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Article

Existence of Common Fixed Points of Generalized Δ-Implicit Locally Contractive Mappings on Closed Ball in Multiplicative G-metric Spaces with Applications

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Abstract: In this paper, we introduce a generalized Δ-implicit locally contractive condition and give some examples to support it and show its significance in fixed point th eory. We prove that the mappings satisfying generalized Δ-implicit locally contractive condition admit a common fixed point, where the ordered multiplicative $G_{\mathcal{M}}-$metric space is chosen as underlying space. The obtained fixed point theorems generalize many earlier fixed point theorems on implicit locally contractive mappings. In addition, some nontrivial and interesting examples are provided to support our findings. To demonstrate the originality of our new main result, we apply it to show existence of solutions to a system of nonlinear -Volterra type- integral equations.

Keywords: ordered complete multiplicative $G_{\mathcal{M}}-$metric space; closed ball; integral equations; locally generalized Δ-implicit contraction

1. Introduction

In the subject of functional analysis, fixed point theory (FPT) plays a vibrant, fascinating, and vital role. Banach (1922) [5] provided a foundational principle that has become a significant instrument in the field of metric fixed point theory to ensure the existence and uniqueness of the fixed point (FP). The Banach fixed-point theorem (also known as contraction mapping theorem) is the core principle in the metric fixed point theory. Because of its benefits, numerous authors have demonstrated various improvements and expansions of this theorem in diverse distance spaces (see [2,4–6,8–11,13,17,19–23,25,26,29–31,34,38]).

Bashirov et al. [6] presented the concept of multiplicative calculus and proved its foundational theorem with certain fundamental features. Multiplicative calculus has vast area of applications and it deals with only positive functions instead of the calculus of Newton and Leibniz. They showed that multiplicative calculus becomes an important mathematical tool for economics and finance because of the interpretation given to multiplicative derivative. Furthermore, they proved multiplicative differential and multiplicative integral equations by using the notion of multiplicative distance space. The research work on the properties of multiplicative metric space was done in [7,14–16]. In 2012, Özavsar et al. [32] came up with the definition of multiplicative contraction mappings on multiplicative

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metric space by using the multiplicative triangle inequality instead of the usual triangular inequality and obtained different existence results of fixed-point beside various topological characteristics of multiplicative metric space. For other examples of fixed point theorems in multiplicative metric space, see weak commutative mappings, locally contractive mappings, $\mathcal{E}.\mathcal{A}$-property, compatible-type mappings, and generalized contraction mappings with cyclic (α, β)-admissible mapping ([2,3,19,42–44]). In 2016, Nagpal *et al.*[28] introduced the concept of multiplicative generalized metric space and studied the notion of weakly commuting compatible maps and its variants by using $(\mathcal{CLR})$ and $(\mathcal{E}.\mathcal{A})$ properties in multiplicative metric space.

Rasham *et al.* [37] recently presented fixed point results for a pair of dominated fuzzy maps in multiplicative metric space on a closed ball and discussed relevant applications to graph theory, integral equations, and functional equations. For additional information on closed ball (see [36,39,40]).

According to this perspective, the main objective of this paper is to establish some new fixed point results on a closed ball in an ordered multiplicative $\delta_{\mathcal{M}}-$metric space that satisfies a new generalized Δ-implicit contraction. To support new results, we present various nontrivial examples and an application for nonlinear — Volterra type — integral equations. The choice of multiplicative $\delta_{\mathcal{M}}-$metric is based on the concept of generality. The corresponding results in multiplicative metric space are special cases of the obtained results in multiplicative $\delta_{\mathcal{M}}-$metric space. We think that any new idea regarding contraction and fixed point theorem should be investigated in a most general metric space so that corresponding results can be derived as special cases. The paper is organized as follows. In Section 2, we state basic notions related to fixed point theorems and multiplicative metric spaces. In Section 3, we present many fixed point theorems and reltaed corollaries and examples for explanations of the stated results. In Section 4, we present two applications of the obtained fixed point results in Section 2, moreover, some numerical examples are given.

2. Preliminaries

Now, we recall some well-known notations and definitions that will be used in our subsequent discussion.

Definition 2.1. [6] Consider a non-empty set $\Re$ and let $\mathcal{L}_{\mathcal{M}} :\Re \times \Re \longrightarrow R^{+}$ be a function satisfying the following properties:

$(\mathcal{L}_1)$ $\mathcal{L}_{\mathcal{M}}(\check{e}, \varsigma) \geq 1, \quad \forall\, \check{e}, \varsigma \in \Re$;

$(\mathcal{L}_2)$ $\mathcal{L}_{\mathcal{M}}(\check{e}, \varsigma) = 1$ if and only if $\check{e} = \varsigma$;

$(\mathcal{L}_3)$ $\mathcal{L}_{\mathcal{M}}(\check{e}, \varsigma) = \mathcal{L}_{\mathcal{M}}(\varsigma, \check{e})$ (symmetry);

$(\mathcal{L}_4)$ $\mathcal{L}_{\mathcal{M}}(\check{e}, \varsigma) \leq \mathcal{L}_{\mathcal{M}}(\check{e}, \hbar).\mathcal{L}_{\mathcal{M}}(\hbar, \varsigma) \quad \forall\, \check{e}, \varsigma, \hbar \in \Re$ (multiplicative triangle inequality).

Then, $\mathcal{L}_{\mathcal{M}}$ is a multiplicative metric on $\Re$ and the pair $(\Re, \mathcal{L}_{\mathcal{M}})$ is a multiplicative metric space.

Definition 2.2. [24] Let $\Re$ be a nonempty set and the function $\mathcal{L} : \Re^3 \rightarrow [0, +\infty)$ satisfies the following conditions:

(1) $\mathcal{L}(\bar{u}, \check{e}, z) = 0$ iff $\bar{u} = \check{e} = \acute{s}$;

(2) $0 < \mathcal{L}(\bar{u}, \bar{u}, \check{e})$ for all $\bar{u}, \check{e} \in \Re$ with $\bar{u} = \check{e}$;

(3) $\mathcal{L}(\bar{u}, \bar{u}, \check{e}) \leq \mathcal{L}(\bar{u}, \check{e}, \acute{s})$ for all $\bar{u}, \check{e}, \acute{s} \in \Re$ with $\check{e} = \acute{s}$;

(4) $\mathcal{L}(\bar{u}, \check{e}, \acute{s}) = \mathcal{L}(\acute{s}, \bar{u}, \check{e}) = \mathcal{L}(\check{e}, \acute{s}, \bar{u})$

(5) $\mathcal{L}(\bar{u}, \check{e}, \acute{s}) \leq \mathcal{L}(\bar{u}, a, a) + \mathcal{L}(a, \check{e}, \acute{s})$for all $\bar{u}, \check{e}, \acute{s}, a \in \Re$.

Then $\mathcal{L}$ is said to be an $\mathcal{L}$-metric on $\Re$ and $(\Re, \mathcal{L})$ is called a $\mathcal{L}$-metric space.

Definition 2.3. [28] Suppose that $\Re$ is a non-empty set and $\delta_{\mathcal{M}}$: $\Re^3 \longrightarrow R^{+}$ is a function satisfying the following conditions:

$(\delta_{\mathcal{M}_1})$ $\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar) = 1$ if $\check{e} = \varsigma = \hbar$;

$(\delta_{\mathcal{M}_2})$ $1 < \delta_{\mathcal{M}}(\check{e}, \check{e}, \varsigma) \quad \forall\, \check{e}, \varsigma \in \Re$ with $\check{e} \neq \varsigma$;

$(\delta_{\mathcal{M}_3})$ $\delta_{\mathcal{M}}(\check{e}, \check{e}, \varsigma) \leq \delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar) \quad \forall\, \check{e}, \varsigma, \hbar \in \Re$ with $\varsigma \neq \hbar$;

$(\delta_{\mathcal{M}_4})$ $\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar) = \delta_{\mathcal{M}}(\check{e}, \hbar, \varsigma) = \delta_{\mathcal{M}}(\varsigma, \hbar, \check{e}) = ...$ (symmetry);

$(\delta_{\mathcal{M}_5})$ $\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar) \leq \delta_{\mathcal{M}}(\check{e},\check{\tau},\check{\tau}).\delta_{\mathcal{M}}(\check{\tau},\varsigma,\hbar)$ $\forall$ $\check{e},\varsigma,\hbar,\check{\tau} \in \Re$, (rectangular inequality).

Then, the function $\delta_{\mathcal{M}}$ is called a multiplicative generalized metric or, more accurately, multiplicative $\delta_{\mathcal{M}}-$metric on $\Re$ and the pair $(\Re,\delta_{\mathcal{M}})$ is called a multiplicative $\delta_{\mathcal{M}}$-metric space.

We note that $\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar) = e^{\mathcal{L}(\check{e},\varsigma,\hbar)}$ $\forall$ $\check{e},\varsigma,\hbar \in \Re$. The $\delta_{\mathcal{M}}$-ball with centre $\check{e}_0$ and radius $\gamma > 0$ is defined by

$$\overline{\odot_{\gamma}(\check{e}_0,\gamma)} = \{\varrho \in \Re : \ \delta_{\mathcal{M}}(\check{e}_0,\varrho,\varrho) \leq \gamma\}.$$

Assume that $(\Re, d)$ is a usual metric space and $\delta_{\mathcal{M}}$: $\Re^3 \longrightarrow R^+$ is defined by $\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar) = a^{d(\check{e},\varsigma)+d(\varsigma,\hbar)+d(\hbar,\check{e})}$ $\forall$ $\check{e},\varsigma,\hbar \in \Re$, where $a > 1$ is any fixed real number. Then for each a, $\delta_{\mathcal{M}}$ is a multiplicative $\delta_{\mathcal{M}}-$metric on $\Re$ and $(\Re,\delta_{\mathcal{M}})$ is called a multiplicative $\delta_{\mathcal{M}}$-metric space. Note that multiplicative $\delta_{\mathcal{M}}-$metric is not a multiplicative metric space nor a $\mathcal{L}$-metric space. Moreover, multiplicative metric space is usually different from metric space (see [37]).

Lemma 2.1. *[28] Let $(\mathcal{R},\delta_{\mathcal{M}})$ be a multiplicative $\delta_{\mathcal{M}}$-metric space. Then for all $\check{v},\check{\varpi},\check{\vartheta},\check{\tau} \in \mathcal{R}$, the following conditions hold:*
(1) $\delta_{\mathcal{M}}(\check{v},\check{\varpi},\check{\vartheta}) = 1$ if $\check{v} = \check{\varpi} = \check{\vartheta}$;
(2) $\delta_{\mathcal{M}}(\check{v},\check{\varpi},\check{\vartheta}) \leq \delta_{\mathcal{M}}(\check{v},\check{\tau},\check{\tau})\ \delta_{\mathcal{M}}(\check{\varpi},\check{\tau},\check{\tau})\ \delta_{\mathcal{M}}(\check{\vartheta},\check{\tau},\check{\tau})$;
(3) $\delta_{\mathcal{M}}(\check{v},\check{\varpi},\check{\vartheta}) \leq \delta_{\mathcal{M}}(\check{v},\check{v},\check{\varpi})\ \delta_{\mathcal{M}}(\check{v},\check{v},\check{\vartheta})$;
(4) $\delta_{\mathcal{M}}(\check{v},\check{\varpi},\check{\varpi}) \leq \delta^2_{\mathcal{M}}(\check{\varpi},\check{v},\check{v})$.

Lemma 2.2. *[28] Let $\{\check{e}_k\}$ be a sequence in a $(\Re,\delta_{\mathcal{M}})$. If the sequence $\{\check{e}_k\}$ is multiplicative $\delta_{\mathcal{M}}$-convergent then it is multiplicative $\delta_{\mathcal{M}}$-Cauchy sequence.*

Lemma 2.3. *[28] Let $\{\check{e}_k\}$ be a sequence in a $(\Re,\delta_{\mathcal{M}})$. The sequence $\{\check{e}_k\}$ in $\Re$ is multiplicative $\delta_{\mathcal{M}}$-convergent to $p \in \Re$ iff $\delta_{\mathcal{M}}(\check{e}_k,p,p) \longrightarrow 1$ as $k \longrightarrow +\infty$.*

Now, we start our main results with illustrative examples.

3. Main results

The requirements for the presence of a fixed-point of mapping $\Im : \xi \longrightarrow \xi$ are stated in the following theorem.

Theorem 3.1. *Let $(\xi,\preceq,\delta_{\mathcal{M}})$ be an ordered complete multiplicative $\delta_{\mathcal{M}}$-metric space. Suppose that the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0,1)$ $\gamma > 0$, and $m \geq 1$ satisfy the following,*

$$\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e},\Im\varsigma,\Im\hbar)} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)}\right]^{\eta}, \tag{3.1}$$

and

$$\delta_{\mathcal{M}}(\check{e}_0,\Im\check{e}_0,\Im\check{e}_0) \leq (1-\eta)\,\gamma, \tag{3.2}$$

for $\check{e},\varsigma,\hbar \in \overline{\odot_{\gamma}(\check{e}_0,\gamma)}$. If for a non-increasing sequence $\{\check{e}_n\} \longrightarrow s \in \overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ such that $s \preceq \check{e}_n$, then, there exists a point $\check{e}^*$ in $\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ so that $\check{e}^* = \Im\check{e}^*$ and $\delta_{\mathcal{M}}(\check{e}^*,\check{e}^*,\check{e}^*) = 1$. Moreover, if for any three points $\check{e},\varsigma$ and $\bar{e}$ in $\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ and there exists a point $t \in \overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ such that $\tilde{u} \preceq \check{e}$, $\tilde{u} \preceq \varsigma$ and $\tilde{u} \preceq \varsigma$ that is every three elements in $\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ has a lower bound (LB), then, the point $\check{e}^*$ is unique in $\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$.

Proof. Let $\check{e}_0$ be any arbitrary point in ξ and $\check{e}_{j+1} = \Im\check{e}_j \preceq \check{e}_j$ for all $n \in N \cup \{0\}$. From inequality (3.2), we get

$$\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1) \leq (1-\eta)\gamma \leq \gamma,$$

implying thereby $\check{e}_1 \in \overline{\odot_{\gamma}(\check{e}_0,\gamma)}$. By multiplicative triangle inequality, we have

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0,\check{e}_2,\check{e}_2)} \leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)}\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_1,\check{e}_2,\check{e}_2)}$$

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$$
\begin{aligned}
&= \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)}\, \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_0, \Im\check{e}_1, \Im\check{e}_1)} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)} \right]^{1+\eta},
\end{aligned}
$$

that is

$$
\begin{aligned}
\delta_{\mathcal{M}}(\check{e}_0, \check{e}_2, \check{e}_2) &\leq \left[\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0) \right]^{1+\eta} \\
&\leq \left[(1-\eta)\gamma \right]^{1+\eta} \leq \gamma.
\end{aligned}
$$

Then, $\check{e}_2 \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$. Consider $\check{e}_3, \check{e}_4, ..., \check{e}_q$ for every $q \in N$. Taking (3.1) in consideration, we obtain

$$
\begin{aligned}
\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \check{e}_{q+1}, \check{e}_{q+1})} &= \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_{q-1}, \Im\check{e}_q, \Im\check{e}_q)} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1}, \check{e}_q, \check{e}_q)} \right]^{\eta} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-2}, \check{e}_{q-1}, \check{e}_{q-1})} \right]^{\eta^2} \\
&\vdots \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)} \right]^{\eta^q}. \qquad (3.3)
\end{aligned}
$$

Using (3.1) and (3.3), we find

$$
\begin{aligned}
\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_{q+1}, \check{e}_{q+1})} &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)}.\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_1, \check{e}_2, \check{e}_2)}.....\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \check{e}_{q+1}, \check{e}_{q+1})} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)} \right]^{1+\eta+...+\eta^q},
\end{aligned}
$$

that becomes

$$
\begin{aligned}
\delta_{\mathcal{M}}(\check{e}_0, \check{e}_{q+1}, \check{e}_{q+1}) &\leq \left[\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0) \right]^{\dfrac{1-\eta^{q+1}}{1-\eta}} \\
&\leq \left[(1-\eta)\,\gamma \right]^{\dfrac{1-\eta^{q+1}}{1-\eta}} \leq \gamma.
\end{aligned}
$$

Hence, $\check{e}_{q+1} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$. Thus, $\check{e}_j \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ for all $j \in N$. Consequently, (3.3) converts to

$$
\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j, \check{e}_{j+1}, \check{e}_{j+1})} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)} \right]^{\eta^j}. \qquad (3.4)
$$

From inequality (3.4), we have

$$
\begin{aligned}
&\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j, \check{e}_{j+k}, \check{e}_{j+k})} \\
&\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j, \check{e}_{j+1}, \check{e}_{j+1})}.\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j+1}, \check{e}_{j+2}, \check{e}_{j+2})}.....\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j+k-1}, \check{e}_{j+k}, \check{e}_{j+k})} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)} \right]^{\eta^j \dfrac{1-\eta^k}{1-\eta}} \longrightarrow 1, \quad j \longrightarrow +\infty.
\end{aligned}
$$

This means that the sequence $\{\check{e}_j\}$ is a $M^\circ\, \delta_{\mathcal{M}} - C^\bullet$ sequence in $(\overline{\odot_\gamma(\check{e}_0, \gamma)},\ \delta_{\mathcal{M}})$. Furthermore, there exists $\check{e}^* \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ with

$$
\lim_{j \longrightarrow +\infty} \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j, \check{e}^*, \check{e}^*)} = \lim_{j \longrightarrow +\infty} \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}_j, \check{e}_j)} = 1. \qquad (3.5)
$$

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Now, assume that $\check{e}^* \preceq \check{e}_j \preceq \check{e}_{j-1}$,

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \Im\check{e}^*, \Im\check{e}^*)} &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}_j, \check{e}_j)}\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j, \Im\check{e}^*, \Im\check{e}^*)} \\ &= \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}_j, \check{e}_j)}\, \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_{j-1}, \Im\check{e}^*, \Im\check{e}^*)} \\ &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}_j, \check{e}_j)} \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j-1}, \check{e}^*, \check{e}^*)} \right]^{\eta} \\ &\leq \lim_{j \longrightarrow +\infty} \left(\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}_j, \check{e}_j)} \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j-1}, \check{e}^*, \check{e}^*)} \right]^{\eta} \right) = 1,\end{aligned}$$

which is a contradiction. Then, $\check{e}^* = \Im\check{e}^*$. By a similar method, $\delta_{\mathcal{M}}(\Im\check{e}^*, \Im\check{e}^*, \check{e}^*) = 1$ and hence $\Im\check{e}^* = \check{e}^*$. Now,

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}^*, \check{e}^*)} = \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}^*, \Im\check{e}^*, \Im\check{e}^*)} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \check{e}^*, \check{e}^*)} \right]^{\eta}$$

which is a contradiction, since $\eta \in [0, 1)$. Thus, $\delta_{\mathcal{M}}(\check{e}^*, \check{e}^*, \check{e}^*) = 1$.

Uniqueness:

Consider ς^* as another point in $\overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ such that $\varsigma^* = F\varsigma^*$. If $\check{e}^*$ and ς^* are comparable, then

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*)} = \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}^*, \Im\varsigma^*, \Im\varsigma^*)} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*)} \right]^{\eta},$$

which is a contradiction and thus,

$$\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) = 1 \quad \text{implies} \quad \check{e}^* = \varsigma^*.$$

Similarly, we can prove $\delta_{\mathcal{M}}(\varsigma^*, \varsigma^*, \check{e}^*) = 1$.

On the other hand, If $\check{e}^*$ and ς^* are not comparable then there is a point $\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ which is the lower bound of $\check{e}^*$ and ς^* that is $\tilde{u} \preceq \check{e}^*$ and $\tilde{u} \preceq \varsigma^*$. Furthermore, by argument $\check{e}^* \preceq \check{e}_n$ as $\check{e}_n \longrightarrow \check{e}^*$. Thus, $\tilde{u} \preceq \check{e}^* \preceq \check{e}_n \preceq ... \preceq \check{e}_0$.

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\tilde{u}, \Im\tilde{u})} &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)}.\, \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_1, \Im\tilde{u}, \Im\tilde{u})} \\ &= \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0)}.\, \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_0, \Im\tilde{u}, \Im\tilde{u})} \\ &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0)}. \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})} \right]^{\eta},\end{aligned}$$

that is

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_0, \Im\tilde{u}, \Im\tilde{u}) &\leq \delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0). \left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u}) \right]^{\eta} \\ &\leq (1-\eta)\gamma \left[(1-\eta)\gamma \right]^{\eta} \leq \gamma \text{ (by (3.1) and (3.2))}\end{aligned}$$

where $\check{e}_0, \tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ and this means that $\Im\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$.

Now, we show that $\Im^j \tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ by using mathematical induction.

Suppose that $\Im^2 \tilde{u}, \Im^3 \tilde{u}, ..., \Im^q \tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ for all $q \in N$. As $\Im^q \tilde{u} \preceq \Im^{q-1} \tilde{u} \preceq ... \preceq \tilde{u} \preceq \check{e}^* \preceq \check{e}_n \preceq ... \preceq \check{e}_0$, then

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1} \tilde{u}, \Im^{q+1} \tilde{u})} &= \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_q, \Im(\Im^q \tilde{u}), \Im(\Im^q \tilde{u}))} \\ &\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \Im^q \tilde{u}, \Im^q \tilde{u})} \right]^{\eta} \leq ... \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \Im^q \tilde{u}, \Im^q \tilde{u})} \right]^{\eta^{q+1}}.\end{aligned}$$

It follows that

$$\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1} \tilde{u}, \Im^{q+1} \tilde{u}) \leq \left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u}) \right]^{\eta^{q+1}}. \tag{3.6}$$

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Now,

$$\delta_{\mathcal{M}}(\check{e}_0, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u}) \leq \delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)...\delta_{\mathcal{M}}(\check{e}_q, \check{e}_{q+1}, \check{e}_{q+1}).\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u})$$

$$\leq \delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)...\Big[\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)\Big]^{\eta^q}\Big[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\Big]^{\eta^{q+1}}$$

$$\leq \Big[\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)\Big]^{1+\eta+...+\eta^q}\Big[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\Big]^{\eta^{q+1}}$$

$$\leq \Big[(1-\eta)\gamma\Big]^{\frac{1-\eta^{q+1}}{1-\eta}}\Big[(1-\eta)\gamma\Big]^{\eta^{q+1}}$$

$$\leq \Big[(1-\eta)\gamma\Big]^{\frac{1-\eta^{q+2}}{1-\eta}} \leq \gamma.$$

It means that $\Im^{q+1}\tilde{u} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ and so $\Im^j\tilde{u} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ for every $j \in N$. Further

$$\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) \leq \delta_{\mathcal{M}}(\Im^j\check{e}^*, \Im^{j-1}\tilde{u}, \Im^{j-1}\tilde{u}).\delta_{\mathcal{M}}(\Im^{j-1}\tilde{u}, \Im^j\varsigma^*, \Im^j\varsigma^*)$$

$$= \delta_{\mathcal{M}}(\Im(\Im^{j-1}\check{e}^*), \Im(\Im^{j-2}\tilde{u}), \Im(\Im^{j-2}\tilde{u})).\delta_{\mathcal{M}}(\Im(\Im^{j-2}\tilde{u}), \Im(\Im^{j-1}\varsigma^*), \Im(\Im^{j-1}\varsigma^*))$$

$$\leq \Big[\delta_{\mathcal{M}}(\Im^{j-1}\check{e}^*, F^{j-2}\tilde{u}, \Im^{j-2}\tilde{u})\Big]^{\eta}\Big[\delta_{\mathcal{M}}(\Im^{j-2}\tilde{u}, \Im^{j-1}\varsigma^*, \Im^{j-1}\varsigma^*)\Big]^{\eta}$$

$$\vdots$$

$$\leq \Big[\delta_{\mathcal{M}}(\check{e}^*, \Im\tilde{u}, \Im\tilde{u})\Big]^{\eta^j}\Big[\delta_{\mathcal{M}}(\Im\tilde{u}, \varsigma^*, \varsigma^*)\Big]^{\eta^j} \longrightarrow 1, \quad \text{when } j \longrightarrow +\infty.$$

Hence, $\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) = 1 \Longrightarrow \check{e}^* = \varsigma^*$. By a similar method

$$\delta_{\mathcal{M}}(\varsigma^*, \varsigma^*, \check{e}^*) = 1 \text{ implies } \varsigma^* = \check{e}^*.$$

Therefore, a point $\check{e}^*$ is unique in ξ.

Corollary 3.1. *Let $(\xi, \preceq, \delta_{\mathcal{M}})$ be an ordered complete multiplicative $\delta_{\mathcal{M}}$ metric space. Suppose the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0, 1)$ and $\gamma > 0$ satisfying the following,*

$$\delta_{\mathcal{M}}(\Im\check{e}, \Im\varsigma, \Im\hbar) \leq [G_{\mathcal{M}}(\check{e}, \varsigma, \hbar)]^{\eta}, \tag{3.7}$$

for $\check{e}, \varsigma, \hbar \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$, with the condition (3.2) .

If for a non-increasing sequence $\{\check{e}_n\} \longrightarrow s \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $s \preceq \check{e}_n$, then, there exists a point $\check{e}^*$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ so that $\check{e}^* = \Im\check{e}^*$ and $\delta_{\mathcal{M}}(\check{e}^*, \check{e}^*, \check{e}^*) = 1$. Moreover, if for any three points $\check{e}, \varsigma$ and $\bar{e}$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ then there exists a point $\tilde{u} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $\tilde{u} \preceq \check{e}, \tilde{u} \preceq \varsigma$ and $\tilde{u} \preceq \varsigma$, that is every two points in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ has a lower bound. Then, the point $\check{e}^*$ is unique.

Example 1. Let ξ be a set of non-negative rationals with $\delta_{\mathcal{M}}$: $\xi^3 \longrightarrow \xi$ be a multiplicative $\delta_{\mathcal{M}}-$metric on ξ is defined as follow:

$$\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar) = e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}.$$

Also, let $\Im : \xi \longrightarrow \xi$ be defined as

$$\Im\check{e} = \begin{cases} \dfrac{\check{e}}{4} & \text{if} \quad \check{e} \in \Big[0, \dfrac{1}{3}\Big); \\ \check{e} - \dfrac{1}{3} & \text{if} \quad \check{e} \in \Big[\dfrac{1}{3}, \infty\Big). \end{cases}$$

For $\check{e}_0 = \frac{1}{3}, \gamma = \frac{11}{2}, \eta = \frac{5}{8}$ and $\overline{\odot_\gamma(\check{e}_0, \gamma)} = \Big[0, \frac{11}{2}\Big]$, we have

$$(1-\eta)\gamma = \frac{3}{8}.\frac{11}{2} = \frac{33}{16} = 2.0625,$$

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and

$$
\begin{aligned}
\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0) &= \delta_{\mathcal{M}}(\frac{1}{3}, \Im\frac{1}{3}, \Im\frac{1}{3}) = \delta_{\mathcal{M}}(\frac{1}{3}, 0, 0) \\
&= e^{2/3} = 1.9477 \\
&\leq (1-\eta)\gamma.
\end{aligned}
$$

Step 1: (when the points are in a closed ball) If $\check{e}, \varsigma, \hbar \in \left[0, \frac{1}{3}\right) \subseteq \overline{\odot_\gamma(\check{e}_0, \gamma)} = \left[0, \frac{11}{2}\right]$, we get

$$
\begin{aligned}
\delta_{\mathcal{M}}(\Im\check{e}, \Im\varsigma, \Im\hbar) &= e^{\frac{1}{4}(|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|)} \\
&\leq e^{\frac{5}{8}(|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|)} = [\delta_{\mathcal{M}}(x, y, z)]^\eta.
\end{aligned}
$$

Step 2: (when the points are not in a closed ball) If $\check{e}, \varsigma, \hbar \in \left[\frac{1}{3}, \infty\right)$, we have

$$
\begin{aligned}
\delta_{\mathcal{M}}(\Im x, \Im y, \Im z) &= e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|} \\
&\geq e^{\frac{5}{8}(|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|)} = [\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar)]^\eta.
\end{aligned}
$$

Clearly, the contractive condition doesn't satisfy in ξ and is satisfied in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$. Hence, all the conditions of Corollary 3.1 is verified in case of $\check{e}, \varsigma, \hbar \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$.

Since every multiplicative δ_M metric space generates multiplicative d_M metric space, we get the following corollaries.

Corollary 3.2. *Let $(\xi, \preceq, d_{\mathcal{M}})$ be an ordered complete multiplicative $d_{\mathcal{M}}-$metric space. Suppose the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0, 1)$ and $\gamma > 0$ satisfying the following,*

$$
\sqrt[m]{d_{\mathcal{M}}(\Im\check{e}, \Im\varsigma)} \leq \left[\sqrt[m]{d_{\mathcal{M}}(\check{e}, \varsigma)}\right]^\eta, \tag{3.8}
$$

and

$$
d_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0) \leq (1-\eta)\gamma, \tag{3.9}
$$

for $\check{e}, \varsigma \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$. If for a non $-$ inc sequence $\{\check{e}_n\} \longrightarrow s \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $s \preceq \check{e}_n$, then, there exists a point $\check{e}^*$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ so that $\check{e}^* = \Im\check{e}^*$ and $d_{\mathcal{M}}(\check{e}^*, \check{e}^*) = 1$. Moreover, if for any two points $\check{e}, \varsigma$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ then there exists a point $\tilde{u} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $\tilde{u} \preceq \check{e}$ and $\tilde{u} \preceq \varsigma$, that is every two points in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ has a lower bound. Then, $\check{e}^*$ is the unique point in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$.

Corollary 3.3. *Consider $(\xi, \preceq, d_{\mathcal{M}})$ as an ordered complete multiplicative d_M metric space. Suppose that the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0, 1)$ and $\gamma > 0$ satisfying the following,*

$$
d_M(\Im\check{e}, \Im\varsigma) \leq [d_M(\check{e}, \varsigma)]^\eta, \tag{3.10}
$$

for $\check{e}, \varsigma \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$, with the condition (3.9).

If for a non-increasing sequence $\{\check{e}_n\} \longrightarrow s \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ implies that $s \preceq \check{e}_n$, then, there is a point $\check{e}^*$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $\check{e}^* = \Im\check{e}^*$ and $d_{\mathcal{M}}(\check{e}^*, \check{e}^*) = 1$. Moreover, if for any two points $\check{e}, \varsigma$ in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ then, there exists a point $\tilde{u} \in \overline{\odot_\gamma(\check{e}_0, \gamma)}$ such that $\tilde{u} \preceq \check{e}$ and $\tilde{u} \preceq \varsigma$, that is every two points in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$ has a lower bound, then, a fixed point $\check{e}^*$ is unique in $\overline{\odot_\gamma(\check{e}_0, \gamma)}$.

Theorem 3.2. *Let $(\xi, \preceq, \delta_{\mathcal{M}})$ be an ordered complete multiplicative δ_M-metric space. Suppose that the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0, 1)$ and $\gamma > 0$ satisfying the following,*

$$
\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}, \Im\varsigma, \Im\hbar)} \leq \mathcal{M}, \tag{3.11}
$$

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since

$$\mathcal{M}=\left[\max\left\{\begin{array}{c}\sqrt[m]{\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e},\Im\check{e},\Im\check{e})},\\ \sqrt[m]{\delta_{\mathcal{M}}(\varsigma,\Im\varsigma,\Im\varsigma)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e},\Im\varsigma,\Im\varsigma)},\\ \sqrt[m]{\min\left\{\delta_{\mathcal{M}}(\hbar,\Im\check{e},\Im\check{e}),\delta_{\mathcal{M}}(\check{e},\hbar,\hbar)\right\}}\end{array}\right\}\right]^{\eta},$$

and

$$G_{\mathcal{M}}(\check{e}_0,\Im\check{e}_0,\Im\check{e}_0)\leq(1-\eta)\,\gamma, \tag{3.12}$$

for $\check{e},\varsigma,\hbar\in\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$. *If for a non-increasing sequence* $\{\check{e}_n\}$ *in* $\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ *and* $\{\check{e}_n\}\longrightarrow v\in\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ *so that* $v\preceq\check{e}_n$, *then, there exists a unique fixed point* $\check{e}^*$ *such that* $\delta_{\mathcal{M}}(\check{e}^*,\check{e}^*,\check{e}^*)=1$ *and* $\check{e}^*=\Im\check{e}^*$.

Proof. Consider an arbitrary point $\check{e}_0$ in ξ and $\check{e}_{q+1}=\Im\check{e}_q\preceq\check{e}_q$ for all $n\in N\cup\{0\}$. From inequality (3.12), we find

$$\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\leq(1-\eta)\gamma\leq\gamma,$$

for all $j\in N\cup\{0\}$. Now, from inequality (3.12), we obtain $\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\leq\gamma$ and $\delta_{\mathcal{M}}(\check{e}_1,\check{e}_2,\check{e}_2)\leq\gamma$, which tends to $\check{e}_1,\check{e}_2\in\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$. Similarly $\check{e}_3,...,\check{e}_q\in\overline{\odot_{\gamma}(\check{e}_0,\gamma)}$ for all $q\in N$. Now,

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})}=\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_{q-1},\Im\check{e}_q,\Im\check{e}_q)}$$

$$\leq\left[\max\left\{\begin{array}{c}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\Im\check{e}_{q-1},\Im\check{e}_{q-1})},\\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\Im\check{e}_q,\Im\check{e}_q)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\Im\check{e}_q,\Im\check{e}_q)},\\ \sqrt[m]{\min\left\{\delta_{\mathcal{M}}(\check{e}_q,\Im\check{e}_{q-1},\Im\check{e}_{q-1}),\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)\right\}}\end{array}\right\}\right]^{\eta}$$

$$\leq\left[\max\left\{\begin{array}{c}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)},\\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_{q+1},\check{e}_{q+1})},\\ \sqrt[m]{\min\left\{G_{\mathcal{M}}(\check{e}_q,\check{e}_q,\check{e}_q),\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)\right\}}\end{array}\right\}\right]^{\eta}$$

$$\leq\left[\max\left\{\begin{array}{c}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)},\\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})},\\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)}.\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})},1\end{array}\right\}\right]^{\eta}\text{(using }(\delta_{M_1})\text{ and }(\delta_{M_5}))$$

implying thereby,

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})}\leq\left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)}.\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})}\right]^{\eta},$$

that is,

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$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1}) &\leq \Big[\delta_{\mathcal{M}}(\check{e}_{q-1},\check{e}_q,\check{e}_q)\Big]^{\mu}\\ &\leq \Big[\delta_{\mathcal{M}}(\check{e}_{q-2},\check{e}_{q-1},\check{e}_{q-1})\Big]^{\mu^2}\\ &\vdots\\ &\leq \Big[\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\Big]^{\mu^q},\end{aligned} \tag{3.13}$$

where $0<\mu=\dfrac{\eta}{1-\eta}<\dfrac{1}{2}$. Taking (3.11) and (3.12) in consideration, we get

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_0,\check{e}_{q+1},\check{e}_{q+1}) &\leq \delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\delta_{\mathcal{M}}(\check{e}_1,\check{e}_2,\check{e}_2)\\ &\cdots\ \delta_{\mathcal{M}}(\check{e}_q,\check{e}_{q+1},\check{e}_{q+1})\\ &\leq \Big[\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\Big]^{\frac{1-\mu^{q+1}}{1-\mu}}\\ &\leq \Big[(1-\eta)\,\gamma\Big]^{\frac{1-\mu^{q+1}}{1-\mu}} \leq \gamma.\end{aligned}$$

Then, $\check{e}_{q+1}\in\overline{\odot_\gamma(\check{e}_0,\gamma)}$. Thus, $\check{e}_j\in\overline{\odot_\gamma(\check{e}_0,\gamma)}$ for every $j\in N$. Now, inequality (3.13) became

$$\delta_{\mathcal{M}}(\check{e}_j,\check{e}_{j+1},\check{e}_{j+1})\leq\Big[\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\Big]^{\mu^j}. \tag{3.14}$$

From inequality (3.14), we find

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_j,\check{e}_{j+k},\check{e}_{j+k}) &\leq \delta_{\mathcal{M}}(\check{e}_j,\check{e}_{j+1},\check{e}_{j+1})\delta_{\mathcal{M}}(\check{e}_{j+1},\check{e}_{j+2},\check{e}_{j+2})\cdots\delta_{\mathcal{M}}(\check{e}_{j+k-1},\check{e}_{j+k},\check{e}_{j+k})\\ &\leq \Big[\delta_{\mathcal{M}}(\check{e}_0,\check{e}_1,\check{e}_1)\Big]^{\mu^j\frac{1-\mu^k}{1-\mu}}\longrightarrow 1,\quad j\longrightarrow+\infty.\end{aligned}$$

This shows that the sequence $\{\check{e}_j\}$ is a $M^\circ\ \delta_{\mathcal{M}}-C^\bullet$ sequence in $(\overline{\odot_\gamma(\check{e}_0,\gamma)},\delta_{\mathcal{M}})$. Then, there exists $\check{e}^*\in\overline{\odot_\gamma(\check{e}_0,\gamma)}$ with (3.5) is verified.

Now, suppose that $\check{e}^*\leq\check{e}_j\leq\check{e}_{j-1}$,

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\Im\check{e}^*,\Im\check{e}^*)} &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}_j,\check{e}_j)}\,\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_j,\Im\check{e}^*,\Im\check{e}^*)}\\ &= \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}_j,\check{e}_j)}\,\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_{j-1},\Im\check{e}^*,\Im\check{e}^*)}\\ &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}_j,\check{e}_j)}\Big[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j-1},\check{e}^*,\check{e}^*)}\Big]^{\eta}\\ &\leq \lim_{j\longrightarrow\infty}\left(\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}_j,\check{e}_j)}\Big[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{j-1},\check{e}^*,\check{e}^*)}\Big]^{\eta}\right)=1,\end{aligned}$$

which is a contradiction. Then, $\check{e}^*=\Im\check{e}^*$. By a similar method, $\delta_{\mathcal{M}}(\Im\check{e}^*,\Im\check{e}^*,\check{e}^*)=1$ and hence $\Im\check{e}^*=\check{e}^*$. Now,

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}^*,\check{e}^*)}=\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}^*,\Im\check{e}^*,\Im\check{e}^*)}\leq\Big[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\check{e}^*,\check{e}^*)}\Big]^{\eta}$$

which is a contradiction, since $\eta\in[0,1)$. Thus, $\delta_{\mathcal{M}}(\check{e}^*,\check{e}^*,\check{e}^*)=1$.

Uniqueness:

Let ς^* be another point in $\overline{\odot_\gamma(\check{e}_0,\gamma)}$ such that $\varsigma^*=\Im\varsigma^*$. If $\check{e}^*$ and ς^* are comparable, then

$$\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\varsigma^*,\varsigma^*)}=\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}^*,\Im\varsigma^*,\Im\varsigma^*)}\leq\Big[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}^*,\varsigma^*,\varsigma^*)}\Big]^{\eta}$$

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which is a contradiction that tend us to

$$\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) = 1 \quad \text{implies} \quad \check{e}^* = \varsigma^*.$$

Similarly, we can prove $\delta_{\mathcal{M}}(\varsigma^*, \varsigma^*, \check{e}^*) = 1$.

On the other hand, if $\check{e}^*$ and ς^* are not comparable then there exists a point $\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ which is the lower bound of $\check{e}^*$ and ς^* that is $\tilde{u} \preceq \check{e}^*$ and $\tilde{u} \preceq \varsigma^*$. Furthermore, $\check{e}^* \preceq \check{e}_n$ as $\check{e}_n \longrightarrow \check{e}^*$, $\tilde{u} \preceq \check{e}^* \preceq \check{e}_n \preceq ... \preceq \check{e}_0$.

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\tilde{u}, \Im\tilde{u})} &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)}.\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_1, \Im\tilde{u}, \Im\tilde{u})}\\ &= \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0)}\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_0, \Im\tilde{u}, \Im\tilde{u})}\\ &\leq \sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0)}.\left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})}\right]^{\eta},\end{aligned}$$

that is,

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_0, \Im\tilde{u}, \Im\tilde{u}) &\leq \delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0).\left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\right]^{\eta}\\ &\leq (1-\eta)\gamma\left[(1-\eta)\gamma\right]^{\eta} \leq \gamma \text{ (by (3.12))}\end{aligned}$$

where $\check{e}_0, \tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ and this means that $\Im\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$.

Now, we prove that $\Im^j\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ by using mathematical induction. Suppose $\Im^2\tilde{u}, \Im^3\tilde{u}, ..., \Im^q\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ for all $q \in N$. As $\Im^q\tilde{u} \preceq \Im^{q-1}\tilde{u} \preceq ... \preceq \tilde{u} \preceq \check{e}^* \preceq \check{e}_n \preceq ... \preceq \check{e}_0$, then

$$\begin{aligned}\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u})} &= \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}_q, \Im(\Im^q\tilde{u}), \Im(\Im^q\tilde{u}))}\\ &\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \Im^q\tilde{u}, \Im^q\tilde{u})}\right]^{\eta} \leq ... \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}_q, \Im^q\tilde{u}, \Im^q\tilde{u})}\right]^{\eta^{q+1}},\end{aligned}$$

it follows that

$$\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u}) \leq \left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\right]^{\eta^{q+1}}. \tag{3.15}$$

Now,

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}_0, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u}) &\leq \delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)...\delta_{\mathcal{M}}(\check{e}_q, \check{e}_{q+1}, \check{e}_{q+1}).\delta_{\mathcal{M}}(\check{e}_{q+1}, \Im^{q+1}\tilde{u}, \Im^{q+1}\tilde{u})\\ &\leq \delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)...\left[\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)\right]^{\eta^q}\left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\right]^{\eta^{q+1}}\\ &\leq \left[\delta_{\mathcal{M}}(\check{e}_0, \check{e}_1, \check{e}_1)\right]^{1+\eta+...+\eta^q}\left[\delta_{\mathcal{M}}(\check{e}_0, \tilde{u}, \tilde{u})\right]^{\eta^{q+1}}\\ &\leq \left[(1-\eta)\gamma\right]^{\frac{1-\eta^{q+1}}{1-\eta}}\left[(1-\eta)\gamma\right]^{\eta^{q+1}}\\ &\leq \left[(1-\eta)\gamma\right]^{\frac{1-\eta^{q+2}}{1-\eta}} \leq \gamma.\end{aligned}$$

It follows that $\Im^{q+1}\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ and so $\Im^j\tilde{u} \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$ for every $j \in N$. Furthermore

$$\begin{aligned}\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) &\leq \delta_{\mathcal{M}}(\Im^j\check{e}^*, \Im^{j-1}\tilde{u}, \Im^{j-1}\tilde{u}).\delta_{\mathcal{M}}(\Im^{j-1}\tilde{u}, \Im^j\varsigma^*, \Im^j\varsigma^*)\\ &= \delta_{\mathcal{M}}(\Im(\Im^{j-1}\check{e}^*), \Im(\Im^{j-2}\tilde{u}), \Im(\Im^{j-2}\tilde{u})).\delta_{\mathcal{M}}(\Im(\Im^{j-2}\tilde{u}), \Im(\Im^{j-1}\varsigma^*), \Im(\Im^{j-1}\varsigma^*))\\ &\leq \left[\delta_{\mathcal{M}}(\Im^{j-1}\check{e}^*, \Im^{j-2}\tilde{u}, \Im^{j-2}\tilde{u})\right]^{\eta}.\left[\delta_{\mathcal{M}}(\Im^{j-2}\tilde{u}, \Im^{j-1}\varsigma^*, \Im^{j-1}\varsigma^*)\right]^{\eta}\\ &\vdots\end{aligned}$$

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$$\leq \left[\delta_{\mathcal{M}}(\check{e}^*, \Im\tilde{u}, \Im\tilde{u})\right]^{\eta^j}\left[\delta_{\mathcal{M}}(\Im\tilde{u}, \varsigma^*, \varsigma^*)\right]^{\eta^j} \longrightarrow 1, \quad \text{where } j \longrightarrow +\infty.$$

Hence, $\delta_{\mathcal{M}}(\check{e}^*, \varsigma^*, \varsigma^*) = 1 \Longrightarrow \check{e}^* = \varsigma^*$. Similarly,

$$\delta_{\mathcal{M}}(\varsigma^*, \varsigma^*, \check{e}^*) = 1 \text{ implies } \varsigma^* = \check{e}^*.$$

Therefore, a point $\check{e}^*$ is unique in ξ.

As illustrated, Theorem 3.1 is a corollary to Theorem 3.6.

Example 2. Consider $\xi = R^+ \cup \{0\}$ with $\delta_{\mathcal{M}}: \xi^3 \longrightarrow \xi$ be a multiplicative $G_{\mathcal{M}}-$metric on ξ is defined by

$$\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar) = e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}.$$

Also, let the mapping $\Im : \xi \longrightarrow \xi$ be defined as

$$\Im\check{e} = \begin{cases} \dfrac{\check{e}}{6} & \text{if } \check{e} \in \left(0, \dfrac{1}{5}\right) \cap \xi; \\ \check{e} - \dfrac{1}{8} & \text{if } \check{e} \in \left[\dfrac{1}{5}, \infty\right) \cap \xi, \end{cases}$$

and

$$\mathcal{M} = \left[\max\left\{\begin{array}{c} \sqrt[m]{\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar)},\ \sqrt[m]{G_{\mathcal{M}}(\check{e}, \Im\check{e}, \Im\check{e})}, \\ \sqrt[m]{\delta_{\mathcal{M}}(\varsigma, \Im\varsigma, \Im\varsigma)},\ \sqrt[m]{\delta_{\mathcal{M}}(\check{e}, \Im\varsigma, \Im\varsigma)}, \\ \sqrt[m]{\min\{\delta_{\mathcal{M}}(\hbar, \Im\check{e}, \Im\check{e}), \delta_{\mathcal{M}}(\check{e}, \hbar, \hbar)\}} \end{array}\right\}\right]^{\eta}.$$

For $\check{e}_0 = \dfrac{1}{4}, \gamma = \dfrac{13}{2}, \eta = \dfrac{2}{3}$ and $\overline{\odot_{\gamma}(\check{e}_0, \gamma)} = \left[0, \dfrac{13}{2}\right]$, we have

$$(1-\eta)\,\gamma = \frac{1}{3}.\frac{13}{2} = \frac{13}{6} = 2.16,$$

and

$$\begin{aligned} \delta_{\mathcal{M}}(\check{e}_0, \Im\check{e}_0, \Im\check{e}_0) &= \delta_{\mathcal{M}}(\frac{1}{4}, \Im\frac{1}{4}, \Im\frac{1}{4}) = \delta_{\mathcal{M}}(\frac{1}{4}, \frac{1}{16}, \frac{1}{16}) \\ &= e^{6/16} = 1.4533 \\ &\leq (1-\eta)\,\gamma. \end{aligned}$$

Step 1: If $\check{e}, \varsigma, \hbar \in \left(0, \dfrac{1}{5}\right) \cap \xi \subseteq \overline{\odot_{\gamma}(\check{e}_0, \gamma)} = \left[0, \dfrac{13}{2}\right]$, we obtain

$$\begin{aligned} \sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}, \Im\varsigma, \Im\hbar)} &= \sqrt[m]{e^{\frac{1}{2}(|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|)}} \\ &\leq \left[\max\left\{\begin{array}{l} \sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}},\ \sqrt[m]{e^{|\check{e}|}}, \\ \sqrt[m]{e^{|\varsigma|}},\ \sqrt[m]{e^{|\varsigma-2\check{e}|}}, \\ \sqrt[m]{\min\{e^{|\check{e}-2\hbar|}, e^{2\,|\check{e}-\hbar|}\}} \end{array}\right\}\right]^{\eta} \\ &= \left[\sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}}\right]^{\eta} \\ &= \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar)}\right]^{\eta}. \end{aligned}$$

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Step 2: If $\check{e}, \varsigma, \hbar \in \left[\frac{1}{5}, \infty\right) \cap \xi$, we have

$$\begin{aligned}
\sqrt[m]{\delta_{\mathcal{M}}(\Im x, \Im y, \Im z)} &= \sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}} \\
&\geq \left[\max \left\{ \begin{array}{c} \sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}}, \sqrt[m]{e^{\frac{1}{2}}}, \\ \sqrt[m]{e^{\frac{1}{2}}}, \sqrt[m]{e^{2|\check{e}-\varsigma+\frac{1}{4}|}}, \\ \sqrt[m]{\min\left\{e^{2|\hbar-\check{e}+\frac{1}{4}|}, e^{2|\check{e}-\hbar|}\right\}} \end{array} \right\} \right]^{\eta} \\
&= \left[\sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}} : \right]^{\eta} \\
&= \left[\sqrt[m]{e^{|\check{e}-\varsigma| + |\varsigma-\hbar| + |\hbar-\check{e}|}} \right]^{5/8}.
\end{aligned}$$

Clearly, the contractive condition doesn't verify in $\left[\frac{1}{5}, \infty\right) \cap \xi$ and is verified in $\overline{\odot_{\gamma}(\check{e}_0, \gamma)}$. Hence, all the assertions of Theorem 3.6 is satisfied in case of $\check{e}, \varsigma, \hbar \in \overline{\odot_{\gamma}(\check{e}_0, \gamma)}$.

4. Application for nonlinear Voltera type integral equations

Clearly, many researchers justified many kinds of linear and nonliear Volterra and Fredhlom type integral equations by using various contractions principle. Rasham et al. [35] proved an expressive fixed point results for sufficient conditions to solve two systems of nonlinear integral equations. For further fixed point results with applications related to integral equations (see[12,18,27,33,41]).

Theorem 4.1. *Let $(\xi, \preceq, \delta_{\mathcal{M}})$ be an ordered complete multiplicative $\delta_{\mathcal{M}}$-metric space. Suppose the mapping $\Im : \xi \longrightarrow \xi$ with $\eta \in [0,1)$ and $\gamma > 0$ satisfies the following,*

$$\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e}, \Im\varsigma, \Im\hbar)} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e}, \varsigma, \hbar)}\right]^{\eta}.$$

Then every non-increasing sequence $\{\check{e}_n\}$ in multiplicative δ_M-metric space converges to $\check{e}^*$. Moreover, $\check{e}^*$ is the fixed point of the mapping $\Im$.

Proof. The proof of the Theorem 4.1 is similar to the Theorem 3.1. Consider the nonlinear Volterra type integral equations as follow:

$$\check{e}(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_1(\tilde{u}, h, \check{e})\, dh, \quad (4.1)$$

$$\varsigma(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_2(\tilde{u}, h, \varsigma)\, dh, \quad (4.2)$$

$$\hbar(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_3(\tilde{u}, h, \hbar)\, dh, \quad (4.3)$$

for all $\tilde{u} \in [0,1]$, and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 : [0,1] \times [0,1] \times \mathcal{C}([0,1], R_+) \longrightarrow R_+$. We prove the existence of solution of (4.1), (4.2) and (4.3). For $\check{e} \in \mathcal{C}([0,1], R_+)$, define norm as:

$$\|\check{e}\|_{\tau} = \sup_{\tilde{u}\in[0,1]} \left\{e^{|\check{e}(\tilde{u})|}\right\}.$$

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Then, define

$$\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)=\left[\sup_{\tilde{u}\in[0,1]}\left\{e^{|\check{e}-\varsigma|+|\varsigma-\hbar|+|\hbar-\check{e}|}\right\}\right]=e^{\|\check{e}-\varsigma\|_{\tau}+\|\varsigma-\hbar\|_{\tau}+\|\hbar-\check{e}\|_{\tau}},$$

where $\tau>0$, for all $\check{e},\varsigma$ and $\hbar\in\mathcal{C}([0,1],R_{+})$. Whith these setting $\left(\mathcal{C}([0,1],R_{+}),\delta_{\mathcal{M}}\right)$ becomes a complete multiplicative δ_{M}-metric space.

Now, we prove the following theorem to show the existence of the solution to integral equations.

Theorem 4.2. *Suppose the followings are satisfied:*
$(i)\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3:[0,1]\times[0,1]\times\mathcal{C}([0,1],R_{+})\longrightarrow R_{+}$;
(ii) Define

$$\begin{aligned}(\Im\check{e})(\tilde{u}) &= \int_0^{\tilde{u}}\mathcal{H}_1(\tilde{u},h,\check{e})dh;\\ (\Im\varsigma)(\tilde{u}) &= \int_0^{\tilde{u}}\mathcal{H}_2(\tilde{u},h,\varsigma)dh;\\ (\Im\hbar)(\tilde{u}) &= \int_0^{\tilde{u}}\mathcal{H}_3(\tilde{u},h,\hbar)dh.\end{aligned}$$

If there exists

$$\left(e^{\sqrt[m]{\int_0^{\tilde{u}}\left(|\mathcal{H}_1(\tilde{u},h,\check{e})-\mathcal{H}_2(\tilde{u},h,\varsigma)|+|\mathcal{H}_2(\tilde{u},h,\varsigma)-\mathcal{H}_3(\tilde{u},h,\hbar)|+|\mathcal{H}_3(\tilde{u},h,\hbar)-\mathcal{H}_1(\tilde{u},h,\check{e})|\right)dh}}\right)^{\eta}$$
$$\leq\int_0^{\tilde{u}}\left(e^{\sqrt[m]{|\check{e}-\varsigma|+|\varsigma-\hbar|+|\hbar-\check{e}|}}\right)^{\eta}dh.$$

For every $\tilde{u},h\in[0,1]$ and $\check{e},\varsigma,\hbar\in\mathcal{C}([0,1],R_{+})$. Then, the integral equations (4.1), (4.2) and (4.3) have one solution in $\mathcal{C}([0,1],R_{+})$.

Proof. By (ii)

$$\begin{aligned}&\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e},\Im\varsigma,\Im\hbar)}\\ &= e^{\sqrt[m]{\int_0^{\tilde{u}}\left(|\Im\check{e}-\Im\varsigma|+|\Im\varsigma-\Im\hbar|+|\Im\hbar-\Im\check{e}|\right)dh}}\\ &= \left(e^{\sqrt[m]{\int_0^{\tilde{u}}\left(|\mathcal{H}_1(\tilde{u},h,\check{e})-\mathcal{H}_2(\tilde{u},h,\varsigma)|+|\mathcal{H}_2(\tilde{u},h,\varsigma)-\mathcal{H}_3(\tilde{u},h,\hbar)|+|\mathcal{H}_3(\tilde{u},h,\hbar)-\mathcal{H}_1(\tilde{u},h,\check{e})|\right)dh}}\right)^{\eta}\\ &\leq \int_0^{\tilde{u}}\left(e^{\sqrt[m]{|\check{e}-\varsigma|+|\varsigma-\hbar|+|\hbar-\check{e}|}}\right)^{\eta}dh\\ &\leq \left(\sqrt[m]{\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)}\right)^{\eta}.\end{aligned}$$

So, all the conditions of Theorem 4.1 are satisfied. Hence, the integral equations (4.1), (4.2) and (4.3) have unique solution.

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Example 3. Take $E = [0,1]$. If we put $\tilde{u} = 1$ in (4.1), (4.2) and (4.3), then we get the following integral equations

$$(\Im\check{e})(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_1(\tilde{u},h,\check{e})dh = \int_0^1 \frac{4}{9(\tilde{u}+1+\check{e}(h))}dh \quad (4.4)$$

$$(\Im\varsigma)(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_2(\tilde{u},h,\varsigma)dh = \int_0^1 \frac{4}{9(\tilde{u}+1+\varsigma(h))}dh \quad (4.5)$$

$$(\Im\hbar)(\tilde{u}) = \int_0^{\tilde{u}} \mathcal{H}_3(\tilde{u},h,\hbar)dh = \int_0^1 \frac{4}{9(\tilde{u}+1+\hbar(h))}dh. \quad (4.6)$$

The equation (4.4)-(4.6) are the special case of (4.1)-(4.3) respectively where $\tilde{u} \in [0,1]$.

$$\begin{aligned}
&\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e},\Im\varsigma,\Im\hbar)} \\
&= \sqrt[m]{\exp\{\|\Im\check{e}-\Im\varsigma\| + \|\Im\varsigma-\Im\hbar\| + \|\Im\hbar-\Im\check{e}\|} \\
&= \sqrt[m]{\int_0^1 \exp\left\{ \begin{array}{c} \|\mathcal{H}_1(\tilde{u},h,\check{e})-\mathcal{H}_2(\tilde{u},h,\varsigma)\| + \|\mathcal{H}_2(\tilde{u},h,\varsigma)-\mathcal{H}_3(\tilde{u},h,\hbar)\| \\ \|\mathcal{H}_3(\tilde{u},h,\hbar)-\mathcal{H}_1(\tilde{u},h,\check{e})\| \end{array} \right\} dh} \\
&= \sqrt[m]{\int_0^1 \exp\left\{ \begin{array}{c} \left\|\frac{4}{9(\tilde{u}+1+\check{e}(h))} - \frac{4}{9(\tilde{u}+1+\varsigma(h))}\right\| + \left\|\frac{4}{9(\tilde{u}+1+\varsigma(h))} - \frac{4}{9(\tilde{u}+1+\hbar(h))}\right\| \\ + \left\|\frac{4}{9(\tilde{u}+1+\hbar(h))} - \frac{4}{9(\tilde{u}+1+\check{e}(h))}\right\| \end{array} \right\} dh} \\
&= \sqrt[m]{e^{\frac{2}{3}} \left[\int_0^1 \exp\left\{ \begin{array}{c} \left\|\frac{1}{(\tilde{u}+1+\check{e}(h))} - \frac{1}{(\tilde{u}+1+\varsigma(h))}\right\| + \left\|\frac{1}{(\tilde{u}+1+\varsigma(h))} - \frac{1}{(\tilde{u}+1+\hbar(h))}\right\| \\ + \left\|\frac{1}{(\tilde{u}+1+\hbar(h))} - \frac{1}{(\tilde{u}+1+\check{e}(h))}\right\| \end{array} \right\} dh \right]} \\
&= \sqrt[m]{e^{\frac{2}{3}} \left[\int_0^1 \exp\left\{ \begin{array}{c} \left\|\frac{\varsigma(h)-\check{e}(h)}{(\tilde{u}+1+\check{e}(h))(\tilde{u}+1+\varsigma(h))}\right\| + \left\|\frac{\hbar(h)-\varsigma(h)}{(\tilde{u}+1+\varsigma(h))(\tilde{u}+1+\hbar(h))}\right\| \\ + \left\|\frac{\check{e}(h)-\hbar(h)}{(\tilde{u}+1+\hbar(h))(\tilde{u}+1+\check{e}(h)}\right\| \end{array} \right\} dh \right]} \\
&= \sqrt[m]{e^{\frac{2}{3}} \left[\exp\left\{ \begin{array}{c} \int_0^1 \left\|\frac{\varsigma(h)-\check{e}(h)}{(\tilde{u}+1+\check{e}(h))(\tilde{u}+1+\varsigma(h))}\right\| dh + \int_0^1 \left\|\frac{\hbar(h)-\varsigma(h)}{(\tilde{u}+1+\varsigma(h))(\tilde{u}+1+\hbar(h))}\right\| dh \\ + \int_0^1 \left\|\frac{\check{e}(h)-\hbar(h)}{(\tilde{u}+1+\hbar(h))(\tilde{u}+1+\check{e}(h)}\right\| dh \end{array} \right\} \right]} \\
&= \sqrt[m]{e^{\frac{2}{3}}[\exp\{\|\varsigma(h)-\check{e}(h)\| + \|\hbar(h)-\varsigma(h)\| + \|\check{e}(h)-\hbar(h)\|\}]} \\
&\leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)}\right]^{\eta} \qquad \eta = \frac{2}{3} \in [0,1].
\end{aligned}$$

It follows that

$$\sqrt[m]{\delta_{\mathcal{M}}(\Im\check{e},\Im\varsigma,\Im\hbar)} \leq \left[\sqrt[m]{\delta_{\mathcal{M}}(\check{e},\varsigma,\hbar)}\right]^{\eta}.$$

Hence, all conditions of Theorem 4.1 hold. The integral equations (4.4), (4.5) and (4.6) have a unique solution by using Theorem 4.1.

5. Conclusions

We provided some novel fixed point results in an ordered complete multiplicative $\delta_{\mathcal{M}}$-metric space that satisfies a generalized locally Δ-implicit contractive mappings. In these spaces, some new definitions and examples are presented. Furthermore, we provided examples to support our new findings. To demonstrate the originality of main theorems, we apply them to show the existence of the solutions to a system of nonlinear integral equations. The obtained results improve and generalize the corresponding results in the ordered metric space, ordered dislocated metric space, ordered $G-$metric space, dislocated $G-$metric space, ordered partial metric space, multiplicative metric space, ordered multiplicative metric space and multiplicative $D-$metric space. The research work done in this paper, in future, will set a direction to work on multivalued mappings, fuzzy mappings, bipolar fuzzy mappings, L-fuzzy mappings, and intuitionistic fuzzy mappings.

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