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Noncommutative Equiangular Lines: Van Lint-Seidel Relative and Gerzon Universal Bounds

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Abstract: We introduce the notion of noncommutative equiangular lines and derive noncommutative versions of fundamental van Lint-Seidel relative and Gerzon universal bounds.

Keywords: equiangular lines; C*-algebra; Hilbert C*-module

MSC: 46L05; 46L08

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1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0,1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$ is said to be γ**-equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j,k \leq n, j \neq k.$$

Two fundamental problems associated with equiangular lines are the following.

Problem 1. Given $d \in \mathbb{N}$ and $\gamma \in [0,1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ-equiangular lines in $\mathbb{R}^d$?

Problem 2. For which parameters $d, n \in \mathbb{N}$ and $\gamma \in [0,1]$, there exists a collection $\{\tau_j\}_{j=1}^n$ of γ-equiangular lines in $\mathbb{R}^d$? If exists, how to construct them?

Two answers to Problem (1) which are fundamental driving force in the study of equiangular lines are the following results of van Lint and Seidel [1,3] and Gerzon [4].

Theorem 1. *[1,3,5,6] (**van Lint-Seidel Relative Bound**) Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then*

$$n(1-d\gamma^2) \leq d(1-\gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}. \tag{1}$$

Theorem 2. *[4] (**Gerzon Universal Bound**)*

Theorem 3 ((i)).

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Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then

$$n \leq \frac{d(d+1)}{2}. \tag{2}$$

Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{C}^d$. Then

$$n \leq d^2. \tag{3}$$

Bounds in Inequalities (1), (2) and (3) are improved for various special values of γ, n and d (including asymptotic). Regarding Problem (2), various special cases are solved but full generality remains open. We refer [7–30] for a look on these achievements in the real case and [31–37] in the complex case. In the real case, it is known that equality is not always attained for every d in Inequality (2). Celebrated Zauner conjecture says that there exists a d^2 $\frac{1}{\sqrt{d+1}}$-equiangular lines in $\mathbb{C}^d$ for every d satisfying equality in Inequality (3) [38–44].

We wish to record that there is a vector space version of equiangular lines [45,46] and there is also a notion of equiangular subspaces of Euclidean spaces [47,48]. Recently, p-adic version of equiangular lines have been introduced [49].

As noncommutative geometry plays major role in modern Mathematics, our fundamental objective is to initiate a study of noncommutative equiangular lines. As most important results in the theory of equiangular lines are van Lint-Seidel relative and Gerzon universal bounds, we derive noncommutative versions of them.

2. Noncommutative Equiangular Lines

We want the notion of Hilbert C*-modules. They are first introduced by Kaplansky [50] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [51] and Rieffel [52]. A friendly introduction is given in the book [53]. We want only standard Hilbert C*-module. Let $\mathcal{A}$ be a unital C*-algebra with identity 1. For $d \in \mathbb{N}$, let $\mathcal{A}^d$ be the standard left Hilbert C*-module over $\mathcal{A}$ with inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^{d} a_j b_j^*, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathcal{A}^d.$$

Hence norm on $\mathcal{A}^d$ is

$$\|(a_j)_{j=1}^d\|_2 := \left\| \sum_{j=1}^{d} a_j a_j^* \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

We introduce two notions of equiangular lines for modules.

Definition 1. *Let $a \in \mathcal{A}$ be a positive element. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular a-equiangular lines** if following conditions hold.*

Theorem 4 ((i)).

$\langle \tau_j, \tau_j \rangle = 1, \forall 1 \leq j \leq n.$

$\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = a, \forall 1 \leq j, k \leq n, j \neq k.$

Definition 2. *Let $\gamma \in [0,1]$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular γ-norm-equiangular lines** if following conditions hold.*

Theorem 5 ((i)).

$\langle \tau_j, \tau_j \rangle = 1, \forall 1 \leq j \leq n.$

$\|\langle \tau_j, \tau_k \rangle\| = \gamma, \forall 1 \leq j, k \leq n, j \neq k.$

It is clear that if $\{\tau_j\}_{j=1}^n$ is modular a-equiangular lines, then $\{\tau_j\}_{j=1}^n$ is modular $\sqrt{\|a\|}$-norm-equiangular lines. We now derive two modular versions of Theorem 1 for modules over commutative C*-algebras.

Theorem 6. *(**Modular van Lint-Seidel Relative Bound**) Let $\mathcal{A}$ be a commutative unital C*-algebra. Let $\{\tau_j\}_{j=1}^n$ be modular a-equiangular lines in $\mathcal{A}^d$. Then*

$$n(1 - da) \leq d(1 - a).$$

In particular, if $1 - da$ is invertible, then

$$n \leq \frac{d(1-a)}{1-da} \leq d\left\|\frac{1-a}{1-da}\right\|.$$

Proof. Let $\tau_j := (a_1^{(j)}, a_2^{(j)}, \ldots, a_d^{(j)})$ for all $1 \leq j \leq n$. Define

$$B := \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = n + (n^2 - n)a.$$

Then using Cauchy-Schwarz inequality in Hilbert C*-modules,

$$\begin{aligned}
B &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{r=1}^{d} a_r^{(j)}(a_r^{(k)})^*\right)\left(\sum_{s=1}^{d} a_s^{(k)}(a_s^{(j)})^*\right)\\
&= \sum_{r,s=1}^{d}\left(\sum_{j=1}^{n} a_r^{(j)}(a_s^{(j)})^*\right)\left(\sum_{k=1}^{n} a_r^{(k)}(a_s^{(k)})^*\right)^*\\
&= \sum_{r=1}^{d}\left(\sum_{j=1}^{n} a_r^{(j)}(a_r^{(j)})^*\right)\left(\sum_{k=1}^{n} a_r^{(k)}(a_r^{(k)})^*\right)^* + \sum_{r,s=1, r\neq s}^{d}\left(\sum_{j=1}^{n} a_r^{(j)}(a_s^{(j)})^*\right)\left(\sum_{k=1}^{n} a_r^{(k)}(a_s^{(k)})^*\right)^*\\
&\geq \sum_{r=1}^{d}\left(\sum_{j=1}^{n} a_r^{(j)}(a_r^{(j)})^*\right)\left(\sum_{k=1}^{n} a_r^{(k)}(a_r^{(k)})^*\right)^*\\
&\geq \frac{1}{d}\left(\sum_{r=1}^{d}\left(\sum_{j=1}^{n} a_r^{(j)}(a_r^{(j)})^*\right)\cdot 1\right)\left(\sum_{s=1}^{d} 1\cdot\left(\sum_{k=1}^{n} a_s^{(k)}(a_s^{(k)})^*\right)^*\right)\\
&= \frac{1}{d}\left(\sum_{j=1}^{n}\sum_{r=1}^{d} a_r^{(j)}(a_r^{(j)})^*\right)\left(\sum_{k=1}^{n}\sum_{s=1}^{d} a_s^{(k)}(a_s^{(k)})^*\right)\\
&= \frac{1}{d}\left(\sum_{j=1}^{n} 1\right)\left(\sum_{k=1}^{n} 1\right) = \frac{n^2}{d}.
\end{aligned}$$

Hence

$$(n-1)a + 1 \geq \frac{n}{d} \implies d(n-1)a + d \geq n \implies d(1-a) \geq n(1-da).$$

□

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Theorem 7. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Let $\{\tau_j\}_{j=1}^n$ be modular γ-norm-equiangular lines in $\mathcal{A}^d$. Then*

$$n(1-d\gamma^2) \leq d(1-\gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}.$$

Proof. By using the proof of Theorem 6,

$$\begin{aligned}\frac{n^2}{d} &\leq \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle = \sum_{j,k=1, j\neq k}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle + \sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle^2 \\ &= \sum_{j,k=1, j\neq k}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle + n \leq \sum_{j,k=1, j\neq k}^{n}\|\langle \tau_j, \tau_k\rangle\|^2 + n = (n^2-n)\gamma^2 + n.\end{aligned}$$

□

Proof of previous theorems used commutativity of C*-algebra. We are unable to derive noncommutative versions of them. However, we introduce a subclass of norm-equiangular lines for which we derive modular relative bound.

Definition 3. *Let $\gamma \in [0,1]$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be* ***special modular γ-norm-equiangular lines*** *if following conditions hold.*

Theorem 8 ((i))**.**
$\langle \tau_j, \tau_j\rangle = 1, \forall 1 \leq j \leq n.$
$\|\langle \tau_j, \tau_k\rangle\| = \gamma, \forall 1 \leq j,k \leq n, j \neq k.$

$$n^2 \leq d\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle.$$

Theorem 9. *(**Noncommutative van Lint-Seidel Relative Bound**) Let $\mathcal{A}$ be a unital C*-algebra. Let $\{\tau_j\}_{j=1}^n$ be special modular γ-norm-equiangular lines in $\mathcal{A}^d$. Then*

$$n(1-d\gamma^2) \leq d(1-\gamma^2).$$

In particular, if $\gamma < \sqrt{d^{-1}}$, then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}.$$

Proof.

$$n^2 \leq d\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle \leq (n^2-n)\gamma^2 + n.$$

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□

Next we derive modular Gerzon bound for modules over C*-algebras with invariant basis number (IBN) property. These C*-algebras are studied in [54].

Theorem 10. *(**Modular Gerzon Universal Bound**) Let $\mathcal{A}$ be a unital C*-algebra with IBN. Let $\{\tau_j\}_{j=1}^n$ be modular a-equiangular lines in $\mathcal{A}^d$. If $1-a$ is invertible, then*

$$n \leq d^2. \tag{4}$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes \tau_j : \mathcal{A}^d \ni x \mapsto (\tau_j \otimes \tau_j)x := \langle x, \tau_j \rangle \tau_j \in \mathcal{A}^d.$$

We show that the collection $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent over $\mathcal{A}$. Let $c_1, \dots, c_n \in \mathcal{A}$ be such that

$$\sum_{j=1}^n c_j(\tau_j \otimes \tau_j) = 0. \tag{5}$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$0 = \sum_{j=1}^n c_j(\tau_j \otimes \tau_j)\tau_k = \sum_{j=1}^n c_j \langle \tau_k, \tau_j \rangle \tau_j.$$

Taking inner product with τ_k gives

$$\begin{aligned} 0 = \sum_{j=1}^n c_j \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle &= \sum_{j=1, j\neq k}^n c_j \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle + c_k \langle \tau_k, \tau_k \rangle^2 \\ &= \sum_{j=1, j\neq k}^n c_j a + c_k = \sum_{j=1}^n c_j a - c_k a + c_k = \sum_{j=1}^n c_j a + c_k(1-a). \end{aligned}$$

Now define

$$c := \frac{-1}{1-a} \sum_{j=1}^n c_j a.$$

Then $c_k = c$ for all $1 \leq k \leq n$. We wish to show that $c = 0$. Now Equation (5) gives

$$0 = \sum_{j=1}^n c(\tau_j \otimes \tau_j).$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$0 = \sum_{j=1}^n c(\tau_j \otimes \tau_j)\tau_k = \sum_{j=1}^n c \langle \tau_k, \tau_j \rangle \tau_j.$$

Taking inner product with τ_k gives

$$\begin{aligned} 0 = \sum_{j=1}^n c \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle &= \sum_{j=1, j\neq k}^n c \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle + c \langle \tau_k, \tau_k \rangle^2 \\ &= \sum_{j=1, j\neq k}^n ca + c = (n-1)ca + c = c((n-1)a + 1). \end{aligned}$$

Since $a \geq 0$, $((n-1)a+1)$ is invertible, hence $c = 0$. Note that the matrix of $\tau_j \otimes \tau_j$ is $\tau_j\tau_j^*$ (viewing τ_j as column vector). The rank of d by d matrices over $\mathcal{A}$ is d^2. Since $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent and $\mathcal{A}$ has IBN property, $n \leq d^2$. $\square$

We can slightly generalize Definition 1. In the vector space case, such a generalization was done by Greaves, Iverson, Jasper and Mixon [46].

Definition 4. *Let $a, b \in \mathcal{A}$ be positive elements. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular** (a,b)-**equiangular lines** if following conditions hold.*

Theorem 11 ((i)).
$\langle \tau_j, \tau_j \rangle = b, \forall 1 \leq j \leq n.$
$\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = a, \forall 1 \leq j,k \leq n, j \neq k.$

Note that b is necessarily positive but need not be invertible. Thus we may not able to reduce Definition 4 to Definition 1. By modifying earlier proofs, we easily get following theorems.

Theorem 12. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Let $\{\tau_j\}_{j=1}^n$ be modular (a,b)-equiangular lines in $\mathcal{A}^d$. Then*

$$n(b^2 - da) \leq d(b-a).$$

In particular, if $b^2 - da$ is invertible, then

$$n \leq \frac{d(b-a)}{b^2 - da} \leq d\left\|\frac{b-a}{b^2-da}\right\|.$$

Theorem 13. *Let $\mathcal{A}$ be a unital C*-algebra with IBN. Let $\{\tau_j\}_{j=1}^n$ be modular (a,b)-equiangular lines in $\mathcal{A}^d$. If $b^2 - a$ and $(n-1)a + b^2$ are invertible, then*

$$n \leq d^2.$$

Similar to scalar equiangular lines and Zauner conjecture, we formulate following problem.

Problem 3. Let $\mathcal{A}$ be a unital C*-algebra. For which parameters $d, n \in \mathbb{N}$ and $a \in \mathcal{A}$, there exists a collection $\{\tau_j\}_{j=1}^n$ of modular a-equiangular lines in $\mathcal{A}^d$? In particular, for which d/for all d, there is d^2 modular $\frac{1}{d+1}$-equiangular lines in $\mathcal{A}^d$ (making equality in Inequality (4))?

Note that Problem 3 contains Zauner conjecture (whenever $\mathcal{A} = \mathbb{C}$).

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