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Article

Wild Character Varieties, Painlevé III_{D₆}, and Positivity Constraints Toward the Riemann Hypothesis

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Abstract

We investigate a potential route to the Riemann Hypothesis based on de Branges positivity and wild isomonodromic geometry, focusing on Painlevé III of type D_6 . Rather than proposing a proof, we reduce any such route to four explicit conditions (C1)–(C4), isolating a single analytic bottleneck: the existence of a global positivity normalization for the associated wild Riemann–Hilbert problem. Using the decorated character variety framework of Chekhov–Mazzocco–Rubtsov and the embedding $t = s(1 - s)$, we show that symmetry, gauge freedom, and growth constraints of the completed zeta function are all compatible with this setting. We further perform a quantitative density test based on the Weyl–Levinson law for canonical systems, showing that the zeta-induced spectral growth is highly selective yet not excluded by the Painlevé III_{D₆} Hamiltonian. The result is a falsifiable and discriminating framework that identifies where a de Branges-based realization of the Riemann Hypothesis must succeed or fail. We further analyze the analytic regularity condition (C4), show that the symmetry-compatibility condition (C3) is automatically satisfied for the natural embedding $t = s(1 - s)$, and isolate the global positivity condition (C2) as the decisive remaining analytic obstacle. In particular, we reduce (C2) to the absence of a single explicit Weyl–Herglotz obstruction for the associated canonical system, and develop falsifiable diagnostics, including a quantitative density test based on Weyl–Levinson asymptotics.

Keywords: Riemann Hypothesis; Painlevé equations; wild isomonodromy; decorated character varieties; de Branges spaces; canonical systems; Riemann–Hilbert problems; spectral density

1. Introduction

The Riemann Hypothesis (RH) remains one of the central open problems of mathematics. Over the past century, a wide variety of approaches have been proposed, ranging from analytic number theory to spectral theory, random matrix models, and quantum–mechanical interpretations. Despite deep partial results, no approach has yet produced a convincing structural explanation for the rigidity of the nontrivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

A recurrent theme in many proposed routes is the search for an underlying *positivity mechanism* capable of enforcing zero localization. Among the few general frameworks that provably yield such rigidity is the theory of de Branges spaces of entire functions, where Hermite–Biehler positivity implies that all zeros of an associated entire function lie on a prescribed symmetry line [1]. However, attempts to connect de Branges theory directly to the Riemann zeta function have historically encountered serious obstructions, often related to insufficient symmetry, lack of a natural canonical system, or incompatibility with the functional equation.

In parallel developments, the theory of isomonodromic deformations and Painlevé equations has revealed a rich interplay between nonlinear differential equations, Riemann–Hilbert (RH) problems, and canonical Hamiltonian systems [2,3]. In particular, Painlevé tau-functions arise naturally as determinants associated with monodromy-preserving deformations, and their analytic properties are governed by underlying RH data. Recent work has clarified the geometric structure of such systems

through the notion of *wild character varieties*, which encode Stokes data associated with irregular singularities and go beyond ordinary (tame) character varieties [4].

A further refinement of this perspective is provided by the theory of *decorated character varieties*, developed notably by Chekhov, Mazzocco, and Rubtsov. In this framework, cluster coordinates, cusp data, and positivity structures are incorporated directly into the moduli space, making it possible to distinguish between Painlevé equations that share the same Fricke polynomial but differ in their degree of wildness [5]. This distinction is essential when addressing positivity questions, since the number of cusps and Stokes sectors controls the available real forms and admissible positivity constraints.

The present paper explores the possibility that a de Branges-type positivity mechanism, if it is to play any role in RH, must arise in a genuinely *wild* isomonodromic setting. Rather than proposing a direct identification between the Riemann zeta function and a Painlevé tau-function, our goal is structural and eliminative: to identify precise geometric and analytic conditions that any de Branges-based route to RH would have to satisfy, and to test whether the simplest viable wild candidate, namely Painlevé III of type D_6 , can accommodate these conditions.

A key technical device in this analysis is the embedding

$$t = s(1 - s),$$

which maps the critical line $\Re(s) = \frac{1}{2}$ onto a positive real ray in the deformation parameter t . This embedding is compatible with the involution $s \mapsto 1 - \bar{s}$ and allows one to translate questions about zero localization in the s -plane into spectral properties of canonical systems parametrized by t . Within this framework, we reduce any potential de Branges-based realization of RH to four explicit conditions, denoted (C1)–(C4), isolating a single decisive analytic bottleneck: the existence of a global positivity normalization for the associated wild Riemann–Hilbert problem.

The main contribution of this work is not a proof of the Riemann Hypothesis, nor a claim that such a proof is imminent. Instead, we provide a coherent and falsifiable framework that: (i) identifies where a de Branges-type route to RH must reside if it exists, (ii) rules out large classes of spectral or geometric models by quantitative density constraints derived from canonical system theory [6,7], and (iii) reduces the problem to a small number of sharply formulated analytic questions. In particular, using Weyl–Levinson asymptotics for canonical systems, we show that the logarithmically corrected zero density induced by the zeta function is highly selective yet not immediately incompatible with the Hamiltonian structure of Painlevé III $_{D_6}$.

Structure of the paper.

Section 2 recalls the de Branges positivity mechanism and formulates the zero-localization principle that underlies the entire approach. Section 3 reviews the Hamiltonian, tau-function, and σ -form of the Painlevé III $_{D_6}$ equation, fixing conventions and highlighting the structures relevant for local exponents and large- t asymptotics. Section 4 introduces the decorated character variety associated with PIII $_{D_6}$ and explains why this equation is the minimal genuinely wild Painlevé system compatible with a positivity-based strategy. Section 5 defines a positive decorated slice and reduces the applicability of the de Branges mechanism to four explicit conditions (C1)–(C4); in particular, Section 5 establishes the geometric feasibility condition (C1), shows that the symmetry-compatibility condition (C3) is automatically satisfied for the natural embedding $t = s(1 - s)$, and provides a detailed analysis of the analytic regularity condition (C4). Section 6 formulates the Bridge Conjecture connecting the de Branges entire function to the completed Riemann zeta function and develops falsifiable diagnostics, including a quantitative density test based on Weyl–Levinson asymptotics. Section 7 summarizes the results from a neutral–eliminative perspective and situates the present approach relative to earlier arithmetic, operator-theoretic, and quantum-statistical formulations of the Riemann Hypothesis. Appendix A explores the global positivity condition (C2) for PIII $_{D_6}$, isolates the precise analytic obstruction to a de Branges realization, and Appendix B outlines a concrete attack plan for proving or refuting (C2).

2. A De Branges-Type Positivity Lemma

A central ingredient in the present work is a general positivity mechanism that enforces zero localization for entire functions. This mechanism originates in the theory of de Branges spaces and canonical systems and is one of the few known frameworks in which a symmetry alone is sufficient to force all zeros onto a prescribed line.

We briefly recall the relevant principle in a form adapted to the present context. Let $E(s)$ be an entire function satisfying the Hermite–Biehler condition [1]:

$$|E(s)| > |E(\bar{s})| \quad \text{for } \Im(s) > 0.$$

Associated with such a function is a canonical system whose Hamiltonian is positive almost everywhere, and whose Weyl–Titchmarsh function is Herglotz (Pick) in the upper half-plane [6,7].

The following lemma isolates the consequence that is relevant for the Riemann Hypothesis.

Lemma 1 (de Branges positivity and zero localization). *Let $E(s)$ be an entire function of finite order satisfying the involutive symmetry*

$$E(s) = e^{Q(s)} \overline{E(1 - \bar{s})},$$

where $Q(s)$ is a polynomial of degree at most two. Assume that $E(s)$ satisfies the Hermite–Biehler condition in the upper half-plane, equivalently that it arises from a positive canonical system. Then all zeros of $E(s)$ lie on the symmetry line $\Re(s) = \frac{1}{2}$.

Sketch of proof. This is a standard consequence of de Branges theory. The Hermite–Biehler condition implies that the real entire functions

$$A(s) = \frac{1}{2}(E(s) + E(1 - \bar{s})), \quad B(s) = \frac{1}{2i}(E(s) - E(1 - \bar{s}))$$

form a de Branges pair. Positivity of the associated canonical system ensures that the corresponding de Branges space has no nonreal zeros. The symmetry then forces all zeros onto the fixed line $\Re(s) = \frac{1}{2}$. Details can be found in [1,6]. $\square$

Remark 2. *The role of the polynomial $Q(s)$ is purely gauge-theoretic. Multiplication by $e^{Q(s)}$ does not affect the zero set and reflects the natural quadratic ambiguity present both in de Branges theory and in isomonodromic tau-functions.*

Remark 3. *Lemma 1 should be understood as a rigidity principle rather than as a number-theoretic statement. The problem addressed in the remainder of this paper is whether a function $E(s)$ satisfying its hypotheses can arise from a wild isomonodromic Riemann–Hilbert problem in a way that is compatible with the Riemann zeta function.*

3. Painlevé III_{D₆} Hamiltonian, Tau-Function, and σ -Form

We briefly recall the Hamiltonian formulation of the Painlevé III equation of type D_6 , together with the associated tau-function and σ -form. This material is standard, but we include it to fix conventions and to emphasize the structures relevant for positivity and spectral asymptotics.

3.1. Hamiltonian Formulation

The generic third Painlevé equation PIII(D_6) admits a Hamiltonian formulation $\dot{q} = \partial H / \partial p$, $\dot{p} = -\partial H / \partial q$ with an explicit Hamiltonian $H_{\text{PIII}_{D_6}}(q, p, t)$; see, e.g., Okamoto's space-of-initial-conditions approach [3] and an explicit Hamiltonian list in [8,9].

Let $(q(t), p(t))$ be canonical variables with Poisson bracket $\{q, p\} = 1$. The Painlevé III_{D_6} Hamiltonian can be written in the form

$$H_{\text{III}_{D_6}}(q, p, t) = \frac{1}{t} \left(2q^2 p^2 - ((1 + 2\theta_0)q - 2\kappa_0 t + 2\kappa_\infty t q^2)p + (\theta_0 + \theta_\infty)\kappa_\infty t q \right), \quad (1)$$

where θ_0, θ_∞ are formal monodromy exponents and κ_0, κ_∞ encode the irregular type at 0 and ∞ .

The corresponding Hamilton equations

$$\dot{q} = \frac{\partial H_{\text{III}_{D_6}}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{\text{III}_{D_6}}}{\partial q}$$

are equivalent to the Painlevé III_{D_6} differential equation. Different but equivalent Hamiltonian normalizations appear in the literature; the form (1) is adapted to the isomonodromic interpretation and to the discussion of tau-functions.

3.2. Tau-Function

Following Jimbo, Miwa, and Ueno, one associates to an isomonodromic deformation problem a tau-function whose logarithmic derivative reproduces the Hamiltonian [2]. For Painlevé III_{D_6} , the tau-function $\tau_{\text{III}_{D_6}}(t)$ is defined (up to a gauge factor) by

$$\frac{d}{dt} \log \tau_{\text{III}_{D_6}}(t) = H_{\text{III}_{D_6}}(q(t), p(t), t). \quad (2)$$

The tau-function is uniquely determined by the monodromy and Stokes data of the associated Riemann–Hilbert problem, up to multiplication by $\exp(at^2 + bt + c)$, reflecting the natural quadratic gauge freedom of isomonodromic theory.

Tau-functions for Painlevé equations play a central role in integrable systems and often admit interpretations as Fredholm determinants of integrable kernels in suitable regimes [10,11].

3.3. σ -Form

Tau-functions in the Jimbo–Miwa–Ueno sense [2] admit specialized representations and σ -forms for PIII; see [12,13].

It is often convenient to work with the σ -function introduced by Okamoto, defined by

$$\sigma(t) = t \frac{d}{dt} \log \tau_{\text{III}_{D_6}}(t). \quad (3)$$

In terms of $\sigma(t)$, Painlevé III_{D_6} can be written as a second-order, second-degree nonlinear differential equation, the σ -form. One convenient normalization is

$$(t\sigma'')^2 = (2\sigma' - \theta_0 - \theta_\infty)^2(\sigma - t\sigma') + 4\sigma'(\sigma' - \theta_0)(\sigma' - \theta_\infty), \quad (4)$$

where primes denote derivatives with respect to t .

This form makes explicit the dependence on the monodromy exponents and encodes the Hamiltonian structure governing both the local behavior at $t = 0$ and the large- t asymptotics relevant for the density considerations in later sections.

4. Chekhov's Decorated Character Variety for PIII_{D_6}

4.1. From Tame to Wild Isomonodromy: Limitations of PVI and PV^{deg}

A natural question is why the present analysis focuses on Painlevé III of type D_6 , rather than on more classical equations such as PVI or the degenerate Painlevé V equation PV^{deg} . The answer is structural rather than aesthetic.

The equation PVI is entirely tame: all singularities are regular, and the associated monodromy data live in ordinary (tame) character varieties. While PVI admits a rich geometric structure and plays a central role in the theory of isomonodromic deformations, it lacks the additional Stokes degrees of freedom required to support a global positivity mechanism compatible with de Branges theory [4]. In particular, the absence of irregular singularities prevents the emergence of the split real forms and cluster positivity structures needed for a canonical-system interpretation enforcing zero localization.

The equation $PV^{\deg}$ is closer in spirit and even shares the same Fricke polynomial as $PIII_{D_6}$. However, as shown by Chekhov, Mazzocco, and Rubtsov, the corresponding decorated character variety has fewer cusps and hence fewer independent positivity parameters [5]. Although $PV^{\deg}$ is already a wild equation in the sense of possessing irregular singularities, this reduced degree of wildness is insufficient to accommodate the split real forms and logarithmically corrected spectral asymptotics required by the zeta zero density.

Painlevé III_{D_6} is therefore the simplest genuinely wild Painlevé equation that simultaneously combines a nontrivial decorated character variety, cluster positivity, and enough asymptotic flexibility to pass all known structural tests imposed by the Riemann zeta function. In this sense, $PIII_{D_6}$ should be viewed as a minimal candidate rather than as an arbitrary choice.

We now summarize the ingredients of Sec. 5.4 (pp. 16–19) in [5] that are needed to construct an explicit positivity slice.

4.2. Coordinates and the Fricke-Type Cubic

Chekhov, Mazzocco, and Rubtsov obtain the $PIII_{D_6}$ decorated character variety by a confluence limit from PV. More precisely, starting from the shear coordinates (s_i, p_i) associated with PV, they consider the limit [5]

$$s_1 \mapsto s_1 + 2 \log \varepsilon, \quad p_2 \mapsto p_2 - 2 \log \varepsilon, \quad p_1 \mapsto p_1 - 2 \log \varepsilon, \quad \varepsilon \rightarrow 0.$$

Writing $\tilde{G}_i := e^{p_i/2}$ and

$$\tilde{G}_\infty := e^{s_1 + s_2 + s_3 + \frac{p_1 + p_2 + p_3}{2}},$$

the associated geodesic functions (x_1, x_2, x_3) satisfy a Fricke-type cubic relation, which coincides with equation (5.31) of [5]:

$$x_1 x_2 x_3 + x_1^2 + x_2^2 - (\tilde{G}_1 \tilde{G}_\infty + \tilde{G}_2 \tilde{G}_3) x_1 - (\tilde{G}_2 \tilde{G}_\infty + \tilde{G}_1 \tilde{G}_3) x_2 + \tilde{G}_1 \tilde{G}_2 \tilde{G}_3 \tilde{G}_\infty = 0. \quad (5)$$

This cubic polynomial encodes the tame part of the monodromy data; the essential difference between $PV^{\deg}$ and $PIII_{D_6}$ lies not in the polynomial itself, but in the additional decoration by cusp data.

4.3. Cluster/Lambda-Length Variables and Casimirs

After suitable flips of the fat graph, Chekhov et al. introduce arc (lambda-length) functions

$$\{a, b, c, d, e, f, g, h\},$$

which serve as cluster coordinates on the decorated character variety. In the transformed chart these variables become monomials, and the induced Poisson algebra is homogeneous [5, eq. (5.37)]. This cluster structure provides a natural notion of positivity that is absent in ordinary character varieties.

In addition, two Casimir elements are identified,

$$de, \quad hg, \quad (6)$$

which can be interpreted geometrically as the perimeters of two boundary components of the decorated surface. Fixing these Casimirs selects symplectic leaves of the Poisson structure and plays a key role in the construction of positive real slices compatible with canonical systems.

5. An Explicit Positivity Slice and an Involution Compatible with Lemma 1

5.1. Positive Decorated Slice

We define the *positive decorated slice* of the decorated character variety associated with PIII_{D_6} by requiring positivity of all cluster (lambda-length) coordinates [5,14]:

$$a, b, c, d, e, f, g, h \in \mathbb{R}_{>0}. \quad (7)$$

In addition, we fix positive values of the Casimir elements (6),

$$de = \Lambda_0 > 0, \quad hg = \Lambda_\infty > 0. \quad (8)$$

On this slice, the geodesic functions (x_1, x_2, x_3) are real and satisfy the Fricke-type cubic constraint (5). This positivity condition is intrinsic to the decorated character variety and has no analogue in the ordinary (undecorated) character variety.

5.2. Spectral Embeddings $t = t(s)$ Compatible with $s \mapsto 1 - \bar{s}$

Lemma 1 is formulated for the anti-holomorphic involution $s \mapsto 1 - \bar{s}$. To connect this involution with the isomonodromic deformation parameter, we choose an embedding $t = t(s)$ satisfying

$$t(1 - \bar{s}) = \overline{t(s)}. \quad (9)$$

This condition guarantees that the involution on the spectral variable s induces complex conjugation on the t -plane, and hence preserves real slices of the isomonodromic system.

Two embeddings are particularly natural.

(E1) **Quadratic (polynomial) embedding:**

$$t(s) = s(1 - s). \quad (10)$$

Then (9) holds identically since $(1 - \bar{s})\bar{s} = \overline{s(1 - s)}$. Moreover, the critical line $\Re(s) = \frac{1}{2}$ is mapped to the positive real ray

$$t\left(\frac{1}{2} + iy\right) = \frac{1}{4} + y^2 \in \left[\frac{1}{4}, \infty\right).$$

Thus any positivity structure formulated for $t \in [\frac{1}{4}, \infty)$ is exactly aligned with the zero-localization mechanism of Lemma 1.

(E2) **Centered-square embedding:**

$$t(s) = \left(s - \frac{1}{2}\right)^2. \quad (11)$$

Again (9) holds, and the critical line maps to the nonnegative real ray $t(\frac{1}{2} + iy) = y^2 \geq 0$. This embedding may be useful if the relevant Riemann–Hilbert positivity is available on the nonnegative real axis in t .

Remark 4. Embeddings based on exponentials (for example $t = e^{2\pi is}$) typically send $s \mapsto 1 - \bar{s}$ to $t \mapsto 1/\bar{t}$ rather than to complex conjugation. Such embeddings may still be useful, but require a different real-form formulation. The polynomial embeddings (10)–(11) are the simplest choices compatible with (9).

5.3. Lemma 1 as an Explicit Feasibility and Positivity Check

Within the present framework, the content of Lemma 1 is reduced to the following explicit conditions:

- (C1) **(Geometric feasibility)** The positive decorated slice (7)–(8) corresponds to a real form of the wild monodromy and Stokes data underlying PIII_{D_6} [4].
- (C2) **(RH positivity)** In an appropriate Riemann–Hilbert formulation of PIII_{D_6} , the wild data coming from (7)–(8) yield jump matrices that are unitary or positive on the relevant contour after normalization. This implies a Pick (Herglotz) property for the associated transfer function and hence a Hermite–Biehler structure for the corresponding entire function [1,6].
- (C3) **(Embedding compatibility)** Choose $t = t(s)$ as in (10) or (11) so that the involution $s \mapsto 1 - \bar{s}$ induces $t \mapsto \bar{t}$, preserving the positivity slice.
- (C4) **(Analytic regularity)** The composed function $\tau(s) := \tau_{\text{PIII}_{D_6}}(t(s))$ is entire of finite order after removal of the standard exponential gauge factor.

Proposition 5 (Conditional “Lemma 1” for PIII_{D_6}). *If conditions (C1)–(C4) hold for some embedding $t = t(s)$, then the corresponding entire function satisfies the hypotheses of Lemma 1. In particular, all its zeros lie on the line $\Re(s) = \frac{1}{2}$.*

5.4. Checking (C4) Under the Embedding $t = s(1 - s)$

We now fix the embedding

$$t = t(s) = s(1 - s),$$

which satisfies $t(1 - \bar{s}) = \overline{t(s)}$ and maps the critical line $\Re(s) = \frac{1}{2}$ onto the positive real ray $[1/4, \infty)$. Under this choice, condition (C3) is automatic.

The remaining analytic issue is (C4): the entireness and finite order of the composed function $\tau(s) := \tau_{\text{PIII}_{D_6}}(t(s))$ (up to the allowed exponential gauge).

Lemma 6 (Reduction of (C4) for PIII_{D_6} under $t = s(1 - s)$). *Let $\tau_{\text{PIII}_{D_6}}(t)$ be a Jimbo–Miwa–Ueno tau-function for PIII_{D_6} [2]. Assume its local behavior at $t \rightarrow 0$ is of the form*

$$\tau_{\text{PIII}_{D_6}}(t) = t^\alpha F(t), \quad F(0) \neq 0,$$

for some exponent $\alpha = \alpha(\text{monodromy/Stokes data})$.

1. If $\alpha \in \mathbb{Z}_{\geq 0}$, then $\tau(s) = \tau_{\text{PIII}_{D_6}}(s(1 - s))$ extends holomorphically across $s = 0$ and $s = 1$, and hence is entire (after removing the standard exponential gauge factor). In this case, condition (C4) is satisfied.
2. If $\alpha \notin \mathbb{Z}$, then $\tau_{\text{PIII}_{D_6}}(t)$ is multivalued around $t = 0$, and $\tau(s)$ necessarily has branch points at $s = 0, 1$. In this case, $\tau_{\text{PIII}_{D_6}}(t(s))$ cannot serve directly as the entire function required by Lemma 1.

Remark 7 (Interpretation). *Lemma 6 shows that, once the embedding $t = s(1 - s)$ is fixed, the only obstruction to (C4) is the local exponent α at $t = 0$. This exponent is determined entirely by the wild monodromy and Stokes data of the associated isomonodromic system and is therefore subject to geometric control on the decorated character variety.*

Remark 8 (Canonical-system alternative). *If $\alpha \notin \mathbb{Z}$ on the positive decorated slice, this does not invalidate the positivity route of Lemma 1. In that case, the correct object to consider is not the raw tau-function $\tau_{\text{PIII}_{D_6}}(t(s))$, but the entire Hermite–Biehler function $E(s)$ associated with the underlying Riemann–Hilbert problem or canonical system [1,6]. In this formulation one has*

$$\tau_{\text{PIII}_{D_6}}(t(s)) = e^{Q(s)} E(s),$$

with an explicit quadratic gauge $Q(s)$. Condition (C4) is then satisfied by construction for $E(s)$.

5.5. Local Exponent at $t = 0$ and Its Monodromy Meaning

The obstruction isolated in Lemma 6 is the local exponent α of the PIII_{D_6} tau-function at $t = 0$. In isomonodromic theory, such exponents are determined by monodromy and Stokes invariants (connection data) rather than by the deformation parameter t itself [10].

Proposition 9 (Structure of the $t \rightarrow 0$ exponent). *Fix a Riemann–Hilbert formulation for PIII_{D_6} and a normalization of the Jimbo–Miwa–Ueno tau-function. Then the leading exponent α in*

$$\tau_{\text{PIII}_{D_6}}(t) = t^\alpha F(t), \quad F(0) \neq 0,$$

is a function of:

1. the formal monodromy exponents $(\theta_0, \theta_\infty)$, and
2. a single additional connection invariant σ determined by a trace-like wild monodromy quantity, equivalently by a suitable geodesic or arc function on the decorated character variety.

Remark 10. In tame character varieties, the parameter σ is commonly defined via a relation of the form $\text{tr}(M) = 2 \cos(\pi\sigma)$ for an appropriate product of monodromy matrices. In the wild and decorated setting relevant here, the role of $\text{tr}(M)$ is played by a decorated geodesic function built from Chekhov’s arc variables, so that σ is read off from the corresponding real value on the positive decorated slice [5].

Proposition 11 (Local exponent for PIII_{D_6} tau-functions). *Fix a standard Jimbo–Miwa–Ueno normalization of the PIII_{D_6} tau-function $\tau_{\text{PIII}_{D_6}}(t)$ and consider its local behavior as $t \rightarrow 0$,*

$$\tau_{\text{PIII}_{D_6}}(t) = t^\alpha F(t), \quad F(0) \neq 0.$$

Then the exponent α depends only on the formal monodromy exponents $(\theta_0, \theta_\infty)$ and on a single additional connection invariant σ , and is expected to take the quadratic form

$$\alpha = \frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_\infty^2), \quad (12)$$

up to an additive constant arising from the conventional quadratic gauge freedom of the tau-function.

Quadratic dependence of the leading tau-function exponent on monodromy data is a standard feature of isomonodromic theory and follows from the structure of the Jimbo–Miwa–Ueno 1-form and its local residue calculus [2,3]. For Painlevé equations of types VI, V, and III, explicit local asymptotic formulas of this form are derived or implicitly contained in classical analyses and in more recent treatments of Painlevé tau-functions [10].

Remark 12. Formulas of the type (12) are well known for PVI and PV and arise from explicit Riemann–Hilbert asymptotics or from the Hamiltonian structure of the corresponding isomonodromic systems [2,3]. For PIII_{D_6} , the same quadratic dependence is expected and is consistent with the general theory of monodromy dependence of tau-functions, although a complete derivation in the precise normalization adopted here would require a detailed local Riemann–Hilbert analysis [10].

5.6. Condition (C1): From Chekhov Positivity to a Real Form of Wild Monodromy Data

We now address (C1), the geometric feasibility condition: the positive decorated slice (7)–(6) should correspond to a real form of the wild monodromy and Stokes data of the PIII_{D_6} irregular connection.

Fix a standard rank-2 Riemann–Hilbert formulation for PIII_{D_6} with two Stokes factors at each irregular singularity (at 0 and ∞), and formal monodromy exponents $\theta_0, \theta_\infty \in \mathbb{C}$. Let $\mathcal{M}_{\text{wild}}$ denote the resulting wild monodromy manifold (Stokes factors, formal monodromy, and connection matrix, modulo conjugation) [4].

On the other hand, let $\mathcal{X}_{\text{PIII}_{D_6}}^{\text{dec}}$ be the decorated character variety of Chekhov–Mazzocco–Rubtsov in the cluster/lambda-length chart $(a, \dots, h)$ with Casimirs de and hg [5].

Proposition 13 (Geometric feasibility of the positive decorated slice (C1)). *Assume that the irregular Riemann–Hilbert correspondence identifies a Zariski-open subset of $\mathcal{X}_{\text{III}_{D_6}}^{\text{dec}}$ with a Zariski-open subset of $\mathcal{M}_{\text{wild}}$ via a birational map Φ .*

Then the positive decorated slice

$$a, \dots, h \in \mathbb{R}_{>0}, \quad de = \Lambda_0 > 0, \quad hg = \Lambda_\infty > 0$$

defines a natural real locus in $\mathcal{X}_{\text{III}_{D_6}}^{\text{dec}}$ which is preserved by the anti-holomorphic involution of complex conjugation. Moreover, under Φ this real locus maps to a split real form of the wild monodromy data: after conjugation, one may choose a normalization in which

1. $\theta_0, \theta_\infty \in \mathbb{R}$ (so that the formal monodromy lies in a real form),
2. the Stokes multipliers are real (and, on the positive component, sign-constrained),
3. the connection matrix lies in the corresponding real form (for example $\text{SL}(2, \mathbb{R})$ or $\text{SU}(1, 1)$).

In particular, condition (C1) holds on this slice.

Remark 14. *The key point is that positivity is a property of the decorated character variety. The cusp data (Stokes sectors) supply precisely the degrees of freedom required for real and positive structures. Compact real forms such as $\text{SU}(2)$ are typically incompatible with nontrivial Stokes data in rank 2, whereas split real forms are naturally compatible with cluster positivity.*

6. The ζ – E Bridge: A Precise Conjecture and Falsifiable Diagnostics

A natural criticism of the preceding analysis is that it does not establish any direct connection between the Painlevé III_{D_6} tau-function (or the associated wild Riemann–Hilbert problem) and the Riemann zeta function $\zeta(s)$. This section addresses that gap by formulating a precise bridge conjecture and by outlining diagnostics that render the proposed route testable and falsifiable.

6.1. From τ to a de Branges E -Function

Lemma 1 applies not to an arbitrary tau-function, but to a Hermite–Biehler entire function $E(s)$ arising from a positive canonical system, equivalently from a Riemann–Hilbert problem with positive jumps [1,6]. As discussed above, in wild isomonodromic settings the raw tau-function $\tau_{\text{III}_{D_6}}(t)$ need not itself be entire in the spectral variable s , even after the embedding $t = s(1 - s)$. The correct object for zero localization is therefore the de Branges E -function extracted from the same Riemann–Hilbert data.

We thus distinguish conceptually between:

- the *isomonodromic tau-function* $\tau_{\text{III}_{D_6}}(t)$, defined by the Jimbo–Miwa–Ueno 1-form [2], and
- the *canonical entire function* $E(s)$ obtained from the associated positive canonical system once condition (C2) holds [1,6].

6.2. Bridge Conjecture

We formulate the missing link as a concrete conjecture.

Conjecture 15 (Bridge conjecture ζ – E). *Assume that conditions (C1)–(C4) hold for the PIII_{D_6} Riemann–Hilbert problem on the positive decorated slice, with the embedding $t = s(1 - s)$. Let $E(s)$ be the associated de Branges (Hermite–Biehler) entire function.*

Then there exists an explicit entire function $Q(s)$ of degree at most two such that

$$\tilde{\zeta}(s) = e^{Q(s)} E(s),$$

where $\tilde{\zeta}(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function [15].

This conjecture is deliberately stated at the level of entire functions, not at the level of differential equations or tau-functions. It isolates the exact point at which number-theoretic input must enter the framework.

6.3. Falsifiable Diagnostics

Conjecture 15 admits concrete diagnostic tests that do not presuppose its truth.

Diagnostic 1: reverse engineering from ζ -zeros.

Let $s_n = \frac{1}{2} + i\gamma_n$ be the nontrivial zeros of $\zeta(s)$, and set

$$t_n = s_n(1 - s_n) = \frac{1}{4} + \gamma_n^2.$$

If $E(s)$ arises from the PIII_{D_6} positive decorated slice, then its zeros must lie on $\Re(s) = \frac{1}{2}$, and hence the corresponding spectral data in the t -variable must be compatible with the sequence $\{t_n\}$. Strong incompatibility (for example with asymptotic spacing or monotonicity constraints imposed by the PIII_{D_6} σ -equation) would rule out the present route.

Diagnostic 2: growth and order.

The de Branges function $E(s)$ extracted from a positive canonical system has controlled growth determined by the underlying Hamiltonian [1,6]. Comparing this growth with the known order and type of $\zeta(s)$ provides a stringent necessary condition for Conjecture 15.

Diagnostic 3: intermediate L -functions.

A more modest but decisive test is to ask whether any known L -function (for example a Dirichlet L -function or an automorphic L -function) can be realized as $e^{Q(s)}E(s)$ for a wild isomonodromic E -function satisfying (C1)–(C4) [16]. Success in such a case would strongly support the plausibility of the bridge conjecture, while systematic failure would indicate a fundamental obstruction.

6.4. Interpretation

Conjecture 15 is not assumed in the preceding sections. Rather, the analysis above shows that if the Riemann Hypothesis admits a realization via de Branges positivity and wild isomonodromic geometry, then it must pass through a bridge of the form stated here. In this sense, the present work does not merely propose a route to the Riemann Hypothesis, but also sharply delineates where such a route could fail.

6.5. Heuristic Support for the Bridge Conjecture

The Bridge Conjecture 15 is deliberately stated as a falsifiable hypothesis rather than as a theorem. In this subsection we provide heuristic support for its plausibility by examining symmetry, gauge freedom, and growth, without assuming any identification between Painlevé tau-functions and $\zeta(s)$.

(i) Symmetry compatibility.

The completed Riemann zeta function $\zeta(s)$ is characterized, among entire functions of finite order, by the involutive symmetry

$$\zeta(s) = \zeta(1 - s),$$

together with reality on the critical line [15]. In the present framework, the involution

$$s \mapsto 1 - \bar{s}$$

is built into the geometry from the outset. Indeed, the embedding $t = s(1 - s)$ is invariant under this involution, and the positive decorated slice of the PIII_{D_6} character variety is preserved by complex

conjugation [5]. Consequently, there is no symmetry obstruction to identifying $\zeta(s)$ with a de Branges E -function extracted from the same wild Riemann–Hilbert data.

(ii) Quadratic gauge freedom.

Allowing the factor $e^{Q(s)}$ in Conjecture 15, with Q of degree at most two, is not an ad hoc fitting freedom. Such a quadratic ambiguity appears simultaneously in de Branges theory [1], in isomonodromic theory [2], and in the normalization of the completed zeta function $\zeta(s)$ [15]. Thus the form of $Q(s)$ allowed in Conjecture 15 is both minimal and canonical.

(iii) Growth and order (Diagnostic 2).

A necessary condition for Conjecture 15 is compatibility of growth. The function $\zeta(s)$ is entire of order 1 and finite type [15]. On the isomonodromic side, the growth of the de Branges function $E(s)$ is controlled by the Hamiltonian of the underlying canonical system [6]. Known asymptotics of Painlevé III $_{D_6}$ tau-functions as $t \rightarrow +\infty$ indicate at most exponential growth in t [10]. Under the embedding $t = s(1-s) \sim -s^2$ for large $|s|$, this yields growth of the form $\exp(C|s|^2)$ at worst, which can be absorbed into the quadratic gauge $e^{Q(s)}$. After removing this gauge, the remaining entire function has order at most 1, in agreement with the growth of $\zeta(s)$. Thus, at the level of order and type, no immediate obstruction to Conjecture 15 is visible.

(iv) Why PIII $_{D_6}$.

Among Painlevé equations, PIII $_{D_6}$ is the simplest genuinely wild case admitting a decorated character variety with sufficient cusp structure to support positivity [4,5]. Equations with fewer cusps (such as PV^{deg}) do not provide enough degrees of freedom to realize split real forms compatible with de Branges positivity, while more complicated wild systems introduce unnecessary complexity. If a de Branges-based geometric realization of $\zeta(s)$ exists at all, PIII $_{D_6}$ is therefore a natural minimal candidate.

(v) Interpretation.

The considerations above do not establish Conjecture 15, but they show that it is compatible with all known structural constraints, namely symmetry, gauge freedom, and growth. In particular, there is no obvious obstruction at the level of asymptotics or functional equations. The remaining content of the conjecture lies in the global analytic positivity condition (C2), whose validity or failure for PIII $_{D_6}$ will ultimately decide the fate of the present route to the Riemann Hypothesis.

6.6. Partial Implementation of Diagnostic 1: Zero Monotonicity and Spacing

We now implement a partial version of Diagnostic 1 that does not require solving the Painlevé III $_{D_6}$ equation explicitly. The aim is to test whether the qualitative properties of the zero set induced by the Riemann zeta function are compatible with those expected from a de Branges canonical system arising from the positive decorated slice.

(i) Zeta zeros in the t -variable.

Let $s_n = \frac{1}{2} + i\gamma_n$ denote the nontrivial zeros of $\zeta(s)$, ordered by $0 < \gamma_1 < \gamma_2 < \dots$. Under the embedding

$$t = s(1-s),$$

these zeros are mapped to the real sequence

$$t_n = \frac{1}{4} + \gamma_n^2.$$

This sequence is strictly increasing and unbounded. Moreover, its asymptotic density is governed by the classical estimate for γ_n [15, Ch. 9]:

$$\gamma_n \sim \frac{2\pi n}{\log n},$$

which implies

$$t_n \sim \gamma_n^2 \sim \frac{4\pi^2 n^2}{(\log n)^2}.$$

In particular, the spacing $t_{n+1} - t_n$ grows slowly but monotonically with n .

(ii) Zeros of de Branges entire functions.

Let $E(s)$ be a Hermite–Biehler entire function arising from a positive canonical system. It is a general consequence of de Branges theory that:

- all zeros of the associated real entire functions lie on the symmetry line;
- the zeros are simple and form a strictly monotone sequence in the spectral parameter;
- the asymptotic spacing of the zeros is controlled by the Hamiltonian of the canonical system and obeys Weyl-type growth laws.

We refer to [1,6] for the canonical-system formulation and its spectral consequences.

(iii) Compatibility test.

Comparing (i) and (ii), we observe that the sequence $\{t_n\}$ induced by the nontrivial zeros of $\zeta(s)$ has the following qualitative properties:

- strict monotonicity;
- unbounded growth;
- asymptotically increasing spacing.

These properties are consistent with the general behavior of zero sequences of Hermite–Biehler entire functions arising from positive canonical systems. In particular, there is no immediate contradiction at the level of monotonicity or asymptotic spacing between the sequence $\{t_n\}$ and what could arise from a PIII_{D_6} -based canonical system.

(iv) Interpretation.

This comparison does not establish that the zero sequence $\{t_n\}$ is realized by any specific PIII_{D_6} system, nor does it imply Conjecture 15. However, it shows that the qualitative spectral properties forced by the Riemann zeta zeros are not incompatible with those of a de Branges spectrum arising from the positive decorated slice. Consequently, Diagnostic 1 does not rule out the present route at the level of monotonicity and spacing.

(v) Quantitative density test.

Beyond monotonicity and spacing, de Branges theory imposes quantitative constraints on the asymptotic zero density. For a Hermite–Biehler entire function $E(s)$ of order 1 associated with a positive canonical system, the zero counting function

$$N_E(R) := \#\{s : |s| \leq R, E(s) = 0\}$$

satisfies a Weyl–Levinson type asymptotic law of the form [1,6,7]

$$N_E(R) \sim \frac{1}{\pi} \int_0^R \sqrt{H(u)} du, \quad R \rightarrow \infty,$$

where $H(u)$ is the Hamiltonian density of the canonical system. In particular, the growth of $N_E(R)$ is governed by the large- u behavior of $H(u)$ and cannot be arbitrary.

For the Riemann zeta function, the nontrivial zeros $s_n = \frac{1}{2} + i\gamma_n$ satisfy the classical asymptotic law [15][Ch. 9]

$$\gamma_n \sim \frac{2\pi n}{\log n}, \quad n \rightarrow \infty.$$

Under the embedding $t = s(1-s)$, this yields

$$t_n = \frac{1}{4} + \gamma_n^2 \sim \frac{4\pi^2 n^2}{(\log n)^2}.$$

Using the Riemann–von Mangoldt formula for the number of nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ with $0 < \gamma \leq T$,

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T),$$

and the embedding $t = \frac{1}{4} + \gamma^2$ (so that $T \sim \sqrt{t}$), we obtain the induced counting function in the t -variable:

$$N_\zeta(t) = \frac{\sqrt{t}}{2\pi} \log\left(\frac{\sqrt{t}}{2\pi}\right) - \frac{\sqrt{t}}{2\pi} + O(\log t) = \frac{\sqrt{t}}{4\pi} \log t + O(\sqrt{t}).$$

In particular, the zeta-induced spectrum exhibits a logarithmically corrected Weyl law.

This asymptotic is neither purely polynomial (as would arise from a Hamiltonian with regular power-law growth) nor purely logarithmic. Instead, it exhibits a *logarithmically corrected Weyl law*, lying at the boundary between standard Weyl behavior and slower spectral growth. Such behavior is highly restrictive and is not reproduced by generic Sturm–Liouville operators or random-matrix-inspired models.

Consequently, any de Branges realization of $\xi(s)$ must arise from a canonical system whose Hamiltonian exhibits slow logarithmic variation at infinity. The Painlevé III_{D₆} Hamiltonian is one of the few integrable systems known to admit such logarithmic corrections through its asymptotic structure. Therefore, while this comparison does not establish Conjecture 15, it shows that the zeta-induced density constraint is compatible with, and in fact highly selective for, the class of canonical systems arising from PIII_{D₆}.

7. Outlook

The analysis developed in this paper reframes a potential route toward the Riemann Hypothesis in terms of wild isomonodromic geometry, decorated character varieties, and de Branges-type positivity. Starting from Lemma 1, we have progressively reduced the problem to a small number of explicit and conceptually distinct conditions.

By fixing the embedding $t = s(1-s)$, condition (C3) becomes automatic, and condition (C4) is reduced either to a local exponent constraint at $t = 0$ or, alternatively, to the construction of a canonical-system entire function extracted from the same Riemann–Hilbert data. Using the decorated character variety framework of Chekhov–Mazzocco–Rubtsov, we have argued that condition (C1) is naturally satisfied on the positive decorated slice, which corresponds to a split real form of the wild monodromy and Stokes data for PIII_{D₆} [4,5].

The decisive remaining issue is therefore condition (C2): whether the positive decorated slice yields, after a suitable global normalization, a Riemann–Hilbert problem with positive jumps and hence a Herglotz (Pick) transfer function. The exploratory analysis in Appendix A shows that no local or algebraic obstruction to this positivity is visible; the difficulty is global and analytic, involving the existence of a normalization valid on the full contour.

This places the present approach at a clear decision point. If condition (C2) can be established for PIII_{D₆}, then the de Branges mechanism applies and Lemma 1 enforces zero localization on $\Re(s) = \frac{1}{2}$. If, on the contrary, (C2) fails globally, then the present route is ruled out for PIII_{D₆}, and one is led either to seek a different wild isomonodromic system with richer positivity properties or to abandon the de Branges paradigm in favor of a different rigidity mechanism.

In either case, the framework developed here makes explicit where the Riemann Hypothesis would have to reside within wild isomonodromic geometry, and where such an approach may ultimately succeed or fail.

By contrast, many proposed spectral or geometric approaches to the Riemann Hypothesis, including those based on polynomial Hamiltonians or random-matrix-type growth, fail to reproduce the logarithmically corrected Weyl law required by the zeta zero counting function [6,15]. Such approaches are therefore excluded already at the level of spectral density. The present framework consequently narrows the search for a de Branges-based realization of the Riemann Hypothesis to a very small and explicit class of wild isomonodromic systems.

Central open problem (positivity) and logical status.

The central open problem isolated by this work is condition (C2): the existence of a *global* normalization of the PIII_{D_6} wild Riemann–Hilbert problem on the positive decorated slice yielding a Herglotz (Pick) transfer function, equivalently a positive canonical system in the sense of de Branges. This is a genuinely global analytic positivity problem: local positivity of individual Stokes factors is not sufficient, since the normalization must be valid on the full contour and compatible with the wild monodromy constraints. If (C2) holds, then the de Branges mechanism produces an explicit Hermite–Biehler function $E(s)$ whose zeros lie on $\Re(s) = \frac{1}{2}$; the remaining step toward the Riemann Hypothesis is then the Bridge Conjecture $\zeta(s) = e^{Q(s)} E(s)$. If (C2) fails globally for PIII_{D_6} , the present route excludes this wild system as a de Branges candidate and forces either a search for a lower-dimensional positive sublocus or a transition to a different wild isomonodromic system with richer positivity degrees of freedom.

A key new structural result of this work is Proposition A17, which shows that condition (C2) is equivalent to the absence of a single, explicit analytic obstruction, namely the failure of the associated Weyl–Titchmarsh function to be Herglotz on the upper half-plane. This reduction eliminates hidden or diffuse sources of failure and places the positivity problem squarely within standard operator-theoretic frameworks. Consequently, any proof or disproof of (C2) must manifest through a concrete, verifiable mechanism.

Why the present approach differs from earlier attempts.

The author’s earlier work on the Riemann Hypothesis via Chebyshev renormalization, Chebyshev bias inequalities, and quantum statistical mechanics encountered fundamental obstructions at the level of global analysis: in the arithmetic setting, the oscillatory error terms could not be fully absorbed into a single positive structure, while in the quantum-statistical setting the low-temperature regime required for RH remained analytically inaccessible. The present wild isomonodromic approach differs in a crucial respect: the positivity condition (C2) is formulated on a concrete geometric object, namely the positive decorated slice of a wild character variety, and can be analyzed independently of solving the full nonlinear Painlevé equation. Moreover, the attack plan of Appendix A reduces (C2) to standard operator-theoretic criteria. If (C2) fails, it fails for a precise, verifiable reason; if it succeeds, the remaining Bridge Conjecture becomes a number-theoretic problem independent of the geometric construction. This makes the present route more falsifiable and more tractable than previous approaches.

Comparison to Previous Work

Table 1 illustrates conceptual correspondence between four positivity-based approaches to the Riemann Hypothesis that we worked on over time.

Table 1. Conceptual correspondence between four positivity-based approaches to the Riemann Hypothesis developed over time. In each case, the oscillatory contribution of the nontrivial zeros of $\zeta(s)$ (or of Dirichlet L -functions) is first absorbed into a rigid global structure, and the hypothesis is reformulated as a positivity or monotonicity condition on that structure.

Aspect	PCF/Chebyshev	GRH via Chebyshev Bias	Quantum Statistical Mechanics	Wild Isomonodromy/de Branges
Primary object	Prime counting function $\pi(x)$	Prime counting in progressions $\pi(x;q,a)$	KMS states of a C^* -dynamical system	Zeros of an entire function
Raw oscillatory data	$\sum_{\rho} x^{\rho} / \rho$	$\sum_{\rho_{\chi}} x^{\rho_{\chi}} / \rho_{\chi}$	Spectral phases of the Hamiltonian	Monodromy and Stokes data of a RHP
Renormalized variable or structure	Chebyshev function $\psi(x)$	Generalized Chebyshev function $\psi(x;q,a)$	Partition function / Gibbs expectation	Canonical Hamiltonian or Weyl function
Rigid global framework	Arithmetic renormalization	Character-indexed arithmetic structure	Quantum statistical mechanics (Bost–Connes)	Wild Riemann–Hilbert problem / canonical system
Positivity or rigidity mechanism	Monotonicity / Robin-type criteria	Inequalities equivalent to GRH	Positivity of KMS states	de Branges (Hermite–Biehler) positivity
Hypothesis reformulated as	RH	GRH (mod q)	RH	RH (via Bridge Conjecture)
Main obstruction or bottleneck	Oscillatory error terms	Character dependence	Low-temperature regime	Global positivity condition (C2)

Relation to prime counting and Chebyshev renormalization.

At a conceptual level, the strategy developed here resonates with earlier work on the prime counting function based on the Chebyshev function $\psi(x)$. In that setting, the oscillatory contribution of the nontrivial zeros of $\zeta(s)$ is not treated perturbatively, but is absorbed into a renormalized variable $\psi(x)$ before positivity or monotonicity arguments are applied. The present approach follows an analogous philosophy at a deeper analytic–geometric level: rather than controlling zeta zeros individually, their collective influence is encoded into the monodromy and Stokes data of a wild Riemann–Hilbert problem, or equivalently into the canonical Hamiltonian underlying a de Branges space. In both cases, rigidity arises only after the oscillatory content of $\zeta(s)$ has been incorporated into the structure from the outset, rather than added a posteriori [17].

Perspective on the generalized Riemann hypothesis.

Although the present work is formulated for the Riemann zeta function, its underlying philosophy is compatible with extensions to the generalized Riemann hypothesis. In earlier work on prime number races and Chebyshev bias, inequalities involving suitable renormalizations of prime-counting functions were shown to be equivalent to GRH for a given modulus [18]. From the present viewpoint, such renormalizations play a role analogous to that of the canonical Hamiltonian or Weyl function in a de Branges realization: in both cases, the oscillatory contribution of nontrivial zeros is incorporated into a rigid structure before positivity or monotonicity arguments are applied. A potential extension of the present approach to GRH would therefore require a family of wild Riemann–Hilbert problems or canonical systems indexed by Dirichlet characters, with positivity conditions encoding the corresponding L -functions. We do not pursue this direction here.

Relation to quantum statistical formulations of RH.

The present framework also resonates with earlier formulations of the Riemann Hypothesis in terms of quantum statistical mechanics. In the Bost–Connes system, the Riemann zeta function appears as a partition function, and RH can be reformulated as a positivity inequality satisfied by suitable Kubo–Martin–Schwinger (KMS) states at low temperature. From the present viewpoint, such inequalities play a role analogous to de Branges positivity: in both cases, the oscillatory contribution of the zeta zeros is encoded into a global operator-theoretic object (KMS states or canonical systems), and RH is equivalent to a positivity condition imposed on that object rather than to a term-by-term control of zeros. The present wild isomonodromic approach may thus be viewed as a geometric refinement of this philosophy, in which the relevant positivity problem is transferred from quantum statistical states to a Riemann–Hilbert problem and its associated canonical Hamiltonian [19].

Remark 16 (A positivity paradigm across arithmetic, operator theory, and wild isomonodromy). *A common structural feature of several previously proposed routes to (G)RH is that the oscillatory contribution of the nontrivial zeros is first absorbed into a rigid object, after which one seeks a positivity or monotonicity principle. In the arithmetic setting, renormalization by the Chebyshev function $\psi(x)$ (and its character variants $\psi(x; q, a)$) leads to inequalities that are equivalent to GRH for a given modulus [18]. In the quantum-statistical setting, the Bost–Connes system realizes $\zeta(s)$ as a partition function, and RH-related statements can be expressed as positivity properties of KMS states [19]. In the present framework, the analogous rigid object is the canonical Hamiltonian (or Weyl function) attached to a wild Riemann–Hilbert problem, and the decisive step is the global positivity condition (C2). From this viewpoint, (C2) may be read as a geometric–analytic counterpart of the positivity mechanisms appearing in earlier arithmetic and operator-theoretic formulations.*

Historical remark.

It is worth noting that de Branges himself expressed the conviction that his theory could be applied to prove the Riemann Hypothesis. The present work adopts a deliberately different stance: rather than relying on such convictions, it isolates the precise analytic condition—namely (C2)—on which any de Branges-based approach must stand or fall. Whether this condition holds in the wild isomonodromic context considered here remains an open and verifiable question.

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Appendix A Exploratory Test of the Positivity Condition for PIII_{D_6}

The purpose of this Appendix is not to prove the positivity condition (C2), but to push the construction far enough to identify where it may succeed or fail. This makes explicit the analytic bottleneck of the approach.

Appendix A.1 Fixing an RH Formulation

We fix a standard rank-2 Riemann–Hilbert formulation for PIII_{D_6} associated with a meromorphic connection having irregular singularities of Poincaré rank 1 at 0 and ∞ . In such a formulation there are two Stokes factors at each end,

$$S_0^\pm = I + s_0^\pm E_\pm, \quad S_\infty^\pm = I + s_\infty^\pm E_\pm,$$

together with formal monodromy matrices

$$M_0^{\text{for}} = e^{\pi i \theta_0 \sigma_3}, \quad M_\infty^{\text{for}} = e^{\pi i \theta_\infty \sigma_3},$$

and a connection matrix C relating canonical solutions at 0 and ∞ [2,4].

Appendix A.2 The Positive Decorated Slice

Following Chekhov–Mazzocco–Rubtsov, we consider the positive decorated slice of the PIII_{D_6} character variety, defined by

$$a, b, c, d, e, f, g, h \in \mathbb{R}_{>0}, \quad de = \Lambda_0 > 0, \quad hg = \Lambda_\infty > 0$$

[5,14]. This slice is preserved by complex conjugation and corresponds geometrically to a decorated Teichmüller-type real form.

Assuming the irregular Riemann–Hilbert correspondence, these data define a real locus in the wild monodromy manifold. On this locus one may take

$$\theta_0, \theta_\infty \in \mathbb{R},$$

and Stokes multipliers $s_0^\pm, s_\infty^\pm$ real (up to normalization) [4].

Appendix A.3 Jump Matrices and Positivity

Let Σ denote the RH contour in the z -plane for the chosen formulation. The jump matrix $J(z)$ is a product of Stokes factors and formal monodromy contributions. On the positive decorated slice, the jump matrices take values in a split real form of $\text{SL}(2, \mathbb{C})$ [4].

The key analytic question is whether there exists a normalization (gauge) in which, for $t > 0$, the jump matrix satisfies a positivity or unitarity condition of the form

$$J(z)^\dagger = J(z) \quad \text{or} \quad J(z)^\dagger = J(z)^{-1}$$

on the relevant parts of Σ . If such a normalization exists, then the associated transfer (Weyl–Titchmarsh) function is Herglotz, and the corresponding entire function is of Hermite–Biehler type [1,6].

Appendix A.4 Interaction with the Embedding $t = s(1 - s)$

Under the embedding $t = s(1 - s)$, the critical line $\Re(s) = \frac{1}{2}$ maps to the positive real ray $t \in [1/4, \infty)$. Thus the positivity condition need only be verified for t on this ray. If positivity fails already on this ray, then condition (C2) is violated for the embedding (10), and the present route cannot lead to Lemma 1.

Appendix A.5 Diagnostic Outcomes

There are three logically distinct outcomes of this test:

1. Positivity holds on the positive decorated slice for $t > 0$: the route via Lemma 1 remains viable.
2. Positivity fails for generic positive decorated data but holds on a lower-dimensional sublocus: the route is viable only under additional constraints.
3. Positivity fails identically on the positive decorated slice: the PIII_{D_6} wild isomonodromic route is ruled out for the embedding (10).

In all cases, the analysis makes explicit where the approach succeeds or fails, and no hidden assumptions remain.

Appendix A.6 A Single-Obstruction Reformulation of (C2)

The purpose of this subsection is to make the bottleneck (C2) maximally concrete. Rather than viewing (C2) as an amorphous “global positivity” requirement, we isolate the unique analytic obstruction that can prevent a positive canonical-system realization once (C1), (C3) and the analytic part of (C4) are in place.

Proposition A17 (Reduction of (C2) to a Weyl/Herglotz obstruction). *Assume (C1) and fix the embedding $t = s(1 - s)$, so (C3) holds. Assume moreover that the PIII_{D_6} Riemann–Hilbert data on the positive decorated slice admit a normalization for which the associated transfer matrix $T(\lambda)$ (or, equivalently, the Evans/de Branges pair extracted from the canonical system) is defined meromorphically in λ and depends continuously on t for $t \in [1/4, \infty)$.*

Then condition (C2) holds for this normalization if and only if the associated Weyl–Titchmarsh (Nevanlinna) function $m(\lambda)$ is a Herglotz function, i.e.

$$\Im(m(\lambda)) > 0 \quad \text{for all } \lambda \in \mathbb{C}_+. \quad (\text{A13})$$

Equivalently, (C2) fails if and only if one encounters a Herglotz obstruction, namely at least one of the following occurs:

- (O1) **(Pole obstruction)** $m(\lambda)$ has a pole in $\mathbb{C}_+$ (breakdown of definitizability/positivity).
- (O2) **(Sign obstruction)** there exists $\lambda \in \mathbb{C}_+$ such that $\Im(m(\lambda)) \leq 0$ (loss of Nevanlinna property).
- (O3) **(Boundary singularity obstruction)** $m(\lambda)$ fails to admit non-tangential boundary values on a set of positive measure of $\mathbb{R}$, preventing the standard Herglotz representation and hence a de Branges space.

In particular, once a candidate normalization is fixed, (C2) is reduced to verifying the single analytic property (A13); any failure of (C2) must manifest through (O1)–(O3).

Remark A18 (Interpretation and operator-theoretic meaning). *Proposition A17 does not prove (C2); rather, it clarifies that (C2) is equivalent to the absence of a Herglotz obstruction for the Weyl function. This equivalence is standard in the theory of canonical systems and de Branges spaces [1,6,20]. In Kreĭn space language, the obstructions (O1)–(O3) correspond to the failure of definitizability or to a sign change of the associated Nevanlinna function [21,22]. Consequently:*

- a proof of (C2) amounts to constructing a global normalization for which $m(\lambda)$ is Herglotz;
- a disproof of (C2) amounts to exhibiting one obstruction among (O1)–(O3).

This explains why the bottleneck is genuinely global: it concerns analyticity and positivity of $m(\lambda)$ on the full upper half-plane, not merely local algebraic positivity of individual jump factors.

Conclusion of the exploratory test.

Under the positive decorated slice of Chekhov–Mazzocco–Rubtsov, the wild monodromy and Stokes data for PIII_{D_6} naturally lie in a split real form. For the embedding $t = s(1 - s)$, the critical line $\Re(s) = \frac{1}{2}$ is mapped to the positive real ray $t \in [1/4, \infty)$. Along this ray, the exponential phases entering the Riemann–Hilbert formulation are real, and the individual Stokes jump matrices become positive triangular matrices when the Stokes multipliers are positive. Products of such matrices remain positive, so that, after an appropriate normalization, the Riemann–Hilbert problem falls within the class of positive canonical systems. At this level, no local or algebraic obstruction to the positivity condition (C2) is visible. The remaining issues are global analytic in nature, namely the construction of a global normalization yielding a Herglotz (Pick) transfer function and the identification of the corresponding Hermite–Biehler entire function with the object required in Lemma 1.

At the present stage, the analysis supports a neutral–eliminative standpoint: condition (C2) is the sole remaining analytic obstruction to a de Branges realization in the wild isomonodromic setting, and the framework developed here neither assumes nor presupposes its validity. The purpose of the present work is therefore to localize the problem precisely, not to anticipate its resolution.

Appendix B A Concrete Attack Plan for Condition (C2)

This Appendix outlines a concrete program to prove or disprove condition (C2) for the PIII_{D_6} wild Riemann–Hilbert problem on the positive decorated slice. The aim is to reduce (C2) to standard positivity criteria for canonical systems and matrix-valued Herglotz (Nevanlinna) functions, thereby placing the problem within established operator-theoretic and Riemann–Hilbert frameworks.

Appendix B.1 Reformulating (C2) as a Herglotz Property

Fix a normalized Riemann–Hilbert formulation for PIII_{D_6} and let $Y(z; t)$ denote the corresponding piecewise-holomorphic solution with jumps $Y_+ = Y_- J$ on the contour Σ . Condition (C2) is equivalent to the existence of a normalization $Y \mapsto \hat{Y} = GY$ such that the associated Weyl–Titchmarsh function $m(\lambda)$ is Herglotz:

$$\Im(m(\lambda)) > 0 \quad (\Im(\lambda) > 0),$$

and hence defines a positive canonical system in the sense of de Branges [1,6,20].

Appendix B.2 Step 1: Choice of a J -Unitary Real Form

On the positive decorated slice, the wild monodromy data lie in a split real form, typically $\text{SL}(2, \mathbb{R})$ or $\text{SU}(1, 1)$ after conjugation. The first task is to identify a fixed Hermitian form J of signature $(1, 1)$ such that the normalized jump matrices satisfy a J -unitarity condition

$$\hat{J}(z)^* J \hat{J}(z) = J \quad (z \in \Sigma).$$

This places the problem in a Kreĭn space setting and is a prerequisite for Herglotz positivity of the Weyl function [21,22].

Appendix B.3 Step 2: Potapov Factorization of the Jumps

A classical route to positivity is to factor each jump matrix into elementary J -contractive or J -unitary factors (Potapov factors). One seeks a factorization

$$\hat{J}(z) = \prod_k \hat{J}_k(z),$$

where each factor $\hat{J}_k$ extends analytically to the upper half-plane and is J -contractive there. The positive decorated slice supplies positivity of the cluster parameters, but it remains to show that these parameters induce a Potapov factorization compatible with the global wild monodromy constraints [23,24].

Appendix B.4 Step 3: Global Normalization as a Matrix Riesz Factorization

Even if each jump admits a local positive factorization, condition (C2) requires a *single* global normalization valid on the full contour. This can be formulated as a matrix Riesz factorization or harmonic majorant problem: find an analytic G such that the transformed jumps are J -unitary or J -contractive everywhere on Σ . Failure of such a global factorization would constitute a genuine obstruction to (C2) [20,24].

Appendix B.5 Step 4: Extraction of the Canonical Hamiltonian

Assuming Steps 1–3 succeed, one can identify the canonical Hamiltonian $H(t)$ associated with the normalized Riemann–Hilbert problem and analyze its large- t behavior. This yields a direct test of compatibility with the logarithmically corrected Weyl law required by the zeta zero density,

$$N_\zeta(t) \sim \frac{\sqrt{t}}{4\pi} \log t,$$

as discussed in Section 6.6. We write $N_\zeta(t) := N(\sqrt{t})$, where $N(T) = \#\{\rho = \frac{1}{2} + i\gamma : 0 < \gamma \leq T\}$.

At this stage the problem becomes quantitative and can be compared directly with Weyl–Levinson asymptotics for canonical systems [6,7].

Appendix B.6 Step 5: Disproof Strategy

To disprove (C2), it suffices to identify a global obstruction, such as:

- a topological obstruction to a J -contractive global factorization compatible with the wild monodromy constraints;
- a sign-changing invariant (for example a Kreĭn signature or Maslov-type index) that cannot remain positive along the full contour;
- a forced violation of the Herglotz property of the Weyl function ($\Im(m(\lambda)) \leq 0$ at some point in $\mathbb{C}^+$).

Any such obstruction rules out the de Branges positivity route for PIII_{D_6} (at least for the embedding $t = s(1 - s)$) and indicates that a different wild isomonodromic system or a different rigidity mechanism is required.

Appendix B.7 Minimal Deliverables

The attack plan outlined above constitutes a research program whose implementation is reserved for future work. Even partial progress on any of the following would constitute a substantial result:

1. a proof of J -unitarity for a canonical normalization on the positive decorated slice;
2. a Potapov-type factorization for the full jump matrix on $t \in [1/4, \infty)$;
3. an explicit Weyl function $m(\lambda)$ with a rigorous Herglotz verification;
4. a rigorous obstruction implying that (C2) fails generically.

Items (1)-(2) are analytic and require geometric input from the decorated character variety. Item (3) is computational and can be approached using existing Painlevé solvers combined with Riemann–Hilbert software. Item (4) could be established through index-theoretic or topological arguments. A subsequent publication will pursue these directions systematically.

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