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The Singular Connection Between a Family of Exponential Functions and the Fermat–Wiles' Theorem. Version 3

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Abstract

The advances related to Modular Elliptic Curves and the Fermat – Wiles' theorem in recent years has opened the door to entirely new approaches to numerous problems and techniques. This paper studies the linking between a family of exponential function and the Fermat – Wiles' theorem. In this family, the convergence of the coordinates of the abscissas of the maxima and the equation determined by the fulfilment of the maximum condition leads to a singular connection with the Fermat – Wiles' theorem.

Keywords: connection; Fermat – Wiles' theorem; family of exponential functions; sequence; maxima

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1. Introduction

In 1637 Pierre Fermat formulated the conjecture known as his Last Theorem. In 1995, Andrew Wiles presented the first successful proof of Fermat's conjecture [1]. In this work, the semistable case of the Taniyama-Shimura theorem - previously a conjecture - which links modular forms and elliptic curves, was demonstrated. From this progress, combined with ideas from Frey and Ribet's theorem, the proof of the conjecture was obtained. This result was hailed as an astonishing breakthrough that opened the door to entirely new approaches to numerous problems and techniques [2]. The number of publications in recent years and the new links with other fields have confirmed this statement [3–5].

In parallel, research into exponential functions has evolved from the study of their basic properties towards deep problems connecting number theory, complex analysis, dynamical systems, and applied mathematics, keeping it at the frontier of mathematical knowledge: Schanuel's Conjecture, Dynamics of the Exponential Function, Transcendence of Hybrid Expressions and others [6,7]. Moreover, the exponential function is the natural language for describing a vast range of natural and human processes involving proportional change, making it a cornerstone of modern scientific modelling: Nuclear and Atomic Physics, Biology and Demography, Geology and Palaeoclimatology, Economics and Finance, Psychology and Neuroscience and many others [8]. It is therefore interesting to explore the possible links between Fermat - Wiles' theorem and exponential functions.

In this work, a singular connection between a family of exponential function and the Fermat – Wiles' theorem (FWT) is revealed. This family depends on the variable $x \in \Re \,/\, x \geq -1$ and a parameter $n \in \mathcal{Z}^{+}$. The bases were chosen as integers, and an additional restriction was imposed on them. The functions have a unique maximum for each value of n, which is determined by an equation relating the bases. On the other hand, the sequence x_n, whose terms are the abscissas of the maxima for different values of n, has a sup x_n. By analysing the limit of the sequence and the additional linking equation between the bases, a singular connection with the Fermat – Wiles' theorem is found.

Below, we proceed to carry out the analysis that reveals the aforementioned connection.

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2. Analysis of the Connection

2.1. Family of Exponential Functions

Let the set of exponential functions (1) be described by the equations

$$y_n(x) = (na)^x + (nb)^x - (nc)^x = n^x(a^x + b^x - c^x) = n^x y_1(x) \tag{1}$$

$$\mathbf{n = 1, 2, 3, \ldots, n, \ldots}$$

where $a, b, c \in \mathbb{Z}^+$

$$c > b > a > 0 \tag{2}$$

and the sets A, B and C, whose elements are the prime numbers present in the prime factorisation of a, b and c respectively, satisfy the condition that they only have one as a common prime factor

$$A \cap B = A \cap C = B \cap C = \{1\} \tag{3}$$

The integers a, b and c are used as bases to define (1) in the interval $x \in (-1, +\infty)$. Additionally, the restriction (4) will be imposed on (1), where $p \in \mathbb{Z}^+ / \, p \geq 3$

$$y_n(p) = n^p(a^p + b^p - c^p) = 0 \tag{4}$$

Next, we proceed to study the consequences of imposing restriction (4) on variation of functions (1). As a first step in the analysis, it will be shown that the functions (1) have a maximum in the interval (0, p).

Taking into account that $n^p > 0$, from (4) follows that

$$na + nb = \frac{(na)^p}{(na)^{p-1}} + \frac{(nb)^p}{(nb)^{p-1}} > \frac{(na)^p}{(nc)^{p-1}} + \frac{(nb)^p}{(nc)^{p-1}} = nc \tag{5}$$

Considering (2) and (5) it follows that $\exists \, d \in \mathbb{Z}^+ /$

$$na + nb = nc + d \tag{6}$$

and it will be shown that $d \equiv 0$ (mod p). Taking into consideration (4) and (6), Fermat's little theorem $(m^p \equiv m \; (mod\; p))$ will be applied to equality

$$(na)^p - (na) + (nb)^p - (nb) = (nc)^p - (nc) - d \tag{7}$$

from which it is concluded that d is a multiple of p, consequently, $d = q \cdot p$ where $q \in \mathbb{Z}^+ (q \geqslant 1)$. Therefore, $d \geqslant p$, from here it is inferred that

$$na + nb - nc - qp = 0 \rightarrow na + nb - nc \geq p \tag{8}$$

Therefore

$$y_n(1) = na + nb - nc \geq p \tag{9}$$

Bearing in mind (4) and (9) as well the theorem of Intermediate Values for continuous functions, then $\exists \; x_n^* \in [1, p] / \; y_n(x_n^*) = 1$. Hence, at the endpoints of the interval $[0, \; x_n^*]$, it holds that $y_n(0) = y_n(x_n^*) = 1$. Consequently, by Rolle's theorem for continuous and differentiable functions, there exists at least one point $x_n \in [0, \; x_n^*]$ such that $y_n'(x_n) = 0$, and $y_n(x_n)$ has an extremum at that value $x_n \; (x_n < x_n^* < p)$.

From (1) it follows that

$$\begin{aligned}\frac{dy_n(x)}{dx} &= (In\,(nc)) \left[(na)^x \left(\frac{In\,(na)}{In\,(nc)}\right) + (nb)^x \left(\frac{In\,(nb))}{In\,(nc))}\right) - (nc)^x\right] = \\ &= n^x \, In \left(\frac{(na)^{a^x} (nb)^{b^x}}{(nc)^{c^x}}\right) = n^x \, In \left[\left(\frac{a^{a^x} b^{b^x}}{c^{c^x}}\right) \cdot \left(\frac{n^{a^x} n^{b^x}}{n^{c^x}}\right)\right]\end{aligned} \tag{10}$$

$$\frac{d^2 y_n(x)}{dx^2} = (In\,(nc))^2 \left[(na)^x \left(\frac{In\,(na)}{In\,(nc)}\right)^2 + (nb)^x \left(\frac{In\,(nb)}{In\,(nc)}\right)^2 - (nc)^x\right] \tag{11}$$

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If $y_n'(x)$ is evaluated at $x = x_n$ and is considered that $n^{x_n} > 0$ it is obtained that:

$$(na)^{a^{x_n}} (nb)^{b^{x_n}} = (nc)^{c^{x_n}} \tag{12}$$

Relation (10) determines that $y_n(x)$ has only one extremum. And if (2), (10) and (11) are taken into account, it follows that $y_n''(x_n) < 0$, so the extremum at $x = x_n$ is a maximum.

From *(12)*, it results that

$$a^{a^{x_n}} b^{b^{x_n}} c^{-c^{x_n}} = n^{-(a^{x_n}+b^{x_n}-c^{x_n})} = n^{-y_1(x_n)} = n^{-\frac{y_n(x_n)}{n^{x_n}}} \tag{13}$$

Note that the abscissas x_n of the maxima of the functions $y_1(x)$, $y_2(x)$,… and $y_n(x)$… are not equal.

In the next stage, the sequence x_n and the variation of the equality (13) as a function of n will be investigated.

2.2. Sequence of the Abscissas x_n of the Maxima

It will then be shown that the coordinates of the abscissas x_n of maxima of the family functions $y_n(x)$, form a monotonically increasing sequence that has a limit.

From (4) it results that

$$\lim_{x\to p} y_n(x) = \lim_{x\to p} n^x (a^x + b^x - c^x) = \lim_{x\to p} n^x y_1(x) = 0 \tag{14}$$

The derivative of $y_n(x)$ is equal to

$$\frac{dy_n(x)}{dx} = n^x \left[y_1'(x) + y_1(x)\, In\,(n)\right] \tag{15}$$

Given $n = k$, at the coordinate $x = x_k$ of the maximum of $y_k(x)$ it is true that $y_k'(x_k) = 0$ and as $n^k > 0$ that implies that

$$y_1'(x_k) = -y_1(x_k)\, In\,(k) \tag{16}$$

If $y_{k+1}'(x)$ is evaluated at $x = x_k$, it results in

$$y_{k+1}'(x_k) = (k+1)^{x_k}\left[y_1'(x_k) + y_1(x_k)\, In\,(k+1)\right] \tag{17}$$

The next step is to replace in (17) $y_1'(x_k)$ by its expression (16) obtaining

$$\begin{aligned} y_{k+1}'(x_k) &= (k+1)^{x_k}[-y_1(x_k)\, In\,(k) + y_1(x_k)\, In\,(k+1)] = \\ &= (k+1)^{x_k} y_1(x_k) In \left(\frac{k+1}{k}\right) \end{aligned} \tag{18}$$

Since $(k+1)^{x_k} > 0$, $y_1(x_k) > 0$ and $In\left(\frac{k+1}{k}\right) > 0$, this means that $y_{k+1}'(x_k) > 0$. Therefore, $y_{k+1}(x)$ is growing at $x = x_k$. Considering (14) and $y_{k+1}'(x_k) > 0$, then the function $y_{k+1}(x)$ must reach a maximum at some point $x_{k+1} > x_k$. Therefore, the sequence formed by the coordinates of the maxima of $x_1, x_2, x_3, \dots, x_n, \dots$ turns out to be monotonically increasing. Furthermore, considering that $x_n < x_n^* < p$, this is bounded by $x = p$

$$x_1 < x_2 < \dots < x_k < x_{k+1} \dots < p \tag{19}$$

This sequence is monotonically increasing and being bounded it converges to a supremum according to the Monotonic Convergence theorem [9]

$$T = \sup \{x_n\} \tag{20}$$

So

$$\lim_{n\to\infty} x_n = T \le p \tag{21}$$

2.3. The Connection with the Fermat – Wiles' Theorem (FWT)

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In some cases, the use of the concept of limit allows to transcend the discrete and individual nature of integers to discern connections in the case of numerical functions [10]. This tool will now be used to determine the behaviour of $y_n'(x_n)$ when n $\rightarrow \infty$.

The case where T $<$ p will be analysed next. From (13) it is obtained that

$$\boldsymbol{In\,\frac{a^{a^{x_n}}b^{b^{x_n}}}{c^{c^{x_n}}} = -(a^{x_n}+b^{x_n}-c^{x_n})\;\cdot\;In\,n} \tag{22}$$

Calculating the limits of both sides of (22) results in

$$\boldsymbol{\lim_{n\to\infty} In\,\frac{a^{a^{x_n}}b^{b^{x_n}}}{c^{c^{x_n}}} = -\lim_{n\to\infty}(a^{x_n}+b^{x_n}-c^{x_n})\cdot\lim_{n\to\infty} n}$$

$$\boldsymbol{Constant = In\,\frac{a^{a^p}b^{b^p}}{c^{c^p}} = -(a^T+b^T-c^T)\,In\,n = Constant\cdot\infty} \tag{23}$$

Note that from (2) and the hypothetical fulfilment of (4) it follows that

$$\boldsymbol{a^T + b^T > \frac{a^p}{c^{p-T}} + \frac{b^p}{c^{p-T}} = c^T \rightarrow \;(a^T+b^T-c^T) > 0} \tag{24}$$

This reasoning leads to a contradiction in (23) when $\lim_{n\to\infty} x_n = T < p$. Since this alternative contradicts the adoption of condition (4), we will then analyze the case where the limit of sequence (21) is p.

From the previous analysis it is known that

$$\boldsymbol{\forall n,\; y_n'(x_n) = 0} \tag{25}$$

Also, expressions (4) and (10) leads to

$$\boldsymbol{\forall n,\; y_n'(p) = n^p\; In\left[\left(\frac{a^{a^p}b^{b^p}}{c^{c^p}}\right)\cdot\left(\frac{n^{a^p}n^{b^p}}{n^{c^p}}\right)\right] = n^p\; In\left[\left(\frac{a^{a^p}b^{b^p}}{c^{c^p}}\right)\right]} \tag{26}$$

Therefore, from (25) and (26) is follows

$$\boldsymbol{\forall n,\; y_n'(p) - y_n'(x_n) = n^p\; In\left(\frac{a^{a^p}b^{b^p}}{c^{c^p}}\right)} \tag{27}$$

If the limit of both sides of the equality (27) is calculated and is taken in account $x_n \rightarrow p$ (21) as n $\rightarrow \infty$ is obtained

$$\boldsymbol{\lim_{n\to\infty}\left(y_n'(p) - y_n'(x_n)\right) = In\left(\frac{a^{a^p}b^{b^p}}{c^{c^p}}\right)\cdot\lim_{n\to\infty} n^p} \tag{28}$$

$$\mathbf{0 = Constant\cdot\infty}$$

The only alternative to solve this contradiction is to assume that the constant present in (26) is equal to zero

$$\boldsymbol{In\left(\frac{a^{a^p}b^{b^p}}{c^{c^p}}\right) = 0} \tag{29}$$

Therefore

$$\boldsymbol{a^{a^p}\,b^{b^p} = c^{c^p} = c^{a^p}\,c^{b^p}} \tag{30}$$

The relations (30) obtained by evaluating the variation of the functions $y_n(x)$ contradict assumptions adopted in (2), (3) and (4). The analysis performed by its essence represents a demonstration by reductio ad absurdum, that shows that if the family $y_n(x)$ has a, b, c $\in Z^+$ bases, then it cannot have roots $p \in Z^+/p \geq 3$.

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3. Conclusions

The previous analysis demonstrates a singular connection between a family of exponential functions and the Fermat – Wiles' theorem. If it is assumed that the bases of the family of exponential functions $y_n(x)$ are integers and the existence of an entire root is imposed, then the convergence of the abscissas x_n of the maxima and the equation determined by the fulfilment of the maximum condition leads to contradictions. This analysis indicates that if the family $y_n(x)$ has integer bases, then it cannot have roots $p \in Z^{+} / p \geq 3$.

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