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Is Time a Cyclic Dimension? Canonical Quantization Implicit in Classical Cyclic Dynamics

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Abstract: If "quantization is an art" then it can be greatly refined by adopting cyclic time formalism. In past papers we have proven the effectiveness of a formulation of physics based on cyclic relativistic time. Now we can demonstrate in a general way, by using theorems of Geometric Quantization, that the Poisson brackets of intrinsically *cyclic* time dynamics directly imply the ordinary canonical commutation relations and the other Dirac's rules of canonical Quantum Mechanics. In other words, according to our result, canonical quantization is an implicit way of imposing intrinsically cyclic time dynamics without explicitly saying that time is a cyclic dimension.

Keywords: foundations of physics; foundations of quantum mechanics; geometric quantization; canonical quantization; relativistic time; time crystal

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1. Introduction

"Quantization is an art"[1] whose canon has been defined by Dirac about a century ago through mathematical axioms in the attempt to map classical dynamics into the corresponding quantum dynamics [2]. Canonical quantization has been confirmed with surprising precision by countless experiments during the years. This article doesn't intend to question its veracity — including Bell's inequalities. However its physical origin remains a mystery. Here we find a one-to-one correspondence between a particular class of classical dynamics and canonical quantum dynamics. The classical dynamics which will exhibit such equivalence are intrinsically *cyclic* dynamics.

The reader will wonder how it is possible to obtain a consistent description of physics in terms of elementary cycles. The answer is very simple, even though it requires special care to be implemented. The recurrences that we will impose as constraints to generic classical systems are none other than those naturally defined by the dual wave description of classical dynamics, as known from Hamilton's optico-mechanical analogy (Hamilton-Jacobian analytical mechanics) [3–5], and which determine the energies and the momenta in Quantum Mechanics (QM) through the Planck constant. Of course these recurrences have a local character. They vary during the system evolution depending on the interactions, in analogy with eikonal waves or modulated signals. Essentially, these are the ordinary recurrences used every day, implicitly in classical mechanics and explicitly in QM, in terms of undulatory mechanics. This is enough to guarantee causality and locality in the resulting description.

Our *ansatz* has deep physical motivations widely exposed in previous articles where its overall consistency is demonstrated with rigor and detail for the various branches of physics [6–14]. Among the various physical motivations we mention the deep affinity with *time crystals* [15–17], which evolve with intrinsic time periodicity even when they are in their ground states. Such phenomenology seems to be the manifestation of a fundamental principle at the base of QM according to our analysis.

We have known for almost two centuries, that is, from the Hamilton-Jacobi analysis, that these local recurrences can be implicitly associated to classical systems, and we have known, explicitly and also experimentally for about 90 years, that is, starting from de Broglie's PhD thesis, that these local recurrences exist and play a fundamental role in QM. The substantial novelty with respect to undulatory mechanics is that here these natural recurrences associated to classical and quantum

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[1] Expression used by W. L. Faddeev in 2009 after E. Witten's talk, as reported in [1].

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dynamics are consistently imposed as constraints directly into the topology of the symplectic manifold describing the physical system in question. As for a vibrating string, when periodicity is imposed as constraint we get an infinite set of degenerate solutions (the harmonics) rather then a single periodic solution. This is essentially the mechanism that will give rise to quantization. In the past papers we have proven its equivalence to QM in the Lagrangian formalism (Feynman path integral) for the specific case of relativistic elementary scalar particles [6,7,11–14]. Here we are able to prove the equivalence in the Hamiltonian formalism (canonical quantization) for classical-relativistic systems in general.

This article has a mathematical character but we will use a formalism familiar to physicists. Our strategy will be the following: *i*) we introduce a Dirac constraint of intrinsic periodicity, first in Lagrangian form and then in Hamiltonian form [18,19], which projects ordinary Hamiltonian dynamics (non compact manifold) into related intrinsically *cyclic* dynamics (compact manifold); *ii*) we prove, by using theorems of Geometric Quantization (GQ) [1,20–24], that the resulting intrinsically *cyclic* dynamics naturally satisfies Dirac's rules of canonical quantization — without postulating them. In short, the canonical quantization is equivalent to a local transformation from ordinary non-compact manifolds into corresponding intrinsically compact manifolds, see also [25,26].

2. Boundary Conditions

We start our investigation about the physical origin of canonical quantization by considering a generic system described by the action

$$S = \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t)) \,. \tag{1}$$

We assume $U(1)$ symmetry. The variational principle yields the Euler-Lagrange equations/Equations of Motion (EoMs) whereas the variation of the boundary term

$$\frac{\partial L}{\partial \dot{q}} \delta q \bigg|_{t_i}^{t_f} = 0 \,, \tag{2}$$

is typically vanished by requiring zero synchronous (*synch*) variations at the boundary [3],

$$\delta q(t_i) = \delta q(t_f) = 0 \,. \tag{3}$$

This leads to the ordinary classical integrals of the system, denoted by $q(t)$ and named here *synch* solutions.

It is known for instance from string theory, however, that other types of boundary conditions (BCs) are equally allowed by the variational principle. The essential requirement is that the BCs must vanish the boundary term of the action variation, eq.(2). With this requirement the BCs preserve all the symmetries of the action (including, for instance, causality as we will see) while the solutions satisfy the same Euler-Lagrange equations of above. So we will be in the framework of classical mechanics.

Let us introduce a related local action

$$\tilde{S} = \int_{t_x}^{t_x + T(x)} dt L(\tilde{q}(t), \dot{\tilde{q}}(t)) \,, \tag{4}$$

where the meaning of the dependence on x of the integration interval $T(x)$ will be clarified below. We choose to vanish its boundary variation by imposing periodic BCs (PBCs), as allowed by the variational principle,

$$\tilde{q}(t_x) = \tilde{q}(t_x + T(x)) \,. \tag{5}$$

The PBCs select a particular class of solutions $\tilde{q}(t)$ named here *cyclic* solutions — this doesn't mean that the domain of the solutions $\tilde{q}(t)$ is limited in $t \in (t_x, t_x + T(x)]$, it can in principle be extended to

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the same temporal domain of the *synch* solutions. Having the same Lagrangian they satisfy the same classical EoMs as $q(t)$.

For the sake of simplicity here we limit our study to PBCs but, besides all the possible combinations of Dirichlet $q|_\Sigma = 0$ and Neumann BCs $\frac{\partial L}{\partial \dot{q}}|_\Sigma = 0$, other possibilities are anti-PBCs $\tilde{q}(t_x) = -\tilde{q}(t_x + T(x))$ or, more in general twisted BCs $\tilde{q}(t_x) = \exp[-i2\pi\beta]\tilde{q}(t_x + T(x))$ with $\beta \in [0,1]$, or those related to the Orbifold $(\mathbb{S}^1/\mathcal{Z})$ — as long as we do not add boundary terms to the action and the *degrees of freedom* (*d.o.f.*) transform as scalars, [27].

3. Elementary space-time cycles

We want to investigate the dynamics of the *cyclic* solutions $\tilde{q}(t)$ which are characterized by intrinsic recurrence $T(x)$ at time t_x, eq.(5). The *ansatz* of intrinsic periodicity, as any other *ansatz*, is valid as long as we can obtain meaningful results from it. Now we will see how to define the locally temporal recurrence $T(x)$ according to the integral lines of $\tilde{q}(t)$ in order to preserve causality. We will follow the dual wave description of classical mechanics (or QM).

First of all, it should be noticed that the intrinsic recurrence of the temporal dimension, eq.(5), induces through the EoMs an effective intrinsic recurrence on the spatial dimensions, *i.e.* a corresponding local recurrence on the position space

$$\tilde{q}(t_x) = \tilde{q}(t_x + T(x)) = \tilde{q}(t_x) + \lambda(x)\,. \tag{6}$$

For this reason will speak about space-time cycles. The action $\tilde{S}$ expressed in terms of the Lagrangian density would have space-time boundary whose temporal and spatial intervals, $T(x)$ and $\lambda(x)$ respectively, form a contravariant space-time vector [13]. See for instance Elementary Cycles Theory (ECT) which is an application to elementary particles of the general result presented in this paper [6,7,11–14].

According to the so called Hamilton's optico-mechanical analogy [3–5], at each point x of a generic *synch* solution $q(t)$ it is possible to associate the evolution of a wave of related local wave-number $\bar{k}(x)$ and angular velocity $\bar{\omega}(x)$, proportional to the momentum and to the energy of the *synch* solution — see also action-angle coordinates. We want to equal the space-time recurrences $\lambda(x)$ and $T(x)$ of the *cyclic* solutions $\tilde{q}(t)$ — *i.e.* the boundary of $\tilde{S}$ — with the space-time recurrences of the wave associated to the *synch* solution $q(t)$, at each time t_x.

It is a general property of Hamiltonian dynamics that if this identification holds in a given point x, then it holds in every other point of the system evolution. In fact the spatial cycle of length $\lambda(x)$ forms a vortex tube along the integral curves of the system (Stoke's Lemma), [3,4]. The same conclusions hold for the time cycle $T(x)$. That is, the local space-time cycles $\lambda(x)$ and $T(x)$ of $\tilde{S}$ define vortex tubes such that, point by point,

$$\hbar \int_{q_x}^{q_x+\lambda(x)} \bar{k} dq = \int_{q_x}^{q_x+\lambda(x)} \tilde{p}_1 dq = 2\pi\hbar\,, \tag{7}$$

$$\hbar \int_{t_x}^{t_x+T(x)} \bar{\omega} dt = \int_{t_x}^{t_x+T(x)} \tilde{E}_1 dt = 2\pi\hbar, \tag{8}$$

respectively. We have made use of the Planck constant to introduce the "quantities" $\tilde{p}_1(x)$ and $\tilde{E}_1(x)$. This is a general property of symplectic 2-forms (integrable systems).

In doing so we are explicitly describing the *cyclic* solutions $\tilde{q}(t)$ as waves, or better as *Elementary Cycle Strings* vibrating in space-time — see [6] for the case of elementary particles. All in all, this identification of the local space-time recurrences is also at the base of the wave-particle duality, where the local space-time recurrence of undulatory mechanics evolves consistently with the local energy and momentum of the corresponding system.

By construction the *cyclic* solution $\tilde{q}(t)$ of the local action $\tilde{S}$ is, similarly to eikonal waves, a locally modulated solution defined $\forall t \in \mathbb{R}$ which in each point x is characterized by modulated temporal and spatial recurrences. These local recurrences of Hamilton-Jacobian mechanics act however as constraints

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for the *cyclic* solution $\tilde{q}(t)$. The (space-time) boundary of $\tilde{S}$ forms vortex tubes along the integral curves of the system which evolve consistently with the energy and momentum of the system itself. Such local modulation of space-time recurrences therefore is perfectly consistent with the causality of the system. Saying that local intrinsically cyclic dynamics cannot describe the complexity of nature is like stating that musical instruments cannot play elaborated symphonies, being based on the intrinsically cyclic phenomena of standing waves/vibrating strings, [6–14].

4. Hamiltonian Analysis

This point is inspired by the Dirac analysis of Hamiltonian constrains. However our purpose is different from Dirac's [18,19]. Dirac brackets are used to establish the correct commutation relations that must be postulated for a classical constrained system in order to quantize it. On the contrary, in analogy with GQ, here we want to show that the Dirac rules of canonical quantization, in particular the relation between Poisson brackets and commutation relations of QM, are mathematical consequences of the constraint of intrinsically *cyclic* dynamics.

The canonical momenta for the two classical dynamics are defined by:

$$\begin{aligned} p &= \frac{\partial L(q,\dot{q})}{\partial \dot{q}}\,, \\ \tilde{p} &= \frac{\partial L(\tilde{q},\dot{\tilde{q}})}{\partial \dot{\tilde{q}}}\,. \end{aligned} \tag{9}$$

We compare the Legendre transformations resulting from the variations of the *synch* and *cyclic* actions, with reference to the same local integration intervals [30],

$$\begin{aligned} \oint_{T(x)} L dt &= \oint_{\lambda(x)} p dq - \oint_{T(x)} H dt\,, \\ \oint_{T(x)} \tilde{L} dt &= \oint_{\lambda(x)} \tilde{p} d\tilde{q} - \oint_{T(x)} \left(H - i\hbar \frac{d}{dt}\tilde{\Phi} \right) dt\,, \end{aligned} \tag{10}$$

where the integrals over the local time cycle and the local space cycle are denoted by:

$$\oint_{T(x)} dt \doteq \int_{t_x}^{t_x+T(x)} dt \quad , \qquad \oint_{\lambda(x)} dx \doteq \int_{q_x}^{q_x+\lambda(x)} dx\,, \tag{11}$$

respectively. The *synch* and *cyclic* solutions have the same EoMs and therefore the same Hamiltonian H. However in the latter case we have added the Hamiltonian constraint $\tilde{\Phi}$ — relate to $\tilde{q}(x)$ — in order to obtain *cyclic* dynamics for the solutions:

$$\oint_{T(x)} \frac{d}{dt}\tilde{\Phi}(\tilde{q},\tilde{p}) dt \equiv 0\,, \tag{12}$$

and

$$\tilde{L} = L + i\hbar \frac{d\tilde{\Phi}}{dt}\,. \tag{13}$$

The multiplier $i\hbar$ will be justified when we study the consistency conditions for the constraint, at the end of Sec.(5).

The reduced phase-space

$$\Gamma_{QM} \subseteq \Gamma_{CM} \tag{14}$$

of the *cyclic* dynamics is contained in the phase-space of the *synch* dynamics Γ_{CM}. The former is a symplectic manifold of even dimensions equipped with a non-degenerate, closed 2-form which can be written in terms of local canonical coordinates as (Darboux's theorem)

$$\tilde{\Omega} = d\tilde{p} \wedge d\tilde{q}\,. \tag{15}$$

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Closed means $d\tilde{\Omega} = dd\tilde{\theta} = 0$ with 1-form $\tilde{\theta} = \tilde{p}d\tilde{q}$, see *e.g.* [1,3].

We say that two quantities, in general not equivalent on Γ_{CM}, are *weakly* equivalent ($\approx$) if they are equivalent on Γ_{QM} when averaged over a local cycle, *e.g.*

$$\oint_{T(x)} \frac{d}{dt}\tilde{\Phi}(q,p)dt \approx 0\,, \tag{16}$$

and

$$\oint_{T(x)} \tilde{L}(q,\dot{q})dt \approx \oint_{T(x)} L(q,\dot{q})dt\,. \tag{17}$$

From the requirement of the average over a local cycle we notice that this notation is not completely equivalent to Dirac's weak equivalence [18].

The domain of definition of the constrained system can be now extended from Γ_{QM} to Γ_{CM}. We introduce the total Hamiltonian $\tilde{H}$, such that

$$\oint_{T(x)} dt\tilde{H}(p,q) = \oint_{T(x)} dt\left(H(p,q) - i\hbar\frac{d}{dt}\tilde{\Phi}(p,q)\right). \tag{18}$$

It will be used to associate an Hamiltonian flux of related intrinsic periodicity to every point x of the *synch* trajectories (clearly, in the limit $\hbar \to 0$ we get $\Gamma_{QM} \to \Gamma_{CM}$ and $\tilde{q}(t) \to q(t)$).

Notice that we are working under general hypothesis for the symplectic manifold associated to the phase-space of our system, therefore our analysis is not limited to non-relativistic mechanics. It is in fact possible to adopt the so called *extended configuration space* which includes the time coordinate as configuration coordinate, or the *doubly extended phase-space* which includes the time coordinate and the energy (the conjugate variable of the time coordinate) as phase-space coordinates. In this way it is easy to see that our analysis includes relativistic classical mechanics, *i.e.* the covariant relativistic space-time physics of Minkowskian manifolds, the curved space-time of pseudo-Riemannian manifolds and the infinite-dimensional manifold of fields, see *e.g.* [28] for a review.

5. Hilbert Space

The constraint of local space-time recurrence implies an infinite set of locally modulated *cyclic* eigenmodes. For instance the time cycle $T(x)$, eq.(5), implies general *cyclic* solutions of the form of eikonal waves or modulated signals:

$$\tilde{q}(t) = \sum_{n\in\mathbb{Z}} \tilde{q}_n(0)e^{-\frac{i}{\hbar}\int_0^t \tilde{E}_n dt'}\,. \tag{19}$$

The local eigenvalues $\tilde{E}_n(t)$ are fixed, up to a phase factor, see the twisted BCs mentioned above, by the local condition of intrinsic periodicity eq.(5):

$$e^{\frac{i}{\hbar}\oint_{T(x)}\tilde{E}_n dt'} \equiv e^{i2\pi\beta}\,, \tag{20}$$

and thus

$$\oint_{T(x)} \tilde{E}_n dt' = 2\pi\hbar(n+\beta)\,. \tag{21}$$

Similarly, for the intrinsic spatial periodicity, eq.(6) we get a generalized *Bohr-Sommerfeld quantization condition* (see Einstein-Brillouin-Keller method) [29]:

$$e^{\frac{i}{\hbar}\oint_{\lambda(x)}\tilde{p}_n dq'} \equiv e^{i2\pi\beta}\,, \tag{22}$$

and thus

$$\oint_{\lambda(x)} \tilde{p}_n dq' = 2\pi\hbar(n+\beta)\,. \tag{23}$$

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Here we will assume PBCs ($\beta = 0$), which will result in normal ordered QM[2]. Hence, the 2-form $\tilde{\Omega}$ satisfies the so-called *Weil integrability condition*, fundamental requirement for GQ: the integral of $\tilde{\Omega}/2\pi\hbar$ over the cyclic paths of Γ_{QM} must be an integer number, [1,3].

Periodic phenomena such as $\tilde{\Phi}$, similarly to $\tilde{q}$, can be naturally described on a corresponding Hilbert space. In fact Hilbert spaces are not a prerogative of QM. They can also be adopted in classical mechanics to describe, for example, harmonic systems. We promote all the classical quantities to Hilbert operators. The determination of $p \to \hat{p}$ and $H \to \hat{H}$ will be the main subject of our investigation.

We associate $\tilde{\Phi}$ to a Hilbert "ket" state $|\tilde{\Phi}\rangle$ named *cyclic* physical state, such that

$$\tilde{q}(x) = \langle x|\tilde{\Phi}\rangle\,, \tag{24}$$

in order to constrain the *cyclic* solutions $\tilde{q}$ to have intrinsic periodicity, eq.(5). In addition we will use the *cyclic* physical state $|\tilde{\Phi}\rangle$ to project functions into Γ_{QM}, *e.g.* :

$$A(p,q)|\tilde{\Phi}\rangle = A(\tilde{p},\tilde{q})|\tilde{\Phi}\rangle\,, \tag{25}$$

so that the constraint is

$$\oint_{T(x)} dt \frac{d}{dt}|\tilde{\Phi}\rangle = 0\,. \tag{26}$$

In this way we avoid the use of the weak identities and we mimic the Dirac notation of QM.

Finally eq.(18) can be written in the Hilbert space as

$$\oint_{T(x)} dt \tilde{H}(p,q)|\tilde{\Phi}\rangle = \oint_{T(x)} dt \big(\hat{H}(p,q) - i\hbar X_{\tilde{H}}\big)|\tilde{\Phi}\rangle\,, \tag{27}$$

where the *the Hamiltonian vector field of* $\tilde{H}$ is defined as

$$X_{\tilde{H}} = \{\,\cdot\,,\tilde{H}\} = \frac{d}{dt}\,, \tag{28}$$

in terms of the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$.

In analogy with the $\tilde{q}(t)$ described above, the physical state $|\tilde{\Phi}\rangle$ is an intrinsically periodic phenomenon whose evolution is evidently described by the *Schrödinger equation* on the reduced phase-space, as confirmed by the consistency condition of the constraint with multiplier $i\hbar$, [18,19,30]:

$$\oint_{T(x)} dt \frac{d}{dt}|\tilde{\Phi}\rangle dt = \oint_{T(x)} dt\{\,\cdot\,,\hat{H} - i\hbar\frac{d}{dt}\}|\tilde{\Phi}\rangle = 0\,. \tag{29}$$

6. Gauge Invariance

It is very unconvenient to work with an action whose boundary varies locally, point by point. As proven in [13] and shortly reviewed below, it is possible to write an equivalent formulation of $\tilde{S}$, *i.e.* with the same solutions $\tilde{q}(t)$, in which however the boundary is fixed to a global reference value $T_{free} = const$ (up to a gauge transformation).

We rewrite the general *cyclic* solution of the locally modulated case in terms of Hilbert operators:

$$\tilde{q}(t) = \tilde{q}(0)e^{-\frac{i}{\hbar}\int_0^t \tilde{H}(t)dt}\,. \tag{30}$$

2 Once that the phase-space variables will be written, as we will see, as non-commutating operators, the symmetric ordering reproduces the zero-point energy of ordinary canonical quantum mechanics. Thus, its origin is not on the factor β of the twisted BCs, but it comes from the ordering of the operators, exactly as in ordinary QM.

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Then we compare it with the analogous solution of the corresponding free case, which is quite trivial. If the Lagrangian L_{free} describes the system free of interactions then the recurrence of the space-time boundary will be globally constant:

$$\tilde{S}_{free} = \int_{t_x}^{t_x+T_{free}} dt L_{free}(\tilde{q}_{free}, \dot{\tilde{q}}_{free}) \,, \tag{31}$$

with $\tilde{E}_{1,free} = 2\pi\hbar/T_{free} = const$. The spatial recurrence is global as well. Due to the global PBCs the general solutions will be of the form

$$\tilde{q}_{free}(t) = \tilde{q}(0)e^{-\frac{i}{\hbar}\tilde{H}_{free}t} \,. \tag{32}$$

The free and the interacting solutions are related by *parallel transport* (Wilson line)

$$\tilde{V}(t) = e^{-\frac{i}{\hbar}\int_0^t \Xi(t)dt} \,, \tag{33}$$

such that

$$\tilde{q}(t) = \tilde{V}(t)\tilde{q}_{free}(t) \,, \tag{34}$$

where $\Xi(t)$ denotes an interaction term:

$$\tilde{H}(t) = \tilde{H}_{free} + \Xi(t) \,. \tag{35}$$

If we want to cast a locally modulated recurrence into an action with globally fixed boundary we must modify the derivative terms, which are the only relevant terms to the BCs, according to the variational principle. The strategy is to use the parallel transport $\tilde{V}(t)$ to "tune" the local recurrences of $\tilde{q}$ to the global boundary of $\tilde{q}_{free}$. Let us consider the free Lagrangian and write the free solution in terms of the interacting solution by means of the "tuning" eq.(34):

$$L_{free}(\tilde{q}_{free}, \partial_t \tilde{q}_{free}) \quad = \quad L_{free}(\tilde{V}^{-1}\tilde{q}, \partial_t \tilde{V}^{-1}\tilde{q}) = L_{free}(\tilde{V}^{-1}\tilde{q}, \tilde{V}^{-1}D_t\tilde{q}) \,. \tag{36}$$

The *covariant derivative* is

$$D_t = \partial_t + \frac{i}{\hbar}\Xi(t) \,. \tag{37}$$

Finally, by considering the $U(1)$ symmetry, the locally modulated *cyclic* solution $\tilde{q}(t)$ is equivalently solution of the gauged free action equipped with global recurrence of the boundary

$$\tilde{S} = \int_{t_x}^{t_x+T(X)} dt L(\tilde{q}, \dot{\tilde{q}}) = \int_{t_x}^{t_x+T_{free}} dt L_{free}(\tilde{q}, D_t\tilde{q}) \,. \tag{38}$$

We have shown that a theory based on intrinsic periodicity naturally exhibits *gauge invariance*. In fact we are free to add a local phase in the *parallel transport*,

$$\tilde{V}'(t) = e^{-\frac{i}{\hbar}\zeta(t)}\tilde{V}(t) \tag{39}$$

corresponding to a total derivative term in the Hamiltonian

$$\tilde{H}(t) = \tilde{H}_{free} + \Xi(t) + \partial_t\zeta(t) \,. \tag{40}$$

It defines the gauge transformation

$$\Xi \to \Xi' = \Xi + \partial_t\zeta \,. \tag{41}$$

This was expected: the intrinsic time periodicity of the *cyclic* solutions — similarly to a dimensional compactification — implies *holonomy* ("the boundary of the boundary is zero"), which in turn identifies

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classes of isometries (gauge orbits) associated to the same physical system, *e.g.* [31]. We have used similar arguments in [13] to prove the geometrodynamical origin of gauge interactions in particle physics, in perfect analogy with the geometrodynamical origin of gravitational interaction of general relativity.

7. Pre-quantum Operators

The gauged free Lagrangian $L_{free}(\tilde{q}, D_t\tilde{q})$ and the Lagrangian $L(\tilde{q}, \dot{\tilde{q}})$ lead to the same Hamiltonian $\tilde{H}$ after Legendre transformation, having the same EoM (same solutions).

Let us consider the particular gauge

$$\partial_t \zeta(t) \equiv -\tilde{H}_{free}\,, \tag{42}$$

so that we have

$$\Xi(t) = \tilde{H}(t)\,. \tag{43}$$

By using the formalism of symplectic geometry we can use the contraction $\tilde{\theta}(X_H)$ of the 1-form $\tilde{\theta}$ with the *Hamiltonian vector field* $X_{\tilde{H}}$, eq.(28), and write

$$\Xi(t) = \tilde{\theta}(X_{\tilde{H}})\,. \tag{44}$$

Hence, in this particular gauge,

$$\tilde{V} = e^{-\frac{i}{\hbar}\int_0^t \tilde{\theta}(X_{\tilde{H}})dt'}\,, \tag{45}$$

we find that

$$i\hbar D_t = i\hbar X_{\tilde{H}} - \tilde{\theta}(X_H) = i\hbar \nabla^{\tilde{\theta}}_{X_{\tilde{H}}}\,, \tag{46}$$

where $\nabla^{\tilde{\theta}}_{X_{\tilde{H}}}$ denotes the covariant derivative along $X_{\tilde{H}}$ in the gauge $\tilde{\theta}$. Notice that, in general, the gauge transformations on $\tilde{V}$ and $\nabla^{\tilde{\theta}}_{X_{\tilde{H}}}$ act as required by GQ ($\tilde{\theta}' = \tilde{\theta} + d\zeta$).

We can now apply the "tuning" method to the Hamiltonian formulation. Again, the only problematic term which must be "tuned" is the time total derivative, see eq.(38),

$$\oint_{T(x)} dt \frac{d}{dt}|\tilde{\Phi}\rangle = \oint_{T_{free}} dt \nabla^{\tilde{\theta}}_{X_{\tilde{H}'}}|\tilde{\Phi}\rangle\,, \tag{47}$$

and therefore eq. (27) can be written as (we assume $U(1)$ invariance when acting with a "bra" state)

$$\oint_{T_{free}} dt \tilde{H}|\tilde{\Phi}\rangle = \oint_{T_{free}} dt \Big(\hat{H} - i\hbar\tilde{\nabla}^{\tilde{\theta}}_{X_{\tilde{H}}}\Big)|\tilde{\Phi}\rangle\,. \tag{48}$$

We have proven that the constraint of intrinsic periodicity associates to each classical (*synch*) Hamiltonian H the same pre-quantum operator prescribed by GQ:

$$\hat{H} \;=\; \tilde{H} + i\hbar X_{\tilde{H}} - \tilde{\theta}(X_{\tilde{H}}) = \tilde{H} + i\hbar X_{\tilde{H}} - \Xi(t) = \tilde{H} + i\hbar\nabla^{\tilde{\theta}}_{X_{\tilde{H}}}\,. \tag{49}$$

Having obtained the Hamiltonian pre-quantum operator, it is straightforward to infer the other possible operators by comparing the Hamiltonian fluxes of the *synch* and *cyclic* dynamics. For the canonical phase-space variables we have

$$\begin{aligned}\{p, \tilde{H}\}|\tilde{\Phi}\rangle &= \Big(\{p, \hat{H}\} - i\hbar\{p, \frac{d}{dt}\} + \{p, \tilde{\theta}(X_{\tilde{H}})\}\Big)|\tilde{\Phi}\rangle\,, \\ \{q, \tilde{H}\}|\tilde{\Phi}\rangle &= \Big(\{q, \hat{H}\} - i\hbar\{q, \frac{d}{dt}\} + \{q, \tilde{\theta}(X_{\tilde{H}})\}\Big)|\tilde{\Phi}\rangle\,,\end{aligned} \tag{50}$$

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which lead, by integrating over a generic time interval, to

$$\begin{aligned} \int_0^t dt \dot{\tilde{p}} |\tilde{\Phi}\rangle &= \int_0^t dt \left(\dot{\hat{p}} + i\hbar \frac{d}{dt} \frac{\partial}{\partial q} \tilde{V}^{-1} \right) |\tilde{\Phi}\rangle \,, \\ \int_0^t dt \dot{\tilde{q}} |\tilde{\Phi}\rangle &= \int_0^t dt \left(\dot{\hat{q}} - i\hbar \frac{d}{dt} \frac{\partial}{\partial p} \tilde{V}^{-1} \right) |\tilde{\Phi}\rangle \,, \end{aligned} \tag{51}$$

where now

$$\tilde{V}^{-1} |\tilde{\Phi}\rangle = e^{-\frac{i}{\hbar} \int_0^q \tilde{p} dq'} |\tilde{\Phi}\rangle = |\tilde{\Phi}_{free}\rangle \,, \tag{52}$$

up to a gauge transformation. The integrals above can be expressed in terms of the covariant derivatives of the phase-space variables and easily integrated:

$$\begin{aligned} p \quad &\to \quad \hat{p} \quad = \quad \tilde{p} - i\hbar X_{\tilde{p}} + \tilde{\theta}(X_{\tilde{p}}) = \tilde{p} - i\hbar \nabla^{\tilde{\theta}}_{X_{\tilde{p}}} \,, \\ q \quad &\to \quad \hat{q} \quad = \quad \tilde{q} - i\hbar X_{\tilde{q}} + \tilde{\theta}(X_{\tilde{q}}) = \tilde{q} - i\hbar \nabla^{\tilde{\theta}}_{X_{\tilde{q}}} \,. \end{aligned} \tag{53}$$

We have obtained the correct pre-quantum operators prescribed by GQ. Thus, by using the properties of GQ, we have inferred the *canonical Dirac rule* directly from the first principle of intrinsic periodicity:

$$\boxed{\text{Cyclic Time Dimension} \quad \Rightarrow \quad [\hat{q}, \hat{p}] = i\hbar \widehat{\{q, p\}} = i\hbar \mathbb{I}} \,. \tag{54}$$

8. Implicit Quantization

We may check that the number of the *d.o.f.* of the *cyclic* dynamics phase-space is actually reduced to an half of the original one. In fact the momentum $\tilde{p}$ is not a free parameter, it cannot be arbitrarily varied in the *cyclic* dynamics. It is fixed geometrodynamically by the boundary of the configuration space through the Planck constant (de Broglie-Planck relation). The momentum is determined point by point by the local boundary of the theory ($\tilde{\Phi}$ is formally a generating function of first kind). Furthermore:

$$\frac{\partial}{\partial p} \hat{V}^{-1} |\tilde{\Phi}\rangle = \frac{\partial}{\partial p} |\tilde{\Phi}_{free}\rangle = 0 \,, \tag{55}$$

since $p_{free} = 2\pi\hbar/\lambda_{free} = const$ for the free system. No further prescriptions such as the "polarization" of GQ are required, see *e.g.* [1], to reduce the *d.o.f.* of the phase-space.

We have finally shown that the above pre-quantum operators, eq.(49) and eqs.(53), can be expressed in the familiar form of canonical QM, see eqs.(51), eq.(43) and eq.(55):

$$\hat{H} |\tilde{\Phi}\rangle = i\hbar \frac{\partial}{\partial t} |\tilde{\Phi}\rangle \,; \quad \hat{p} |\tilde{\Phi}\rangle = -i\hbar \frac{\partial}{\partial q} |\tilde{\Phi}\rangle \,; \quad \hat{q} |\tilde{\Phi}\rangle = q |\tilde{\Phi}\rangle \,.$$

We can now repeat the same demonstration of above for any generic function $F(p, q)$ obtaining the related pre-quantum operator,

$$F \quad \to \quad \hat{F} = \tilde{F} - i\hbar X_{\tilde{F}} + \theta(X_{\tilde{F}}) = \tilde{F} - i\hbar \nabla^{\tilde{\theta}}_{X_{\tilde{F}}} \,. \tag{56}$$

For two generic functions F and G Dirac's rules of canonical quantization are automatically satisfied:

$$\boxed{\text{Cyclic Time Dimension} \quad \Rightarrow \quad [\hat{F}, \hat{G}] = i\hbar \widehat{\{F, G\}}} \,. \tag{57}$$

The canonical commutation relations are here *inferred* from the constraint of intrinsically *cyclic* dynamics, rather than postulated as for canonical QM.

GQ guarantees that eq.(56) satisfies all the requirements of canonical quantization, see *e.g.* [1]. In particular we have the linearity rule $\widehat{aF + bG} = a\hat{F} + b\hat{G}$ where a, b are constants, the power rule $\widehat{F^n} = \hat{F}^n$, and the identity rule $\hat{1} = \mathbb{I}$, besides the Dirac rule above[3].

9. Conclusions

We have proven, with all the generality allowed by Hamiltonian mechanics, that to every classical system (*synch* motions) representable by a symplectic manifold (including the Minkowskian manifold of special relativity, the pseudo-Riemannian manifold of general relativity and infinite-dimensional manifold of field theory) it is possible to associate classical *cyclic* dynamics, which in turn are fully equivalent to the canonical quantum dynamics of the system itself.

The Poisson brackets of ECT automatically implies the canonical commutation relations of QM, as direct consequence of its formulation in compact time, in confirmation of the results in [6–14].

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[3] GQ suggests that, actually, QM should be based on some kind of compact support such as circles, cylinders, spheres or tori, and that the momenta should be fixed by some geometrical condition as for "polarization", [1,20–24].

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