutions of the obtained system is proved. Using this algorithm, the optimal coefficients of quadrature formulas with a weight function are found.

In the case where the weight function is equal to one, the coefficients of the well - known Euler - Maclolean quadrature formula are obtained from the general formula for the optimal coefficients.

The approximate solutions of specific linear Fredholm integral equations of the second kind are found by the constructed optimal quadrature formula, and they are compared with the results of [29] - [46].

2. Compound quadrature formulas of Hermite type

Let us consider quadrature formulas of the form

$$\int_0^1 p(x) \varphi(x) dx \cong \sum_{\beta=0}^N \sum_{\nu=0}^t C_{\beta}^{(\nu)} \varphi^{(\nu)}(x_{\beta}) \quad (3)$$

in the space $L_2^{(m)}(0, 1)$. Here $L_2^{(m)}(0, 1)$ - the space of functions whose m - th generalized derivative sums to square on the interval $[0, 1]$, $p(x)$ - the weight function whose

$$\int_0^1 p(x) dx < \infty,$$

$C_{\beta}^{(\nu)}$ - coefficients, x_{β} - nodes of quadrature formulas, $\beta = \overline{0, N}$, $\nu = \overline{0, t}$, $t = \overline{0, m-1}$. Here, the integral is considered to be regular, singular, fractional and strongly oscillating.

The error of the quadrature formula is the difference of

$$(\ell_N^{(t)}, \varphi) = \int_0^1 p(x) \varphi(x) dx - \sum_{\beta=0}^N \sum_{\nu=0}^t C_{\beta}^{(\nu)} \varphi^{(\nu)}(x_{\beta}) = \int_0^1 \ell_N^{(t)}(x) \varphi(x) dx,$$

where

$$\ell_N^{(t)}(x) = \mathcal{E}_{[0,1]}(x) p(x) - \sum_{\beta=0}^N \sum_{\nu=0}^t (-1)^{\nu} C_{\beta}^{(\nu)} \delta^{(\nu)}(x - x_{\beta}), \quad (4)$$

$\mathcal{E}_{[0,1]}$ - index of the segment $[0,1]$, $\delta(x)$ - Dirac delta function, $\ell_N^{(t)}$ - is the error functional of the quadrature formula (3).

The functional $\ell_N^{(t)}(x)$ of the form (4) is defined in the space $L_2^{(m)}(0,1)$, i.e., this functional belongs to the conjugate space $L_2^{(m)*}(0,1)$, then we have

$$(\ell_N^{(t)}, x^\alpha) = 0, \alpha = 0, 1, \dots, m-1. \quad (5)$$

The problem of constructing an optimal quadrature formula of the form (3) with the error functional (4) in the space $L_2^{(m)}(0,1)$ consists in finding the value of

$$\|\ell_N^{(t)}\|_{L_2^{(m)*}(0,1)}^2 = \inf_{C_\beta^{(v)}} |(\ell_N^{(t)}, \psi_\ell)| \quad (6)$$

at fixed nodes x_β .

In formula (6), $\psi_\ell(x)$ - is the extremal function of the quadrature formula (3) in the space $L_2^{(m)}(0,1)$.

Theorem 1. The extremal function of the error functional $\ell_N^{(t)}(x)$ in the space $L_2^{(m)}(0,1)$ is of the form

$$\psi_\ell(x) = (-1)^m \ell_N^{(t)}(x) * G_m(x) + P_{m-1}(x), \quad (7)$$

where $G_m(x)$ - the Green's function of the operator $\frac{d^{2m}}{dx^{2m}}$, i.e.

$$G_m(x) = \frac{x^{2m-1} \text{sign}(x)}{2(2m-1)!},$$

$P_{m-1}(x)$ - some polynomial of degree $m-1$.

At $t=0$, i.e. for the functional

$$\ell(x) = \mathcal{E}_{[0,1]}(x) - \sum_{\beta=0}^N C_\beta \delta(x - h\beta)$$

the extremal function was found in [1] and [2]. For any $0 \leq t \leq m-1$ theorem 1 is proved in [8].

Since $L_2^{(m)}(0,1)$ - Hilbert space, the norm of the error functional $\ell_N^{(t)}$ and the function $\psi_\ell(x)$ are related by the relation

$$\|\ell_N^{(t)}\|_{L_2^{(m)*}(0,1)}^2 = \int_0^1 (\psi_\ell^{(m)}(x))^2 dx. \quad (8)$$

In addition, there is an equality

$$\|\ell_N^{(t)}\|_{L_2^{(m)*}(0,1)}^2 = (\ell, \psi_\ell). \quad (9)$$

Substituting the extremal function defined by formula (7) into equality (9) and considering (5), after some calculations for the square of the norm of the error functional (4) of the quadrature formula (3) we obtain

$$\begin{aligned} \|\ell_N^{(t)}\|_{L_2^{(m)*}(0,1)}^2 = & (-1)^m \left[\sum_{\beta=0}^N \sum_{\beta'=0}^N \sum_{v=0}^t \sum_{v'=0}^t C_\beta^{(v)} C_{\beta'}^{(v')} G_m^{(v+v')}(x_\beta - x_{\beta'}) - \right. \\ & - 2 \sum_{\beta=0}^N \sum_{v=0}^t C_\beta^{(v)} \int_0^1 p(x) G_m^{(v)}(x - x_\beta) dx + \\ & \left. + \int_0^1 \int_0^1 p(x) p(y) G_m(x - y) dx dy \right] \equiv F(C), \end{aligned} \quad (10)$$

where

$$G_m^{(k)}(x) = \frac{x^{2m-1-k} \text{sign}(x)}{2(2m-1-k)!}. \quad (11)$$

Recall that the coefficients $C_\beta^{(v)}$ in equality (10) must satisfy the system of linear equations

$$\sum_{\beta=0}^N \sum_{v=0}^t C_\beta^{(v)} x_\beta^{(k-v)} = \int_0^1 p(x) x^k dx, k = \overline{0, m-1} \quad (12)$$

equivalent to the accuracy conditions (5) of the quadrature formula on polynomials of degree lower than m . In system (12)

$$x_\beta^{(k-v)} = \begin{cases} x_\beta^{(k-v)}, & \text{if } k-v \geq 0, \\ 0, & \text{if } k-v < 0. \end{cases}$$

Now let us formulate the conditions under which the quadratic function $F(C)$ on the set of vectors

$$C = (C_0^{(0)}, C_1^{(0)}, \dots, C_N^{(0)}, C_0^{(1)}, C_1^{(1)}, \dots, C_N^{(1)}, \dots, C_0^{(t)}, C_1^{(t)}, \dots, C_N^{(t)})$$

subject to relations (5). For this purpose, we apply the method of Lagrange indefinite multipliers.

Let us compose the Lagrange function

$$F(C, \lambda) = F(C) + 2 \sum_{k=0}^{m-1} \lambda_k (\ell_N^{(t)}, x^k).$$

Equating to zero the partial derivatives of $F(C, \lambda)$ by $C_\beta^{(v)}$ and λ_k , we obtain

$$\sum_{\beta=0}^N \sum_{v=0}^t C_\beta^{(v)} G_m^{(v+v')} (x_{\beta'} - x_\beta) + \sum_{k=0}^{m-1} \lambda_k x_{\beta'}^{(k-v')} \cdot \frac{k!}{(k-v')!} = f_{\beta'}^{(v')}, \quad (13)$$

$$v' = \overline{0, t}, \beta' = \overline{0, N},$$

$$\sum_{\beta=0}^N \sum_{v=0}^t \frac{k!}{(k-v)!} C_\beta^{k-v} = g_k, k = \overline{0, m-1}, \quad (14)$$

where

$$f_{\beta'}^{(v')} = \int_0^1 p(x) G_m^{(v')} (x - x_{\beta'}) dx, v' = \overline{0, t}, \beta' = \overline{0, N},$$

$$g_k = \int_0^1 p(x) x^k dx, k = \overline{0, m-1}.$$

In system (13) - (14), the unknowns are $C_\beta^{(v)}$ and λ_k . The solution of this system is the stationary point of the function $F(C, \lambda)$, which we denote as $\overset{\circ}{C}_\beta^{(v)}$ and $\overset{\circ}{\lambda}_k$. From the theory of conditional extremal, we know a sufficient condition under which this solution gives a conditional minimum of $F(C)$ on the manifold (5). It consists in positive definiteness of the quadratic form

$$\phi(C) = \sum_{\beta=0}^N \sum_{\beta'=0}^N \sum_{v=0}^t \sum_{v'=0}^t \frac{\partial^2 F(C, \lambda)}{\partial C_\beta^{(v)} \partial C_{\beta'}^{(v')}} C_\beta^{(v)} C_{\beta'}^{(v')}, \quad (15)$$

On the set of vectors $C_\beta^{(\nu)}$ obtaining the requirement

$$\sum_{\beta=0}^N \sum_{\nu=0}^t C_\beta^{(\nu)} x_\beta^{(k-\nu)} \frac{k!}{(k-\nu)!} = 0, \quad k = \overline{0, m-1}. \quad (16)$$

In matrix form the system (16) has the form

$$SC = 0. \quad (17)$$

We proceed to prove that in the considered case the quadratic form (15) is positive definite.

Lemma 1. For any nonzero vector C lying in subspace $SC = 0$, the function $\Phi(C)$ is strictly positive.

Proof of lemma 1. From the definition of the Lagrange function $F(C, \lambda)$ and from equality (15) it follows that

$$\Phi(C) = \sum_{\beta=0}^N \sum_{\beta'=0}^N \sum_{\nu=0}^t \sum_{\nu'=0}^t C_\beta^{(\nu)} C_{\beta'}^{(\nu')} G_m^{(\nu+\nu')}(x_\beta - x_{\beta'}). \quad (18)$$

Consider the following functional

$$\mu(C) = \sum_{\beta=0}^N \sum_{\nu=0}^t (-1)^\nu C_\beta^{(\nu)} \delta^{(\nu)}(x - x_\beta). \quad (19)$$

It is known that, by condition (6), this functional belongs to $L_2^{(m)*}$, i.e., $\mu(C) \in L_2^{(m)*}(0, 1)$.

For this functional, there corresponds an extremal function $\psi_\mu(x) \in L_2^{(m)}(0, 1)$, which is a solution of Eq.

$$\frac{d^{2m}}{dx^{2m}} \psi_\mu(x) = (-1)^m \ell_\mu(x). \quad (20)$$

The solution of equation (20) has the form

$$\psi_\mu(x) = (-1)^m \ell_\mu(x) * G_m(x) = (-1)^m \sum_{\beta=0}^N \sum_{\nu=0}^t (-1)^\nu G_m^{(\nu)}(x - x_\beta). \quad (21)$$

The square of the norm of the function $\psi_\mu(x)$ in $L_2^{(m)}(0, 1)$ coincides with the form $\phi(C)$

$$\begin{aligned} \|\psi_\mu / L_2^{(m)}\|^2 &= (\ell_\mu(x), \psi_\mu(x)) = \\ &= \sum_{\beta=0}^N \sum_{\beta'=0}^N \sum_{\nu=0}^t \sum_{\nu'=0}^t (-1)^{\nu+\nu'} C_\beta^{(\nu)} C_{\beta'}^{(\nu')} G_m^{(\nu+\nu')}(x_\beta - x_{\beta'}). \end{aligned}$$

It follows that for nonzero $C(C_\beta^{(\nu)}, C_{\beta'}^{(\nu')})$ the function $\phi(C)$ is strictly positive, i.e., the positivity of $\phi(C)$ for such C follows from the positivity of the norm $\psi_\mu(x)$ in $L_2^{(m)}(0, 1)$.

Lemma 1 is proved completely. $\square$

Lemma 2. If the matrix S of the system (16) has a right inverse, then the matrix Q of the system (13) - (14) is non degenerate.

Proof of Lemma 2. Let us write the homogeneous system corresponding to the system (13) - (14) in the following form

$$\sum_{\beta=0}^N \sum_{\nu=0}^t \bar{C}_\beta^{(\nu)} G_m^{(\nu+\nu')}(x_\beta - x_{\beta'}) + \sum_{k=\nu'}^{m-1} \bar{\lambda}_k \frac{k!}{(k-\nu')!} x_\beta^{(k-\nu')} = 0, \quad (22)$$

$$\beta' = \overline{0, N}, \nu' = \overline{0, t},$$

$$\sum_{\beta=0}^N \sum_{\nu=0}^t \frac{k!}{(k-\nu)!} \overline{C}_\beta^{(\nu)} x_\beta^{(k-\nu)} = 0, k = \overline{0, m-1}, \quad (23)$$

where

$$x_\beta^{(k-\nu)} = \begin{cases} x_\beta^{(k-\nu)} & \text{if } k - \nu \geq 0 \\ 0 & \text{if } k - \nu < 0. \end{cases}$$

Let's denote by G the matrix of quadratic form (18), and write the homogeneous system (22) - (23) in the following form

$$G \begin{pmatrix} \overline{C} \\ \overline{\lambda} \end{pmatrix} = \begin{pmatrix} G & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} \overline{C} \\ \overline{\lambda} \end{pmatrix} = 0. \quad (24)$$

Now we prove that the only solution of the homogeneous system (24) is identically zero, i.e., $\overline{C} = 0$ and $\overline{\lambda} = 0$.

Let $\overline{C}, \overline{\lambda}$ - be the solution of the system (24).

Consider the functional corresponding to the vector $\overline{C}$

$$\mu_{\overline{C}}(x) = \sum_{\beta=0}^N \sum_{\nu=0}^t (-1)^\nu \overline{C}_\beta^{(\nu)} \delta^{(\nu)}(x - x_\beta).$$

Clearly, $\mu_{\overline{C}}(x) \in L_2^{(m)*}(0, 1)$.

Let us take the following as the extremal function for the functional $\mu_{\overline{C}}(x)$:

$$U_{\overline{C}}(x) = (-1)^m \mu_{\overline{C}}(x) * G_m(x) + \sum_{k=\nu'}^{m-1} \overline{\lambda}_k \frac{k!}{(k-\nu')!} x^{(k-\nu')}.$$

This is possible because $U_{\overline{C}}(x) \in L_2^{(m)}(0, 1)$ and is a solution of Eq.

$$\frac{d^{2m} U_{\overline{C}}(x)}{dx^{2m}} = (-1)^m \mu_{\overline{C}}(x).$$

The system of equations (24) means that $U_{\overline{C}}^{(\nu)}(x)$ takes zero values at all nodes x_β , i.e., $U_{\overline{C}}^{(\nu)}(x_\beta) = 0$, when $\beta = \overline{0, N}, \nu = \overline{0, t}$. Then with respect to the norm in $L_2^{(m)*}(0, 1)$ of the functional $\mu_{\overline{C}}(x)$ we have

$$\|\mu_{\overline{C}}/L_2^{(m)*}\|^2 = (\mu_{\overline{C}}(x), U_{\overline{C}}(x)) = \sum_{\beta=0}^N \sum_{\nu=0}^t C_\beta^{(\nu)} U_{\overline{C}}^{(\nu)}(x_\beta) = 0,$$

which is possible only at $\overline{C} = 0$. Taking this into account from (24) we obtain

$$S^* \overline{\lambda} = 0. \quad (25)$$

By convention, the matrix S has a right inverse, then S^* has a left inverse. Hence and from (25) it follows that $\overline{\lambda} = 0$.

Lemma 2 is proved completely. $\square$

Thus, the system (13) - (14) has a single solution. Thus, the vector $\overset{\circ}{C}$ delivers a local minimum to the quadratic function $F(C)$ on the set of solutions of the system (5). The following theorem follows directly from Theorem 1 and Lemma 1, 2.

Theorem 2. Let the error functional of the quadrature formula (3) $\ell_N^{(t)}(x)$ be defined in the space $L_2^{(m)}(0, 1)$, i.e., its values for all polynomials of degree $m - 1$ are zero and optimal, i.e., among all functionals of the form (4) with given nodes x_β it has the smallest norm in $L_2^{(m)}(0, 1)$. Then there exists a solution $\psi_\ell(x)$ of Eq.

$$\frac{d^{2m}}{dx^{2m}}\psi_\ell(x) = (-1)^m \ell_N^{(t)}(x),$$

which goes to zero with its derivatives of order ν ($\nu = \overline{0, t}, t = \overline{0, m-1}$) at nodes x_β , i.e., $\psi_\ell^{(\nu)}(x_\beta) = 0$ and belongs to $L_2^{(m)}(0, 1)$.

Theorem 2 generalizes the theorem of I. Babushka [28], i.e. it is proved that the extremal function for the error functional goes to zero at nodes x_β .

3. On one method of construction of weighted optimal quadrature formulas with derivatives

In this section we will consider the following quadrature formula

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=0}^N \sum_{\nu=0}^{m-1} C_\beta^{(\nu)} \varphi^{(\nu)}(x_\beta), \quad (26)$$

i.e., the case $t = m - 1$ in formula (3). The error of this formula is

$$\begin{aligned} (\ell_N^{(m-1)}, \varphi) &= \int_0^1 p(x)\varphi(x)dx - \sum_{\beta=0}^N \sum_{\nu=0}^{m-1} C_\beta^{(\nu)} \varphi^{(\nu)}(x_\beta) = \\ &= \int_{-\infty}^{+\infty} \ell_N^{(m-1)}(x)\varphi(x)dx, \end{aligned}$$

where

$$\ell_N^{(m-1)}(x) = \mathcal{E}_{[0,1]}(x)p(x) - \sum_{\nu=0}^{m-1} \sum_{\beta=0}^N (-1)^\nu C_\beta^{(\nu)} \delta^{(\nu)}(x - x_\beta). \quad (27)$$

The quadrature formula (26) with error functional (27), considered in the space $L_2^{(m)}(0, 1)$, can be characterized in two ways. On the one hand, it is defined by the coefficients $C_\beta^{(\nu)} (\nu = \overline{0, m-1}, \beta = \overline{0, N})$ subject to the conditions:

$$(\ell_N^{(m-1)}(x), x^\alpha) = 0, \quad \alpha = \overline{0, m-1}, \quad (28)$$

and, on the other hand, the extreme function $\psi_\ell(x)$ of the quadrature formula, which is obtained as a solution of Eq.

$$\frac{d^{2m}U(x)}{dx^{2m}} = (-1)^m \ell_N^{(m-1)}(x)$$

and can be written in the form

$$\psi_\ell(x) = (-1)^m \ell_N^{(m-1)}(x) * G_m(x) + P_{m-1}(x), \quad (29)$$

where

$$G_m(x) = \frac{x^{2m-1} \text{sign}(x)}{2(2m-1)!},$$

$P_{m-1}(x)$ - some polynomial of degree $m - 1$.

In this case, the square of the norm of the error functional $\ell_N^{(m-1)}(x)$ is calculated by the formula

$$\|\ell_N^{(m-1)}\|^2 = (\ell_N^{(m-1)}, \psi_\ell). \quad (30)$$

The function $\psi_\ell(x)$ is expressed, as we have seen, by formula (29), so calculating the square of the norm of the error functional using (30), we obtain

$$\begin{aligned} \|\ell_N^{(m-1)}\|^2 &= (-1)^m \left[\sum_{\beta=0}^N \sum_{\beta'=0}^N \sum_{v=0}^{m-1} \sum_{v'=0}^{m-1} (-1)^{v+v'} C_\beta^{(v)} C_{\beta'}^{v'} G_m^{(v+v')}(x_\beta - x_{\beta'}) - \right. \\ &\quad \left. - 2 \sum_{v=0}^{m-1} \sum_{\beta=0}^N (-1)^v C_\beta^{(v)} \int_0^1 p(x) G_m^{(v)}(x - x_\beta) dx + \right. \\ &\quad \left. + \int_0^1 \int_0^1 p(x) p(y) G_m(x - y) dx dy, \right. \end{aligned} \quad (31)$$

where

$$G_m^{(k)}(x) = \frac{x^{2m-1-k}}{2(2m-1-k)!}, \quad k \leq 2m-1.$$

4. Method of construction of weighted optimal quadrature formulas derivative formulas

Let's $x_\beta = [\beta], h = \frac{1}{N}, N = \overline{1, N}, N \geq m, C_\beta^{(v)} = C^{(v)}[\beta], [\beta] = h\beta$.

Our method for constructing optimal quadrature formulas with derivatives is as follows. First, for $m = 1$, i.e., in the space $L_2^{(1)}(0, 1)$ minimizing the square of the norm of the error functional (31) by the coefficients $C_\beta^{(0)} (\beta = \overline{0, N})$, under conditions (28), we obtain the following system for finding $C_\beta^{(0)}$:

$$\begin{aligned} \sum_{\gamma=0}^N \overset{\circ}{C}_\gamma^{(0)} \frac{(h\beta - h\gamma) \text{sign}(h\beta - h\gamma)}{2} + \lambda_0 &= \\ = \int_0^1 p(x) \frac{(x - h\beta) \text{sign}(x - h\beta)}{2} dx, \quad \beta = \overline{0, N}, \end{aligned} \quad (32)$$

$$\sum_{\gamma=0}^N \overset{\circ}{C}_\gamma^{(0)} = \int_0^1 p(x) dx. \quad (33)$$

Here (33) is obtained from (28) when $m = 1$. The system (32) - (33) is solved in [8], i.e. here we find the optimal coefficients $\overset{\circ}{C}_\gamma^{(0)}$ in the space $L_2^{(1)}(0, 1)$. The application of this quadrature formula to linear integral equations is given in [40].

Next, let us consider the case $m=2$. For this purpose, by substituting the found optimal coefficients into (31), then minimizing the square of the norm on the coefficients $\overset{\circ}{C}_\gamma^{(1)}$ in the space $L_2^{(2)}(0, 1)$, we obtain the optimal coefficients $\overset{\circ}{C}_\gamma^{(1)} (\beta = \overline{0, N})$.

Continuing this method, we successively find the optimal coefficients $\overset{\circ}{C}_\gamma^{(0)}, \overset{\circ}{C}_\gamma^{(1)}, \dots, \overset{\circ}{C}_\gamma^{(k-1)}$. Substituting these coefficients into (31) and minimizing the square of the norm on the coefficients of $\overset{\circ}{C}_\gamma^{(k-1)}$ in the space $L_2^{(k)}(0, 1)$, we obtain a system for finding the optimal coefficients of $\overset{\circ}{C}_\gamma^{(k-1)} (\beta = \overline{0, N})$:

$$\sum_{\gamma=0}^N \overset{\circ}{C}_\gamma^{(k)} \frac{(h\beta - h\gamma) \text{sign}(h\beta - h\gamma)}{2} + (-1)^k k! \lambda_k = (-1)^k F_{k\beta}, \quad (34)$$

$$\sum_{\gamma=0}^N \overset{\circ}{C}_\gamma^{(k)} = \frac{g_k}{k!} - \sum_{i=0}^{k-1} \sum_{\gamma=0}^N \overset{\circ}{C}_\gamma^{(i)} \frac{(h\gamma)^{k-i}}{(k-i)!}, \quad \beta = \overline{0, N}, \quad (35)$$

Here

$$F_{k\beta} = f_{k\beta} - \sum_{i=0}^{k-1} \sum_{\gamma=0}^N (-1)^i \overset{\circ}{C}_{\gamma}^{(i)} \frac{(h\beta - h\gamma)^{k-i+1} \text{sign}(h\beta - h\gamma)}{2(k-i+1)!}, \quad (36)$$

$$f_{k\beta} = (-1)^k \int_0^1 p(x) \frac{(x - h\beta)^{k+1} \text{sign}(x - h\beta)}{2(k+1)!} dx,$$

$$g_k = \int_0^1 p(x) x^k dx, \quad k = \overline{0, m-1}.$$

5. Optimal coefficients of weighted quadrature derivative formulas

We now solve the system of linear algebraic equations (34) - (35) with respect to $\overset{\circ}{C}_{\beta}^{(k)}$ ($\beta = \overline{0, N}, k = \overline{0, m-1}$) the optimal coefficients of weight quadrature formulas with derivatives.

Let us rewrite the system (34) - (35) in the following form

$$\sum_{\gamma=0}^N \overset{\circ}{C}_{\gamma}^{(k)} G_1[\beta](h\beta - h\gamma) + (-1)^k k! \lambda_k = (-1)^k F_{k\beta}, \quad (37)$$

$$\sum_{\gamma=0}^N \overset{\circ}{C}_{\gamma}^{(k)} = p_k, \quad \beta = \overline{0, N}, k = \overline{0, m-1}. \quad (38)$$

Here λ_k - constant, $F_{k\beta}$ is determined by the equality (36),

$$p_k = \frac{g_k}{k!} - \sum_{i=0}^{k-1} \sum_{\gamma=0}^N \overset{\circ}{C}_{\gamma}^{(i)} (h\gamma)^{k-i} (k-i)!, \quad G_1(h\beta) = \frac{|h\beta|}{2}, \quad k = \overline{0, m-1}.$$

The coefficients of the first term of the first equation depend only on the difference $(h\beta - h\gamma)$. This kind of equations in the continuous case, where instead of the sum there are integrals, are called Wiener - Hopf equations. As is usually done for Wiener - Hopf equations, we assume that $\overset{\circ}{C}_{\gamma}^{(k)}$ is defined everywhere, i.e., $\gamma \in \mathbb{Z}$ and equal to zero if $h\gamma \notin [0, 1]$.

Let us further assume $\overset{\circ}{C}_{\beta}^{(k)} = \overset{\circ}{C}^{(k)}[\beta]$, $G_1[\beta] = G_1(h\beta)$, $F_{k\beta} = F_k[\beta]$. Then the system (37) - (38) is written in the form of convolution equations

$$G_1[\beta] * \overset{\circ}{C}^{(k)}[\beta] + (-1)^k k! \lambda_k = (-1)^k F_k[\beta], \quad \beta = \overline{0, N}, \quad (39)$$

$$\sum_{\gamma=0}^N \overset{\circ}{C}^{(k)}[\gamma] = p_k, \quad (40)$$

$$\overset{\circ}{C}^{(k)}[\beta] = 0, \quad [\beta] \notin [0, 1]. \quad (41)$$

We now proceed to the actual solution of the system (39) - (41). For this purpose, instead of $\overset{\circ}{C}^{(k)}[\beta]$ we introduce the following function

$$u[\beta] = G_1[\beta] * \overset{\circ}{C}^{(k)}[\beta] + (-1)^k k! \lambda_k.$$

Next, we define the function $u[\beta]$ when $\beta \leq 0$ and $\beta \geq N$.

Let $\beta \leq 0$, then by virtue of (40) we obtain

$$u[\beta] = -\frac{h\beta p_k}{2} + a_k^-,$$

a_k^- - unknown.

Let $\beta \geq N$, then

$$u[\beta] = \frac{h\beta p_k}{2} + a_k^+,$$

a_k^+ - unknown.

So, we have defined a function $u[\beta]$ for all values of $[\beta] \in \mathbb{Z}$:

$$u[\beta] = \begin{cases} -\frac{h\beta p_k}{2} + a_k^- & \text{if } \beta \leq 0, \\ (-1)^k F_k[\beta] & \text{if } 0 \leq \beta \leq N, \\ \frac{h\beta p_k}{2} + a_k^+ & \text{if } \beta \geq N. \end{cases} \quad (42)$$

Since in (42) at $\beta = 0$ and $\beta = N$ the left and right parts coincide, so we have

$$a_k^- = (-1)^k F_k[0], \quad a_k^+ = (-1)^k F_k[N] - \frac{p_k}{2}.$$

Then we obtain a new representation of the function $u[\beta]$ in $[\beta] \in \mathbb{Z}$:

$$u[\beta] = \begin{cases} -\frac{h\beta p_k}{2} + (-1)^k F_k[0] & \text{if } \beta \leq 0, \\ (-1)^k F_k[\beta] & \text{if } 0 \leq \beta \leq N, \\ \frac{h\beta p_k}{2} + (-1)^k F_k[N] - \frac{p_k}{2} & \text{if } \beta \geq N. \end{cases} \quad (43)$$

Now we will need the well-known formula [9]:

$$hD_1[\beta] * G_1[\beta] = \delta[\beta], \quad (44)$$

where

$$D_1[\beta] = \begin{cases} 0 & \text{if } [\beta] \geq 2, \\ h^{-2} & \text{if } [\beta] = 1, \\ -2h^{-2} & \text{if } [\beta] = 0, \end{cases} \quad (45)$$

$$\delta[\beta] = \begin{cases} 0 & \text{at } [\beta] \neq 0, \\ 1 & \text{at } [\beta] = 0. \end{cases} \quad (46)$$

By virtue of formulas (43) and (44), we find the optimal coefficients $\overset{\circ}{C}^{(k)}[\beta]$ at $\beta = \overline{0, N}$:

$$\begin{aligned} \overset{\circ}{C}^{(k)}[\beta] &= hD_1[\beta] * u[\beta] = h \sum_{\gamma=-\infty}^{\infty} D_1[\beta - \gamma] u[\gamma] = \\ &= h[(-1)^k \sum_{\gamma=0}^N D_1[\beta - \gamma] F_k[\gamma] + \\ &+ \sum_{\gamma=-\infty}^{-1} D_1[\beta - \gamma] (-\frac{h\gamma p_k}{2} + (-1)^k F_k[0]) + \\ &+ \sum_{\gamma=N+1}^{\infty} D_1[\beta - \gamma] (\frac{h\gamma p_k}{2} + (-1)^k F_k[N] - \frac{p_k}{2})], k = \overline{0, m-1}. \end{aligned}$$

Hence, using formula (45), we obtain

$$\overset{\circ}{C}^k[0] = \frac{p_k}{2} + (-1)^k h^{-1} [F_k[1] - F_k[0]], \quad (47)$$

$$\overset{\circ}{C}^k[\beta] = (-1)^k h^{-1} [F_k[\beta - 1] - 2F_k[\beta] + F_k[\beta + 1]], \beta = \overline{0, N}, \quad (48)$$

$$\overset{\circ}{C}^k[N] = \frac{p_k}{2} + (-1)^k h^{-1} [F_k[N - 1] - F_k[N]], k = \overline{0, m - 1} \quad (49)$$

Thus, the following theorem is proved.

Theorem 3. The optimal coefficients of the quadrature formula of the form (26) in the Sobolev space $L_2^{(m)}(0, 1)$ are determined by formulas (47) - (49).

From Theorem 3, when $p(x) \equiv 1$ we obtain:

Corollary 1: The optimal coefficients of the quadrature formula of the form (26) at $p(x) \equiv 1$ in the Sobolev space $L_2^{(m)}(0, 1)$ are defined by the formulas

$$\overset{\circ}{C}^{(0)}[0] = \frac{h}{2}, \overset{\circ}{C}^{(0)}[\beta] = h, \overset{\circ}{C}^{(0)}[N] = \frac{h}{2},$$

$$\overset{\circ}{C}^{(k)}[0] = \frac{(-1)^k h^{k+1} B_{k+1}}{(k+1)!},$$

$$\overset{\circ}{C}^{(k)}[\beta] = 0, \beta = \overline{1, N-1},$$

$$\overset{\circ}{C}^{(k)}[N] = \frac{(-1)^k h^{k+1} B_{k+1}}{(k+1)!}, k = \overline{1, m-1},$$

where B_{k+1} - the Bernoulli numbers.

Corollary 1 shows that the optimal coefficients are the coefficients of the well-known Euler - Macloean quadrature formula. The optimality of the Euler - Macloean quadrature formula in the space $L_2^{(m)}(0, 1)$ is proved in [8,9].

For completeness we give the following theorem.

Theorem 4. Euler - Macloean quadrature formula

$$\int_0^1 \varphi(x) dx \cong h \sum_{n=1}^N \varphi(nh) + \sum_{\alpha=0}^{m-1} (-1)^\alpha h^{\alpha+1} \frac{B_{\alpha+1}}{(\alpha+1)!} (\varphi^{(\alpha)}(1) - \varphi^{(\alpha)}(0))$$

with the error functional

$$\ell(x) = \mathcal{E}_{[0,1]}(x) - h \sum_{n=1}^N \delta(x - hn) - \sum_{\alpha=0}^{m-1} \frac{h^{\alpha+1} B_{\alpha+1}}{(\alpha+1)!} (\delta^{(\alpha)}(x-1) - \delta^{(\alpha)}(x))$$

is the optimal quadrature formula in Sobolev space $L_2^{(m)}(0, 1)$. The square of the norm of the error functional of the optimal Euler - Macloean quadrature formula is defined by the following equality

$$\|\ell\|_{L_2^{(m)*}(0,1)}^2 = \left(\frac{h}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{\gamma^{2m}}.$$

Here $h = \frac{1}{N}$, $N = 2, 3, \dots$, $B_{\alpha+1}$ - Bernoulli numbers, $B_1 = -\frac{1}{2}$,

$$\delta[\beta] = \begin{cases} 0 & \text{at } [\beta] \neq 0, \\ 1 & \text{at } [\beta] = 0. \end{cases}$$

$$\frac{B_{\alpha+1}}{(\alpha+1)!} = \begin{cases} -(-1)^{\frac{\alpha+1}{2}} \sum_{\gamma \neq 0} \frac{1}{(2\pi\gamma)^{\alpha+1}} & \text{if } \alpha+1 - \text{even number,} \\ 0 & \text{if } \alpha+1 - \text{odd number.} \end{cases}$$

From Theorem 3, after some simplifications, we obtain the following results in the spaces $L_2^{(1)}(0,1), L_2^{(2)}(0,1), L_2^{(3)}(0,1)$.

Theorem 5. In the Sobolev space $L_2^{(1)}(0,1)$, the following quadrature formula is the optimal quadrature formula

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=0}^N \overset{\circ}{C}^{(0)}[\beta]\varphi[\beta],$$

where

$$\begin{aligned} \overset{\circ}{C}^{(0)}[0] &= h^{-1} \int_0^h p(x)(h-x)dx, \\ \overset{\circ}{C}^{(0)}[\beta] &= h^{-1} \int_{h(\beta-1)}^{h(\beta+1)} p(x)[(h\beta-x)\text{sign}(x-h\beta)+h]dx, \beta = \overline{1, N-1}, \\ \overset{\circ}{C}^{(0)}[N] &= h^{-1} \int_{1-h}^1 p(x)(x-1+h)dx. \end{aligned}$$

Theorem 6. In the Sobolev space $L_2^{(2)}(0,1)$, the optimal quadrature formula is the following formula

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=0}^N \left(\overset{\circ}{C}^{(0)}[\beta]\varphi[\beta] + \overset{\circ}{C}^{(1)}[\beta]\varphi'[\beta] \right).$$

Here $\overset{\circ}{C}^{(0)}[\beta]$ - are determined from Theorem 5, and the optimal coefficients $\overset{\circ}{C}^{(1)}[\beta]$ are computed by the following formulas below:

$$\begin{aligned} \overset{\circ}{C}^{(1)}[0] &= -\frac{h^{-1}}{2} \int_0^h p(x)(x-h)xdx, \\ \overset{\circ}{C}^{(1)}[\beta] &= \frac{h^{-1}}{2} \int_{h(\beta-1)}^{h(\beta+1)} p(x)[(x-h\beta)^2\text{sign}(h\beta-x)+h(x-h\beta)]dx, \beta = \overline{1, N-1}, \\ \overset{\circ}{C}^{(1)}[N] &= \frac{h^{-1}}{2} \int_{1-h}^1 p(x)(x-1+h)(x-1)dx. \end{aligned}$$

Theorem 7. The optimal coefficients $\overset{\circ}{C}^{(2)}[\beta]$ of the quadrature formula of the form

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=0}^N \left(\overset{\circ}{C}^{(0)}[\beta]\varphi[\beta] + \overset{\circ}{C}^{(1)}[\beta]\varphi'[\beta] + \overset{\circ}{C}^{(2)}[\beta]\varphi''[\beta] \right),$$

in the space $L_2^{(3)}(0,1)$ are defined by the formulas

$$\begin{aligned} \overset{\circ}{C}^{(2)}[0] &= \frac{h^{-1}}{12} \int_0^h p(x)x(h-x)(2x-h)xdx, \\ &\beta = \overline{1, N-1}, \\ \overset{\circ}{C}^{(2)}[\beta] &= \frac{h^{-1}}{12} \int_{h(\beta-1)}^{h(\beta+1)} p(x)[(2(x-h\beta)^3+h^2(x-h\beta))\text{sign}(h\beta-x)+3h(x-h\beta)^2]dx, \\ \overset{\circ}{C}^{(2)}[N] &= \frac{h^{-1}}{12} \int_{1-h}^1 p(x)(x-1)(2x+h-2)(x+h-1)dx. \end{aligned}$$

6. Application of the quadrature formula with derivatives to linear Fredholm equations of the second kind

Consider the following linear Fredholm equation of the second kind

$$y(x) - \lambda \int_0^1 K(x, s)y(s)ds, x \in [0, 1], \quad (50)$$

where $K(x, s)$ - the kernel of the integral equation, $f(x)$ - the right-hand side, λ - the parameter of the integral equation, $y(x)$ - an unknown function to be determined.

Several numerical methods have been described for the numerical approximation of the solution of (50) (collocation methods, projection methods, Galerkin methods, etc.) and have been extensively investigated in terms of stability and convergence in suitable function spaces, also according to the smoothness properties of the kernel K and the right-hand side f ; (see. [48]-[66]).

To solve equation (50) we apply the quadrature formula (26) and pass the difference grid on the argument x

$$y_i - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta}^{(k)} = f_i, i = \overline{0, N}, m = 1, 2, \dots, \quad (51)$$

Here $C_{\beta}^{(k)}$ - optimal coefficients of the quadrature formula, $f_i = f(x_i)$, $y_i = y(x_i)$, $y_i^{(k)} = y^{(k)}(x_i)$, $x_i = i * h$, $i = \overline{0, N}$, h - grid spacing.

In the system of equations (51) the number of equations is $N + 1$, and the number of unknowns is $(N + 1) * m$, i.e., in addition to the unknown function, its derivatives at nodal points participate in the system of equations. To solve this problem, we differentiate equation (50) $m - 1$ times by the argument x , we have

$$\begin{aligned} y(x) - \lambda \int_0^1 K(x, s)y(s)ds &= f(x), x \in [0, 1], \\ y'(x) - \lambda \int_0^1 K'_x(x, s)y(s)ds &= f'(x), \\ &\dots \\ y^{(m-1)}(x) - \lambda \int_0^1 K_x^{(m-1)}(x, s)y(s)ds &= f^{(m-1)}(x), m = 1, 2, \dots. \end{aligned} \quad (52)$$

Now applying the quadrature formula to the system (52), we obtain

$$\begin{aligned} y_i - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta}^{(k)} &= f_i, i = \overline{0, N}, \\ y'_i - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta}^{(k)} &= f'_i, \\ &\dots \\ y_i^{(m-1)} - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta}^{(k)} &= f_i^{(m-1)}, m = 1, 2, \dots. \end{aligned} \quad (53)$$

Thus, we have a system of linear algebraic equations with respect $y_i^{(k)}$ ($i = \overline{0, N}, k = \overline{0, m-1}$). The values of the desired function $y_i = y_i^{(0)}$ ($i = \overline{0, N}$).

7. Numerical results

The following examples are to solve integral equations by the quadrature method using the optimal quadrature formula with derivatives at $m = 2, 4, 8$. Compare the results with the exact solution and the results of foreign researchers using the absolute error:

$$E_m = \max_x |y(x) - y_{ex}(x)|.$$

Here E_m - maximum absolute error, $y(x)$ - approximate solution, $y_{ex}(x)$ -exact solution.

It should be noted, according to the above algorithm, a program in Maple language has been made. All calculations are performed using 32 significant digits.

Example 1. In (50) $K(x, s) = -x \cdot (e^{x \cdot s} - 1), f(x) = e^x - x, \lambda = 1, a = 0, b = 1$. Then the integral equation (6.1) will take the following form:

$$y(x) + \int_0^1 x \cdot (e^{x \cdot s} - 1) \cdot y(s) ds = e^x - x, x \in [0, 1] \tag{54}$$

Exact solution of the integral equation (54):

$$y_{ex}(x) = 1.$$

The results for this example are summarized in Table 1.

Table 1

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0.0	1.000	1.000	1.000	1.000	0.00E + 00	0.00E + 00	0.00E + 00
0.1	1.000	1.000	1.000	1.000	1.60E - 31	0.00E + 00	1.80E - 31
0.2	1.000	1.000	1.000	1.000	4.00E - 32	8.00E - 32	4.00E - 31
0.3	1.000	1.000	1.000	1.000	1.70E - 31	2.10E - 31	1.40E - 31
0.4	1.000	1.000	1.000	1.000	1.00E - 31	1.00E - 31	1.00E - 31
0.5	1.000	1.000	1.000	1.000	2.00E - 32	0.00E + 00	5.00E - 32
0.6	1.000	1.000	1.000	1.000	1.00E - 31	2.00E - 31	2.80E - 31
0.7	1.000	1.000	1.000	1.000	0.00E + 00	1.00E - 31	3.00E - 31
0.8	1.000	1.000	1.000	1.000	5.00E - 32	1.60E - 31	2.00E - 31
0.9	1.000	1.000	1.000	1.000	1.00E - 31	1.00E - 31	1.00E - 31
1.0	1.000	1.000	1.000	1.000	1.00E - 31	1.00E - 31	1.00E - 31
				MAE	1.70E - 31	3.00E - 31	4.00E - 31

Here ES-Exact solution, OQF-Optimal quadrature formulas, MAE-Maximum absolute error.

The authors of [29] solved this example using a modified multistage average integral method and obtained a result with maximum absolute error $E_m = 3.44 \cdot 10^{-15}$ for equidistant collocation nodes $N = 13$.

On the basis of using the method of integral replacement with the twelfth-order quadrature formula, the authors of [45] obtained the result with the maximum absolute error $E_m = 4.44 \cdot 10^{-16}$ at the number of integration intervals $N = 20$.

Our method gave the result with the value of the maximum absolute error $E_m = 4.0 \cdot 10^{-31}$ at $m = 1$.

Example 2. In (50), $K(x, s) = x \cdot s, f(x) = \cos(x), \lambda = 1, a = 0, b = 1$. Then the integral equation (50) takes the following form:

$$y(x) - \int_0^1 x \cdot s \cdot y(s) ds = \cos(x), x \in [0, 1]. \tag{55}$$

Exact solution of the integral equation (55):

$$y_{ex}(x) = \cos(x) + 1.5 \cdot (\sin(1) + \cos(1) - 1) \cdot x.$$

The results for this example are shown in Table 2.

To solve the Fredholm integral equation of the second kind, the authors of [30] developed a method using spline technology. Using this method, they obtained the result for this example. The maximum absolute error was $E_m = 2.0 \cdot 10^{-5}$ at $N = 11, n = 10$. Our method gave the result with the value of maximum absolute error $E_m = 1.62 \cdot 10^{-8} - 2.43 \cdot 10^{-19}$ at $m = \overline{3, 8}$.

Table 2

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	1,000	1,000	1,000	1,000	0,00E + 00	0,00E + 00	0,00E + 00
0,1	1,052	1,052	1,052	1,052	1,62E - 09	3,85E - 13	2,43E - 20
0,2	1,095	1,095	1,095	1,095	3,23E - 09	7,69E - 13	4,86E - 20
0,3	1,127	1,127	1,127	1,127	4,85E - 09	1,15E - 12	7,29E - 20
0,4	1,150	1,150	1,150	1,150	6,46E - 09	1,54E - 12	9,72E - 20
0,5	1,164	1,164	1,164	1,164	8,08E - 09	1,92E - 12	1,21E - 19
0,6	1,169	1,169	1,169	1,169	9,69E - 09	2,31E - 12	1,46E - 19
0,7	1,166	1,166	1,166	1,166	1,13E - 08	2,69E - 12	1,70E - 19
0,8	1,155	1,155	1,155	1,155	1,29E - 08	3,08E - 12	1,94E - 19
0,9	1,137	1,137	1,137	1,137	1,45E - 08	3,46E - 12	2,19E - 19
1,0	1,113	1,113	1,113	1,113	1,62E - 08	3,85E - 12	2,43E - 19
				MAE	1,62E - 08	3,85E - 12	2,43E - 19

Example 3. In (50), $K(x, s) = x^2 - x - s^2 + s$, $f(x) = -2x^3 + 3x^2 - x$, $\lambda = 1$, $a = 0$, $b = 1$. Then the integral equation (50) will take the following form:

$$y(x) + \int_0^1 (x^2 - x - s^2 + s)y(s)ds = -2x^3 + 3x^2 - x, x \in [0, 1]. \tag{56}$$

Exact solution of the integral equation (56):

$$y_{ex}(x) = -2x^3 + 3x^2 - x.$$

The results for this example are given in Table 3.

Table 3

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	0,000	0,000	0,000	0,000	4,30E - 33	2,47E - 32	2,47E - 32
0,1	-0,072	-0,072	-0,072	-0,072	1,10E - 32	5,00E - 33	5,00E - 33
0,2	-0,096	-0,096	-0,096	-0,096	7,00E - 33	4,00E - 33	4,00E - 33
0,3	-0,084	-0,084	-0,084	-0,084	6,00E - 33	1,20E - 32	1,20E - 32
0,4	-0,048	-0,048	-0,048	-0,048	1,00E - 33	3,00E - 33	3,00E - 33
0,5	0,000	0,000	0,000	0,000	7,80E - 34	1,46E - 32	1,46E - 32
0,6	0,048	0,048	0,048	0,048	2,00E - 33	3,00E - 33	3,00E - 33
0,7	0,084	0,084	0,084	0,084	1,70E - 32	4,00E - 33	4,00E - 33
0,8	0,096	0,096	0,096	0,096	1,00E - 32	1,30E - 32	1,30E - 32
0,9	0,072	0,072	0,072	0,072	1,00E - 33	8,00E - 33	8,00E - 33
1,0	0,000	0,000	0,000	0,000	4,38E - 33	2,47E - 32	2,47E - 32
				MAE	1,70E - 32	2,47E - 32	2,47E - 32

The authors of [42] developed a graph-theoretic polynomial using Hosoi polynomials and solved the integral equation (56). They obtained the result with the maximum absolute error $E_m = 8.88 \cdot 10^{-16}$ at $n = 3$.

The authors of [43] solved the integral equation (56) and obtained the result with the maximum absolute error $E_m = 4.97 \cdot 10^{-4}$ at $N = 8$.

Our method gave the result with the value of the maximum absolute error $E_m = 1.0 \cdot 10^{-32}$ at $m = 1$.

Example 4. In (50) $K(x, s) = x \cdot e^s$, $f(x) = e^{-x}$, $\lambda = 1$, $a = 0$, $b = 1$. Then the integral equation (50) will take the following form:

$$y(x) + \int_0^1 x \cdot e^s \cdot y(s)ds = e^{-x}, x \in [0, 1]. \tag{57}$$

Exact solution of the integral equation (57):

$$y_{ex}(x) = e^{-x} - \frac{x}{2}.$$

The results for this example are given in Table 4.

To solve (57) integral equation, the author [32] of the paper applied polynomial method using Boo-Baker polynomials and obtained the result with maximum absolute error $E_m = 7.1 \cdot 10^{-3}$ at $N = 11$.

Table 4

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	1,000	1,000	1,000	1,000	0,00E + 00	0,00E + 00	0,00E + 00
0,0	1,000	1,000	1,000	1,000	0,00E + 00	0,00E + 00	0,00E + 00
0,1	0,855	0,855	0,855	0,855	1,39E - 08	3,31E - 12	2,09E - 19
0,2	0,719	0,719	0,719	0,719	2,78E - 08	6,62E - 12	4,18E - 19
0,3	0,591	0,591	0,591	0,591	4,17E - 08	9,92E - 12	6,27E - 19
0,4	0,470	0,470	0,470	0,470	5,56E - 08	1,32E - 11	8,35E - 19
0,5	0,357	0,357	0,357	0,357	6,95E - 08	1,65E - 11	1,04E - 18
0,6	0,249	0,249	0,249	0,249	8,34E - 08	1,98E - 11	1,25E - 18
0,7	0,147	0,147	0,147	0,147	9,73E - 08	2,32E - 11	1,46E - 18
0,8	0,049	0,049	0,049	0,049	1,11E - 07	2,65E - 11	1,67E - 18
0,9	-0,043	-0,043	-0,043	-0,043	1,25E - 07	2,98E - 11	1,88E - 18
1,0	-0,132	-0,132	-0,132	-0,132	1,39E - 07	3,31E - 11	2,09E - 18
				MAE	1,39E - 07	3,31E - 11	2,09E - 18

The authors of [36] also applied the polynomial method. They used the Tushar polynomial and obtained the result with the maximum absolute error $E_m = 2.0 \cdot 10^{-3}$ at $N = 11$ for $n = 2$.

Using the Gauss - Lobatto quadrature formula, the author of [41] obtained good results. The maximum absolute error was $E_m = 1.28 \cdot 10^{-15}$ at $N = 11$ and $n = 6$.

The authors of [44] applied the polynomial method using Bernstein polynomials to solve the integral equation (57) and obtained the result with the maximum absolute error $E_m = 3.5 \cdot 10^{-3}$ at $N = 11$ and $n = 6$.

Our method gave the result with the value of maximum absolute error $E_m = 2.09 \cdot 10^{-18}$ at $m = 8$.

Example 5. In (50), $K(x, s) = e^{x+s}, f(x) = e^{-x}, \lambda = 1, a = 0, b = 1$. Then the integral equation (6.1) will take the following form:

$$y(x) + \int_0^1 e^{x+s} \cdot y(s)ds = e^{-x}, x \in [0, 1]$$

(58)

Exact solution of the integral equation (58):

$$y_{ex}(x) = e^{-x} + \frac{2e^x}{3 - e^2}.$$

The results for this example are shown in Table 5.

Table 5

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	0,544	0,544	0,544	0,544	1,27E - 07	3,01E - 11	1,90E - 18
0,1	0,401	0,401	0,401	0,401	1,40E - 07	3,33E - 11	2,10E - 18
0,2	0,262	0,262	0,262	0,262	1,55E - 07	3,68E - 11	2,32E - 18
0,3	0,126	0,126	0,126	0,126	1,71E - 07	4,07E - 11	2,57E - 18
0,4	-0,009	-0,009	-0,009	-0,009	1,89E - 07	4,50E - 11	2,84E - 18
0,5	-0,145	-0,145	-0,145	-0,145	2,09E - 07	4,97E - 11	3,14E - 18
0,6	-0,281	-0,281	-0,281	-0,281	2,31E - 07	5,49E - 11	3,47E - 18
0,7	-0,421	-0,421	-0,421	-0,421	2,55E - 07	6,07E - 11	3,83E - 18
0,8	-0,565	-0,565	-0,565	-0,565	2,82E - 07	6,71E - 11	4,24E - 18
0,9	-0,714	-0,714	-0,714	-0,714	3,11E - 07	7,42E - 11	4,68E - 18
1,0	-0,871	-0,871	-0,871	-0,871	3,44E - 07	8,19E - 11	5,17E - 18
				MAE	3,44E - 07	8,19E - 11	5,17E - 18

The authors of [33] solved the integral equation (58) using the integral method of mean values and obtained the result with the maximum absolute error $E_m = 2.0 \cdot 10^{-5}$ at $N = 11$.

Our method gave the result with the value of maximum absolute error $E_m = 3.44 \cdot 10^{-7} - 5.17 \cdot 10^{-18}$ at $m = \overline{2, 8}$.

Example 6. In (50) $K(x,s) = e^{x-s-12}, f(x) = \cos(x) - \frac{e^{-x-12}}{2}[e \cdot \sin(1) + e \cdot \cos(1) - 1], \lambda = 1, a = 0, b = 1$. Then the integral equation (50) will take the following form:

$$y(x) + \int_0^1 e^{x-s-12} \cdot y(s)ds = \cos(x) - \frac{e^{-x-12}}{2} \cdot (e \cdot \sin(1) + e \cdot \cos(1) - 1), x \in [0, 1]. \tag{59}$$

Exact solution of the integral equation (59):

$$y_{ex}(x) = \cos(x).$$

The results for this example are given in Table 6.

Table 6

x_i	ES	QQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	1,000	1,000	1,000	1,000	4,02E - 13	9,56E - 17	6,04E - 24
0,1	0,995	0,995	0,995	0,995	3,63E - 13	8,65E - 17	5,46E - 24
0,2	0,980	0,980	0,980	0,980	3,29E - 13	7,83E - 17	4,94E - 24
0,3	0,955	0,955	0,955	0,955	2,98E - 13	7,08E - 17	4,47E - 24
0,4	0,921	0,921	0,921	0,921	2,69E - 13	6,41E - 17	4,05E - 24
0,5	0,878	0,878	0,878	0,878	2,44E - 13	5,80E - 17	3,66E - 24
0,6	0,825	0,825	0,825	0,825	2,20E - 13	5,25E - 17	3,31E - 24
0,7	0,765	0,765	0,765	0,765	1,99E - 13	4,75E - 17	3,00E - 24
0,8	0,697	0,697	0,697	0,697	1,80E - 13	4,30E - 17	2,71E - 24
0,9	0,622	0,622	0,622	0,622	1,63E - 13	3,89E - 17	2,45E - 24
1,0	0,540	0,540	0,540	0,540	1,48E - 13	3,52E - 17	2,22E - 24
				MAE	4,02E - 13	9,56E - 17	6,04E - 24

The authors of [34] solved the integral equation (59) by applying a new type of spline function of fractional order and obtained the result with the maximum absolute error $E_m = 3.1 \cdot 10^{-6}$ at $N = 11$ and $n = 2$. Our method gave the result with the value of maximum absolute error $E_m = 7.06 \cdot 10^{-9} - 6.04 \cdot 10^{-24}$ at $m = \overline{1,8}$.

Example 7. In (50) $K(x,s) = e^{2x-\frac{5s}{3}}, f(x) = e^{2x+\frac{1}{3}}, \lambda = 1, a = 0, b = 1$. Then the integral equation (50) will take the following form:

$$y(x) + \frac{1}{3} \int_0^1 e^{2x-\frac{5s}{3}} \cdot y(s)ds = e^{2x+\frac{1}{3}}, x \in [0,1]. \tag{60}$$

Exact solution of the integral equation (60):

$$y_{ex}(x) = e^{2x}.$$

The results for this example are given in Table 7.

Table 7

x_i	ES	QQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	1,000	1,000	1,000	1,000	1,16E-06	1,10E-09	1,11E-15
0,1	1,221	1,221	1,221	1,221	1,41E-06	1,34E-09	1,36E-15
0,2	1,492	1,492	1,492	1,492	1,72E-06	1,64E-09	1,66E-15
0,3	1,822	1,822	1,822	1,822	2,11E-06	2,01E-09	2,03E-15
0,4	2,226	2,226	2,226	2,226	2,57E-06	2,45E-09	2,48E-15
0,5	2,718	2,718	2,718	2,718	3,14E-06	2,99E-09	3,02E-15
0,6	3,320	3,320	3,320	3,320	3,84E-06	3,66E-09	3,69E-15
0,7	4,055	4,055	4,055	4,055	4,69E-06	4,47E-09	4,51E-15
0,8	4,953	4,953	4,953	4,953	5,73E-06	5,45E-09	5,51E-15
0,9	6,050	6,050	6,050	6,050	6,99E-06	6,66E-09	6,73E-15
1,0	7,389	7,389	7,389	7,389	8,54E-06	8,14E-09	8,22E-15
				MAE	8,54E-06	8,14E-09	8,22E-15

The authors of [35] solved the integral equation (60) based on a special representation of vector forms of triangular functions and obtained the result with the maximum absolute error $E_m = 6.4 \cdot 10^{-7}$ at $m = 1024$.

The authors of [39] using a combination of Taylor series and block-plus functions solved the same integral equation (60). They obtained the result with the maximum absolute error $E_m = 4.6 \cdot 10^{-4}$ at $N = 80$.

The authors of [40] solved the integral equation (60) using Chebyshev polynomial approximation and obtained the result with the maximum absolute error $E_m = 0.49 \cdot 10^{-4}$ at $N = 10$.

With the scheme based on Legendre polynomials and Legendre wavelets, the authors of [43] solved the integral equation (60) and obtained the result with the maximum absolute error $E_m = 10^{-4}$ at $N = 11, M = 8$ and $k = 3$.

Our method gave the result with the maximum absolute error from $E_m = 8.14 \cdot 10^{-9}$ to $E_m = 8.22 \cdot 10^{-15}$ at $m = 4, 8$.

Example 8. In (50) $K(x,s) = x \cdot s, f(x) = e^x - x, \lambda = 1, a = 0, b = 1$. Then the integral equation (50) will take the following form:

$$y(x) - \int_0^1 x \cdot s \cdot y(s)ds = e^x - x, x \in [0,1] \tag{61}$$

Exact solution of the integral equation (61):

$$y_{ex}(x) = e^x.$$

The results for this example are shown in Table 8.

The author [37] solved the integral equation (61) using the Pell-Lucas series method and obtained the result with the maximum absolute error $E_m = 7.5 \cdot 10^{-8}$ at $N = 11$.

Table 8

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	-0,186	-0,186	-0,186	-0,186	6,00E - 32	2,30E - 31	2,70E - 31
0,1	-0,205	-0,205	-0,205	-0,205	0,00E + 00	6,00E - 32	1,00E - 31
0,2	-0,227	-0,227	-0,227	-0,227	1,00E - 32	2,90E - 31	3,00E - 31
0,3	-0,250	-0,250	-0,250	-0,250	2,20E - 31	2,60E - 31	1,80E - 31
0,4	-0,277	-0,277	-0,277	-0,277	4,90E - 31	6,50E - 31	6,60E - 31
0,5	-0,306	-0,306	-0,306	-0,306	3,40E - 31	5,30E - 31	6,10E - 31
0,6	-0,338	-0,338	-0,338	-0,338	3,30E - 31	5,20E - 31	6,60E - 31
0,7	-0,374	-0,374	-0,374	-0,374	1,20E - 31	5,00E - 31	4,90E - 31
0,8	-0,413	-0,413	-0,413	-0,413	1,20E - 31	6,00E - 32	1,70E - 31
0,9	-0,456	-0,456	-0,456	-0,456	5,00E - 32	5,00E - 32	1,30E - 31
1,0	-0,504	-0,504	-0,504	-0,504	6,00E - 32	8,00E - 32	6,00E - 32
				MAE	4,90E - 31	6,50E - 31	6,60E - 31

Our method gave the result with the maximum absolute error $E_m = 3.57 \cdot 10^{-11} - 2.25 \cdot 10^{-18}$ at $m = \overline{4,8}$.

Example 9. In (50) $K(x,s) = e^{\frac{x-s}{2}}, f(x) = x, \lambda = \frac{1}{2}, a = 0, b = 1$. Then the integral equation (50) will take the following form:

$$y(x) - \frac{1}{2} \int_0^1 e^{\frac{x-s}{2}} y(s) ds = x, x \in [0,1].$$

(62)

Exact solution of the integral equation (62):

$$y_{ex}(x) = x + (4 \cdot \sqrt{e} - 6) \cdot e^{\frac{x-1}{2}}.$$

The results for this example are shown in Table 9.

Table 9

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	1,000	1,000	1,000	1,000	0,00E + 00	0,00E + 00	0,00E + 00
0,1	1,105	1,105	1,105	1,105	1,50E - 08	3,57E - 12	2,25E - 19
0,2	1,221	1,221	1,221	1,221	3,00E - 08	7,14E - 12	4,51E - 19
0,3	1,350	1,350	1,350	1,350	4,50E - 08	1,07E - 11	6,76E - 19
0,4	1,492	1,492	1,492	1,492	6,00E - 08	1,43E - 11	9,01E - 19
0,5	1,649	1,649	1,649	1,649	7,50E - 08	1,78E - 11	1,13E - 18
0,6	1,822	1,822	1,822	1,822	8,99E - 08	2,14E - 11	1,35E - 18
0,7	2,014	2,014	2,014	2,014	1,05E - 07	2,50E - 11	1,58E - 18
0,8	2,226	2,226	2,226	2,226	1,20E - 07	2,86E - 11	1,80E - 18
0,9	2,460	2,460	2,460	2,460	1,35E - 07	3,21E - 11	2,03E - 18
1,0	2,718	2,718	2,718	2,718	1,50E - 07	3,57E - 11	2,25E - 18
				MAE	1,50E - 07	3,57E - 11	2,25E - 18

The authors of [38] solved the integral equation (62) using the Taylor series expansion method and obtained the result with the maximum absolute error $E_m = 8.88 \cdot 10^{-16}$ in $N = 11$ and $p = 12$.

Our method gave the result with the maximum absolute error $E_m = 3.84 \cdot 10^{-17} - 2.43 \cdot 10^{-21}$ at $m = \overline{6,8}$.

Example 10. In (50) $K(x, s) = x + s, f(x) = e^x + (1 - e) \cdot x - 1, \lambda = 1, a = 0, b = 1$. Then the integral equation (50) will take the following form:

$$y(x) - \int_0^1 (x + s) \cdot y(s) ds = e^x + (1 - e) \cdot x - 1, x \in [0, 1]. \tag{63}$$

Exact solution of the integral equation (63):

$$y_{ex}(x) = e^x.$$

The results for this example are shown in Table 10.

Table 10

x_i	ES	OQF			Absolute error		
		$m = 2$	$m = 4$	$m = 8$	$m = 2$	$m = 4$	$m = 8$
0,0	0,361	0,361	0,361	0,361	$6,26E - 09$	$3,73E - 13$	$1,47E - 21$
0,1	0,479	0,479	0,479	0,479	$6,59E - 09$	$3,92E - 13$	$1,55E - 21$
0,2	0,599	0,599	0,599	0,599	$6,92E - 09$	$4,12E - 13$	$1,63E - 21$
0,3	0,719	0,719	0,719	0,719	$7,28E - 09$	$4,33E - 13$	$1,71E - 21$
0,4	0,841	0,841	0,841	0,841	$7,65E - 09$	$4,55E - 13$	$1,80E - 21$
0,5	0,963	0,963	0,963	0,963	$8,04E - 09$	$4,79E - 13$	$1,89E - 21$
0,6	1,087	1,087	1,087	1,087	$8,46E - 09$	$5,03E - 13$	$1,99E - 21$
0,7	1,212	1,212	1,212	1,212	$8,89E - 09$	$5,29E - 13$	$2,09E - 21$
0,8	1,338	1,338	1,338	1,338	$9,35E - 09$	$5,56E - 13$	$2,19E - 21$
0,9	1,466	1,466	1,466	1,466	$9,83E - 09$	$5,85E - 13$	$2,31E - 21$
1,0	1,595	1,595	1,595	1,595	$1,03E - 08$	$6,15E - 13$	$2,43E - 21$
				MAE	$1,03E - 08$	$6,15E - 13$	$2,43E - 21$

The authors of [38] solved the integral equation (63) and obtained the result with the maximum absolute error $E_m = 5.6 \cdot 10^{-9}$ in $N = 11$ and $p = 8$.

The authors of [39], in addition to the integral equation (60), solved the integral equation (63) and obtained the result with maximum absolute error $E_m = 2.84 \cdot 10^{-6}$ at $N = 80$.

The authors of [46] solved the integral equation (63) using general Legendre wavelets and obtained the result with the highest absolute error $E_m = 0.7 \cdot 10^{-13}$ in $M = 11$ and $N = 4$. Our method gave the result with the maximum absolute error $E_m = 8.82 \cdot 10^{-18}$ at $m = 8$.

8. Conclusion

The present work is devoted to the solution of the optimization problem of weight quadrature formulas with derivatives in the Sobolev space and to determine by means of examples on linear Fredholm integral equations of the second kind the theoretical validity of the constructed quadrature formula comparing the results of works performed by other authors.

The main results of the study are as follows:

- the square of the norm of the error functional of the considered quadrature formula was calculated using the extremal function;
- minimized this norm by the coefficients of the quadrature formula and obtained a system of linear algebraic equations for finding the optimal coefficients of the quadrature formula;
- the singularity of solutions of this system is proved;
- an algorithm for the solution of this system is constructed with the help of which the optimal coefficients of weight quadrature formulas with derivatives are found;
- an algorithm of application of weight quadrature formulas with derivatives for an approximate solution of linear Fredholm integral equations of the second kind is given;
- simplified formulas for the optimal coefficients of the quadrature formula with derivatives in the spaces $L_2^{(1)}(0, 1), L_2^{(2)}(0, 1)$ and $L_2^{(3)}(0, 1)$ were obtained;

- in the end, the numerical results are compared with the results of other authors.

References

1. Sobolev S.L. *Introduction to the theory of cubature formulas*. Moscow: Nauka, 1974, p. 808.
2. Sobolev S.L., Vaskevich V.L. *Cubature formulas*. -Novosibirsk: izvo IM SB RAS, 1996, p. 484.
3. Bakhvalov N.S. *About optimal methods of solving problems*. Appl. Math. V. 1, 1968, Vol. 3.
4. Polovinkin V.I. *Weighted cubature formulas in the periodic case*. Mat. Zametki. 1968, Vol. 3, No. 3, pp. 319-326.
5. Polovinkin V.I. *Weighted cubature formulas*. DAN USSR. 1968. Vol. 179, No 3, pp. 42-44
6. Ramazanov M. D. *Theory of lattice cubature formulas with a bounded boundary layer*. Ufa 2009, 178 p.
7. Noskov M. V. *About Cartesian products of cubature formulas*. Kn. *Theory of cubature formulas and computational mathematics*. Novosibirsk. Nauka. 1980, pp. 114-116.
8. Shadimetov, Kh.M. *On the optimal lattice quadrature and cubature formulas*. Doklady Akademii Nauk, 2001, 376(5), pp. 597-600.
9. Shadimetov, Kh.M. *Optimal lattice quadrature and cubature formulas*. Doklady Mathematics, 2001, 63(1), pp. 92-94.
10. Ramazanov, M.D., Shadimetov, Kh.M. *Weight optimal formulas in Sobolev's periodic space*. Doklady Akademii Nauk, 1999, 368(4), pp. 453-455.
11. Ramazanov, M.D., Shadimetov, Kh.M. *Weight Optimal Cubature Formulas in Periodic Sobolev Space*. Doklady Mathematics, 1999, 60(2), pp. 217-219.
12. Sard A. *Best approximate integration formulas, best approximate formulas*. // American J. of Math. 1949. -LXXI. pp. 80-91.
13. Nikolsky S.M. *Quadrature formulas*. -Moscow: Nauka, 1988. p. 256.
14. Lanzara F. *On optimal quadrature formulae* // Journal of Ineq. Appl. 2000. vol.5. pp. 201-225.
15. Boltaev A. Hayotov A.R. Shadimetov Kh.M. *Construction of Optimal Quadrature Formulas Exact for Exponential-trigonometric Functions by Sobolev's Method*. // Acta Mathematica Sinica, English Series. 2021. -vol. 37. No. 7. -pp. 1066-1088.
16. Shadimetov Kh., Boltaev A., Parovik R. *Optimization of the approximate integration for mula using the discrete analogue of a high-order differential operator*, // Mathematics, 2023, vol. 11, no.14, 3114. DOI:https://doi.org/10.3390/math11143114.
17. Hayotov A.R., Milovanovic G.V., Shadimetov Kh.M. *Optimal quadratures in the sense of Sard in a Hilbert space* // Applied Mathematics and Computation. Elsevier, 2015. Vol.259. -pp.637-653.
18. Shadimetov Kh.M., Akhmedov D.M. *Approximate Solution of a Singular Integral Equation Using the Sobolev Method*. // Lobachevskii Journal of Mathematics. 2022. -vol. 43, issue. 2. -pp. 496-505. ISSN 1995-0802.
19. Shadimetov, K.M., Hayotov, A.R., Azamov, S.S. *Optimal quadrature formula in $K_2(P_2)$ space*. Applied Numerical Mathematics, 2012, 62(12), pp. 1893-1909.
20. Cabada, A., Hayotov, A.R., Shadimetov, K.M. *Construction of D^m -splines in $L_2^{(m)}(0,1)$ space by Sobolev method*. Applied Mathematics and Computation, 2014, 244, pp. 542-551.
21. Shadimetov, K.M., Nuraliev, F.A. *Optimal formulas of numerical integration with derivatives in Sobolev space*. Journal of Siberian Federal University – Mathematics and Physics, 2018, 11(6), pp.
22. Shadimetov, K.M., Hayotov, A.R., Nuraliev, F.A. *Optimal interpolation formulas with derivative in the space $L_2^{(m)}(0,1)$* . Filomat, 2019, 33(17), pp. 5661-5675.
23. Shadimetov, K.M., Boltaev, A.K. *An exponential-trigonometric spline minimizing a seminorm in a Hilbert space*. Advances in Difference Equations, 2020, Vol. 2020, 352.
24. Hayotov, A.R., Jeon, S., Lee, C.-O., Shadimetov, Kh.M. *Optimal Quadrature Formulas for Non-periodic Functions in Sobolev Space and Its Application to CT Image Reconstruction*. Filomat, 2021, 35(12), pp. 4177-4195.
25. Akhmedov, D.M., Shadimetov, K.M. *Optimal quadrature formulas for approximate solution of the first kind singular integral equation with Cauchy kernal*. Studia Universitatis Babes-Bolyai Mathematica, 2022, 67(3), pp. 633-651.
26. Shadimetov, K.M., Daliev, B.S. *Optimal formulas for the approximate- analytical solution of the general Abel integral equation in the Sobolev space*. Results in Applied Mathematics, 2022, 15, 100276.
27. Shadimetov, K.M., Boltaev, A.K. *An Exponential-Trigonometric Optimal Interpolation Formula*. Lobachevskii Journal of Mathematics, 2023, 44(10), pp. 4379-4392.
28. Babushka I. *Optimal quadrature formulas*. DAN SSSR, 149, 1963, pp. 227-230.
29. Ahmad Molabahrani¹, Sakineh Niasadegh, *Multistage Integral Mean Value Method for the Fredholm Integral Equations of the Second Kind*, Thai Journal of Mathematics, Vol. 18, Number 2, 2020, pp 751-763.

30. Ahmed Flayih Jaber, Adil Alrammahi. *Spline technique for second type Fredholm integral equation*. Journal of Kufa for Mathematics and Computer, vol. 7, No. 1, Mar., 2020, pp. 31-40.
31. M. Arrai, C. Allouch, H. Bouda, M. Tahrichi. *Legendre-Kantorovich method for Fredholm integral equations of the second kind*. Mathematical modeling and computing, 2022, Vol. 9, No. 3, pp. 471–482.
32. Balasim Taha Abdilrazak. *Approximate solutions for solving two types linear integral equations by using Boubaker polynomials method*, Journal of Engineering and Sustainable Development Vol. 21, No. 6, 2017.
33. Bulent Yilmaz, Yasin Cetin. *Numerical solutions of the Fredholm integral equations of the second type*. NTMSCI 5, No. 3, 284-292 (2017).
34. Faraidun K. Hamasalih, Rahel J. Qadir. *On the numerical solution of Volterra and Fredgholm integral equations using the fractional spline function method*. Journal of numerical analysis and approximation theory, vol. 51 (2022) no. 2, pp. 167–180.
35. S. Hatamzadeh-Varmazyar, Z. Masouri. *Numerical solution of second kind Volterra and Fredholm integral equations based on a direct method via triangular functions*, Int. J. Industrial mathematics, Vol. 11, No. 2, 2019, pp. 79-87.
36. Jalil Talab Abdullah1, Ali Hussein Shuaa Al-Taie. *A comparison of Numerical Solutions for Linear Fredholm Integral Equation of the Second Kind*. IOP Conf. Series: Journal of Physics: Conf. Series 1279 (2019) 012067, p. 9.
37. Lukonde Peter Alpha. *Pell-Lucas Series Solution for Fredholm Integral Equations of the Second Kind*. International Journal of Science and Research ISSN: 2319-7064 SJIF (2020): 7.803, pp. 1376-1383.
38. Majid Amirfakhrian, Seyyed Mahmood Mirzaei. *A modified Taylor-series for solving a Fredholm integral equation of the second kind*. Mathematical Analysis and its Contemporary Applications, Volume 4, Issue 4, 2022, pp. 39–48.
39. K. Maleknejad, Y. Mahmoudi. *Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block - Pulse functions*, Applied Mathematics and Computation, 149(2004), pp. 799-806.
40. M. Gulsu, Y. Ozturk. *Chebyshev polynomial approximation for solving the second kind linear Fredholm integral equation*, Erciyes University Journal of Institute of Science and Technology, 26(3), pp. 203-216(2010).
41. Peng Guo, *Numerical Simulation for Fredholm Integral Equation of the Second Kind*, Journal of Applied Mathematics and Physics, 2020, 8, pp. 2438-2446.
42. H. S. Ramane, S.C. Shiralashetti, R. A. Mundewadi, R. B. Jummanner. *Numerical Solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs*, American Journal of Numerical Analysis, 2017, Vol. 5, No. 1, 11-15.
43. Ravikiran A. Mundewadi, Bhaskar A. Mundewadi. *Legendre Wavelet Collocation Method for the Numerical Solution of Integral and Integro-Differential Equations*. International Journal of Advanced in Management, Technology and Engineering Sciences. Volume 8, Issue 1, JAN/2018, pp.151-170.
44. Shihab S. N., Asmaa A. A., Mohammed Ali M. N. *Collocation Orthonormal Bernstein Polynomials method for Solving Integral Equations*. Mathematical Theory and Modeling, Vol. 5, No 12, 2015, p. 9.
45. N. K. Volosova, K. A. Volosov, A. K. Volosova, M. I. Karlov, D. F. Pastuhov, Yu. F. Pastuhov. *Solution of the Fredholm Integral Equations Method of Replacing the Integral by a Quadrature with the Twelveth Order of Error in Matrix Form*, Bulletin of Perm University. Mathematics. Mechanics. Computer Science, 2022, 4(59), pp. 9–17.
46. Xing Tao Wang and Yuan Min Li. *Numerical Solution of Fredholm Integral Equation of the Second Kind by General Legendre Wavelets*, International Journal of Innovative Computing, Information and Control, Vol. 8, No. 1(B), Jan. 2012, pp. 799-805.
47. Kholmat Shadimetov, Khojiakbar Usmanov. *Quadrature Method for Linear Integral Equations in $L_2^{(1)}(0,1)$* . AIP Conference Proceedings, 3004, 060050 (2024)
48. Atkinson, K.E. *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge (Series number 4), 2009, 551 p.
49. Davis, P.J., Rabinowitz, P. *Methods of Numerical Integration Second Edition*. Dover Publications, Mineola, New York, 2007, 612 p.
50. DeBonis, M.C., Laurita, C. *Numerical treatment of second kind Fredholm integral equations systems on bounded intervals*. J. Comput. Appl. Math. Vol. 217, 2008, pp. 64-87
51. De Bonis, M.C., Mastroianni, G. *Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations*. SIAM J. Numer. Anal. Vol. 44, 2006, pp. 1351–1374
52. Fermo, L., Russo, M.G. *Numerical methods for Fredholm integral equations with singular right-hand sides*. Adv. Comput. Math. Vol. 33, 2010, pp. 305–330

53. Golberg M.A. *Solution Methods for Integral Equations: Theory and Applications*. Springer Science and Business Media, 2013, 350 p.
54. Kress, R. *Linear Integral Equations*, Applied Mathematical Sciences. Springer Science and Business Media, 2012, 299 p.
55. Mastroianni, G., Milovanovic, G.V.: *Interpolation Processes: Basic Theory and Applications*. Springer Monographs in Mathematics. Springer, Berlin, 2008, 446 p.
56. Occorsio, D., Russo, M.G. *Numerical methods for Fredholm integral equations on the square*. Appl. Math. Comput. Vol. 218, 2011, pp. 2318–2333
57. Prossdorf, S., Silbermann, B.: *Numerical Analysis for Integral and Related Operator Equations*. Akademie-Verlag and Birkhauser Verlag, Berlin, Basel, 1991, 542 p.
58. Sloan, I.H. *A quadrature-based approach to improving the collocation method*. Numer. Math. Vol. 54, 1988, pp. 41–56
59. Sloan, I.H., Spence, A. *Projection methods for solving integral equations on the half line*. IMA J. Numer. Anal. Vol. 6, 1986, pp. 153–172
60. Sloan, I.H., Thomee, V.: *Superconvergence of the Galerkin iterates for integral equations of the second kind*. J. Integral Equ. Vol. 9, 1985, pp. 1–23
61. C.A. Micchelli, Y. Xu and Y. Zhao. *Galerkin methods for second-kind integral equations*, J. Comp. Appl. Math. Vol. 86, 1997, pp. 251–270.
62. Z. Chen, C.A. Micchelli and Y. Xu. *Multiscale methods for Fredholm integral equations*, Cambridge University Press, Cambridge, 2015, 537 p.
63. Z. Avazzadeh, M. Heydari, G.B. Loghmani. *Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem*, Applied Mathematical Modelling, 2011, Vol. 35, issue 5, pp. 2374–2383.
64. X.Z. Liang, M.C. Liu, X.J. Che. *Solving second kind integral equations by Galerkin methods with continuous orthogonal wavelets*, J. Comput. Appl. Math, 2001, Vol. 136, pp. 149–161.
65. K. Maleknejad, M. Tavassoli Kajani. *Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions*, Appl. Math. Comput., 2003, Vol. 145, pp. 623–629.
66. A. Golbabai, S. Seifollahi. *Numerical solution of the second kind integral equations using radial basis function networks*, Appl. Math. Comput., 2006, Vol. 174, pp. 877–883.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.