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Probability Bracket Notation for Probability Modeling

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Abstract: Following Dirac's notation in Quantum Mechanics (QM), we propose the Probability Bracket Notation (PBN), by defining a probability-bra (P-bra), P-ket, P-bracket, P-identity, etc. . . Using the PBN, many formulas, such as normalizations and expectations in systems of one or more random variables, can now be written in abstract basis-independent expressions, easy to expand by inserting a proper P-identity. The time evolution of homogeneous Markov processes can also be formatted in such a way. Our system P-kets are identified with probability vectors and our P-bra system is comparable with Doi's state function or Peliti's standard bra. In the Heisenberg picture of the PBN, a random variable becomes a stochastic process, and the Chapman-Kolmogorov equations are obtained by inserting a time-dependent P-identity. Also, some QM expressions in Dirac notation are naturally transformed to probability expressions in PBN by a special Wick rotation. Potential Applications show the usefulness of the PBN beyond the constrained domain and range of Hermitian operators on Hilbert Spaces in QM all the way to IT.

Keywords: bra-ket notation; chapman-kolmogorov theorem; information technology; markov processes; quantum mechanics; wick rotation

1. Introduction

The postulates of Quantum Mechanics (QM) were established in terms of Hilbert Spaces and linear Hermitian operators. Values of physical observables such as energy and momentum were considered as eigenvalues, more precisely as spectral values of linear operators in a Hilbert space. Dirac's vector bracket notation (VBN) is a very powerful tool to manipulate vectors in Hilbert spaces. It has been widely used in QM and quantum field theories. The main beauty of the VBN is that many formulas can be presented in an abstract symbolic fashion, independent of state expansions or basis selections, which, when needed, can be easily done by inserting an appropriate v-identity operator [1, p.96]

$$\text{v-Bracket}: \quad \langle\psi_A|\psi_B\rangle \quad \Rightarrow \quad \text{v-bra}: \quad \langle\psi_A|\,, \quad \text{v-ket}: \quad |\psi_B\rangle \tag{1}$$

$$\text{where}: \quad \langle\psi_A|\psi_B\rangle \quad = \quad \langle\psi_B|\psi_A\rangle^*, \quad \langle\psi_A| \;=\; |\psi_A\rangle^\dagger \tag{2}$$

$$\text{v-basis \& v-identity:} \quad \hat{H}|\varepsilon_i\rangle \quad = \quad \varepsilon_i|\varepsilon_i\rangle, \quad \langle\varepsilon_i|\varepsilon_j\rangle = \delta_{ij}, \quad \hat{I}_H = \sum_i |\,\varepsilon_i\rangle\,\langle\varepsilon_i| \tag{3}$$

$$\text{Normalization:} \quad 1 = \langle\Psi|\Psi\rangle \quad = \quad \langle\Psi|\hat{I}_H|\Psi\rangle = \sum\nolimits_i \langle\Psi|\varepsilon_i\rangle\langle\varepsilon_i|\Psi\rangle = \sum\nolimits_i |c_i|^2 \tag{4}$$

$$\text{Expectation:} \; \langle H\rangle \;\equiv\; \bar{H} \quad \equiv \quad \langle\Psi|\hat{H}|\Psi\rangle = \sum\nolimits_i \langle\Psi|\hat{H}|\varepsilon_i\rangle\langle\varepsilon_i|\,\Psi\rangle \;=\; \sum\nolimits_i \varepsilon_i\,|c_i|^2 \tag{5}$$

where † denotes the Hermitian adjoint and $*$ is the complex conjugate. However, when applying operators, the more mathematically minded worried if the transformation yielded a result contained within a Hilbert Space or subspace. They had to consider if the operator was bounded or unbounded, etc. . . One had to develop an entire spectral theory for Hermitian operators in a Hilbert space[2].

Inspired by the great success of the VBN in QM, we now propose the *Probability Bracket Notation* (PBN). The latter relies on a *sample space* which is less constrained than a Hilbert space. Assuming X is

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a random variable $(R.V)$, Ω is the set of all its outcomes and $P(x_i)$ is the probability of $x_i \in \Omega$. Then we can make an expression like Equation (4):

$$\sum_i P(x_i) = \sum_i P(x_i|\Omega) = \sum_i P(\Omega|x_i)(x_i|\Omega) = P(\Omega| \left\{ \sum_i | x_i)P(x_i| \right\} |\Omega) = P(\Omega|\Omega) = 1$$

Here we have used the definition of conditional probability for $A, B \subseteq \Omega$ [3, p.91].

$$P(A|B) \equiv P(A \cap B)/P(B), \quad \therefore P(x_i|\Omega) = P(x_i),\ P(\Omega|x_i) = 1 \text{ for } \forall x_i \in \Omega \tag{6}$$

Therefore, we seem to have discovered a probability "identity operator":

$$I_X \equiv \sum_i | x_i)P(x_i| \ \Rightarrow\ P(\Omega|\Omega) = P(\Omega|I_X|\Omega) = \sum_i P(\Omega|x_i)P(x_i|\Omega) \underset{(6)}{=} \sum_i P(x_i) = 1$$

Then, following Dirac's notation, we define the probability bra (P-bra), P-ket, P-bracket (as conditional probabilities by nature), P-basis, the system P-ket, P-identity, normalization, expectation and more, similar but not identical to their counterparts in Equations (1) - (5). In Section 2, for systems of one $R.V$, we show that the PBN has an advantage similar to that of the VBN: miscellaneous probability expressions [3–5] now can be presented in an abstract way, independent of P-basis and can expanded by inserting a suitable P-identity.

Next, in Section 3, we investigate the time evolution of homogeneous Markov chains (HMC) [3–5]. We realize that the time evolution of a continuous-time HMC can be written in a symbolic abstract expression (in Section 3.2), just like the stationary Schrödinger equation in the VBN:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle &= \hat{H}|\Psi(t)\rangle = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V(x)\right]|\Psi(t)\rangle \\ |\Psi(t)\rangle = U(t)|\Psi(0)\rangle &= \exp\left[\frac{-i}{\hbar}\hat{H}\,t\right]|\Psi(0)\rangle \end{aligned} \tag{7}$$

We also find that our time-dependent system P-kets can be identified with *probability vectors* ([3]). Our system P-bra is closely related to the *state function* or *standard bra* introduced in *Doi-Peliti Techniques* [6–8]. We show that by transforming from the Schrödinger picture to the *Heisenberg* picture in the PBN, the time-dependence of a system P-ket relocates to the $R.V$, which becomes a stochastic process; the *Chapman-Kolmogorov Theorem* [4,5,9] for transition probabilities can be derived by just inserting a time-dependent P-identity. Section 5 shows that a Schrödinger equation transforms to a master equation by making a special Wick rotation. Section 6 showcases the potential applications of the PBN, such as handling non-Hermitian operators and clustering text datasets. Discussion and concluding remarks are made at the end.

2. Probability Bracket Notation and Random Variable $(R.V)$

2.1. Discrete random variable

We define a probability space (Ω, X, P) of a discrete random variable ($R.V$, or observable) X as follows: the set of all elementary events ω, associated with a discrete random variable X, is the sample space Ω, and

$$\text{For } \forall \omega_i \in \Omega,\ X(\omega_i) = x_i \in \Re, \quad P : \omega_i \mapsto P(\omega_i) = m(\omega_i) \geq 0, \sum_i m(\omega_i) = 1 \tag{8}$$

Definition 1. (Probability event-bra and evidence-ket) **:** *Let $A \subseteq \Omega$ and, $B \subseteq \Omega$,*

1. *The symbol $P(A|$ represents a probability event bra, or P-bra;*
2. *The symbol $|B)$ represents a probability evidence ket, or P-ket.*

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Definition 2. *(*Probability Event-Evidence Bracket*): The* conditional probability *(CP) of event A given evidence B in the sample space* Ω *can be treated as a* P-bracket*, and it can be split into a P-bra and a P-ket, similar to a Dirac bracket. For* $A, B \subseteq \Omega$*, we define:*

$$\text{P-bracket: } P(A|B) \quad \Rightarrow \quad \text{P-bra: } P(A|\,, \quad \text{P-ket: } |B) \quad \text{Note: } P(A| \neq |A)^{\dagger}$$
$$\text{Here: } P(A|B) \equiv \frac{P(A \cap B)}{P(B)} \quad \text{(a conditional probability by nature)} \tag{9}$$

As a CP, the P-bracket has the following properties for $A, B \subseteq \Omega$:

$$P(A|\,B) = 1 \quad \text{if } A \supseteq B \supset \varnothing \tag{10}$$
$$P(A|\,B) = 0 \quad \text{if } A \cap B = \varnothing \tag{11}$$

Definition 3. ***System P-ket:*** *For any subset, the probability* $P(E)$ *can be written as a conditional probability, or a P-bracket:*

$$P(E) = P(E|\Omega) \tag{12}$$

Here $|\Omega)$ *is called the* **system P-ket***. The P-bracket defined in Equations (9) now becomes:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B|\Omega)}{P(B|\Omega)} \tag{13}$$

We have the following important property expressed in *PBN*:

$$\text{For } \forall B \subseteq \Omega \text{ and } B \neq \varnothing, \quad P(\Omega|B) = 1 \tag{14}$$

The *Bayes formula* (see [3, Sec.(2.1)]) now can be expressed as:

$$P(A|B) \equiv \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A|\Omega)}{P(B|\Omega)} \tag{15}$$

The set of all elementary events in Ω forms a complete mutually disjoint set:

$$\bigcup\nolimits_{\omega_i \in \Omega} \omega_i = \Omega, \quad \omega_i \cap \omega_j = \delta_{ij}\, \omega_i, \quad \sum\nolimits_i m(\omega_i) = 1\,. \tag{16}$$

Definition 4. *(*Discrete **P-Basis** and **P-Identity***): Using Equations (8-11), we have the following properties for basis elements in* (Ω, X, P)*:*

$$X(\omega_j) = x_j \;\rightarrow X|\omega_j) = x_j|\omega_j), \quad P(\Omega|\omega_j) = 1, \quad P(\omega_i|\Omega) = m(\omega_i) \tag{17}$$

In view of the one-to-one correlation between x_j and ω_j, from now on, we will use x_j to label basis elements, just like labeling eigenstates in Equation (3) in the VBN for QM:

$$X(x_j) = x_j \;\rightarrow X|x_j) = x_j|x_j), \quad P(\Omega|x_j) = 1, \quad P(x_i|\Omega) = m(x_i) \tag{18}$$

Here X behaves like a right-acting operator. The complete mutually-disjoint events in (16-18) form a *probability basis* (or *P-basis*) and a *P-identity*, similar to Equation (3) in QM:

$$P(x_i|\,x_k) = \delta_{i\,k}, \quad \sum\nolimits_{x \in \Omega} |\,x)P(x| = \sum\nolimits_i |x_i)P(x_i| = I_X. \tag{19}$$

The system P-ket, $|\Omega)$, now can be expanded from left as:

$$|\Omega) = I_X|\Omega) = \sum_i |\,x_i)P(x_i|\Omega) = \sum_i m(x_i)|x_i) \tag{20}$$

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While the *system P-bra*, has its expansion from right as:

$$P(\Omega| \;=\; P(\Omega|\, I_X = \sum_i (\Omega|\, x_i)P(x_i| \underset{(14)}{=} \sum_i P(x_i| \tag{21}$$

The two expansions are quite different, and $P(\Omega| \neq [|\Omega)]^{\dagger}$. But their P-bracket is consistent with the requirement of normalization, similar to Equation (4) in the VBN:

$$1 \;=\; P(\Omega) \equiv P(\Omega|\,\Omega) \;=\; \sum_{i,j=1}^{N} P(x_i|m(x_j)|x_j) \;=\; \sum_{i,j=1}^{N} m(x_j)\,\delta_{ij} = \sum_{i=1}^{N} m(x_i) \tag{22}$$

Definition 5. (Expectation Value)*: Analogous to Equation (5) in QM, the expected value of the R.V or observable X in (Ω, X, P) now can be expressed as:*

$$\langle X\rangle \;\equiv\; \bar{X} \;\equiv E[X] = P(\Omega|X|\Omega) = \sum_{x\in\Omega} P(\Omega|X|x)P(x|\,\Omega) \;=\; \sum_{x\in\Omega} x\, m(x) \tag{23}$$

If $f(X)$ is a continuous function of observable X, then it is easy to show that:

$$\langle f(X)\rangle \;\equiv\; E[f(X)] \;\equiv\; P(\Omega|f(X)|\Omega) \;=\; \sum_{x\in\Omega} f(x)\, m(x) \tag{24}$$

2.2. Independent random variables

Let $\vec{X} = \{X_1, X_2, \ldots X_n\}$ be a vector of *independent* random variables and the sample space (i.e., the set of possible outcomes) of X_i is the set Ω_i. Then the *joint probability distribution* can be denoted as:

$$P(x_1, ..., x_N|\Omega) = P(X_1 = x_1, ..., X_n = x_n|\Omega), \quad |\Omega) = \;|\Omega_1 \otimes \ldots \otimes \Omega_n) \tag{25}$$

The joint probability of *independent R.V* are factorable, e.g.:

$$P(x_i, x_k|\Omega) \;=\; P(x_i|\Omega_i)\, P(x_k|\Omega_k) = P(x_i)\, P(x_k) \tag{26}$$

The factorable system have the following factorable expectation, for example:

$$P(\Omega|X_i X_j X_k|\Omega) \;=\; P(\Omega_i|X_i|\Omega_i)P(\Omega_j|X_j|\Omega_i)P(\Omega_k|X_k|\Omega_k) \tag{27}$$

As an example, in *Fock space*, we have the following basis from the *occupation numbers*

$$\langle N_i\rangle \equiv P(\Omega|N_i|\Omega) = \; P(\Omega_i|N_i|\Omega_i) = P(\Omega_i|N_i I_i|\Omega_i) = \sum_{n_i} n_i\, P(n_i|\Omega_i) \tag{28}$$

The expectation value of an occupation number now is given by:

$$\langle N_i\rangle \equiv P(\Omega|N_i|\Omega) = \; P(\Omega_i|N_i|\Omega_i) = P(\Omega|N_i I_i|\Omega_i) = \sum_{n_i} n_i\, P(n_i|\Omega_i) \tag{29}$$

Moreover, if sets A and B are mutually independent in Ω, we have following equivalence:

$$P(A|\, B) \;=\; P(A|\Omega) \quad \Leftrightarrow \quad P(A\cap B|\Omega) = P(A|\Omega)P(B|\Omega) \tag{30}$$

2.3. Continuous P-basis and P-Identity

Equations (17-19) can be extended to the probability space (Ω, X, P) of a *continuous* random variable X,

$$X(x) = x \;\rightarrow X|x) \;=\; x|x), \quad P(\Omega|x) = 1, \quad P : x \mapsto P(x) \equiv\; P(x|\Omega) \tag{31}$$

$$P(x|x') \;=\; \delta(x - x'), \quad \int_{x\in\Omega} |\, x)\, dx\, P(x| \;=\; I_X \tag{32}$$

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We see that it is consistent with the normalization requirement:

$$P(\Omega|\Omega) \;=\; P(\Omega|I_X|\Omega) \;=\int P(\Omega|x)\,dx\,P(x|\Omega) = \int_{x\in\Omega} dx\,P(x|\Omega) = \int_{x\in\Omega} dx\,P(x) = 1 \tag{33}$$

The expected value $E(X)$ can be easily extended from (23):

$$\langle X\rangle \;\equiv\; \bar{X} \equiv E[X] = P(\Omega|X|\Omega) \;=\; \int_{x\in\Omega} P(\Omega|X|x)dx\,P(x|\Omega) = \int_{x\in\Omega} dx\,x\,P(x) \tag{34}$$

The *basis-independent expressions* in the PBN are similar to those in Dirac VBN, among them are the expectation and normalization formulas in the two notations:

$$PBN: \quad \langle f(X)\rangle \;\;\equiv\;\; E[f(X)] = P(\Omega|f(X)|\Omega), \quad P(\Omega|\Omega) = 1\,, \tag{35}$$

$$VBN: \quad \langle f(\hat{X})\rangle \;\;\equiv\;\; E[f(\hat{X})] = \langle\psi|f(\hat{X})|\psi\rangle, \quad \langle\psi|\psi\rangle = 1\,. \tag{36}$$

Of course, there exist differences between the PBN and Dirac VBN. For example, the expansions of bra, ket and normalization have their own expressions:

$$PBN: \quad |\Omega) \;=\; \int dx|x)P(x|\Omega), \quad P(\Omega| \;=\int dxP(x|, \quad P(\Omega|\Omega) = \int dxP(x|\Omega) = 1\,, \tag{37}$$

$$VBN: \quad |\psi\rangle \;=\; \int dx|x\rangle\langle x|\psi\rangle, \quad \langle\psi| = \int dx\langle\psi|x\rangle\langle x|, \quad \langle\psi|\psi\rangle = \int dx|\langle\psi|x\rangle|^2 = 1\,. \tag{38}$$

2.4. *Conditional probability and expectation*

The conditional expectation of X given $H \subset \Omega$ in the continuous basis (31) can be expressed as ([5, p.61]):

$$E[X|H] \equiv P(\Omega|X|H) = \int P(\Omega|X|x)dx\,P(x|H) = \int x\,dx\,P(x|H) \tag{39}$$

$$\text{where: } P(x|H) \underset{(9)}{=} \frac{P(x\cap H|\Omega)}{P(H|\Omega)} \quad \text{(conditional probability)} \tag{40}$$

To get familiar with the PBN, let us see two simple examples.

2.4.1. Example: *Rolling a Die*, (Ref. [3, Example 2.6-2.8])

A die is rolled once. We let X denote the outcome of this experiment. Then the sample space for this experiment is the 6-element set $\Omega = \{1,2,3,4,5,6\}$. We assumed that the die was fair, and we chose the distribution function defined by $m(i) = 1/6$, for $i = 1,\ldots 6$. Using the PBN, we have the P-identity for this sample space:

$$\sum_{i=1}^{6} |i)P(i\,| \;=\; I_D\,, \tag{41}$$

and because the six outcomes have the same probability p, we can calculate the probability for each outcome:

$$1 \;=\; P(\Omega|\,\Omega) \;=\; \sum_{i=1}^{6} P(\Omega|i)P(i\,|\Omega) \;=\; \sum_{i=1}^{6} P(\,i|\Omega) \;= 6p\,. \tag{42}$$

Hence, the probability for each outcome has the same value:

$$P(i) \;\equiv\; P(i|\Omega) \;=\; p \;=\; \frac{1}{6}\,. \tag{43}$$

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Its expectation value can be readily calculated:

$$
\begin{aligned}
P(\Omega|X|\Omega) &= \sum_{i=1}^{6} P(\Omega|X|i)P(i|\,\Omega) = \sum_{i=1}^{6} P(\Omega|i|i)P(i|\,\Omega) \\
&= \sum_{i=1}^{6} P(\Omega|i)\frac{i}{6} = \sum_{i=1}^{6}\frac{i}{6} = \frac{21}{6} = \frac{7}{2}
\end{aligned}
\tag{44}
$$

And the variance can be calculated as:

$$
\sigma^2 = \langle X^2\rangle - \bar{X}^2 = \sum_{i=1}^{6}\frac{1}{6}i^2 - \frac{49}{4} = \frac{91}{6} - \frac{49}{4} = \frac{45}{4}
\tag{45}
$$

2.4.2. Example: *Rolling a Die* (Examples 2.1 continued,[3, 2.8 continued])

If E is the event that the result of the roll is an even number, then $E = \{2,4,6\}$ and $P(E) = m(2) + m(4) + m(6) = 1/6 + 1/6 + 1/6 = \frac{1}{2}$. Using the PBN, the probability of event E can be easily calculated as:

$$
P(E) \equiv P(E|\Omega) = P(E|\hat{I}|\Omega) = \sum_{i=1}^{6} P(E|i)P(i|\Omega)
\tag{46}
$$

$$
= \sum_{i\in E} P(i|\Omega) = \sum_{i=2,4,6} p = 3\cdot\frac{1}{6} = \frac{1}{2}
\tag{47}
$$

Applying Equation (40), we can calculate the conditional probabilities $P(i|E)$ as follows:

$$
P(i|E) = \frac{P(E\wedge i|\Omega)}{P(E|\Omega)} = \begin{cases} \frac{P(i|\Omega)}{1/2} = \frac{1/6}{1/2} = \frac{1}{3}\,, & (i\ even) \\ 0, & (i\ odd) \end{cases}
\tag{48}
$$

Our discussions above can be easily extended to systems of multiple $R.V$. For example, we can introduce the following system of three independent discrete $R.V$:

$$
\begin{aligned}
X|x,y,z) &= x|x,y,z), \quad Y|x,y,z) = y|x,y,z), \quad Z|x,y,z) = z|x,y,z) \\
P(x,y,z) &= P(x,y,z|\Omega); \quad P(\Omega|\Omega) = 1; \quad P(\Omega|H) = 1 \ \text{ if } H \subseteq \Omega \quad \to P(\Omega|x,y,z) = 1
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\text{Orthonormality:} &\quad P(x,y,z|\,x',y',z') = \delta_{x\,x'}\,\delta_{y\,y'}\,\delta_{z\,z'} \\
\text{Completeness:} &\quad \textstyle\sum_{x,y,z} |\,x,y,z)P(x,y,z| = I_{X,Y,Z}
\end{aligned}
\tag{50}
$$

3. Probability Vectors and Homogeneous Markov Chains (HMCs)

For simplicity, from now on, we will only discuss time evolution for one $R.V$. We assume our probability space (Ω, N, P) has the following stationary discrete P-basis from a random variable (possible for state-labeling or occupation-number counting):

$$
\hat{N}|i) = i|i), \quad P(i|j) = \delta_{ij}, \quad \textstyle\sum_{i=1}^{N} |\,i)P(i| = I
\tag{51}
$$

3.1. Discrete-time HMC

The transition *matrix element* p_{ij} is defined as ([3, p.407]:

$$
p_{ij} \equiv P(N_{t+1} = j|N_t = i) \equiv P(j,t+1|i,t), \quad \textstyle\sum_{j=1}^{N} p_{ij} = 1
\tag{52}
$$

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In matrix form, if we define a *probability row vector* (PRV) at $t = 0$ as $\langle u^{(0)}|$, then matrix P acting on it from right for k times gives the PRV at $time = k$ ([3, theorem 11.2]):

$$\langle u^{(k)}| \;=\; \langle u^{(0)}|P^k, \;\text{ or : }\quad \langle u^{(k)}|i\rangle = u^{(k)}{}_i = u^{(0)}{}_j P^k{}_{ji} \tag{53}$$

Definition 6. Time-dependent System P-ket*: we use the following system P-ket, to represent a* probability column vector *in probability space* (Ω, X, P)*:*

$$|\Omega_t) = \sum_i^N |\, i) P(i|\Omega_t) = \sum_i^N m(i,t)|i), \quad P(\Omega|\Omega_t) = \sum_i^N m(i,t) \;= 1 \tag{54}$$

The time evolution equation (53) now can be written in a basis-independent way:

$$|\Omega_t) = \left(P^T\right)^t |\Omega_0) \equiv \hat{U}(t)|\Omega_0), \quad t \in \mathbb{N}; \quad |\Omega_0) \equiv [\langle u^{(0)}|]^T \tag{55}$$

Definition 7. (Time-dependent Expectation)*: The expectation value of a continuous function f of occupation number $\hat{N}$ in (Ω, N, P) can be expressed as:*

$$\langle f(\hat{N})\rangle = P(\Omega|f(\hat{N})|\Omega_t) = \sum_i P(\Omega|f(i)|i)P(i|\Omega_t) = \sum_i f(\,i)\, m(i,t) \tag{56}$$

We can map the P-bra and P-ket into the Hilbert space by using Dirac's notation:

$$P(\Omega| \;= \sum_i P(i| \;\leftrightarrow \langle\Psi| \;= \sum_i \langle i|, \quad |\Omega_t) \leftrightarrow |\Psi_t\rangle = \sum_i |\, i\rangle\langle i|\, \Psi_t\rangle = \sum_i c(i,t)|i\rangle \tag{57}$$

Then the expectation Equation (56) can be rewritten in Dirac's notation as:

$$\langle f(\hat{N})\rangle \;=\; \langle\Psi_t|f(\hat{N})|\Psi_t\rangle = \sum_i \langle\Psi_t|f(i)|i\rangle\langle i|\Psi_t\rangle = \sum_i f(\,i)\,|c(i,t)|^2 \tag{58}$$

$$\therefore \quad P(i|\Omega_t) \equiv m(i,t) = \;|c(i,t)|^2 \equiv \;|\langle i|\Psi_t\rangle|^2, \quad t \in \mathbb{N} \tag{59}$$

3.2. Continuous-time HMC

The time-evolution equation of a continuous-time HMC with a discrete basis (see Equation (85) or Ref. [5, p.221]) can be written as:

$$\frac{\partial}{\partial t} p_j(t) = \sum_k p_k(t) q_{kj} \;\Rightarrow\; \frac{\partial}{\partial t} P(j|\Omega_t) = \sum_k \hat{Q}^T_{jk} P(k|\Omega_t) \tag{60}$$

$$= \sum_k P(j|\hat{Q}^T|k)P(k|\Omega_t) \;=\; P(j|\hat{Q}^T I|\Omega_t) = P(j|\hat{Q}^T|\Omega_t) \equiv P(j|\hat{L}|\Omega_t) \tag{61}$$

Equations (60-61) lead to a basis-independent **master equation** (see Equation [8, Eq.(2.12)]):

$$\frac{\partial}{\partial t}|\Omega_t) = \hat{L}|\Omega_t), \quad |\Omega_t) = \hat{U}(t)|\Omega_0) = e^{\hat{L}\,t}|\Omega_0), \quad t \geq 0 \tag{62}$$

It looks just like Schrödinger's equation (7) of a conserved quantum system in Dirac' notation. With the discrete P-basis of Fock space in Equation (28), Equations (56-59) now can be written as:

$$|\Omega_t) = \sum_{\vec{n}} m(\vec{n},t)\,|\vec{n}), \quad P(\Omega| \;= \sum_{\vec{n}} (\vec{n}|, \quad \therefore \langle f(\vec{n})\rangle = P(\Omega|f(\vec{n})|\Omega_t) \tag{63}$$

Doi's definition of the state function and state vector ([8,9]) correspond to our system P-bra and P-ket respectively:

$$P(\Omega| \;= \sum_{\vec{n}} P(\vec{n}| \quad \leftrightarrow \quad \langle s\,| \equiv \sum_{\vec{n}} \langle\vec{n}|$$

$$|F(t)\rangle \equiv \sum_{\vec{n}} P(\vec{n},t)|\vec{n}\rangle \quad \leftrightarrow \quad |\Omega_t) = \sum_{\vec{n}} |\vec{n})P(\vec{n}|\,\Omega_t)$$

$$\therefore \langle\hat{B}(\vec{n})\rangle = \langle s|\hat{B}(\vec{n})|F(t)\rangle = P(\Omega|\hat{B}(\vec{n})|\Omega_t) \tag{64}$$

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Note that the vector-basis here corresponds to the P-basis in Equation (28):

$$\hat{n}_i|\vec{n}\rangle = n_i|\vec{n}\rangle, \quad \sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}| = I_{\vec{n}}, \quad \langle\vec{n}|\vec{n}'\rangle = \delta_{\vec{n},\vec{n}'} = \prod_{i=1} \delta_{n_i,n_i'} \tag{65}$$

In Peliti's formalism ([8, p.1472]), the vector-basis (from population operator n) is normalized in a special way, so the expansion of the system P-bra is also changed:

$$\sum_n |n\rangle \frac{1}{n!}\langle n| = I_n, \quad \langle m|n\rangle = n!\delta_{m,n} \tag{66}$$

$$P(\Omega| = P(\Omega|\, I_n = \sum_n^{\infty} P(\Omega|\, n) \frac{1}{n!} P(n| \underset{(2.3)}{=} \sum_n^{\infty} P(n| \frac{1}{n!} \tag{67}$$

Equation (67) can be identified with the *standard bra*, introduced in [8, Eq.(2.27-28)]:

$$\begin{aligned}
P(\Omega| &= \sum_n \frac{1}{n!} P(n| \quad \leftrightarrow \quad \langle\,| \equiv \sum_n \frac{1}{n!}\langle n| \\
|\Omega) &= \sum_n |n\rangle \frac{1}{n!} P(n|\,\Omega) \quad \leftrightarrow \quad |\phi\rangle = \sum_m |m\rangle \frac{1}{m!}\langle m|\phi\rangle \\
\therefore E[\hat{A}] &\equiv \langle\hat{A}\rangle = \langle\,|\,\hat{A}|\phi\rangle = P(\Omega|\hat{A}|\Omega)
\end{aligned} \tag{68}$$

3.3. The Heisenberg Picture

We call Equation (55) and (62) the evolution equations in the *Schrödinger picture*. Now we introduce the *Heisenberg picture* of a *R.V* (or observable), similar to what is used in QM [1, p.541]:

$$|\Omega_t) = \hat{U}(t)|\Omega_0) \quad \Rightarrow \quad \hat{X}(t) = \hat{U}^{-1}(t)\,\hat{X}\,\hat{U}(t) \tag{69}$$

Based on $\hat{U}(t)$, we can introduce following time-dependent P-basis:

$$|x,t) = \hat{U}^{-1}(t)|x), \quad P(x|\hat{U}(t) = P(x,t|, \quad P(x,t|x',t) = P(x|x'), \quad P(\Omega|x,t) = 1 \tag{70}$$

$$P(x',t|\hat{X}(t)|x,t) = P(x'|\hat{U}(t)\hat{U}^{-1}(t)\,\hat{X}\,\hat{U}(t)\hat{U}^{-1}(t)|x) = P(x'|\hat{X}\,|x) = xP(x'\,|x) \tag{71}$$

The probability density now can be interpreted in the two pictures:

$$P(x,t) \equiv P(x|\Omega_t) = P(x|\hat{U}(t)|\Omega_0) = P(x,t|\Omega_0) = P(x,t|\Omega) \tag{72}$$

In the last step, we have used the fact that in the Heisenberg picture, $|\Omega_0) = |\Omega)$.

Definition 8. Time-dependent P-Identity*: Equation (69-71) also provides us with a time-dependent P-identity in the Heisenberg Picture:*

$$\begin{aligned}
\textit{Discrete: } I_X(t) &= \hat{U}^{-1}(t) I_X \hat{U}(t) = \hat{U}^{-1}(t) \sum_i |x_i) P(x_i|\,\hat{U}(t) \underset{(70)}{=} \sum_i |x_i,t) P(x_i,t| \\
\textit{Continuous: } I_X(t) &= \hat{U}^{-1}(t) I_X \hat{U}(t) = \hat{U}^{-1}(t) \int |x)\, dx\, P(x|\,\hat{U}(t) = \int |\,x,t)\, dx\, P(x,t|
\end{aligned} \tag{73}$$

Now the expectation value of the stochastic process $X(t)$ can be expressed as:

$$\begin{aligned}
P(\Omega|\hat{X}(t)|\Omega) &= P(\Omega|\hat{X}(t) I_X(t)|\Omega) = \int dx\, P(\Omega|\hat{X}(t)\,|x,t) P(x,t|\Omega) \\
&= \int dx\, x\, P(x,t|\Omega) = \int dx\, x\, P(x|\Omega_t) = P(\Omega|X|\Omega_t)
\end{aligned} \tag{74}$$

This suggests that a Markov *stochastic process* can be thought as an *operator in the Heisenberg picture*, and its expectation value can be found from its Schrödinger picture. Additionally, if a Markov process

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$X(t) \equiv X_t$ is homogeneous, with independent and stationary increments ([4, p.15]), we can always set $X_0 = 0$, and obtain the following useful property:

$$\begin{aligned} P(X_{t+s} - X_s = x) &= P(X_t - X_0 = x|\Omega) \\ &= P(X_t = x|\Omega) = P(x,t|\Omega) = P(x|\Omega_t) \equiv P(x,t) \end{aligned} \tag{75}$$

Moreover, there is a relation between their transition probability and probability density:

$$P(x_2,t_2|x_1,t_1) = P(x_2 - x_1, t_2 - t_1|\Omega) \equiv P(x_2 - x_1, t_2 - t_1), \quad (t_1 < t_2) \tag{76}$$

3.4. *Chapman-Kolmogorov Equations of transition probability [5,9]*

These can now be easily obtained by inserting our time-dependent P-identity in Equations (73). For the HMC of *discrete R.V* ([5, p.174]):

$$\begin{aligned} {p^{m+n}}_{ij} &\equiv P(j,m+n|i,0) = P(j,m+n|\hat{I}(m)|i,0) = \sum_k P(j,m+n|k,m)P(k,m|i,0) \\ &\underset{\text{HMC}}{=} \sum_k P(j,n|k,0)P(k,m|i,0) \underset{52}{=} \sum_k {p^m}_{ik}\, {p^n}_{kj} \;\; \text{(Discrete time)} \end{aligned} \tag{77}$$

$$\begin{aligned} p_{ij}(t+s) &\equiv P(j,t+s|i,0) = P(j,t+s|\hat{I}(s)|i,0) = \sum_k P(j,t+s|k,s)P(k,s|0,i) \\ &\underset{\text{HMC}}{=} \sum_k P(j,t|k,0)P(k,s|0,i) = \sum_k p_{ik}(s)\; p_{kj}(t) \;\; \text{(Continuous time)} \end{aligned} \tag{78}$$

For a Markov process of both *continuous R.V* and time [9, (Eq.(3.9),p.31]):

$$P(x,t|y,s) = P(x,t|\hat{I}(\tau)|y,s) = \int P(x,t|z,\tau)dz\, P(z,\tau|y,s) \quad \text{where } t > \tau > s \tag{79}$$

3.4.1. Absolute probability distribution (APD)

Likewise, we can find the time evolution of APD for a HMC simply by inserting P-identity $I(t=0) = I(0)$. For the HMC with discrete states ([5, p.174,214]):

$$\begin{aligned} \text{Discrete time}: P(i,m) &= P(i,m|\Omega) = P_i^{(m)}, \quad P(i,0) \equiv p_i^{(0)} \\ \therefore P_i^{(m)} = P(i,m|I(0)|\Omega) &= \sum_k P(i,m|k,0)P(k,0|\Omega) = \sum_k p_k^{(0)} {p^m}_{ki} \end{aligned} \tag{80}$$

Note that Equations (80) are identical to Equation (52):

$$\begin{aligned} \text{Continuous time}: P(x_i,t) &= P(x_i,t|\Omega) = P(x_i|\Omega_t), \quad P(x_i,0) \equiv p_i(0) \\ \therefore P(x_i,t) = P(x_i,t|I(0)|\Omega) &= \sum_k P(x_i,t|x_k,0)P(x_k,0|\Omega) = \sum_k p_k(0) p_{ki}(t) \end{aligned} \tag{81}$$

For a Markov process of both *continuous R.V* and time ([9, p.26]):

$$\begin{aligned} P(x,t) &= P(x,t|\Omega) = P(x|\Omega_t), \quad P(x,0) \equiv p^{(0)}(x) \qquad (82) \\ \therefore P(x,t) &= P(x,t|I(0)|\Omega) = \int dy P(x,t|y,0)P(y,0|\Omega) = \int dy P(x,t|y,0)p^{(0)}(y) \end{aligned}$$

3.5. *Kolmogorov Forward and Backward Equations*

If the Markov chains are *stochastically continuous*, then for infinitesimal h, the transition probability has the Taylor expansions ([4, Sec.(6.8)]; [5, p.217]):

$$p_{ij}(h) = p_{ij}(0) + {p_{ij}}'(0)h + o(h^2) = \delta_{ij} + q_{ij}h + o(h^2) \tag{83}$$

Then, using the Chapman-Kolmogorov Equation (78), we have:

$$\begin{aligned} p_{ij}(t+h) &= \sum_k p_{ik}(t)\ p_{kj}(h) = \sum_k p_{ik}(t)\ (\delta_{kj} + q_{kj}h + o(h^2)) \\ &= p_{ij}(t) + \sum_k p_{ik}(t)\ (q_{kj}h + o(h^2)) \end{aligned} \tag{84}$$

Therefore, we get following Kolmogorov *Forward* equations:

$$p'_{ij}(t) = \lim_{h\to 0} [(p_{ij}(t+h)\ - p_{ij}(t))/h] = \sum_k p_{ik}(t)\ q_{kj} \tag{85}$$

Similarly, we can derive the Kolmogorov *Backward* equations:

$$p_{ij}(h+t) = \sum_k p_{ik}(h)\ p_{kj}(t)\ \Rightarrow\ p'_{ij}(t) = \sum_k q_{ik}\ p_{kj}(t) \tag{86}$$

Using Equations (81) and (83) one can extend Equation (62) to a continuous-time HMC.

3.6. Transition Probability and Path Integrals

From Equation (62) and (70) the transition probability of a Markov process can be expressed as follows (assuming $t_a < t_0 < t_b$):

$$P(x_b, t_b|x_a, t_b) = P(x_b|U(t_b, t_0)U^{-1}(t_a, t_0)|x_a) = P(x_b|U(t_b, t_a)|x_a) = P(x_b|e^{\int_{t_a}^{t_b} dtL}|x_a) \tag{87}$$

We can divide time interval into small pieces and insert the P-identity N times:

$$\Delta t = t_{n+1} - t_n = (t_b - t_a)/(N+1) > 0, \qquad t_N = t_b,\ t_0 = t_a;\ x_N = x_b,\ x_0 = x_a, \tag{88}$$

$$\begin{aligned} P(x_b, t_b|x_a, t_a) = P(x_b, t_b|I(t_N)..I(t_1)|x_a, t_a) &= \prod_{n=1}^N \int_{-\infty}^{\infty} dx_n P(x_n, t_n|x_{n-1}, t_{n-1}) \\ = \prod_{n=1}^N \int_{-\infty}^{\infty} dx_n P(x_n|\hat{U}(t_n, t_{n-1})|x_{n-1}) &= \prod_{n=1}^N \int_{-\infty}^{\infty} dx_n P(x_n|e^{L\,\Delta t}|x_{n-1}) \end{aligned} \tag{89}$$

It perfectly matches the starting equation of Feynman's path integral for the transition amplitude in QM ([10, Eq.(2.4),p.90]):

$$\langle x_b, t_b|x_a, t_a\rangle = \prod_{n=1}^N \int_{-\infty}^{\infty} dx_n \langle x_n|\hat{U}(t_n, t_{n-1})|x_{n-1}\rangle = \prod_{n=1}^N \int_{-\infty}^{\infty} dx_n \langle x_n|e^{-\frac{i}{\hbar}\hat{H}\,\Delta t}|x_{n-1}\rangle \tag{90}$$

For a *free particle*, $V(x) = 0$, the Schrödinger equation and resulted transition amplitude from Feynman's path integral are (see [10, Eq.(2.125)]):

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \langle x|\hat{H}|\Psi(t)\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x), \quad \langle x|\Psi(t)\rangle = \psi(x) \tag{91}$$

$$\langle x_b, t_b|x_a, t_a\rangle = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\ \exp\left[\frac{im\,(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right] \tag{92}$$

4. Examples of Homogeneous Markov Processes

Now let us observe some important examples of the HMC using the PBN.

4.1. Poisson Process ([4, p.250], [5, p.161])

It is a counting process, $N(t)$, having following properties:

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1. $\{N(t), t \geq 0\}$ is non-negative process with independent increments and $N(0) = 0$;
2. It is homogeneous and its probability distribution is given by:

$$
\begin{aligned}
m(k,t) \quad &\equiv \quad P([N(t+s) - N(s) = k]|\Omega) \underset{\substack{independent\\increment}}{=} P([N(t) - N(0) = k]|\Omega) \\
&\underset{N(0)=0}{=} \quad P([N(t) = k]|\Omega) \equiv P(k|\Omega(t)) \underset{\substack{Poisson\\Distribution}}{=} \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\end{aligned} \tag{93}
$$

It can be shown [5, p.161] that:

$$\mu(t) \equiv \bar{N}(t) = \sum_k k\, m(k,t) \ = \lambda t; \quad \sigma^2(t) \equiv P(\Omega|[N(t) - \bar{N}(t)]^2|\Omega) = \lambda t \tag{94}$$

It can be shown (see [5, p.215]; [11, Theorem 1.5,p.6]) that Poisson Process has Markov property, and its transition probability is:

$$
\begin{aligned}
&P([N(t+s) = j]|N(t) = i) = P([N(t+s) - N(t) = j - i]|\Omega) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}, \text{ if } j \geq i \\
&p_{ij}(t) = 0, \text{if } j < i
\end{aligned} \tag{95}
$$

4.2. Wiener-Levy Process (see [5, p.159]; [9, sec.(3.6),p.32])

It is a homogeneous process with *independent and stationary increments* and $W(0) = 0$. Its probability density is a normal (Gaussian) distribution $N(0, t\sigma^2)$:

$$P(x,t) \equiv P([W(t+s) - W(s) = x]|\Omega) \underset{homogeneous}{=} P([W(t) - W(0) = x]|\Omega) \tag{96}$$

Its stationary increment is defined as:

$$P(x_2, t_2|x_1, t_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}\sigma} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)\sigma^2}\right], \quad (t_1 < t_2) \tag{97}$$

We see that the Wiener-Levy process satisfies Equation (76).

$$P(x_2, t_2|x_1, t_1) = P(x_2 - x_1, t_2 - t_1), \quad (t_1 < t_2) \tag{98}$$

It can be verified ([5, p.161]) that it is an $N(0, t\sigma^2)$:

$$\tilde{\mu}(t) \equiv P(\Omega|W(t)|\Omega) \ = 0, \quad \tilde{\sigma}^2(t) \equiv P(\Omega|W(t)^2|\Omega) \ = t\sigma^2 \tag{99}$$

4.3. Brownian motion ([4, Sec.(10.1), p.524]; [9, p.6,42])

The stochastic process $X(t)$ has $X(0) = 0$, has *stationary and independent increments* for $t \geq 0$ and its density function for $t > 0$ is a normal distribution $N(t\mu, t\sigma^2)$, or:

$$P(x,t) \equiv P(x|\Omega(t)) = \frac{1}{\sqrt{2\pi\, t}\,\sigma} \exp\left[-\frac{(x - \mu\, t)^2}{2t\sigma^2}\right] \tag{100}$$

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Brownian motions have *stationary and independent* increments ([4, Sec.(10),p.524,529]), so they are a HMC and satisfy Equation (76). They are the solution of the following master equation (Einstein's diffusion equation, see [9, p.6]) with a drift speed μ:

$$\frac{\partial}{\partial t}P(x,t) = D\frac{\partial^2}{\partial x^2}P(x,t), \quad D = \frac{\sigma}{\sqrt{2}}. \tag{101}$$

Here the constant D is called the diffusion coefficient. Equation (101) can be interpreted as a special HMC case of Equation (62), given in the x-basis:

$$\frac{\partial}{\partial t}P(x|\Omega_t) = \int dx' P(x|\hat{L}|x')P(x'|\Omega_t), \quad P(x|\hat{L}|x') = D\frac{\partial^2}{\partial x^2}\delta(x-x') \tag{102}$$

Equation (102) closely resembling Schrödinger's Equation (7) for a free particle in the x-basis:

$$i\hbar\frac{\partial}{\partial t}\langle x|\Psi(t)\rangle = \int dx' \langle x|\hat{H}|x'\rangle\langle x'|\Psi(t)\rangle, \quad P(x|\hat{H}|x') = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\delta(x-x') \tag{103}$$

5. Special Wick Rotation, Time Evolution and Induced Diffusions

The ***induced microscopic diffusion*** is defined by the following equation:

$$\frac{\partial}{\partial t}P(x,t) = \hat{G}(x)P(x,t), \quad \hat{G} = \frac{1}{2\mu_\hbar}\frac{\partial^2}{\partial x^2} - \mu_\hbar u(x), \quad \mu_\hbar \equiv \frac{m}{\hbar} \tag{104}$$

The ***Special Wick Rotation*** (SWR), caused by the imaginary time rotation *Wick rotation* $(i\,t \to t)$ [12], is defined by:

$$\text{SWR:}i\,t \to t, \quad |\psi(t)\rangle \to |\Omega_t), \quad \langle x_b, t_b|x_a, t_a\rangle \to P(x_b, t_b|x_a, t_a) \tag{105}$$

Under a SWR, Schrödinger Equation (7) is shifted to the induced micro diffusion (104):

$$\frac{\partial}{i\,\partial t}|\psi(t)\rangle = \frac{-1}{\hbar}\hat{H}|\psi(t)\rangle \to \frac{\partial}{\partial t}|\Omega_t) = G|\Omega_t) \tag{106}$$

$$-\frac{1}{\hbar}\hat{H} = \frac{\hbar\partial_x{}^2}{2m} - \frac{m}{\hbar}\frac{V(x)}{m} \equiv \frac{\partial_x{}^2}{2\mu_\hbar} - \mu_\hbar u(x) \to \hat{G} = \frac{1}{2\mu_\hbar}\frac{\partial^2}{\partial x^2} - \mu_\hbar u(x) \tag{107}$$

The simplest example is the ***Induced Micro Einstein-Brown motion*** when $u(x) = 0$ in Equation (104). Applying (105) to the transition amplitude in Equation (92), we get the transition probability:

$$P(x_b, t_b|x_a, t_a) = \sqrt{\frac{1}{4\pi D_\hbar(t_b - t_a)}}\,\exp\left[-\frac{(x_b - x_a)^2}{4D_\hbar(t_b - t_a)}\right] \tag{108}$$

Here we have introduced the *induced micro diffusion coefficient*,

$$D_\hbar \equiv 1/(2\mu_\hbar) = \hbar/(2m) \tag{109}$$

Similarly, applying (105) to the Schrödinger for free particle, Equation (103), we get the Einstein's diffusion equation (102), with $D_\hbar \to D$.

5.0.1. A Special non-Hermitian Case

We can apply Dirac notation and the PBN together to solve the following special quantum system with a non-Hermitian Hamiltonian:

$$\hat{H} = \hat{H}_1 - i\hat{H}_2; \quad \hat{H}_1 = \hat{H}_1^\dagger; \quad \hat{H}_2 = \hat{H}_2^\dagger; \quad [\hat{H}_1, \hat{H}_2] = 0; \quad \partial_t\hat{H} = 0 \tag{110}$$

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Then the time-evolution equation can be written as:

$$i\hbar\partial_t \left[|\Psi_t\rangle|\Omega_t)\right] = \left[\hat{H}_1|\Psi_t\rangle\right]|\Omega_t) + |\Psi_t\rangle\left[-i\hat{H}_2|\Omega_t)\right] \tag{111}$$

It leads to two equations:

$$\partial_t|\Psi_t\rangle = -\frac{i}{\hbar}\hat{H}_1|\Psi_t\rangle, \quad \partial_t|\Omega_t) = \frac{-1}{\hbar}\hat{H}_2|\Omega_t) \underset{(106)}{=} \hat{G}|\Omega_t)) \tag{112}$$

The first is an ordinary Schrodinger equation while the second is a master equation for an induced micro diffusion. The product of their solutions is the solution of Equation (111). Suppose we have $\hat{H}_1(x)$ and $\hat{H}_2(y)$, then the product of Equation (89) and Equation (90) gives the path-integral expression of the system:

$$\begin{aligned}&[\langle x_b, t_b|P(y_b, t_b|]e^{-\int_{t_a}^{t_b} dt \frac{i}{\hbar}[\hat{H}_1(x) - i\hat{H}_2(y)]}|[x_a, t_a\rangle|y_a, t_a)]\\ &= \prod_{m=1}^{N}\int_{-\infty}^{\infty} dx_m \langle x_m|e^{-\frac{i}{\hbar}\hat{H}_1(x)\,\Delta t}|x_{m-1}\rangle \prod_{n=1}^{N}\int_{-\infty}^{\infty} dy_n P(y_n|e^{\hat{G}(y)\,\Delta t}|y_{n-1})\end{aligned} \tag{113}$$

6. Potential Applications

Hermitian operators are sufficient for "pure" eigenvalue states for closed systems where the energies are conserved and real valued. However, for mixed states and in a number of physical circumstances, non-Hermitian operators have had to be considered. It is well known that any non-Hermitian linear operator $\hat{C}$ can be expressed in the form:

$$\hat{C} = \hat{A} + i\hat{B} \tag{114}$$

where $\hat{A}$ and $\hat{B}$ are Hermitian and commute with each other, by letting $\hat{A} = (\hat{C} + \hat{C}^\dagger)/2$ and $\hat{B} = (\hat{C} - \hat{C}^\dagger)/(2i)$. Note that $\hat{B} \to i\hat{B}$ can be treated as a wick rotation in the path-integral application of the PBN. Section 5.0.1 dovetails into the matter of general linear operators where there have been a number of applications. We mention:

1. The method of "complex scaling" applied to quantum mechanical Hamiltonians was a "hot" area in the area of atomic and molecular Physics during the 1970s and early 1980s and involved non-Hermitian linear operators. We cite an application by the Mathematician Barry Simon of complex scaling to non-relativistic Hamiltonians for molecules [13].
2. Another application requiring such an operator was made by Botten *et al.*[14]. They show how to solve a practical problem involving wave scattering using a bi-orthogonal basis, where there is a VPN bra basis and a ket basis consisting of different functions. In a unitary problem, these VPN bra and ket basis functions would be the same. Here, the Helmholtz equation Laplacian $\nabla^2 f + k^2 f = 0$ has a wave number k which is complex. The imaginary part of k indicates loss or gain depending on its sign.
3. Konstantin G. Zloshchastiev has made many applications on the general density operator approach with non-Hermitian Hamiltonians of the form Equation (114) applied to e.g. open dissipative systems, which automatically deals with mixed states (see [15–17]).
4. Consider a rectangular real data matrix $\hat{Q}$. Its similarity matrix (or adjacency matrix) $\hat{S}$ and corresponding row stochastic (Markov) matrix $\hat{R}$ defined by:

$$\hat{S} = \hat{Q} * \hat{Q}^T \quad R_{i,j} = S_{ij} / \sum_k S_{i,k} \tag{115}$$

The symmetric matrix $\hat{S}$ has real eigenvalues as does $\hat{R}$ even though the latter is non-symmetric. $\hat{R}$ is a *transition matrix* in the language of QM but also the key operator in the Meila-Shi algorithm of spectral clustering, and *quantum clustering* with IT applications in science, engineering, (unstructured) text using a "bag-of-words" model [18–21] and even medicine [22]. Probably, the PBN may provide us with some new approaches to quantum clustering.

If the real data matrix $\hat{Q}$ in Equation (115) is huge but very sparse (as is often the case in a "bag-of-words" model[20]), we may use the PBN to find other alternative algorithms for text document clustering. Suppose that the dataset has a vocabulary of N labeled keywords, which serve as the P-basis in Equation (28), and $q_{\mu,k}$ represents the frequency of k^{th} keyword in document Q_μ. Since $q_{\mu,k}$ can be greater than 1, they behave like the occupation numbers of a Boson system in Quantum Field Theory. The conditional probability of finding doc Q_μ given doc Q_ν is:

$$P(Q_\mu|Q_\nu) = \sum_k^N P(Q_\mu|k)(k|Q_\nu) \underset{(14)}{=} \sum_{k\in Q_\mu} P(k|Q_\nu); \quad P(k|Q_\mu) = q_{\mu,k} / \sum_{k\in Q_\mu} q_{\mu,k} \tag{116}$$

Now, for example, we can define the relevance of two docs and get its expression as:

$$R_{\mu,\nu} \equiv \frac{1}{2}[P(Q_\mu|Q_\nu) + P(Q_\nu|Q_\mu)] = \frac{1}{2}\left[\sum_{k\subset Q_\mu} P(k|Q_\nu) + \sum_{k\subset Q_v} P(k|Q_\mu)\right] \tag{117}$$

This algorithm may be very effective for huge data sets with a lot of abstracts of articles published in many journals with different subjects.

7. Summary and Discussion

Inspired by Dirac's notation used in quantum mechanics (QM), we proposed the Probability Bracket Notation (PBN). We demonstrated that the PBN could be a very useful tool for symbolic representation and manipulation in probability modeling, like the various normalization and expectation formulas for systems of single or multiple random variables, as well as the master equations of homogeneous Markov processes. We also show that a stationary Schrödinger Equation (103) naturally becomes a master equation under the special Wick rotation (see Equations 106-107).

We have shown the similarities between many QM expressions in VBN and related probabilistic expressions in PBN, which might provide us with a beneficial bridge connecting the quantum world with the classical one. We also showed how the PBN could be applied to problems which require non-Hermitian operators. We make no pretense that the PBN creates new Physics. Rather it provides a notational interdisciplinary "umbrella" to various statistical and physical processes and thereby opens up the possibility of their synthesis and integration.

8. Conclusions

Of course, more investigations need to be done to verify the consistency (or correctness), usefulness and limitations of our proposal of the PBN but it is is intriguing in its possibility of expressing formulations from classical statistics and QM and handling non-Hermitian operators.

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Abbreviations

The following abbreviations are used in this manuscript:

APD	Absolute Probability Distribution
CP	Conditional Probability
HMC	Homogeneous Markov Chains
IT	Information Technology
PBN	Probability Bracket Notation
P-basis	Probability basis
P-bra	Probability (event) bra
P-identity	Probability identity
P-ket	Probability (event) ket
PRV	Probability Row Vector
QM	Quantum Mechanics
R.V	Random Variable
SWR	Special Wick Rotation
VBN	(Dirac) Vector Bracket Notation

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