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The Generalized Characteristic Polynomial of the $K_{m,n}$-Complement of a Bipartite Graph

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Abstract: The generalized matrix of a graph G is defined as $M(G) = A(G) - tD(G)$ ($t \in \mathbb{R}$, $A(G)$ and $D(G)$ respectively denote the adjacency matrix and the degree matrix of G), and the generalized characteristic polynomial of G is merely the characteristic polynomial of $M(G)$. Let $K_{m,n}$ be the complete bipartite graph. Then the $K_{m,n}$-complement of a subgraph G in $K_{m,n}$ is defined as the graph obtained by removing all edges of an isomorphic copy of G from $K_{m,n}$. In this paper, by using a determinant expansion on the sum of two matrices (one of which is a diagonal matrix), a general method for computing the generalized characteristic polynomial of the $K_{m,n}$-complement of a bipartite subgraph G was provided. Furthermore, when G is a graph with rank no more than 4, the explicit formula for the generalized characteristic polynomial of the $K_{m,n}$-complements of G is given.

Keywords: bipartite graph; $K_{m,n}$-complement; generalized matrix; balanced bipartite subgraph

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1. Introduction

In the present paper, we only consider undirected, simple, and connected graphs unless otherwise stated. Let $G = (V(G), E(G))$ (or shortly (V, E)) be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix of G is defined as $A(G) = (a_{ij})$, where a_{ij} equals the number of edges connecting vertices v_i and v_j when $i \neq j$ and 0 when $i = j$. The rank of a graph G, denoted by $r(G)$, is defined to be the rank of its adjacency matrix $A(G)$. The degree matrix $D(G)$ is defined as the diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$, where $n = |V|$, and d_i equals the number of edges incident to vertex v_i. In the literature [1], Cvetković et al. introduced a bivariate polynomial, $\phi(G; \lambda, t) = \det(\lambda I - (A(G) - tD(G)))$ (abbreviated as $\phi(G)$). Wang et al. [2] referred to it as the generalized characteristic polynomial of G. It is natural to define $A(G) - tD(G)$ in the variable t as the generalized matrix of a graph G, denoted by $M(G) = (m_{ij}(G))_{|V|\times|V|}$. To be specific,it is easy to see

$$m_{ij}(G) = \begin{cases} -td_G(v_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the generalized characteristic polynomial of graph G is exactly the characteristic polynomial of the generalized matrix $M(G)$. That is, the polynomial $\phi(G) = \phi(G; \lambda, t) = \det(\lambda I - M(G)) = \det(\lambda I - (A(G) - tD(G)))$ is referred to as the generalized characteristic polynomial of G. Note that $A(G) - tD(G)$ with $t \in \mathbb{R}$ encodes several well-known graph matrices, such as the adjacency matrix, the Laplacian matrix, and the normalized Laplacian matrix. It is evident that the generalized characteristic polynomial of a graph generalizes several well-known polynomial invariants of graphs, for example:

- The characteristic polynomial of the adjacency matrix of a graph G is given by $\phi(G; \lambda, 0) = \det(\lambda I - A(G))$;
- The characteristic polynomial of the Laplacian matrix $D(G) - A(G)$ of G is $(-1)^{|V|}\phi(G; -\lambda, 1) = \det(-\lambda I - A(G) + D(G))$;
- The characteristic polynomial of the unsigned Laplacian matrix $D(G) + A(G)$ of graph G is $\phi(G; \lambda, -1) = \det(\lambda I - A(G) - D(G))$;

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- The characteristic polynomial of the normalized Laplacian matrix $I - D_G^{-\frac{1}{2}} A_G D_G^{-\frac{1}{2}}$ is $(-1)^{|V|}\phi(G;0,-\lambda+1) = \det(-A(G) - (\lambda-1)D(G))$.

Following [3], let $G = (V,E)$ be an undirected graph, if E is considered as a set of symmetric directed edges, meaning that if $e \in E$, then $\bar{e} \in E$, where $\bar{e}$ is the reverse edge of e, then G can also be viewed as a directed graph. For $e \in E$, let $h(e)$ denote the head of the directed edge e and $t(e)$ the tail of e. A closed walk in G is defined as a sequence of edges $C = (e_1,\ldots,e_k)$ such that $h(e_i) = t(e_{i+1})$ for $i \in \mathbb{Z}/k\mathbb{Z}$. Here $k = |C|$ is the length of C and $cbc(C) = \#\{i \in \{1,\ldots,k\} \mid e_{i+1} = e_i\}$ is called the cyclic bump count of C. The notation $[C]$ is referred to as the equivalence class of the closed walk C under edge permutation, meaning that $(e_1,\ldots,e_k) \sim (e_2,\ldots,e_k,e_1)$. If none of the representatives of $[C]$ can be expressed as C^k (for $k \geq 2$), then the cycle C is said to be irreducible. The set of all irreducible cycles is denoted by $\mathcal{C}$. The Bartholdi zeta function of a graph G is defined as (see [4] for details)

$$Z_G(\lambda,t) = \prod_{[C]\in\mathcal{C}} \frac{1}{1-\lambda^{cbc(C)}t^{|C|}}.$$

The function $Z_G(t) = Z_G(0,t)$ is referred to as the (Ihara–Selberg) zeta function [5], which was introduced by Ihara to study the zeta function of a regular graph and its reciprocal, and the reciprocal of the zeta function of a regular graph was generalized to the reciprocal of the Bartholdi zeta function for a general graph G as below:

$$Z_G(\lambda,t)^{-1} = \Big(1-(1-\lambda)^2t^2\Big)^{|E|-|V|} \det\Big(I - tA_G + (1-\lambda)(D_G-(1-\lambda)I)t^2\Big).$$

In particular, the reciprocal of the zeta function for a general graph G is given by:

$$Z_G(t)^{-1} = (1-t^2)^{|E|-|V|} \det\Big(I - tA_G + t^2(D_G - I)\Big).$$

The zeta function encodes significant structural information about the graph, such as the number of vertices, edges, and loops. Moreover, the number of spanning trees (the complexity of the graph) $\tau(G)$ satisfies the equation [6]:

$$\left.\frac{\partial f_G(t)}{\partial t}\right|_{t=1} = 2(|E|-|V|)\tau(G), \quad \text{where} \quad f_G(t) = \det\Big(I - tA_G + t^2(D_G-I)\Big).$$

For a comprehensive treatment of many aspects of Zeta function, refer to [7].

Zeta functions of certain classes of graphs have received considerable attention, such as the line graph of semi-regular bipartite graphs [8], the middle graph of semi-regular bipartite graphs [9], the cone graph of regular graphs [10], and various special join graphs of regular graphs [11,12]. It is not difficult to prove that $\phi(G;\lambda,t)$ determines the reciprocal of the zeta function, and vice versa [13]. Let $G = (V,E)$ be a subgraph of the complete bipartite graph $K_{m,n}$. The $K_{m,n}$-complement of G is defined as the graph obtained from $K_{m,n}$ by deleting all edges of G in $K_{m,n}$, i.e., $K_{m,n} - E(G)$. In this paper, we shall show a computational method for deriving the formula for the generalized characteristic polynomial of the $K_{m,n}$-complement of any bipartite graph G, and further give an explicit formula for the generalized characteristic polynomials of the $K_{m,n}$-complement of a bipartite graph with rank less than or equal to 4.

2. Notations and Terminology

Let $G = (V,E)$ be a graph with the vertex set $V = \{v_1,v_2,\ldots,v_n\}$ and the edge set $E = \{e_1,\ldots,e_m\}$. For two vertices $v_i,v_j \in V$, if v_i and v_j are adjacent, we denote this as $v_i \sim v_j$. The neighborhood of a vertex v_i in G is defined as $N_G(v_i) = \{v_j \in V \mid v_i \sim v_j\}$, and the degree of vertex v_i in G is denoted by $d_i = d_G(v_i) = |N_G(v_i)|$. The complement of the graph $G = (V,E)$ is denoted as

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$G^c = (V, E^c)$, where $E^c = \{v_iv_j \mid v_i, v_j \in V, v_iv_j \notin E\}$. If $G = (V, E)$ and $G' = (V', E')$ with $V' \subseteq V$ and $E' = \{(u, v) \mid u, v \in V', (u, v) \in E\}$, then G' is referred to as an induced subgraph of G.

A graph $G = (V, E)$ is a bipartite graph if and only if there exists a bipartition of V into (V_1, V_2) (namely $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$) such that no two vertices within V_1 or V_2 are adjacent. If the sizes of the bipartition sets are equal, i.e., $|V_1| = |V_2|$, then G is said to be a balanced bipartite. If $G = (V, E)$ is a bipartite graph with a bipartition (V_1, V_2), the bipartite complement of G, denoted as G^{bc}, has vertex set $V(G^{bc}) = V(G)$ and edge set $E(G^{bc}) = \{xy \mid x \in V_1, y \in V_2, xy \notin E(G)\}$. For a bipartite graph G, its adjacency matrix is given by

$$A(G) = \begin{bmatrix} 0 & B(G) \\ B^T(G) & 0 \end{bmatrix},$$

where $B(G) = (b_{ij})_{m\times n}$ is the bipartite adjacency matrix of G that defines the vertex adjacency relationship between the bipartite sets V_1 and V_2. Specifically,

$$b_{ij} = \begin{cases} 1 & v_i \sim v_j,\ v_i \in V_1,\ v_j \in V_2 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 ([14]). *Let G be a balanced bipartite graph. If G has a unique perfect matching, then the bipartite adjacency matrix $B(G)$ has determinant* 1 *or* -1.

Lemma 2 ([15]). *Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix, and let D be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, i.e., $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. Then*

$$\det(A + D) = \sum_{\theta\subseteq[n]} \det(A_\theta)\det(D_{\overline{\theta}}), \tag{1}$$

where θ is a subset of $[n] = \{1, 2, \ldots, n\}$ and $\overline{\theta}$ is the complement of θ in $[n]$, namely $\overline{\theta} = \{k \mid k \in [n], k \notin \theta\}$; A_θ is the submatrix formed by the rows and columns of A indexed by θ. By convention, $\det(A_\emptyset) = 1$.

The following lemma immediately follows from Lemma 2.

Lemma 3 ([15]). *If D is an invertible matrix, then the determinant of the matrix $A + D$ can be expressed as*

$$\det(A + D) = \det(D) \sum_{\theta\subseteq[n]} \frac{\det(A_\theta)}{\det(D_\theta)}. \tag{2}$$

Lemma 4 ([16]). *Let A be an $n \times n$ matrix. If there exists a $p \times q$ zero submatrix in A such that $p + q \geq n + 1$, then* $\det(A) = 0$.

3. The generalized characteristic polynomial

For the sake of simplicity, the complete graph, cycle, and path on n vertices are denoted by K_n, C_n, and P_n, respectively. Notationally, for $m, n \in \mathbb{Z}$, $[m] = \{1, 2, \ldots, m\}$ and $[m + 1, m + n] = \{m + 1, m + 2, \ldots, m + n\}$; I_n and $J_{m\times n}$ (or J_n) respectively denote the $n \times n$ identity matrix and the $m \times n$ (or $n \times n$) matrix of all ones. For the rest of this paper, we will use $K_{m,n}$ to symbolize the complete bipartite graph with bipartite partition (X, Y), where $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$, and $\mathscr{B}_{m,n}$ to symbolize the set of all bipartite graphs with bipartite partition (X, Y) such that $|X| = m$ and $|Y| = n$. Note that a graph $G \in \mathscr{B}_{m,n}$ if and only if its bipartite complement $G^{bc} \in \mathscr{B}_{m,n}$. Let G be a subgraph of the complete bipartite graph $K_{m,n}$. The $K_{m,n}$-complement of a subgraph G in $K_{m,n}$ is defined as the graph obtained from $K_{m,n}$ by deleting all edges of G in $K_{m,n}$, denoted by $K_{m,n} - G$.

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Theorem 1. *Let $G = (V, E)$ be a subgraph of $K_{m,n}$ with bipartite partition (X, Y) mentioned previously, and G has a bipartite partition (V_1, V_2) such that $V_1 \subseteq X$ ($|V_1| = s$), $V_2 \subseteq Y$ ($|V_2| = t$). Then, we have*

$$\begin{aligned}
&\phi(K_{m,n} - G) \\
&= (nt+\lambda)^{m-s}(mt+\lambda)^{n-t} \prod_{v\in V_1}((n-d_G(v))t+\lambda) \prod_{v\in V_2}((m-d_G(v))t+\lambda) \\
&\cdot\Big[1 + \sum_{Q^{bc}\in\mathcal{G}(H^{bc})} \frac{(-1)^{\frac{|V(Q^{bc})|}{2}}\big(\det(J_k - B(Q^{bc}))\big)^2}{\prod_{v\in V(Q^{bc})\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q^{bc})\cap Y}((m-d_G(v))t+\lambda)}\Big] \\
&= (nt+\lambda)^{m-s}(mt+\lambda)^{n-t} \prod_{v\in V_1}((n-d_G(v))t+\lambda) \prod_{v\in V_2}((m-d_G(v))t+\lambda) \\
&\cdot\Big[1 + \sum_{Q\in\mathcal{G}(K_{m,n}-G)} \frac{(-1)^{\frac{|V(Q)|}{2}}\big(\det(B(Q))\big)^2}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\Big],
\end{aligned} \tag{3}$$

where $\mathcal{G}(H^{bc})$ is the set of all induced balanced bipartite subgraphs Q^{bc} in H such that the bipartite complement Q of Q^{bc} has a nonsingular biadjacency matrix $B(Q)$; $\mathcal{G}(H)$ is the set of all nonempty induced balanced bipartite subgraphs Q in H (i.e., $K_{m,n} - G$) such that $B(Q)$ is nonsingular.

Proof. Let $H = K_{m,n} - G \in \mathcal{B}_{m,n}$. Note that H has the same bipartite partition as $K_{m,n}$, that is, $(X, Y) = \{v_1, v_2, \ldots, v_m\} \cup \{v_{m+1}, \ldots, v_{m+n}\}$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Obviously, $H^{bc} \cong G \cup (m + n - |V|)K_1 \in \mathscr{B}_{m,n}$. The generalized matrix $M(H) = A(H) - tD(H) = (m_{ij}(H))_{(m+n)\times(m+n)}$ is given by

$$m_{ij}(H) = \begin{cases} t(d_{H^{bc}}(v_i) - n) & \text{if } i = j \in [m], \\ t(d_{H^{bc}}(v_i) - m) & \text{if } i = j \in [m+1, m+n], \\ 1 & \text{if } i \neq j \text{ and } v_iv_j \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$M(H) = \begin{bmatrix} -tD_1(H) & J_{m\times n} - B(H^{bc}) \\ J_{n\times m} - B^T(H^{bc}) & -tD_2(H) \end{bmatrix}$$

where $D_1(H) = \mathrm{diag}(n - d_{H^{bc}}(v_1), n - d_{H^{bc}}(v_2), \ldots, n - d_{H^{bc}}(v_m))$ and $D_2(H) = \mathrm{diag}(m - d_{H^{bc}}(v_{m+1}), m - d_{H^{bc}}(v_{m+2}), \ldots, m - d_{H^{bc}}(v_{m+n}))$.

Let $\mathbf{1}_{V_1}$ be the $(m+n) \times 1$ column vector such that the i-th element is 1 for $1 \le i \le m$ and 0 for $m+1 \le i \le m+n$. Similarly, let $\mathbf{1}_{V_2}$ be the $(m+n) \times 1$ vector where the i-th element is 0 for $1 \le i \le m$ and 1 for $m+1 \le i \le m+n$.

$$\begin{aligned}
\lambda I_{m+n} - M(H) &= \begin{bmatrix} tD_1(H) + \lambda I_m & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & tD_2(H) + \lambda I_n \end{bmatrix} \\
&= tD(H) + \lambda I_{m+n} + A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T \\
&= D' + A'
\end{aligned}$$

Let $D' = tD(H) + \lambda I_{m+n}$ and $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$. Then, we have

$$\det(D') = \prod_{v\in X}((n - d_{H^{bc}}(v))t + \lambda) \prod_{v\in Y}((m - d_{H^{bc}}(v))t + \lambda) \neq 0.$$

By Lemma 2, the following equality holds:

$$\det(\lambda I_{m+n} - M(H)) = \det(D' + A') = \det(D') \sum_{\theta\subseteq[m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)}. \tag{4}$$

Now, we claim two facts:

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Fact 1: The first-order principal submatrix of A' is zero. The third-order principal submatrices of A' are in the form:

$$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix},$$

and both of their determinants are zero. The fifth-order principal submatrices of A' are in one of the following forms:

$$\begin{bmatrix} 0 & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & * & * & * & 0 \end{bmatrix},$$

and each of their determinants equals zero. Analogously, the odd-order principal submatrices of A' are in the form:

$$A'_\theta = \begin{bmatrix} 0_{p\times p} & A_1 \\ A_1^T & 0_{q\times q} \end{bmatrix} \quad (p+q = |\theta|).$$

Note that $2p \geq n+1$ or $2q \geq n+1$. According to Lemma 4, we conclude that $\det(A'_\theta) = 0$, indicating that all odd-order principal submatrices of A' are singular.

Fact 2: By definition, $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$ or equivalently

$$A' = \begin{bmatrix} 0 & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & 0 \end{bmatrix}.$$

Suppose θ is a subset of $[m+n]$ such that $\frac{|\theta|}{2} = |\theta \cap [m]| = |\theta \cap [m+1, m+n]|$. Let A'_θ be an even-order principal submatrix of A'. We denote by $A'_{\theta\cap[m],\theta\cap[m+1,m+n]}$ the submatrix of A' formed by the rows indexed by $\theta \cap [m]$ and the columns indexed by $\theta \cap [m+1, m+n]$. Furthermore,

$$\det(A'_\theta) = (-1)^{\frac{|\theta|}{2}\times\frac{|\theta|}{2}} \left(\det(A'_{\theta\cap[m],\theta\cap[m+1,m+n]})\right)^2,$$

This reduces to

$$\begin{aligned}\det(A'_\theta) &= (-1)^{\frac{|\theta|}{2}\times\frac{|\theta|}{2}} \left(\det(B_\theta - J_{\frac{|\theta|}{2}})\right)^2 \\ &= (-1)^{\left(\frac{|\theta|}{2}\right)^2} \left(\det(J_{\frac{|\theta|}{2}} - B_\theta)\right)^2 \\ &= (-1)^{\left(\frac{|\theta|}{2}\right)^2} (\det(\overline{B}_\theta))^2 \\ &= (-1)^{|\theta|} (\det(\overline{B}_\theta))^2,\end{aligned}$$

where B_θ is the $\frac{|\theta|}{2} \times \frac{|\theta|}{2}$ submatrix obtained from $B(H^{bc})$ by deleting the rows that are not in $\theta \cap [m]$ and the columns that are not in $\theta \cap [m+1, m+n]$. Similarly, $\overline{B}_\theta$ is the matrix resulting from $B(H)$ by the same deletion of rows and columns, and it is easy to see $B_\theta + \overline{B}_\theta = J_{\frac{|\theta|}{2}}$.

Set $k = \frac{|\theta|}{2}$, and let Q be a balanced bipartite induced subgraph of H with $2k$ vertices, and the bipartite complement Q^{bc} corresponding to Q is also a balanced bipartite induced subgraph of H^{bc}. Obviously, $B(Q) + B(Q^{bc}) = J_k$. Let $\mathcal{G}_k(H^{bc})$ denote the set of induced balanced bipartite subgraphs Q^{bc} in H with $2k$ vertices such that the bipartite complement Q of Q^{bc} has a nonsingular biadjacency

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matrix $B(Q)$, that is, the rank of $B(Q)$ is k ($k \le r(A')/2$). Moreover, we denote by $\mathcal{G}(H^{bc})$ the set of all induced balanced bipartite subgraphs Q^{bc} in H such that $B(Q)$ is nonsigular. We denote by $\mathcal{G}(H)$ the set of all nonempty induced balanced bipartite subgraphs Q in H (i.e., $K_{m,n}-G$) such that $B(Q)$ is nonsingular; that is to say, if Q is such a nonempty induced balanced bipartite subgraph in H on $2k$ vertices, then the $k\times k$ matrix $B(Q)$ satisfies the condition $r(J_k - B(Q^{bc})) = k$, and it's worth noting that the induced subgraph Q may not necessarily be nonsingular.

Observe that (i) $r(A') = 2r(J_{m\times n} - B(H^{bc})) = 2r(B(H^{bc}) - J_{m\times n}) \le 2r(B(H^{bc})) + 2r(J_{m\times n}) = 2r(B(G)) + 2 = r(A(G)) + 2$, where $r(M)$ symbolizes the rank of the matrix M; (ii) $B(Q) = J_k - B(Q^{bc})$. Then, by Lemma 2.3, we conclude that

$$\begin{aligned}
&\phi(K_{m,n}-G) = \det(D'+A')\\
=&\prod_{v\in X}((n-d_{H^{bc}}(v))t+\lambda)\prod_{v\in Y}((m-d_{H^{bc}}(v))t+\lambda)\\
&\cdot\left[1+\sum_{k=1}^{r(A(G))/2+1}\sum_{Q^{bc}\in\mathcal{G}_k(H^{bc})}\frac{(-1)^{k^2}\left(\det(J_k-B(Q^{bc}))\right)^2}{\prod\limits_{v\in V(Q^{bc})\cap X}((n-d_{H^{bc}}(v))t+\lambda)\prod\limits_{v\in V(Q^{bc})\cap Y}((m-d_{H^{bc}}(v))t+\lambda)}\right]\\
=&(nt+\lambda)^{m-s}(mt+\lambda)^{n-t}\prod_{v\in V_1}((n-d_G(v))t+\lambda)\prod_{v\in V_2}((m-d_G(v))t+\lambda)\\
&\times\left[1+\sum_{Q^{bc}\in\mathcal{G}(H^{bc})}\frac{(-1)^{\frac{|V(Q^{bc})|}{2}}\left(\det(J_k-B(Q^{bc}))\right)^2}{\prod\limits_{v\in V(Q^{bc})\cap X}((n-d_G(v))t+\lambda)\prod\limits_{v\in V(Q^{bc})\cap Y}((m-d_G(v))t+\lambda)}\right]\\
=&(nt+\lambda)^{m-s}(mt+\lambda)^{n-t}\prod_{v\in V_1}((n-d_G(v))t+\lambda)\prod_{v\in V_2}((m-d_G(v))t+\lambda)\\
&\cdot\left[1+\sum_{Q\in\mathcal{G}(K_{m,n}-G)}\frac{(-1)^{\frac{|V(Q)|}{2}}\left(\det(B(Q))\right)^2}{\prod\limits_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod\limits_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\right].
\end{aligned}$$

This completes the proof. □

4. An application

Our main result in this section gives an application of Theorem 1. A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows. Two n by n matrices A and B are said to be permutationally equivalent if there exist n by n permutation matrices P, Q such that $PAQ = B$.

Theorem 2. *With notations mentioned in Theorem 1, if G has its rank $r(A(G)) \le 4$, then*

$$\begin{aligned}
\phi(K_{m,n}-G) = \ &(nt+\lambda)^{m-s}(mt+\lambda)^{n-t}\prod_{v\in V_1}((n-d_G(v))t+\lambda)\prod_{v\in V_2}((m-d_G(v))t+\lambda)\\
&\times\Big[1+\sum_{Q\in\Omega_{\mathcal{F}_1}(K_{m,n}-G)}\frac{1}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\\
&+\sum_{Q\in\Omega_{\mathcal{F}_2}(K_{m,n}-G)}\frac{-1}{\prod_{v\in V(Q)\cap X}((n-d_G(v_i))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v_i))t+\lambda)}\\
&+\sum_{Q\in\Omega_{\{C_6\}}(K_{m,n}-G)}\frac{-2}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\Big],
\end{aligned} \tag{5}$$

where $\Omega_Q(G)$ is the set of all induced bipartite subgraphs of G that are isomorphic to the graph Q, and $\Omega_{\mathcal{F}}(G)$ is the set of all induced bipartite subgraphs of G that are isomorphic to certain graph in the family $\mathcal{F}$. Here, $\mathcal{F}_1 = \{2K_2, Q_7\}$, $\mathcal{F}_2 = \{K_2, 3K_2, K_2\cup P_4, P_6, Q_3, Q_6\}$, where Q_3, Q_6 and Q_7 are illustrated in Figure 1, Figure 2 and Figure 3.

Proof. From the proof of Theorem 1, we know that

$$\det(\lambda I_{m+n} - M(H)) = \det(D' + A') = \det(D') \sum_{\theta \subseteq [m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)} \tag{6}$$

where $H = K_{m,n} - G$, $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$, or

$$A' = \begin{bmatrix} 0 & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & 0 \end{bmatrix}.$$

Hence we only need to study the summation

$$\sum_{\theta \subseteq [m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)} \tag{7}$$

or equivalently

$$1 + \sum_{Q \in \mathcal{G}(K_{m,n} - G)} \frac{(-1)^{\frac{|V(Q)|}{2}} \left(\det(B(Q))\right)^2}{\prod_{v \in V(Q)\cap X} ((n - d_G(v))t + \lambda) \prod_{v \in V(Q)\cap Y} ((m - d_G(v))t + \lambda)} \Bigg]. \tag{8}$$

Observe that $r(A') = 2r(B(H^{bc}) - J_{m\times n}) \leq 2r(B(H^{bc})) + 2r(J_{m\times n}) = 2r(B(G)) + 2 = r(A(G)) + 2 \leq 6$. The following cases need to be discussed:

Case 1: If $\theta = \emptyset$, then

$$\frac{\det(A'_\theta)}{\det(D'_\theta)} = 1.$$

Case 2: If $|\theta| = 2$, then

$$A'_\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consequently, $Q \cong K_2$ and $Q^{bc} \cong K_2^{bc}$ (the trivial graph on two vertices). The contribution of this case to the summation part of Eq. (7) is given by

$$\sum_{Q \in \Omega_{K_2}(K_{m,n} - G)} \frac{-1}{\prod_{v \in V(Q)\cap X} ((n - d_Q(v))t + \lambda) \prod_{v \in V(Q)\cap Y} ((m - d_Q(v))t + \lambda)}.$$

Case 3 If $|\theta| = 4$, then $|\theta \cap [m]| = |\theta \cap [m+1, m+n]| = 2$. Let $\theta = \{i, j, k, l\}$ with $i < j < k < l$, where $i, j \in [m]$ and $k, l \in [m+1, m+n]$. In this case, the matrix A' is in the form

$$A'_\theta = \begin{bmatrix} 0 & 0 & a_{ik} & a_{il} \\ 0 & 0 & a_{jk} & a_{jl} \\ a_{ik} & a_{jk} & 0 & 0 \\ a_{il} & a_{jl} & 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$ is permutationally equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ since $\det(A'_\theta) \neq 0$. This indicates that the induced subgraph Q in $K_{m,n} - G$ satisfies that $Q \cong 2K_2$ or P_4. The contribution of this case to the summation part of Eq. (7) is given by

$$\sum_{Q \in \Omega_{\{2K_2, P_4\}}(K_{m,n} - G)} \frac{1}{\prod_{v \in V(Q)\cap V_1} ((n - d_Q(v))t + \lambda) \prod_{v \in V(Q)\cap V_2} ((m - d_Q(v))t + \lambda)}.$$

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Case 4 If $|\theta| = 6$, then $|\theta \cap [m]| = |\theta \cap [m+1, m+n]| = 3$. Suppose $B_3(Q^{bc})$ is the 3×3 principal submatrix of $B(Q^{bc})$ in A' such that

$$A'_\theta = \begin{bmatrix} 0 & B_3(Q^{bc}) - J_3 \\ (B_3(Q^{bc}) - J_3)^T & 0 \end{bmatrix}.$$

Moreover,

$$\det(A'_\theta) = (-1)^{3\times 3} \det\left(B_3(Q^{bc}) - J_3\right) \det\left((B_3(Q^{bc}) - J_3)^T\right) = -(\det(J_3 - B_3(Q^{bc})))^2.$$

By exhaustive search, there exist 174 nonsingular 0–1 matrices of order 3, each of which is permutationally equivalent to one of the following seven matrices or their transposes:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$B_5 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In [17], it is proved by the authors that B_k corresponds to the induced subgraph Q_k in $K_{m,n} - G$, where $Q_1 \cong 3K_2$, $Q_2 \cong K_2 \cup P_4$, Q_3, $Q_4 \cong P_6$, $Q_5 \cong C_6$, Q_6, or Q_7 (see Figure 1, Figure 2 and Figure 3). By Lemma 1 or simple calculations, we know that $(\det(J_3 - B(Q_k^{bc}))^2 = 1$ for $k = 1,2,3,4,6,7$ and $(\det(J_3 - B(Q_5^{bc})))^2 = 4$. Hence, the contribution of this case to Eq. (7) is given by

$$\sum_{Q\in\Omega_{\{Q_1,Q_2,Q_3,Q_4,Q_6,Q_7\}}(K_{m,n}-G)} \frac{-1}{\prod_{v\in V(Q)\cap X}((n-d_Q(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_Q(v))t+\lambda)}$$
$$+ \sum_{Q\in\Omega_{C_6}(K_{m,n}-G)} \frac{-4}{\prod_{v\in V(Q)\cap X}((n-d_Q(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_Q(v))t+\lambda)}.$$

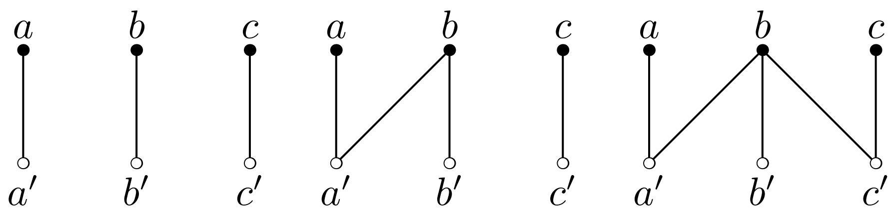

Figure 1. $Q_1 \cong 3K_2$, $Q_2 \cong K_2 \cup P_4$, Q_3.

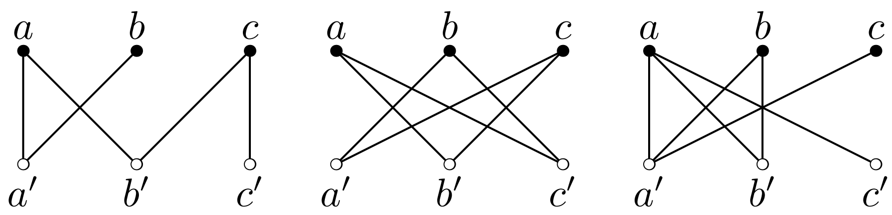

Figure 2. $Q_4 \cong P_6$, $Q_5 \cong C_6$ and Q_6.

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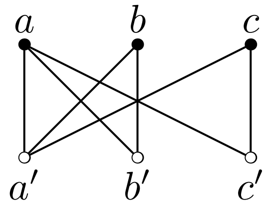

Figure 3. Q_7.

The proof is completed. □

5. Conclusions

In this paper we studied the computation of the generalized characteristic polynomial or equivalently the zeta function of graphs, and derived a general formula for the generalized characteristic polynomial of the $K_{m,n}$-complement of a bipartite graph. As a by-product, we obtained an explicit formula for the generalized characteristic polynomial of the $K_{m,n}$- tite complement of a bipartite graph with rank no more than 4. In a sense, the formulas obtained in this paper are straightforward and only rely on the use of fundamental linear algebra about the biadjacency matrix of the bipartite graph.

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