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A Complete Proof Of The Riemann Hypothesis Based On A New Expression Of $\xi(s)$

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Abstract Based on Hadamard product, a new expression of the completed zeta function $\xi(s)$ is obtained, i.e.,

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

where $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i$ are complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$. Then we have, by the functional equation $\xi(s) = \xi(1-s)$, that

$$\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i} = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

i.e.,

$$\prod_{i=1}^{\infty}\Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i} = \prod_{i=1}^{\infty}\Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}$$

which, by Lemma 3, is equivalent to

$$\alpha_i = \frac{1}{2}, \text{with } i \text{ from 1 to infinity.}$$

Thus, we conclude that the Riemann Hypothesis is true.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

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1 Introduction

It has been almost 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$ being the Jaccobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty}(1 + \frac{s}{n})e^{-s/n} \tag{3}$$

where γ is the Euler-Mascheroni constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: -2, -4, -6, -8, $\cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the critical strip $0 < \Re(s) < 1$. This

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was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the critical line $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4−9].
Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.
In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for the RH are equivalent.

Statement 1 of the RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of the RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote

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the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

On the other hand, many zeros have been calculated by hand or by computers. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [19–20]. Here are the first three zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced from two types of infinite expansions, i.e., infinite polynomial and infinite product as follows

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [21] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) = \frac{1}{2} \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where ρ are precisely the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over the zeros of the completed zeta function $\xi(s)$.

2 A Complete Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2 we know that Statement 1 of the RH is also true. For this purpose, we need the following Lemma 3 and Lemma 4.

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Lemma 3: Given two infinite products

$$f(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{11}$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{12}$$

where s is a complex variable, $\alpha_i \neq 0$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are natural numbers, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.
Then we have that

$$f(s) = f(1-s) \Leftrightarrow \alpha_i = \frac{1}{2}, \text{with } i \text{ from 1 to infinity.} \tag{13}$$

where $'' \Leftrightarrow ''$ is the equivalent sign.

Proof: The proof is based on Mathematical Induction and Transfinite Induction. We first prove that $P(n)$ holds for all natural numbers $n \in \mathbb{N}$, and then prove that $P(\infty)$ holds.

First of all, we have the following fact:

$$\begin{aligned}
&(1 + \frac{(s-\alpha)^2}{\beta^2})^d = (1 + \frac{(1-s-\alpha)^2}{\beta^2})^d \\
&\Leftrightarrow \\
&(s-\alpha)^2 = (1-s-\alpha)^2 \\
&\Leftrightarrow \alpha = \frac{1}{2}
\end{aligned} \tag{14}$$

where $d \geq 1$ is a natural number, $\alpha \neq 0$ and $\beta \neq 0$ are real numbers.

Let $P(n)$ be:

$$\begin{aligned}
&\prod_{i=1}^{n} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
&\Leftrightarrow \\
&\begin{cases}
(1 + \frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1} = (1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\
\cdots \\
(1 + \frac{(s-\alpha_n)^2}{\beta_n^2})^{d_n} = (1 + \frac{(1-s-\alpha_n)^2}{\beta_n^2})^{d_n}
\end{cases} \\
&\Leftrightarrow \\
&\alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, n
\end{aligned} \tag{15}$$

According to Eq.(14), $P(1)$ is an obvious fact as the **Base Case**, i.e.,

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$$\prod_{i=1}^{1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_1}=\prod_{i=1}^{1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_1}$$
$$\Leftrightarrow$$
$$(1+\frac{(s-\alpha_1)^2}{\beta_1^2})=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}) \tag{16}$$
$$\Leftrightarrow$$
$$\alpha_1=\frac{1}{2}$$

For the **Inductive Case**, assume that $P(n)$ hold, we need to show that $P(n+1)$ holds too. We have

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$
$$\Leftrightarrow$$
$$\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}$$
$\Leftrightarrow$ (by Lemma 4
$$\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ (1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\end{cases}$$
$\Leftrightarrow$ (by Eq.(15)
$$\begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\ \cdots\\ (1+\frac{(s-\alpha_n)^2}{\beta_n^2})^{d_n}=(1+\frac{(1-s-\alpha_n)^2}{\beta_n^2})^{d_n}\\ (1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\end{cases}$$
$\Leftrightarrow$ (by Eq.(14)
$$\alpha_i=\frac{1}{2}, i=1,2,3,\cdots,n,n+1 \tag{17}$$

That is to say $P(n+1)$ holds when $P(n)$ is true.
Hence by Mathematical Induction, $P(n)$ is true for all natural numbers n.
Next, we prove that $P(\infty)$ holds by considering well-ordered ordinal set A indexing the family of statements $P(\gamma : \gamma \in A)$, $A = \mathbb{N}\bigcup\{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the first limit ordinal.
It is well-known that $\omega = \bigcup\{\gamma : \gamma < \omega\}$.

To prove that $P(\infty)$ holds, it suffices to prove the **Limit Case**, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.

Now, suppose that $P(\omega)$ does not hold, then $\xi(s)$ has at least one pair of zeros off the critical line, i.e., $\alpha_i \neq \frac{1}{2}, i < \omega$, which contradicts that $P(\gamma < \omega)$ holds. Thus, the assumption that $P(\omega)$ does not hold, is false. Then the limit case is true, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$. Hence we conclude by Transfinite Induction that $P(\infty)$ holds.

That completes the proof of Lemma 3.

Lemma 4: We have that

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$

$$\Leftrightarrow$$

$$\begin{cases} \prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ (1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}} = (1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}} \end{cases}$$

where s is a complex variable, $\alpha_i \neq 0$ and $\beta_i \neq 0$ are real numbers, $n \geq 1, d_i \geq 1$, and i are all natural numbers, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

Proof: Let's consider

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{18}$$

i.e.,

$$\begin{aligned} &\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\ =&\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{aligned} \tag{19}$$

Leaving out the trivial cases

$$\begin{aligned} &\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = 0 \\ &\Leftrightarrow \\ &(s-\alpha_i)^2 = (1-s-\alpha_i)^2 = -\beta_i^2, i = 1,2,3,\cdots,n,n+1 \end{aligned} \tag{20}$$

which directly imply Lemma 4 is true, Eq.(18) is then equivalent to the following Eq.(21).

$$\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}} = \frac{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} \tag{21}$$

Another possibility $\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} = \frac{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}$ is excluded due to that means $\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}$ and $\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}$ are factors of $\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$ and $\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, respectively, i.e., $\beta_{n+1}=\beta_n, \alpha_{n+1}= 1-\alpha_n, 1-\alpha_{n+1}=\alpha_n, d_{n+1}\le d_n$, i.e., the $(n+1)^{th}$ group of zeros $(\alpha_{n+1}\pm j\beta_{n+1}, 1-\alpha_{n+1}\pm j\beta_{n+1})$ with multiplicity d_{n+1} are included in the n^{th} group of zeros $(\alpha_n \pm j\beta_n, 1-\alpha_n \pm j\beta_n)$ with multiplicity d_n. Figure 1 gives the illustration of this situation.

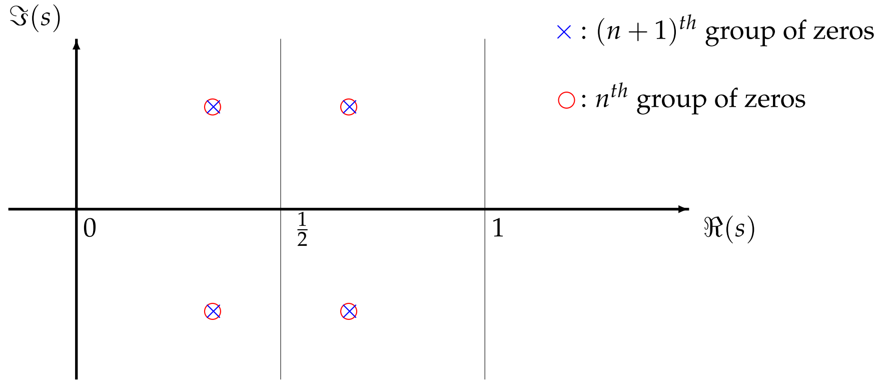

Figure 1 Illustration of the relationship between the n^{th} group of zeros and the $(n+1)^{th}$ group of zeros of $\xi(s)$ while $\beta_{n+1}=\beta_n, \alpha_{n+1}=1-\alpha_n$

Next, without loss of generality, we have from Eq. (21)

$$\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}} = \frac{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} = k \neq 0 \tag{22}$$

where k is a constant to be determined.
Therefore Eq.(21) is equivalent to the following Eq.(23)

$$\begin{cases} \prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = k\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = k\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \tag{23}$$

By comparing the like terms of polynomials in both sides of Eq.(23), we know that Eq.(23) holds if and only if $k=1$, that means Eq.(21) is equivalent to the

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following Eq.(24)

$$\begin{cases} \prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \tag{24}$$

Recalling that Eq.(18) ⇔ Eq.(19) ⇔ Eq.(21), then we have

$$\begin{aligned} &\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ &\Leftrightarrow \\ &\begin{cases} \prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \end{aligned} \tag{25}$$

That completes the proof of Lemma 4.

Proof of the RH: The details are delivered in three steps as follows.

Step 1: Based on the Hadamard product in Eq.(10), we have

$$\begin{aligned} \xi(s) &= \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \end{aligned} \tag{26}$$

where $\xi(0)=\frac{1}{2}$.
The absolute convergence of the infinite product in Eq.(26) in the form

$$\xi(s)=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{\bar{\rho}})=\xi(0)\prod_{\rho}\left(1-\frac{s(2\alpha-s)}{|\rho|^2}\right), 0<\alpha=\Re(\rho)<1$$

depends on the convergence of infinite series $\sum_{\rho}\frac{1}{|\rho|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[22].

Further, taking into account the possibility of multiple zeros in Eq. (26), we have

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{27}$$

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where $d_i \geq 1$ are natural numbers, i are natural numbers from 1 to infinity.
Step 2: Replacing s with $1-s$ in Eq.(27), we obtain the infinite product expression of $\xi(1-s)$

$$\xi(1-s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{28}$$

Step 3: We have by $\xi(s) = \xi(1-s)$ that

$$\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} = \prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{29}$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1 + \frac{(s-\alpha_i)^2}{\beta_i^2})^{d_i} = \prod_{i=1}^{\infty}(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2})^{d_i} \tag{30}$$

And that β_i can be certainly arranged in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.
Then according to Lemma 3, Eq.(30) is equivalent to $\alpha_i = \frac{1}{2}$, with i from 1 to infinity.
Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true; According to Lemma 2, Statement 1 of the RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.
That completes the proof of the RH.

Remarks: By Lemma 1, there are 2 pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$ are all non-trivial zeroes of $\zeta(s)$. With the proof of the RH, i.e., $\alpha = \frac{1}{2}$, these 2 pairs of zeros are actually only one pair, because $\rho = 1-\bar{\rho} = \frac{1}{2}+j\beta, \bar{\rho} = 1-\rho = \frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho = 1-\bar{\rho}, \bar{\rho} = 1-\rho$ are all non-trivial zeroes.

3 Conclusion

The celebrated Riemann Hypothesis is proved to be true based on a new expression of the completed zeta function $\xi(s)$, i.e.,

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \Big(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2} \Big)^{d_i}$$

where $\xi(0) = \frac{1}{2}, \rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i$ are complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

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