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The Jacod-Yor Theorem for Sigma Martingales and the Second Fundamental Theorem of Asset Pricing

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Abstract: In this paper, we prove the Jacod-Yor Theorem for sigma martingales, a class of processes that generalize local martingales and play a pivotal role in financial mathematics. While the Jacod-Yor Theorem has been extensively studied for L^2-martingales, martingales, and local martingales, no prior version exists for sigma martingales. Our result establishes the connection between sigma martingales and their martingale representation properties, addressing a critical gap in the literature. As an application, we prove the Second Fundamental Theorem of Asset Pricing for markets where price processes are modeled as sigma martingales.

Keywords: sigma martingales; predictable representation property; Jacod-Yor Theorem; financial mathematics; fundamental theorem of asset pricing

MSC: 60H05; 91G30; 60G44; 60G07; 60H30

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1. Introduction

The question of determining criteria for a class of processes $\mathcal{A}$ to be representable as a stochastic integral with respect to a given semimartingale X is fundamental to many applications, particularly in modeling, risk management, and decision-making frameworks. For example, martingale representation theorems enable the expression of portfolio wealth in self-financing trading strategies, directly linking the hedgeability of a claim to its representation as a stochastic integral. Beyond finance, these theorems are indispensable in filtering theory for constructing optimal filters for stochastic signals and in stochastic control for characterizing solutions to stochastic optimal control problems.

Most commonly, the martingale representation theorem is studied in the context of Brownian motion. Originally developed by Itô [16], the theory was extended to square-integrable martingales by Kunita and Watanabe [23] and by Davis and Varaiya [5]. Occasionally, representation theorems are also developed for processes other than Brownian motion such as Possion processes ([22] or Lévy processes ([2,34]).

For local martingales, representation theory is notably sparse, with only a few significant contributions, such as [4,9,24]. Many of these works depend on the Jacod–Yor Theorem, introduced by Jacod and Yor [18], which provides necessary and sufficient conditions for the existence of a martingale representation. While this theorem has been extensively studied for L^2-martingales and, in some cases, local martingales, it has not been extended to sigma martingales.

No general martingale representation theorem exists for sigma martingales. In particular, the Jacod–Yor Theorem has not been established for this class of processes. Karandikar and Rao [20] explicitly acknowledge this shortfall:

> *"To the best of our knowledge, the martingale representation property in the framework of sigma martingales is not available in the literature. Indeed, most treatments deal with square-integrable martingales where the notion of orthogonality of martingales is available, which simplifies the treatment."*

Sigma martingales generalize local martingales and are particularly important in financial mathematics. For example, the First Fundamental Theorem of Asset Pricing (FTAP) asserts the existence of an

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equivalent sigma martingale measure under the condition of No Lunch with Vanishing Risk (NFLVR), as shown by Delbaen and Schachermayer [7]. This result states that, under certain no-arbitrage conditions, discounted price processes are sigma martingales.

The lack of a martingale representation theory for sigma martingales also affects the Second Fundamental Theorem of Asset Pricing (Second FTAP), which concerns market completeness. The classical Second FTAP, attributed to Harrison et al. [12–14], is limited to cases where discounted price processes are martingales or local martingales. While versions of the Second FTAP for sigma martingales are presented in [37] and [20], these works deviate from the economically justified approaches commonly associated with the First FTAP.

This paper aims to fill these gaps by providing a version of the Jacod–Yor Theorem for sigma martingales. As an application, we prove the Second Fundamental Theorem of Asset Pricing in a general form, thereby laying the foundation for a unified theory that bridges sigma martingales, their representation properties, and their role in financial mathematics.

2. Definitions and Main Theorems

Sigma martingales were first introduced as processes "de la classe (Σ_m)" in [3] and were later formalized in [10]. They arise naturally in financial mathematics, as any process that can be represented as a stochastic integral with respect to a local martingale is necessarily a sigma martingale.

There are several equivalent definitions of sigma martingales. Our definition aligns with that of Goll and Kallsen [11]. Different definitions are used by Jacod and Shiryaev [19], Protter [35], and Delbaen and Schachermayer [7]. Nevertheless, they are all equivalent, as is shown in Theorem 1.

Definition 1. *Let $\mathcal{P}$ denote the predictable σ-algebra.*

1. *A one-dimensional semimartingale S is called a **sigma martingale** if there exists a sequence of predictable sets $D_n \subset \mathcal{P}$, satisfying $D_n \subset D_{n+1}$ for all n, and $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$. Additionally, for any $n \geq 1$, the process $1_{D_n} \bullet S$ is a uniformly integrable martingale.*
2. *A d-dimensional semimartingale is called a sigma martingale if each of its components is a one-dimensional sigma martingale.*
3. *The vector space of all d-dimensional sigma martingales is denoted by $\mathcal{M}_\sigma^d$.*

We recall that $\mathcal{H}^p$ denotes the space of $\mathcal{H}^p$-martingales (see Appendix A for details).

The following theorem provides multiple equivalent characterizations of sigma martingales.

Theorem 1 (Characterization of Sigma Martingales)**.** *Let $X = (X^1, \ldots, X^d)$ be a d-dimensional semimartingale. The following are equivalent:*

1. *The process X is a sigma martingale.*
2. *There exists a strictly positive predictable process H and an $\mathcal{H}^1$-martingale $M = (M^1, \ldots, M^d)$ such that*
$$X^i = H \bullet M^i \quad \text{for all } i \in \{1, \ldots, d\}.$$
3. *There exists a local martingale $M = (M^1, \ldots, M^d)$ and a predictable process $H = (H^1, \ldots, H^d)$ with*
$$H^i \in L(M^i) \quad \text{and} \quad X^i = H^i \bullet M^i \quad \text{for all } i \in \{1, \ldots, d\}.$$
4. *There exists a sequence $D_n \subset \mathcal{P}$ such that $D_n \subset D_{n+1}$, $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$, and for any $n \geq 1$, $1_{D_n} \bullet X$ is a local martingale.*
5. *There exists a sequence $D_n \subset \mathcal{P}$ such that $D_n \subset D_{n+1}$, $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$, and for any $n \geq 1$, $1_{D_n} \bullet X$ is a sigma martingale.*

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The predictable representation property (PRP) is usually defined for L^2-martingales; see, for example, [35]. Generalizations exist for local martingales as in [15,37]. We extend this concept further to sigma martingales.

Definition 2. *For $X \in \mathcal{M}^d_{\sigma,0}$, define:*

$$\mathcal{I}(X) := \{H \bullet X;\, H \in L(X)\}.$$

We say that X has the ***predictable representation property (PRP)*** *if*

$$\mathcal{I}(X) = \mathcal{M}^1_{\sigma,0}.$$

Furthermore, define

$$\mathcal{I}^1(X) := \mathcal{I}(X) \cap \mathcal{H}^1.$$

The PRP is fundamental in applications like mathematical finance, where it enables us to express contingent claims or wealth processes in terms of integrals with respect to a sigma martingale. This property is also central to the Jacod-Yor Theorem for sigma martingales.

Remark 1. *The set $\mathcal{I}(X)$ depends on the probability measure $\mathbf{P}$. However, for the sake of simplicity, we only write $\mathcal{I}(X)$ instead of $\mathcal{I}(X, \mathbf{P})$.*

The importance of orthogonality in stochastic analysis traces back to foundational work by P. A. Meyer in the 1960s. In his seminal papers [30,31], Meyer laid the groundwork for understanding martingales using Hilbert-space methods. This perspective was further developed by Itô, Motoo and Kunita [17,23,32], who demonstrated that the space of L^2-martingales could be treated as a Hilbert space. These techniques and tools allowed for significant advancements in the theory of stochastic analysis.

Inspired by this classical notion of orthogonality, we extend the concept to sigma martingales.

Definition 3. *Let $X \in \mathcal{M}^d_\sigma$. We say a process $Z \in \mathcal{M}^1_\sigma$ is* ***orthogonal*** *to X if $XZ \in \mathcal{M}^d_\sigma$, and we write $X \perp Z$. Furthermore, we define the set of all processes orthogonal to X as*

$$X^\perp = \{Z \in \mathcal{M}^1_\sigma;\, X \perp Z\}.$$

Remark 2. *Clearly, $X^\perp$ contains all constant processes and for $d = 1$, by partial integration, two sigma martingales X, Y are orthogonal if and only if $[X, Y] \in \mathcal{M}^1_\sigma$.*

Definition 4. *Let $X \in \mathcal{M}^d_\sigma$. Then $\mathfrak{P}_\sigma(X)$ denotes the sets of all probability measures $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ such that X is a sigma martingale under $\mathbf{Q}$. A measure $\mathbf{P} \in \mathfrak{P}_\sigma(X)$ is* ***extremal*** *if, for any $\mathbf{Q}, \mathbf{R} \in \mathfrak{P}_\sigma(X)$ and $\lambda \in (0, 1)$ such that $\mathbf{P} = \lambda\mathbf{Q} + (1 - \lambda)\mathbf{R}$, we have $\mathbf{P} = \mathbf{Q} = \mathbf{R}$.*

The main result of this paper is a generalization of the classical Jacod-Yor Theorem to sigma martingales.

Theorem 2 (Jacod-Yor Theorem for Sigma Martingales)**.** *Let $X \in \mathcal{M}^d_\sigma$. The following are equivalent:*

1. *The process X has the predictable representation property.*
2. *The set $X^\perp$ only contains the constant processes.*
3. *For any $\mathbf{Q} \in \mathfrak{P}_\sigma(X)$ with $\mathbf{Q} \ll \mathbf{P}$, we have $\mathbf{P} = \mathbf{Q}$.*
4. *The probability measure $\mathbf{P}$ is an extremal point of $\mathfrak{P}_\sigma(X)$.*

As a classical martingale representation theorem, the Jacod-Yor Theorem has multiple applications. As probably the most popular one, we are going to prove the Second Fundamental Theorem of Asset

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Pricing for unbounded stochastic processes for a market that satisfies NFLVR. By the First Fundamental Theorem of Asset Pricing [7], we can assume that the discounted price processes are sigma martingales with respect to to some equivalent probability measure.

Our definition of the financial market is essentially the same as in [8] and [7]. For simplicity, we assume discounted processes under an equivalent sigma martingale measure. These simplifications can be justified via the First Fundamental Theorem of Asset Pricing and basic results about general markets. For a general derivation of that setting, starting from a non-discounted setup with a real-world probability measure, we refer to [38].

We consider a financial market consisting of $d+1$ tradeable securities whose price processes are represented by the $d+1$-dimensional process $S_t = (S_t^0, S_t^1, \dots, S_t^d)_{t\in\mathbb{R}_+}$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We make the following assumptions:

- The processes S_t^i are sigma martingales for all $i = 0, \dots, d$.
- The filtration $\mathbb{F}$ satisfies the usual conditions, and the σ-algebra $\mathcal{F}_0$ is trivial; that is, $A \in \mathcal{F}_0$ implies $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Definition 5. *1. We call a $d+1$-dimensional process $\varphi = (\varphi^0, \dots, \varphi^d) \in L(S)$ a **trading strategy**.*

2. *The **wealth process** of the investor is defined as*

$$V_t := \sum_{i=0}^{d} \varphi_t^i S_t^i, \quad t \in \mathbb{R}_+,$$

where φ_t^i represents the number of the i-th security that the investor holds in their portfolio at t.

3. *A trading strategy $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$ is called **self-financing** if the wealth process $V_t(\varphi)$ satisfies*

$$V_t = \varphi \bullet S_t, \quad \forall t \in \mathbb{R}_+. \tag{1}$$

4. *A self-financing strategy $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d) \in L(S)$ is called **admissible** if there exists an $\alpha \in \mathbb{R}_+$ such that*

$$V_t \geq -\alpha \quad \forall t \in [0, \infty)$$

for the corresponding wealth process V.

5. *A **contingent claim** X settled at time T is a non-negative $\mathcal{F}_T$-measurable, integrable random variable.*

We now turn to the Second Fundamental Theorem of Asset Pricing. In the case of finitely many assets, a version of this theorem was first proved for the discrete-time setting in [12]. The proof for the continuous-time case followed in [13,14], but Sigrid Müller showed in [33] that the statement in a higher-dimensional case is not valid for the componentwise stochastic integral without additional assumptions. However, one can bypass the additional assumptions by not restricting the class $L(S)$ to the componentwise-integrable integrands, as shown in [27] and [28]. In all these publications, however, a probability measure is assumed under which the process S^i is a local martingale and not just a sigma martingale.

Theorem 3 (Second Fundamental Theorem of Asset Pricing)**.** *Let $(S_t)_{t\in[0,T]}$ (with $T < \infty$) be a $d+1$-dimensional sigma martingale under* **P**. *The following are equivalent:*

1. *The market is complete, that means for each contingent claim X, there exists an admissible self-financing $\varphi \in L(S)$ such that $\varphi \bullet S$ is a true martingale and $(\varphi \bullet S)_T = X$.*
2. *For each* uniformly integrable *martingale M, there is a $\varphi \in L(S)$ with*

$$M = \varphi \bullet S.$$

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3. *For each $M \in \mathcal{M}^1_{\mathrm{loc}}$, there is a $\varphi \in L(S)$ with*

$$M = \varphi \bullet S.$$

4. *For each $X \in \mathcal{M}^1_{\sigma}$, there is a $\varphi \in L(S)$ with*

$$X = \varphi \bullet S.$$

5. *For any other measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ under which S is a sigma martingale, we must have $\mathbf{Q} = \mathbf{P}$.*

Proof. **(i) $\Rightarrow$ (ii).** Assume the market is complete. Let M be a uniformly integrable martingale. Then $|M|$ is a submartingale with $\sup_{0\le t\le T} \mathbf{E}[|M_t|] \le \mathbf{E}[|M_T|] < \infty$. By the Krickeberg (or Doob–Meyer) decomposition, M can be assumed w.l.o.g. nonnegative. Then $M_T \ge 0$ is a claim. Completeness implies there is a self-financing φ such that $(\varphi \bullet S)$ is a true L^1-martingale with terminal value M_T. Two UI martingales agreeing at time T must coincide for all $t \le T$. Thus $M = \varphi \bullet S$.
(ii) $\Rightarrow$ (iv). Take $X \in \mathcal{M}^1_{\sigma}$. By definition (see [3,10,11] or Theorem 1), there is a strictly positive predictable process H such that $H \cdot X$ is a true (UI) martingale. Hence (ii) says $H \cdot X = \psi \bullet S$ for some $\psi \in L(S)$. Since $H > 0$, we re-scale or rearrange to get $X = \phi \bullet S$. Hence, X also admits a martingale representation of S.
(iv) $\Leftrightarrow$ (v). This follows directly from the Jacod-Yor Theorem (Theorem 2).
(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). These inclusions are obvious.
(ii) $\Rightarrow$ (i). Finally, if every UI martingale is a true martingale stochastic integral in S, then the model is complete. Indeed, let $X \ge 0$ be any claim. Define $M_t := \mathbf{E}[X|\mathcal{F}_t]$, a UI martingale. By (ii), $M = \varphi \bullet S$ for some predictable φ. Since we start the trading strategy at time 0 with wealth $M_0 = \mathbf{E}[X]$, this $\varphi \bullet S$ must be a self-financing wealth process and $(\varphi \bullet S)_T = X$. Thus, X is replicated, proving completeness. $\square$

Remark 3. *In the statement of completeness, we require that any replicating strategy $\varphi \bullet S$ be a* true *(L^1-)martingale rather than just a local martingale (or a supermartingale). Local martingale strategies do not necessarily pin down the terminal value at time T due to possible unbounded variations or jumps, and uniform integrability is essential to guarantee the desired payoff X is matched exactly at T.*

Some authors ensure the true martingale property by requiring that all replicating strategies be bounded (or that φ be bounded), which trivially forces $(\varphi \bullet S)$ to be a UI martingale. However, this is clearly a stronger restriction than ours, as we only require $(\varphi \bullet S)$ itself to be a true martingale.

For an example of a market that admits a hedging strategy replicating a claim but lacks a martingale hedging strategy and, consequently, market completeness, refer to Example 7.3 in [37].

3. Proof of the Jacod-Yor Theorem for Sigma Martingales

By $\mathcal{M}^d$, we denote the space d-dimensional martingales, by $\mathcal{M}^d_\infty$ the space of bounded d-dimensional martingales, and by $\mathcal{M}^d_{\mathrm{loc}}$ the space of d-dimensional local martingales. A subscript 0 as in $\mathcal{M}_{\mathrm{loc},0}$ will further indicate that the process starts in 0, that is $M_0 = 0$ almost surely for all $M \in \mathcal{M}_{\mathrm{loc},0}$.

Lemma 1. *Let $X \in \mathcal{M}^d_{\sigma,0}$, $Y \in \mathcal{M}^1_{\sigma,0}$ and let Y be locally in $\mathcal{I}(X)$. Then we have $Y \in \mathcal{I}(X)$.*

Proof. Let Y be locally in $\mathcal{I}(X)$. That means there exists an increasing sequence of stopping times $(T^n)_{n\in\mathbb{N}}$, tending to infinity, such that there are $H^n \in L(X)$ with $Y^{T^n} = H^n \bullet X$ for all $n \in \mathbb{N}$. By putting $T^0 := 0$ and $H := \sum_{n=1}^{\infty} H^n 1_{T^{n-1},T^n}$, we obtain $H \in L(X)$ and locally we have $Y = H \bullet X$. But then we also have $Y = H \bullet X$ and $Y \in \mathcal{I}(X)$ in general. $\square$

Definition 6. *Let V be a vector space of stochastic processes. We say V is* **stable** *if $X \in V$ implies $X^T \in V$.*

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In the literature, it is often required that a stable subspace is also closed. In such a case, we will speak of a stable, closed vector space.

Lemma 2. *Let $X \in \mathcal{M}^d_{\sigma,0}$. Then, $\mathcal{I}^1(X)$ is a stable and closed subspace of $\mathcal{H}^1$.*

Proof. Because of the linearity of the stochastic integral and since $\mathcal{H}^1$ is a vector space, $\mathcal{I}^1(X)$ is also a vector space. We have for $H \bullet X \in \mathcal{I}^1(X)$ and a stopping time T

$$(H \bullet X)^T = H1_{0,T} \bullet X,$$

where

$$0, T := \{(\omega, t) \in \Omega \times \mathbb{R}_+ ; 0 \leq t \leq T(\omega)\}.$$

Hence $(H \bullet X)^T \in \mathcal{I}(X)$. Since the quadratic variation is an increasing process, we further have

$$\left\|(H \bullet X)^T\right\|_{\mathcal{H}^1} \leq \|H \bullet X\|_{\mathcal{H}^1}$$

and thus $(H \bullet X)^T \in \mathcal{H}^1$. We conclude that $\mathcal{I}^1(X)$ is stable.

We still have to show the closedness. For $H \in L(X)$, we put

$$\|H\|_{\mathcal{I}^1(X)} := \mathbf{E}\left[\sqrt{[H \bullet X, H \bullet X]_\infty}\right].$$

We define $I^1(X)$ to be the quotient space of $\{H \in L(X) ; \|H\|_{\mathcal{I}^1(X)} < \infty\}$ by $\{H \in L(X) ; \|H\|_{\mathcal{I}^1(X)} = 0\}$. By proceeding analogously to the L^p theory,[1] it is easy to see that $I^1(X)$ is a Banach space. Now let $H^n \bullet X$ be a Cauchy sequence in $\mathcal{H}^1$. Then we have

$$\|H^n \bullet X\|_{\mathcal{H}^1_0} = \mathbf{E}\left[([H^n \bullet X, H^n \bullet X]_\infty)^{\frac{1}{2}}\right] = \|H^n\|_{\mathcal{I}^1(X)}.$$

Since $I^1(X)$ is closed, we deduce that $\mathcal{I}^1(X)$ is closed as well. □

Theorem 4. *Let $X \in \mathcal{M}^d_{\sigma,0}$. The following are equivalent:*

1. *The process X has the predictable representation property;*
2. *$\mathcal{M}^1_{\infty,0} \subset \mathcal{I}^1(X)$;*
3. *$\mathcal{H}^1_0 = \mathcal{I}^1(X)$;*
4. *$\mathcal{M}^1_{\mathrm{loc},0} \subset \mathcal{I}(X)$.*

Proof. (i) ⇒ (ii). This is trivial.
(ii) ⇒ (iii). Let $N \in \mathcal{H}^1_0$. By Theorem A2, $\mathcal{M}^\infty_0$ is dense in $\mathcal{H}^1_0$. Hence, there exists a sequence $(M^n)_{n\in\mathbb{N}} \subset \mathcal{M}^\infty_0$ that converges in $\mathcal{H}^1$ to N. By assumption, we have $M^n \in \mathcal{I}^1(X)$ for all n, and from Lemma 2, we know that $\mathcal{I}^1(X)$ is closed. Thus, $N \in \mathcal{I}^1(X)$.
(iii) ⇒ (iv). By Theorem A3, each local martingale is locally in $\mathcal{H}^1$. Hence, by assumption, each local martingale is locally in $\mathcal{I}^1(X)$ and, therefore, in $\mathcal{I}(X)$. Lemma 1 yields $\mathcal{M}^1_{\mathrm{loc},0} \subset \mathcal{I}(X)$.
(iv) ⇒ (i). Let $Y \in \mathcal{M}^1_{\sigma,0}$. By Theorem 1, there exists a one-dimensional martingale M and a process $H \in L(M)$ such that $Y = H \bullet M$. Since $\mathcal{M}^1_{\mathrm{loc},0} \subset \mathcal{I}(X)$, there also exists a process $K \in L(X)$ such that $M = K \bullet X$. Hence, we have

$$Y = H \bullet (K \bullet X) = (HK) \bullet X,$$

and $Y \in \mathcal{I}(X)$. □

[1] compare, for example, [36] Theorem 3.11

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The following lemma is often used in connection with Girsanov's theorem (see, for example, [35] Exercise III.21, [4] Lemma 15.2.1, [19] Proposition 3.8, [21] Lemma 18.16, [25] Lemma 8.9.3, [15] Lemma 12.11, [2] Lemma 5.2.11).

Lemma 3. *Suppose a semimartingale M, $\mathbf{Q} \ll \mathbf{P}$ and let $Z_\infty = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}$ be the Radon-Nikodym derivative with*

$$Z_t = \mathbf{E}_{\mathbf{P}}(Z_\infty \mid \mathcal{F}_t).$$

Then M is a $\mathbf{Q}$-(local) martingale if and only if MZ is a $\mathbf{P}$-(local) martingale.

We are going to investigate a possible generalization to sigma martingale measures.

Lemma 4. *Let X be a semimartingale, $\mathbf{Q} \ll \mathbf{P}$, $Z_\infty = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}$ the Radon-Nikodym derivative and $Z_t = \mathbf{E}_{\mathbf{P}}(Z_\infty \mid \mathcal{F}_t)$.*

1. *If X is a $\mathbf{Q}$ sigma martingale, then XZ is a $\mathbf{P}$-sigma martingale.*
2. *If $\mathbf{Q} \sim \mathbf{P}$ then X is a $\mathbf{Q}$-sigma martingale if and only if XZ is a $\mathbf{P}$-sigma martingale.*

Proof. First we note that by [35] Theorem IV.25 , it is irrelevant whether we calculate the integral under $\mathbf{Q}$ or $\mathbf{P}$. We will therefore only write $\bullet$ instead of $\bullet_{\mathbf{Q}}$ or $\bullet_{\mathbf{P}}$. Without loss of generality, we assume X to be one-dimensional.

For any Borel set D_n, we have

$$\begin{aligned} 1_{D_n} \bullet (XZ) &= 1_{D_n} \bullet (X_- \bullet Z + Z_- \bullet X + [X,Z]) \\ &= 1_{D_n} \bullet (X_- \bullet Z) + 1_{D_n} \bullet (Z_- \bullet X) + 1_{D_n} \bullet [X,Z] \\ &= (1_{D_n} X_-) \bullet Z + (1_{D_n} Z_-) \bullet X + [1_{D_n} \bullet X, Z] \\ &= (1_{D_n} X_-) \bullet Z + (1_{D_n} Z_-) \bullet X + (1_{D_n} \bullet X)Z \\ &\qquad - (1_{D_n} \bullet X)_- \bullet Z - Z_- \bullet (1_{D_n} \bullet X) \\ &= (1_{D_n} X_-) \bullet Z + (1_{D_n} Z_-) \bullet X + (1_{D_n} \bullet X)Z \\ &\qquad - (1_{D_n} \bullet X)_- \bullet Z - (Z_- 1_{D_n}) \bullet X \\ &= (1_{D_n} X_-) \bullet Z + (1_{D_n} \bullet X)Z - (1_{D_n} \bullet X)_- \bullet Z. \end{aligned} \tag{2}$$

Let X be a $\mathbf{Q}$-sigma martingale with localizing sequence $(D_n)_{n\in\mathbb{N}}$. By Lemma 3, $(1_{D_n} \bullet X)Z$ is a local $\mathbf{P}$-martingale. Furthermore, $(1_{D_n} X_-)$ and $(1_{D_n} \bullet X)_-$ are locally bounded. Since Z is a $\mathbf{P}$-martingale, the first and the third summand on the right-hand side of (2) are local $\mathbf{P}$-martingales . Hence, $1_{D_n} \bullet (XZ)$ is also a local $\mathbf{P}$-martingale and $(D_n)_{n\in\mathbb{N}}$ is a localizing sequence for XZ. Thus XZ is a $\mathbf{P}$-sigma martingale.

Now let XZ be a $\mathbf{P}$-sigma martingale and $\mathbf{Q} \sim \mathbf{P}$. Then we have $Z_\infty > 0$ and $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}$ is given by $\frac{1}{Z_\infty}$. Applying the already proven direction yields that $XZ\frac{1}{Z} = X$ is a $\mathbf{Q}$-sigma martingale. □

Lemma 5. *Let $X \in \mathcal{M}^d_\sigma$ and $H \in L(X)$, then there exists a process $M = (M^1, \dots, M^d)^\top$ with $M^i \in \mathcal{H}^1$ for all i and a stochastic process $K = (K^1, \dots, K^d)^\top$ with $K^i \in L(M^i)$ for all i with*

$$H \bullet X = \sum_{i=1}^{d} K^i \bullet M^i.$$

Furthermore, if there exists a stochastic process $Z \in \mathcal{M}^1_\sigma$ such that $Z \perp X$, then we also have $Z \perp M$.

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Proof. By Theorem 1, each X^i can be written as $X^i = \hat{K} \bullet N^i$ with a strictly positive $\hat{K} \in L(N^i)$ and $N^i \in \mathcal{H}^1$ for all i. We set $G^i := H^i \hat{K}$ and obtain $H \bullet X = G \bullet N$ for $G = (G^1, \ldots, G^d)^\top$. Now we put $K := \|G\| \vee 1$ and $J = \frac{G}{K}$ and J is a bounded process and thus $J^i \in L(X^i)$ for each i. We have

$$G \bullet N = K \bullet (J \bullet N) = K \bullet \left(\sum_{i=1}^{d} J^i \bullet N^i \right) = \sum_{i=1}^{d} K \bullet (J^i \bullet N^i).$$

By putting $M^i := J^i \bullet N^i$, we obtain the desired result.

Now let $Z \in \mathcal{M}^1_\sigma$ with $ZX \in \mathcal{M}^d_\sigma$. By partial integration, we get

$$ZX^i = Z(\hat{K} \bullet N^i) = Z_- \bullet (\hat{K} \bullet N^i) + (\hat{K} \bullet N^i)_- \bullet Z + [Z, \hat{K} \bullet N^i].$$

Hence, we obtain $\hat{K} \bullet [Z, N^i] \in \mathcal{M}^1_\sigma$. Since $\hat{K}$ is strictly positive, we also get

$$[Z, N^i] = \left(\frac{1}{\hat{K}} \hat{K} \right) \bullet [Z, N^i] = \frac{1}{\hat{K}} \bullet (\hat{K} \bullet [Z, N^i]) \in \mathcal{M}^1_\sigma.$$

Analogously, we obtain

$$ZM^i = Z_- \bullet M^i + M^i_- \bullet Z + [Z, M^i].$$

Since $[Z, M^i] = [Z, J \bullet N^i] = J \bullet [Z, N^i]$, all three summands on the right-hand side are sigma martingales. We conclude that ZM^i is a sigma martingale as well. □

For the proof of the Jacod-Yor Theorem for sigma martingales, we recall that $\mathcal{BMO}$ denotes the space of BMO martingales (see Appendix A).

Theorem 5 ([26] Theorem 4.2.3). *Let X be a normed space with dual space X', a subspace M and let $y_0 \in X$, such that*

$$d := \inf_{x \in M} \|y_0 - x\| > 0.$$

There exists a $f \in X'$, such that

1. $f(x) = 0$ *for all* $x \in M$,
2. $f(y_0) = 1$,
3. $\|f\|_{X'} = \frac{1}{d}$.

Furthermore, the famous Lemma of Ansel-Stricker ([1,6,39]) is required.

Theorem 6 (Ansel-Stricker). *A one-sided bounded sigma martingale X is a local martingale. If X is bounded from below (resp. above), it is also a supermartingale (resp. submartingale).*

We can now prove the Jacod-Yor Theorem for sigma martingales.

Proof of Theorem 2. **(i) ⇒ (ii).** Suppose $Z \in X^\perp$. By assumption, there exists an $H \in L(X)$ such that $Z = H \bullet X$. Now Lemma 5 yields a martingale $M = (M^1, \ldots, M^d)^\top$ and a process $K = (K^1, \ldots, K^d)^\top$ with $K^i \in L(M^i)$ such that

$$Z = K \bullet M \quad \text{and} \quad MZ \in \mathcal{M}^d_\sigma.$$

We define

$$N := [K \bullet M, K \bullet M] = \sum_{i=1}^{d} K^i \bullet [M^i, K \bullet M] = \sum_{i=1}^{d} K^i \bullet [M^i, Z],$$

where

$$N = \sum_{i=1}^{d} K^i \bullet (M^i Z - M^i_- \bullet Z - Z_- \bullet M^i).$$

Hence, N is a sigma martingale. However, as the quadratic variation of a semimartingale, N is also positive and increasing. By Theorem 6, N is a supermartingale, and thus $\mathbf{E}N_t \leq \mathbf{E}N_0$. Since N is also increasing, we conclude that N is constant, which implies that Z is constant as well.
(ii) ⇒ (iii). Let $\mathbf{Q} \in \mathbf{M}_\sigma(X)$ with $\mathbf{Q} \ll \mathbf{P}$. Define

$$Z_t := \mathbf{E}\left[\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \mid \mathcal{F}_t\right].$$

Since X is a $\mathbf{Q}$-sigma martingale, XZ is a $\mathbf{P}$-sigma martingale by Lemma 4. By assumption, Z is constant. Hence, we have $\mathbf{P} = \mathbf{Q}$.
(iii) ⇒ (iv). Let $\mathbf{Q}, \mathbf{R} \in \mathfrak{P}_\sigma(X)$ and $\lambda \in (0,1)$ such that $\mathbf{P} = \lambda\mathbf{Q} + (1-\lambda)\mathbf{R}$. Clearly, $\mathbf{Q} \ll \mathbf{P}$. By assumption, we have $\mathbf{Q} = \mathbf{P}$. Similarly, $\mathbf{R} = \mathbf{P}$. Thus, $\mathbf{P}$ is an extremal point in $\mathfrak{P}_\sigma(X)$.
(iv) ⇒ (i). By Theorem 4, it suffices to show that $\mathcal{I}^1(X) = \mathcal{H}^1_0$.

Suppose this is not the case. Then, by Lemma 2, $\mathcal{I}^1(X)$ is a stable closed subspace of $\mathcal{H}^1$, and there exists an element $Y_0 \in \mathcal{H}^1$, $Y_0 \notin \mathcal{I}^1(X)$, such that

$$\inf_{Y \in \mathcal{I}^1(X)} \|Y_0 - Y\|_{\mathcal{H}^1} > 0.$$

By Theorem 5, there exists a bounded linear functional $\varphi : \mathcal{H}^1_0 \to \mathbb{R}$ with $\varphi \mid_{\mathcal{I}^1(X)} = 0$ and $\varphi(Y_0) = 1$. By Theorem A5, φ can be represented as

$$\varphi(I) = \mathbf{E}[I_\infty M_\infty],$$

where $M \in \mathcal{BMO}_0$.

Thus, $\mathbf{E}[I_\infty M_\infty] = 0$ for all $I \in \mathcal{I}^1(X)$. Since $\mathcal{I}^1(X)$ is stable, we have

$$\mathbf{E}\left[I_\infty M^T_\infty\right] = 0 \quad \text{for all } I \in \mathcal{I}^1(X). \tag{3}$$

By Theorem A4, M is locally bounded. Thus, there exists a sequence of stopping times $(T^n)_{n\in\mathbb{N}}$ such that $M^{T^n} \in \mathcal{M}^\infty_0$. Define

$$Z^n_t := \mathbf{E}\left[1 + \frac{M^{T^n}_\infty}{2c_n} \mid \mathcal{F}_t\right], \quad Z^{-n}_t := \mathbf{E}\left[1 - \frac{M^{T^n}_\infty}{2c_n} \mid \mathcal{F}_t\right],$$

where c_n is a constant larger than $\sup|M^{T^n}|$.

Clearly, all Z^n and Z^{-n} are bounded martingales. Define probability measures $\mathbf{Q}^n, \mathbf{Q}^{-n}$ by $Z^n_\infty = \frac{\mathrm{d}\mathbf{Q}^n}{\mathrm{d}\mathbf{P}}$ and $Z^{-n}_\infty = \frac{\mathrm{d}\mathbf{Q}^{-n}}{\mathrm{d}\mathbf{P}}$.

Since $XZ^n = X + \frac{XM^{T^n}}{2c_n}$ and $X \in \mathcal{I}^1(X)$, we conclude with (3) that XZ^n is a $\mathbf{P}$-sigma martingale. Similarly, XZ^{-n} is a $\mathbf{P}$-sigma martingale. By Lemma 4, $\mathbf{Q}^n, \mathbf{Q}^{-n} \in \mathfrak{P}_\sigma(X)$.

Because $\mathbf{P} = \frac{1}{2}(\mathbf{Q}^n + \mathbf{Q}^{-n})$, we get, by assumption, $\mathbf{P} = \mathbf{Q}^n = \mathbf{Q}^{-n}$. Thus,

$$1 + \frac{M^{T^n}_\infty}{2c_n} = \frac{\mathrm{d}\mathbf{Q}^n}{\mathrm{d}\mathbf{P}} = 1,$$

which implies $M^{T^n}_\infty = 0$ for all n. Taking $n \to \infty$, we get $M \equiv 0$, and hence $\varphi \equiv 0$.

Therefore, no process $Y_0 \in \mathcal{H}^1$ can exist with $Y_0 \notin \mathcal{I}^1(X)$. Thus, $\mathcal{I}^1(X) = \mathcal{H}^1_0$ follows. □

Appendix A. Martingale Spaces

Definition A1. *For M a martingale and $p \in [1,\infty)$, write*

$$\|M\|_{\mathcal{H}^p} := \|M^*_\infty\|_p = \mathbf{E}\left[\sup_t |M_t|^p\right]^{\frac{1}{p}}.$$

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Here $\|\cdot\|_p$ denotes the norm in L^p. Then $\mathcal{H}^p$ is the space of martingales such that

$$\|M\|_{\mathcal{H}^p} < \infty.$$

Theorem A1. *Let $0 < p \leq \infty$. Then, a càdlàg process X is locally uniformly bounded in L^p if and only if ΔX is locally uniformly bounded in L^p.*

Theorem A2 ([4] Theorem 10.1.7)**.** *The space of bounded martingales (and hence $\mathcal{H}^p$ for any $p > 1$) is dense in $\mathcal{H}^1$.*

Theorem A3 ([15] Theorem 7.30)**.** *Let $M \in \mathcal{M}_{\mathrm{loc},0}$. Then M is locally in $\mathcal{H}^1$.*

The definition of the $\mathcal{BMO}$ given here is taken from [35]. A similar presentation can also be found in [4,15,29], which use equivalent definitions.

Definition A2. *Let $M \in \mathcal{H}^2$. We say $M \in \mathcal{BMO}$, if there exists a constant c such that for any stopping time T we have*

$$\mathbf{E}\Big((M_\infty - M_{T-})^2 \mid \mathcal{F}_T\Big) \leq c^2 \quad a.s. \tag{A1}$$

*The smallest such c is defined to be the $\mathcal{BMO}$-**Norm** of M, and it is written $\|M\|_{\mathcal{BMO}}$. If the constant c does not exist, or if M is not in $\mathcal{H}^2$, then we set $\|M\|_{\mathcal{BMO}} = \infty$.*

For the definition above, we use the convention $M_{0-} = 0$.

Theorem A4 ([4] Lemma A.8.7 and Lemma A.8.5)**.** *Each $\mathcal{BMO}$-martingale is locally bounded, and each bounded martingale is element of $\mathcal{BMO}$.*

Theorem A5 ([4] Theorem A.8.14)**.** *Any continuous linear function $\varphi : \mathcal{H}^1 \to \mathbb{R}$ can be written $\varphi_N(M) := \mathbf{E}(M_\infty N_\infty)$ for some $N \in \mathcal{BMO}$.*

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