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Hassan Bouamoud *

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The Polynomial $t^2(4x-n)^2-2ntx$ Does Not Always Admits a Perfect Square

Hassan Bouamoud

Independent Researcher; habo5671@gmail.com

Abstract

In this article we show that the polynomial $t^2(4x-n)^2-2ntx$ does not always admits a perfect square with $n \geq 2$ and $(x,t) \in (\mathbb{N}^*)^{\not{2}}$. We discuss this when $n=3$ and we show by contradiction that one of x or t (in the expression $t^2(4x-3)^2-6tx$) isn't an integer.

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1. Introduction

In his article : Partial Resolution of the Erdős-Straus, Sierpiński, and Generalized Erdős-Straus Conjectures Using New Analytical Formulas, in the page 13, Philemon Urbain MBALLA pose the following question: how can we prove that the polynomial $t^2(4x-n)^2-2ntx$ or $4t^2(4x-n)^2-8ntx$ always admits a perfect square for all $n \in \mathbb{N}$, with $n \geq 2$?

2. Definitions

Definition: we mean by a positive integer a natural number strictly greater than 1.

3. Principal Resulte

Théorème 3.1. *The polynomial $t^2(4x-n)^2-2ntx$ does not always admits a perfect square with $n \geq 2$ and x,t positive integers.*

Proof. For $n=3$, the polynomial $t^2(4x-n)^2-2ntx$ become:
$t^2(4x-3)^2-6tx$
We need to have a positive integer m such that : $t^2(4x-3)^2-6tx=m^2$
In this expression if we show that x or t isn't a postive integer then the theorem 3.1 is proved.
We have: $t^2(4x-3)^2-6tx=m^2 \quad \Leftrightarrow \quad t^2(4x-3)^2-m^2=6tx$

$$\Leftrightarrow (t(4x-3)-m)(t(4x-3)+m)=6tx$$

The two expression $t(4x-3)-m$ and $t(4x-3)+m$ have the same parity because it deffirencies by 2m, or the parity of it's product is even(because it's product is 6tx) then $t(4x-3)-m$ and $t(4x-3)+m$ are both even.
Let: $t(4x-3)-m=2r$ then: $2r+m=t(4x-3)$ and $r(r+m)=3xt$
So: $x=\frac{3r(r+m)}{4r(r+m)-6r+3m}$
Remark: We need to find (r,m) positive integeres such that x also a positive integer. Now We define the function: $x: t->x(t)$ with $t=r+m$ (then: $t \geq r+1$ because $m \geq 1$)
So: $x(t)=\frac{3rt}{t(4r+3)-9r}$ Or $x \geq 0$, the denominator is positive, we studie x as a function of t with fixed r.
Proof that $x: t->x(t)$ is strictly decreasing:
If we calcule the derivative of x we find:
$x'(t)=\frac{-27r^2}{(t(4r+3)-9r)^2}$
Then $x'(t)<0$ for all $t>\frac{9r}{4r+3}$, then x is strictly decreasing for all $t \geq r+1$.
And the maximum of the function x is atteint in the limit of left so when $t=r+1$ Then

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$x(t = r+1) = \frac{3r(r+1)}{4r^2-2r+3}$ this value of x does not atteint 1: $x(t = r+1) = \frac{3r(r+1)}{4r^2-2r+3} < 1$ for all $r \geq 5$,
We will prove it by studiyng the function $g(r) = \frac{3r(r+1)}{4r^2-2r+3}$ for all $r \geq 5$
The derivative of g is:$g'(r) = \frac{9(-2r^2+2r+1)}{(4r^2-2r+3)}$ this function is negative for all $r \geq \frac{1+\sqrt{3}}{2}$, So g atteint its maximum on the extrem of left mean on $r = 5$
$g(5) = 0.967 < 1$ mean $x(r+1) < 1$ for all $r \geq 5$
So x is not an integer for all value $r \geq 5$.
Remaine the cases when $r = 1, 2, 3, 4$:
For every value of r we solve the equation $x = k \in \mathbb{Z}^+$ with $t \geq r+1$ this is equivalent to:
$t = \frac{9kr}{k(4r+3)-3r}$, we need to have t positive integer and $t \geq r+1$:

For $r = 1$, $t = \frac{9k}{7k-3}$ any $k \geq 1$ integer does not give $t \geq 2$
Consider: $f(x) = \frac{9x}{7x-3}$, the derivative of f is: $f'(x) = -\frac{27}{(7x-3)^2} < 0$ for all $x > \frac{3}{7}$, so f is strictly decreasing for all $x \geq 1$
Then the maximum of f is atteint in the limit of left when $x = 1$ then $f(1) = \frac{9}{4} = 2.25$ and when $x \to +\infty$ then $f(x) \to \frac{9}{7} \approx 1.2857$
Since $\frac{9}{4} = 2.25$ and the next integer less than 2.25 is 2, and $2 > \frac{9}{7}$, we need to check for $f(x) = 2$:
When $f(x) = 2$ then $x = \frac{6}{5} \notin \mathbb{N}$
So no integer $k \geq 1$ gives $t = \frac{9k}{7k-3}$ integer and $t \geq 2$

For $r = 2$, $t = \frac{18k}{11k-6}$ any $k \geq 1$ integer does not give $t \geq 3$
Consider: $f(x) = \frac{18x}{11x-6}$, the derivative of f is: $f'(x) = -\frac{108}{(11x-6)^2} < 0$, so f is strictly decreasing for all $x \geq 1$
Then the maximum of f is atteint in the limit of left when $x = 1$ then $f(1) = \frac{18}{5} = 3.6$ and when $x \to +\infty$ then $f(x) \to \frac{18}{11} \approx 1.636$
Possible integer values of $f(x)$ are 3 and 2 because $3 < 3.6$ and $2 > \frac{18}{11}$
We study just the value of x when $f(x) = 3$ (beacuse $t \geq 3$) then $x = \frac{6}{5} \notin \mathbb{N}$
So no integer $k \geq 1$ gives $t = \frac{18k}{11k-6}$ integer and $t \geq 3$

For $r = 3$, $t = \frac{27k}{15k-9}$ any $k \geq 1$ integer does not give $t \geq 4$
Consider: $f(x) = \frac{27x}{15x-9} = \frac{9x}{5x-3}$, the derivative of f is: $f'(x) = -\frac{27}{(5x-3)^2} < 0$, so f is strictly decreasing for all $x \geq 1$
Then the maximum of f is atteint in the limit of left when $x = 1$ then $f(1) = \frac{9}{2} = 4.5$ and when $x \to +\infty$ then $f(x) \to \frac{9}{5} = 1.8$
Possible integer values of $f(x)$ are 4 and 3 because $4 < 4.5$ and $3 > 1.8$
We study just the value of x when $f(x) = 4$ (beacuse $t \geq 4$) then $x = \frac{12}{11} \notin \mathbb{N}$
So no integer $k \geq 1$ gives $t = \frac{27k}{15k-9}$ integer and $t \geq 4$

For $r = 4$, $t = \frac{36k}{19k-12}$ any $k \geq 1$ integer does not give $t \geq 5$

Consider: $f(x) = \frac{36x}{19x-12}$, the derivative of f is: $f'(x) = -\frac{432}{(19x-12)^2} < 0$, so f is strictly decreasing for all $x \geq 1$
Then the maximum of f is atteint in the limit of left when $x = 1$ then $f(1) = \frac{36}{7} \approx 5.142857$ and when $x \to +\infty$ then $f(x) \to \frac{36}{19} \approx 1.8947$
Possible integer values of $f(x)$ are 5 and 4 because $5 < 5.142857$ and $4 > 1.8947$
We study just the value of x when $f(x) = 5$ (beacuse $t \geq 5$) then $x = \frac{60}{59} \notin \mathbb{N}$
So no integer $k \geq 1$ gives $t = \frac{36k}{19k-12}$ integer and $t \geq 5$

So no couple (r, m) of positive integers give x as a positive integer.
So no couple (x,t) of positive integers exists such that $t^2(4x-3)^2 - 6tx$ to be a perfect square. $\square$

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4. Conclusions

We have prove that the polynomial $t^2(4x-n)^2 - 2ntx$ does not always admits a perfect square with $n \geq 2$ and x, t positive integers.

Reference

1. Philemon Urbain MBALLA, *Partial Resolution of the Erdős-Straus, Sierpiński, and Generalized Erdős-Straus Conjectures Using New Analytical Formulas*, article.