Concept Paper

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Finding a Published Research Paper which Meaningfully Averages the Most Pathalogical Functions (V2)

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Concept Paper

Finding a Published Research Paper Which Meaningfully Averages The Most Pathalogical Functions (v2)

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Abstract: I want to meaningfully average a pathalogical function (i.e., an everywhere surjective function whose graph has zero Hausdorff measure in its dimension). In case this impossible, we wish to average a nowhere continuous function defined on the rationals. We do this by taking the most generalized, satisfying extension of the expected value, w.r.t the Hausdorff measure in its dimension, on bounded functions with domains of finite measure which takes finite values only. As of now, I'm unable to solve this due to limited knowledge of advanced math and most people are too busy to help. Therefore, I'm wondering if anyone knows a research paper which solves my doubts. Unlike the previous paper, "Finding a Research Paper Which Meaningfully Averages Pathalogical Functions" [13]) we add motivations, examples, and change parts of the original paper.

Keywords: Expected Values; Hausdorff measure; Hausdorff dimension; Discretization; Partitions; Sample; Euclidean Distance; Choice Function; Pathalogical Functions; Functions

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1. Intro

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Let $\dim_{\text{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\text{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra.

1.1. *First Special Case of f*

If the graph of f is G, is there an explicit f where:

(1) The function f is everywhere surjective [5]
(2) $\mathcal{H}^{\dim_{\text{H}}(G)}(G) = 0$

1.1.1. Potential Answer

If $A = \mathbb{R}$, using this MathOverflow post [8], define f such that:

Consider a Cantor set $C \subseteq [0,1]$ with Hausdorff dimension 0 [9]. Now consider a countable disjoint union $\cup_{m\in\mathbb{N}} C_m$ such that each C_m is the image of C by some affine map and every open set $O \subseteq [0,1]$ contains C_m for some m. Such a countable collection can be obtained by e.g. at letting C_m be contained in the biggest connected component of $[0,1] \setminus (C_1 \cup \cdots \cup C_{m-1})$ (with the center of C_m being the middle point of the component).

Note that $\cup_m C_m$ has Hausdorff dimension 0, so $(\cup_m C_m) \times [0,1] \subseteq \mathbb{R}^2$ has Hausdorff dimension one [7].

Now, let $g : [0,1] \to \mathbb{R}$ such that $g|_{C_m}$ is a bijection $C_m \to \mathbb{R}$ for all m (all of them can be constructed from a single bijection $C \to \mathbb{R}$, which can be obtained without choice, although it may be ugly to define) and outside $\cup_m C_m$ let g be defined by $g(x) = h(x)$, where $h : [0,1] \to \mathbb{R}$ has a graph with Hausdorff dimension 2 [21] (this doesn't require choice either).

Then the function g has a graph with Hausdorff dimension 2 and is everywhere surjective, but its graph has Lebesgue measure 0 because it is a graph (so it admits uncountably many disjoint vertical translates).

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Note, we can make the construction with union of C_m rather explicit as follows. Split the binary expansion of x as strings of size with a power of two, say $x = 0.1101000010\ldots$ becomes $(s_0, s_1, s_2, \ldots) = (1, 10, 1000, \ldots)$. If this sequence eventually contains only strings of the form $0\cdots0$ or $1\cdots1$, say after s_k, then send it to $y = \sum_{i>0} \epsilon_i 2^{-i}$, where $s_{k+i} = \epsilon_i \cdots \epsilon_i$. Otherwise, send it to the explicit continuous function h given by the linked article [21]. This will give you something from $[0,1) \to [0,1)$

Finally, compose an explicit (reasonable) bijection from $[0,1)$ to $\mathbb{R}$. In your case, the construction can be easily adapted so that the $[0,1]$ or $[0,1)$ target space is actually $(0,1)$, then compose with $t \mapsto (1-2x)/(x^2-x)$.

In case this function is impossible to average, consider the following example:

1.2. Second Special Case of f

Suppose, we define $A = \mathbb{Q}$, where $f : A \to \mathbb{R}$, such that:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{1}$$

In the following sections, we shall see why we chose §1.1 and this function.

1.3. Attempting to Analyze/Average f

Suppose, the expected value of f is:

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, d\mathcal{H}^{\dim_{\mathrm{H}}(A)} \tag{2}$$

Note, using §1.1, explicit f is pathological since it's everywhere surjective and difficult to meaningfully average (i.e., the most *generalized*, *satisfying* (§2) extension of $\mathbb{E}[f]$ is non-finite).

Thus, we want the most *generalized*, *satisfying* extension of $\mathbb{E}[f]$ on bounded f, where the extension takes finite values for all f defined within §1.1. Moreover, suppose:

(1) The sequence of bounded functions is $\overrightarrow{f_s} = (f_{r_s}^{(s)})_{r_s \in \mathbb{N}}$

(2) The sequence of bounded functions converges to f: i.e., $f_{r_s}^{(s)} \to f$

(3) The *generalized*, *satisfying* extension of $\mathbb{E}[f]$ is $\mathbb{E}[f_{r_s}^{(s)}]$: i.e., there exists a $s \in \mathbb{N}$, where $\mathbb{E}[f_{r_s}^{(s)}]$ is finite

(4) There exists $k, v \in \mathbb{N}$ where the expected value of $\overrightarrow{f_k}$ and $\overrightarrow{f_v}$ are finite and non-equivelant: i.e.,

$$-\infty < \mathbb{E}[f_{r_k}^{(k)}] \neq \mathbb{E}[f_{r_v}^{(v)}] < +\infty$$

(Whenever ((4)) is true, ((3)) is *non-unique*.)

1.3.1. Example Proving §1.3 ((1))-((4)) Correct

Using the second case of $f : A \to \mathbb{R}$ in §1.2, where $A = \mathbb{Q}$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{3}$$

suppose:

$$(A_r^{\star})_{r \in \mathbb{N}} = (\{c/r! : c \in \mathbb{Z}, -r \cdot r! \leq c \leq r \cdot r!\})_{r \in \mathbb{N}}$$

and

$$(A_j^{\star\star})_{j \in \mathbb{N}} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq j, -dj \leq c \leq dj\})$$

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where for $f_r^\star : A_r^\star \to \mathbb{R}$,

$$f_r^\star(x) = f(x) \text{ for all } x \in A_r^\star \tag{4}$$

and for $f_j^{\star\star} : A_j^{\star\star} \to \mathbb{R}$

$$f_j^{\star\star}(x) = f(x) \text{ for all } x \in A_j^{\star\star} \tag{5}$$

Note, $f_r^\star$ and $f_j^{\star\star}$ are bounded since f is bounded (i.e., criteria (1) of §1.3 is satisfied). Also, the set-theoretic limit of $(A_r^\star)_{r\in\mathbb{N}}$ and $(A_j^{\star\star})_{j\in\mathbb{N}}$ is $A = \mathbb{Q}$: i.e.,

$$\limsup_{r\to\infty} A_r^\star = \bigcap_{r\geq 1}\bigcup_{q\geq r} A_q^\star$$
$$\liminf_{r\to\infty} A_r^\star = \bigcup_{r\geq 1}\bigcap_{q\geq r} A_q^\star$$

where:

$$\limsup_{r\to\infty} A_r^\star = \liminf_{r\to\infty} A_r^\star = A = \mathbb{Q}$$

(We are not sure how to prove this; however, a mathematician which specializes in rational numbers and set-theoretic limits should be able to verify the former.)

Hence, one can see that $f_r^\star : A_r^\star \to \mathbb{R}$ and $f_j^{\star\star} : A_j^{\star\star} \to \mathbb{R}$ converges to $f : A \to \mathbb{R}$ (i.e., criteria (2) of §1.3 is satisfied).

Now, suppose we want to average $f_r^\star$ and $f_j^{\star\star}$ which we denote $\mathbb{E}[f_r^\star]$ and $\mathbb{E}[f_j^{\star\star}]$. Note, this is the same as computing the following (i.e., the cardinality is $|\cdot|$ and the absolute value is $||\cdot||$):

$$\forall(\epsilon > 0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N \Rightarrow \left|\left|\frac{1}{|A_r^\star|}\int_{A_r^\star} f(x) - \mathbb{E}[f_r^\star]\right|\right| < \epsilon\right) \tag{6}$$

$$\forall(\epsilon > 0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N \Rightarrow \left|\left|\frac{1}{|A_j^{\star\star}|}\int_{A_j^{\star\star}} f(x) - \mathbb{E}[f_j^{\star\star}]\right|\right| < \epsilon\right) \tag{7}$$

If we assume, $\mathbb{E}[f_r^\star] = 1$ in eq. 6, then [15]:

> The integral $\int_{A^\star} f(x)\,dx$ counts the number of fractions with even denominator and odd numerator in set $A^\star$, after cancelling all possible factors of 2 in the fraction. Let's consider the first case. We can write $\left|\left|1 - |A_r^\star|^{-1}\int_{A_r^\star} f\right|\right| = (|A_r^\star| - \int_{A_r^\star} f)/|A_r^\star| = H(r)/|A_r^\star|$, where $H(r)$ counts the fractions $x = c/r!$ in $A_r^\star$ that are not counted in $\int_{A^\star} f$, i.e., for which $f(x) = 0$. This is the case when the denominator is odd after the cancellation of the factors of 2, i.e., when the numerator c has a number of factors of 2 greater than or equal to that of $r!$, which we will denote by $V(r) := v_2(r!)$ a.k.a the 2-valuation of $r!$, oeis:A11371$(r) = r - O(\ln(r))$ [17]. That means, c must be a multiple of $2^{V(r)}$. The number of such c with $-r\cdot r! \leq c \leq r\cdot r$ is simply the length of that interval, equal to $|A_r^\star| = 2r(r!) + 1$, divided by $2^{V(r)}$. Thus, $\left|\left|1 - |A_r^\star|^{-1}\int_{A_r^\star} f\right|\right| = [|A_r^\star|/2^{V(r)}]/|A_r^\star| \sim 1/2^{V(r)} = 1/2^{n-O(\log n)}$. This obviously tends to zero, proving $\mathbb{E}[f_r^\star] = 1$

Since $\mathbb{E}[f_r^\star] = 1$ is finite, this proves §1.3 criteria (3).
Last, we need to show $\mathbb{E}[f_j^{\star\star}] = 1/3$, where $\mathbb{E}[f_r^\star] \neq \mathbb{E}[f_j^{\star\star}]$ proving §1.3 criteria (4).

> Concerning the second case [15], it's again simpler to consider the complementary set of $x \in A_j^{\star\star}$ such that the denominator is odd when all possible factors of 2 are canceled. We can see that for $j = 2p - 1$, and these include obviously all those we had for smaller j. The "new" elements in $A_j^{\star\star}$ with $j = 2p - 1$ are those which have denominator $d = 2p - 1$ when written in lowest terms. Their number is equal to the number of $\kappa < d$, $\gcd(\kappa, d) = 1$, which is given by Euler's ϕ function. Since we also consider negative fractions, we have to multiply this by 2. Including $x = 0$, we have $G(j) = \left|\left\{x \in A_j^{\star\star} | f(x) = 0\right\}\right| = 1 + 2\sum_{0\leq\kappa\leq j/2}\phi(2\kappa+1)$.

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There is no simple explicit expression for this (cf. oeis:A99957 [18]), but we know that $G(j) = 1 + 2 \cdot \text{A99957}(j/2) \sim 2 \cdot 8(j/2)^2/\pi^2 = 4j^2/\pi^2$ [18]. On the other hand, the total number of all elements of $A_j^{\star\star}$ is $|A_j^{\star\star}| = 1 + 2\sum_{1\le\kappa\le j}\phi(\kappa)$, since each time we increase j by 1, we have the additional fractions with the new denominator $d = j$ and numerators being coprime with d, again with $+$ or $-$ sign. From oeis:A002088 [16] we know that $\sum_{1\le\kappa\le j}\phi(\kappa) = 3j^2/\pi^2 + O(j\log j)$, so $|A_j^{\star\star}| \sim 6j^2/\pi^2$, which finally gives $|A_j^{\star\star}|^{-1}\int_{A_j^{\star\star}} f = (|A_j^{\star\star}| - G(j))/|A_j^{\star\star}| \sim (6-4)/6 = 1/3$ as desired.

Hence, $\mathbb{E}[f_j^\star] = 1/3$ and $\mathbb{E}[f_r^\star] \neq \mathbb{E}[f_j^{\star\star}]$ proving §1.3 criteria (4).
Therefore, in §1.3, since:

Theorem 1. *The set of all Borel f, where $\mathbb{E}[f]$ is finite, forms a shy [19] subset of all Borel measurable functions in $\mathbb{R}^A$*

Note 2 (Proof theorem 1 is true). *We follow the argument presented in example 3.6 of this paper [19], take $X := L^0(A)$ (measurable functions over A), let P denote the one-dimensional sub-space of A consisting of constant functions (assuming the Lebesgue measure on A) and let $F := L^0(A) \setminus L^1(A)$ (measurable functions over A with no finite integral). Let λ_P denote the Lebesgue measure over P, for any fixed $f \in F$:*

$$\lambda_P\left(\left\{\alpha \in \mathbb{R} \,\middle|\, \int_A (f+\alpha)d\mu < \infty\right\}\right) = 0$$

Meaning P is a one-dimensional, so f is a 1-prevelant set.

Theorem 3. *The set of all Borel f, where the generalized, satisfying extension of $\mathbb{E}[f]$ is non-unique (§1.3, ((4))), forms a prevalent [19] subset of all Borel measurable functions in $\mathbb{R}^A$*

Note 4 (Possible method to proving theorem 3 true). *Note, we first must prove that only f whose lines of symmetry intersect at one point have the same expected value for any bounded sequence of functions converging to f. Notice, the set of all symmetric Borel functions forms a shy subset of the set of all Borel measurable functions in $\mathbb{R}^A$. Hence, the set of all functions, where their lines of symmetry intersect at one point also forms a shy subset of the set of all Borel measurable functions in $\mathbb{R}^A$ (i.e., a subset of a shy subset is also shy).*

Since thm. 1 and 3 are true, we need to fix the problems the theorems address with the following:

1.3.2. Blockquote

> We want to find an unique, *satisfying* (§3) extension of $\mathbb{E}[f]$, on bounded functions to f which takes finite values only, such that the set of all f with this extension forms:
>
> (1) a prevalent [19] subset of $\mathbb{R}^A$
> (2) If not prevalent then a non-shy (i.e., neither prevalent [19] nor shy [19]) subset of $\mathbb{R}^A$.

For the sake of clarity & precision, we describe examples of "extending $\mathbb{E}[f]$ on all A with positive & finite Hausdorff measure" (§2) and use the examples to define the terms "unique & satisfying" (§3) in the blockquote of this section.

2. Extending the Expected Value w.r.t the Hausdorff Measure

The following are two methods to determining the most *generalized*, *satisfying* extension of $\mathbb{E}[f]$ on all A with a positive and finite Hausdorff measure:

(1) One way is defining a generalized, satisfying extension of the Hausdorff measure on all A with positive & finite measure which takes positive, finite values for all Borel A. This can theoretically be done in the paper "A Multi-Fractal Formalism for New General Fractal Measures"[1] by taking the expected value of f w.r.t the extended Hausdorff measure.

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(2) Another way is finding a *generalized*, *satisfying* average of all A in the fractal setting. This can be done with the papers "Analogues of the Lebesgue Density Theorem for Fractal Sets of Reals and Integers" [3] and "Ratio Geometry, Rigidity and the Scenery Process for Hyperbolic Cantor Sets" [4] where we take the expected value of f w.r.t the densities in [3,4].

(Note, the methods in these papers could be used in §3.2 to answer the blockquote of §1.3.2.)

3. Attempt to Define "Unique and Satisfying" in The Blockquote of §1.3

3.1. Note

Before reading, when §3.2 is unclear, see §5 for clarity. In §5, we define:

(1) "Sequences of bounded functions converging to f" (§5.1)
(2) "Equivalent sequences of bounded functions" (§5.3, def. 7)
(3) "Nonequivalent sequences of bounded functions" (§5.3, def. 9)
(4) The "measure" of a property on a sequence of bounded functions which increases at rate *linear* or *super-linear* to that of "non-equivelant" sequences of bounded functions (§5.4.1, §5.4.2)
(5) The "actual" rate of expansion on a sequence of bounded sets (§5.5)

3.2. Leading Question

To define *unique* and *satisfying* in the blockquote of the §1.3, we take the expected value of a sequence of bounded functions chosen by a choice function. To find the choice function we ask the **leading question**...

If we make sure to:

(A) See §3.1 and (C)-(E) when something is unclear
(B) Take all sequences of bounded functions which converge to f
(C) Define C to be chosen center point of $\mathbb{R}^{n+1}$
(D) Define E to be the chosen, fixed rate of expansion of a sequence on the graph of bounded functions
(E) Define $\mathcal{E}$ to be actual rate of expansion of a sequence on the graph of bounded functions (§5.5)

Does there exist a unique choice function which chooses a unique set of equivalent sequences of bounded functions where:

(1) The chosen, equivelant sequences of bounded functions should satisfy (B).
(2) The "measure" of the graph of all chosen, equivalent sequences of bounded functions which satisfy (B) should increase at a rate *linear* or *superlinear* to that of non-equivelant sequences of bounded functions satisfying (B)
(3) The expected values, defined in the papers of §2, for all equivalent sequences of bounded functions are equivalent and finite
(4) For the chosen, equivalent sequences of bounded functions satisfying ((1)), ((2)) and ((3)), when f is unbounded (i.e, skip when f is bounded):
    - The absolute difference between criteria ((3)) and the $(n+1)$-th coordinate of C is the *less than or equal* to that of non-equivalent sequences of bounded functions satisfying ((1)), ((2)), and ((3))
    - The "rate of divergence" [20, p.275-322] of $||\mathcal{E}-E||$, using the absolute value $||\cdot||$, *is less than or equal* to that of non-equivalent sequences of bounded functions which satisfy ((1)), ((2)), and ((3))
(5) When set $Q \subseteq \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where the choice function chooses all equivalent sequences of bounded functions satisfying ((1)), ((2)), ((3)) and ((4)), then Q is

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(a) a prevelant [19] subset of $\mathbb{R}^A$
(b) If not ((5)a) then a non-shy (i.e., neither prevelant [19] nor shy [19]) subset of $\mathbb{R}^A$.

(6) Out of all choice functions which satisfy ((1)), ((2)), ((3)), ((4)) and ((5)), we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

(In case this is unclear, see §5.)

3.2.1. Explaining Motivation Behind §3.2

(1) When defining "the measure" (§5.4.1,§5.4.2) of a function, we want a bounded sequence of functions with a "high" entropic density (i.e., we aren't sure if this is infact what the "measure" measures.) For example, when $A = \mathbb{R}$ and f is everywhere surjective [5], the "measure" chooses bounded sequence of functions whose lines of symmetry intersect at one point rather than non-symmetrical functions §5.4.1-§6, §8.

(2) Note, the later criteria doesn't apply to bounded functions: e.g., using §1.3.1, when $A = \mathbb{Q}$ and $f : A \to \mathbb{R}$, where:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{8}$$

depending on the sequence of bounded function $(f_r^\star)$ *chosen which converge to* f: $\mathbb{E}[f_r^\star]$ can be any number in $[0,1]$. Since $[0,1]$ is bounded, we need only §3.2.1 crit. (1) to solve the blockquote of §1.3.2.

(3) Using an unbounded function in §1.1 (i.e., an everywhere surjective f whose graph has zero Hausdorff measure in its dimension) *depending on the sequence of bounded functions* $(f_r^\star)_{r\in\mathbb{N}}$ *chosen which converge to* f: $\mathbb{E}[f_r^\star]$ can be any real number (when it exists). To fix this, take all $(f_r^\star)_{r\in\mathbb{N}}$, where the $\mathbb{E}[f_r^\star]$ has smallest absolute difference from the $(n+1)$-th coordinate of a reference point (i.e., the center point $C \in \mathbb{R}^n$). The problem is there exists f, where the expected value of non-equivalent sequences (§5.3, def. 9) of bounded functions have the same minimum absolute difference from $(n+1)$-th coordinate of C.

(4) Thus, we take the sequence of functions whose actual rate of expansion $\mathcal{E}$ from C (§5.5) "diverges" [20, p.275-322] at the smallest rate from the chosen, fixed rate of expansion E from C (i.e., the "rate of divergence of $||\mathcal{E} - E||$, using the absolute value $||\cdot||$, *is less than or equal* to that of all the non-equivalent sequences of bounded functions which satisfy §3.2 criteria ((1)), ((2)), and ((3))).

(5) Finally, since there might still be non-equivalent sequences (§5.3, def. 9) of bounded functions which satisfy §3.2.1 criteria ((1)), ((3)) and ((4)), but their graphs are congruent with different $\mathbb{E}[f_r^\star]$, we use equation T in §6.3.1 eq. 144 to choose a unique set of all equivalent sequences of bounded sets with the same expected value.

I'm convinced the expected values of the sequences of bounded functions chosen by a choice function which answers the *leading question* aren't *unique* nor *satisfying enough* to answer the blockquote of §1.3.2. Still, adjustments are possible by changing the criteria or by adding new criteria to *the question*.

4. Question Regarding My Work

Most don't have time to address everything in my research, hence I ask the following:

> *Is there a research paper which already solves the ideas I'm working on? (Non-published papers, such as mine [12], don't count.)*

Using AI, papers that might answer this question are "Prediction of dynamical systems from time-delayed measurements with self-intersections" [2] and "A Hausdorff measure boundary element method for acoustic scattering by fractal screens" [6].

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Does either of these papers solve the blockquote of §1.3.2?

5. Clarifying §3

Suppose $(f_r)_{r\in\mathbb{N}}$ is a sequence of bounded functions converging to f and $(G_r)_{r\in\mathbb{N}}$ is a sequence of the graph of each f_r. Let $\dim_{\text{H}}(\cdot)$ be the Hausdorff dimension and $\mathcal{H}^{\dim_{\text{H}}(\cdot)}(\cdot)$ be the Hausdorff measure *in its dimension* on the Borel σ-algebra.

See §3.2 once reading §5, and consider the following:

> Is there a simpler version of the definitions below?

5.1. Defining Sequences of Bounded Functions Converging to f

The sequence of bounded functions $(f_r)_{r\in\mathbb{N}}$, where $f_r : A_r \to \mathbb{R}$ and $(A_r)_{r\in\mathbb{N}}$ is a sequence of bounded sets, converges to function $f : A \to \mathbb{R}$ when:

> For any $x \in A$ there exists a sequence $\mathbf{x} \in A_r$ s.t. $\mathbf{x} \to (x_1, \cdots, x_n)$ and $f_r(\mathbf{x}) \to f(x_1, \cdots, x_n)$ (see [10] for info).

Example 5 (**Example of §5.1**). *If $A = \mathbb{R}$ and $f : A \to \mathbb{R}$, where $f(x) = 1/x$, then an example of $(f_r)_{r\in\mathbb{N}}$, such that $f_r : A_r \to \mathbb{R}$ is:*

(1) $(A_r)_{r\in\mathbb{N}} = ([-r, -1/r] \cup [1/r, r])_{r\in\mathbb{N}}$
(2) $f_r(x) = 1/x$ *for* $x \in A_r$

Example 6 (**More Complex Example**). *If $A = \mathbb{R}$ and $f : A \to \mathbb{R}$, where $f(x) = x$, then an example of $(f_r)_{r\in\mathbb{N}}$, such that $f_r : A_r \to \mathbb{R}$ is:*

(1) $(A_r)_{r\in\mathbb{N}} = ([-r, r])_{r\in\mathbb{N}}$
(2) $f_r(x) = x + (1/r)\sin(x)$ *for* $x \in A_r$

5.2. Expected Value of Bounded Sequence of Functions

If (f_r) converges to f (§5.1), the expected value of f w.r.t $(f_r)_{r\in\mathbb{N}}$ is $\mathbb{E}[f_r]$ (when it exists), where the absolute value is $||\cdot||$, such that $\mathbb{E}[f_r]$ satisfies:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{H^{\dim_{\text{H}}(A_r)}(A_r)}\int_{A_r} f_r\, dH^{\dim_{\text{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \tag{9}$$

Note, $\mathbb{E}[f_r]$ can be extended by using §2.

5.2.1. Example

Using example 5, when $(f_r)_{r\in\mathbb{N}} = (\{(x, 1/x) : x \in [-r, -1/r] \cup [1/r, r]\})_{r\in\mathbb{N}}$ where:

(1) $(A_r)_{r\in\mathbb{N}} = ([-r, -1/r] \cup [1/r, r])_{r\in\mathbb{N}}$
(2) $f_r(x) = 1/x$ for $x \in A_r$

If we assume $\mathbb{E}[f_r] = 0$:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}(A_r)}(A_r)}\int_{A_r} f_r\, d\mathcal{H}^{\dim_{\text{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) = \tag{10}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \tag{11}$$

$$\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}([-r,-1/r]\cup[1/r,r])}([-r,-1/r] \cup [1/r,r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x\, d\mathcal{H}^{\dim_{\text{H}}([-r,-1/r]\cup[1/r,r])} - 0\right|\right| < \epsilon\right) = \tag{12}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^1([-r,-1/r] \cup [1/r,r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x\, d\mathcal{H}^1\right|\right| < \epsilon\right) = \tag{13}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(-1/r - (-r)) + (r - 1/r)}\left(\int_{-r}^{-1/r} 1/x\, dx + \int_{1/r}^{r} 1/x\, dx\right)\right|\right| < \epsilon\right) = \tag{14}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(r - 1/r) + (-1/r + r)}\left(\ln(||x||) + C\Big|_{-r}^{-1/r} + \ln(||x||) + C\Big|_{1/r}^{r}\right)\right|\right| < \epsilon\right) = \tag{15}$$

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$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\left|\frac{1}{(r-1/r)+(-1/r+r)}(\ln(||-r||)-\ln(||-1/r||)+\ln(||r||)-\ln(||1/r||))\right|\right|<\epsilon\right)= \tag{16}$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\left|\frac{1}{2r-2/r}\cdot 4\ln(r)\right|\right|<\epsilon\right)= \tag{17}$$

To prove eq. 17 is true, recall:

$$r\ll e^{r/2},\, e^{1/r}\ll e^{r} \tag{18}$$
$$r\ll e^{r/2},\, e^{1/(2r)}\ll e^{r/2} \tag{19}$$
$$re^{1/(2r)}\ll e^{r/2} \tag{20}$$
$$r\ll e^{r/2}/e^{1/(2r)} \tag{21}$$
$$r\ll e^{r/2-1/(2r)} \tag{22}$$
$$\ln(r)\ll r/2-1/(2r) \tag{23}$$
$$4\ln(r)\ll 2r-2/r \tag{24}$$

Hence, for all $\varepsilon>0$

$$4\ln(r)<\varepsilon(2r-2/r) \tag{25}$$
$$\frac{4\ln(r)}{2r-2/r}<\varepsilon \tag{26}$$
$$\left|\left|\frac{4\ln(r)}{2r-2/r}\right|\right|<\varepsilon \tag{27}$$

Since eq. 17 is true, $\mathbb{E}[f_r]=0$. Note, if we simply took the average of f from $(-\infty,\infty)$, using the improper integral, the expected value:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0+,+\infty)}\frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\int_{x_1}^{x_2}\frac{1}{x}\,dx+\int_{x_3}^{x_4}\frac{1}{x}\,dx\right)= \tag{28}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0+,+\infty)}\frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\ln(||x||)+C\Big|_{x_1}^{x_2}+\ln(||x||)+C\Big|_{x_3}^{x_4}\right)= \tag{29}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0+,+\infty)}\frac{1}{(x_4-x_3)+(x_2-x_1)}(\ln(||x_2||)-\ln(||x_1||)+\ln(||x_4||)-\ln(||x_3||)) \tag{30}$$

is $+\infty$ (when $x_2=1/x_1$, $x_3=1/x_4$, and $x_1=\exp(x_4^2)$) or $-\infty$ (when $x_2=1/x_1$, $x_3=1/x_4$, and $x_4=-\exp(x_1^2)$), making $\mathbb{E}[f]$ undefined. (However, using eq. 10-17, we get the $\mathbb{E}[f_r]=0$ instead of an undefined value.)

5.3. *Defining Equivelant and Non-Equivelant Sequences of Bounded Functions*

Let $S\subseteq\mathbb{N}$ be an arbitrary set and define the following sequence of functions:

$$\overrightarrow{f_1}=\{f_{r_1}^{(1)}\}_{r_1\in\mathbb{N}},\overrightarrow{f_2}=\{f_{r_2}^{(2)}\}_{r_2\in\mathbb{N}},\cdots,\overrightarrow{f_s}=\{f_{r_s}^{(s)}\}_{r_s\in\mathbb{N}}$$

Note, the sequences of bounded functions in $\left\{\overrightarrow{f_s}:s\in\mathbb{N}\right\}$ converges to f and the sequences of the graphs of all functions in each former sequence are:

$$\begin{aligned}\overrightarrow{G_1}&=(\text{graph}(f_{r_1}^{(1)}))_{r_1\in\mathbb{N}}=(G_{r_1}^{(1)})_{r_1\in\mathbb{N}},\\ \overrightarrow{G_2}&=(\text{graph}(f_{r_2}^{(2)}))_{r_2\in\mathbb{N}}=(G_{r_2}^{(2)})_{r_2\in\mathbb{N}},\\ &\vdots\\ \overrightarrow{G_s}&=(\text{graph}(f_{r_s}^{(s)}))_{r_s\in\mathbb{N}}=(G_{r_s}^{(s)})_{r_s\in\mathbb{N}}\end{aligned} \tag{31}$$

Definition 7 (Equivelant Sequences of functions). *Suppose $S \subseteq \mathbb{N}$ is an arbitrary set. The sequences of bounded functions in:*

$$\left\{ \overrightarrow{f_s} : s \in S \right\}$$

are equivalent, if for all $k, v \in S$, where $k \neq v$, $\overrightarrow{f_k}$ and $\overrightarrow{f_v}$ are equivelant: i.e., there exists a $N' \in \mathbb{N}$, such for all $r_k \geq N'$, there is a $r_v \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_H(G_{r_k}^{(k)})}(G_{r_k}^{(k)} \Delta G_{r_v}^{(v)}) = 0$$

and for all $r_v \geq N'$, there is a $r_k \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_H(G_{r_v}^{(v)})}(G_{r_k}^{(k)} \Delta G_{r_v}^{(v)}) = 0$$

More, for each $s \in \mathbb{N}$, we denote all equivalent sequences of bounded functions to $\{f_{r_s}^{(s)}\}_{r_s \in \mathbb{N}}$ using the notation

$$\sim \{f_{r_s}^{(s)}\}_{r_s \in \mathbb{N}}$$

5.3.1. Explanation

We define $\overrightarrow{f_k}$ and $\overrightarrow{f_v}$ as equivalent, where $\overrightarrow{A_k}$ and $\overrightarrow{A_v}$ are constant in $f_{r_k}^{(k)} : A_{r_k}^{(k)} \to \mathbb{R}$ and $f_{r_v}^{(v)} : A_{r_v}^{(v)} \to \mathbb{R}$, such for all Borel measurable $f \in \mathbb{R}^A$, when both $\mathbb{E}[f_{r_k}^{(k)}]$ and $\mathbb{E}[f_{r_v}^{(v)}]$ exist:

$$\mathbb{E}[f_{r_k}^{(k)}] = \mathbb{E}[f_{r_v}^{(v)}] \quad (\S 5.2)$$

Hence, consider the following:

Theorem 8. *If the sequence of functions in:*

$$\left\{ \overrightarrow{f_s} : s \in S \right\}$$

are equivalent, then for all $k, v \in S$, where $k \neq v$:

$$\mathbb{E}[f_{r_k}^{(k)}] = \mathbb{E}[f_{r_v}^{(v)}]$$

Note, this explains criteria ((3)) in §3

5.3.2. Example of Equivalent Sequences of Bounded Functions

Suppose we define $\overrightarrow{f_1} = (f_{r_1}^{(1)})_{r_1 \in \mathbb{N}}$, where $f_{r_1}^{(1)} : A_{r_1}^{(1)} \to \mathbb{R}$ and $\overrightarrow{f_2} = (f_{r_2}^{(2)})_{r_2 \in \mathbb{N}}$, where $f_{r_2}^{(2)} : A_{r_2}^{(2)} \to \mathbb{R}$ such that:

(1) $f_{r_1}^{(1)}(x) = x$
(2) $A_{r_1}^{(1)} = [-r_1 - 2, r_1 + 2]$
(3) $G_{r_1}^{(1)} = \{(x, x) : x \in [-r_1 - 2, r_1 + 2]\}$

and

(1) $f_{r_2}^{(2)}(x) = x$
(2) $A_{r_2}^{(1)} = [-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])$
(3) $G_{r_2}^{(2)} = \{(x, x) : x \in [-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])\}$

Note, using def. 7, $S = \{1, 2\}$, $k = 1$, and $v = 2$. In other words, $r_k = r_1$ and $r_v = r_2$. Now, suppose $N = 3$, where $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$. Then,

(1) For all $r_1 \geq N = 3$, there exists a $5 \leq r_2 =: r_1 + 2 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_H(G_{r_1}^{(1)})}(G_{r_1}^{(1)} \Delta G_{r_2}^{(2)}) = 0 \tag{32}$$

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We show this with the following:
In eq. 32, since $\{(x,x) : x \in [-r_1-2, r_1+2]\}$ is a 1-d interval, $\dim_{\mathrm{H}}(G_{r_1}^{(1)}) = 1$. Hence,

$$\mathcal{H}^1(\{(x,x) : x \in [-r_1-2, r_1+2]\}\Delta\{(x,x) : x \in [-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2-1, r_2+1])\}) = \quad (33)$$
$$\mathcal{H}^1(\{(x,x) : x \in [-r_1-2, r_1+2]\Delta([-(r_1+2), (r_1+2)] \cup (\mathbb{Q} \cap [-(r_1+2)-1, (r_1+2)+1])\}) = \quad (34)$$
$$\mathcal{H}^1(\{(x,x) : x \in [-r_1-2, r_1+2]\Delta([-(r_1+2), (r_1+2)] \cup (\mathbb{Q} \cap [-r_1-3, r_1+3]))\}) = \quad (35)$$
$$\mathcal{H}^1(\{(x,x) : x \in \mathbb{Q} \cap ([r_1+2, r_1+3]) \cup (\mathbb{Q} \cap [-r_1-3, r_1-2])\}) = \quad (36)$$
$$0 \quad (37)$$

We also show:

(2) For all $r_2 \geq N = 3$, there exists a $1 \leq r_2 - 2 =: r_1 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(G_{r_2}^{(2)})}(G_{r_1}^{(1)} \Delta G_{r_2}^{(2)}) = 0 \quad (38)$$

We show this with the following:
In eq. 38, since $\dim_{\mathrm{H}}(G_{r_2}^{(2)}) = \dim_{\mathrm{H}}([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - .5, r_2+1])) = 1$:

$$\mathcal{H}^1(\{(x,x) : [-r_1-2, r_1+2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2-1, r_2+1]))\}) = \quad (39)$$
$$\mathcal{H}^1(\{(x,x) : [-(r_2-2)-2, (r_2-2)+2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2-1, r_2+1])\}) = \quad (40)$$
$$\mathcal{H}^1(\{(x,x) : [-r_2, r_2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2-1, r_2+1])\}) = \quad (41)$$
$$\mathcal{H}^1((\mathbb{Q} \cap [-r_2-1, -r_2]) \cup (\mathbb{Q} \cap [r_2, r_2+1])) = \quad (42)$$
$$0 \quad (43)$$

Since crit. ((1)) and ((2)) is true, using def. 7, we have shown $f_{r_1}^{(1)}$ and $f_{r_2}^{(2)}$ are equivalent.

Definition 9 (Non-Equivalent Sequences of functions). *Again, $S \subseteq \mathbb{N}$ is an arbitrary set. Therefore, all sequences of bounded functions in:*

$$\left\{\overrightarrow{f_s} : s \in S\right\}$$

are non-equivalent, if def. 7 is false, meaning for some $k, v \in S$, where $k \neq v$, $\overrightarrow{f_k}$ and $\overrightarrow{f_v}$ are non-equivelant: there is a $N' \in \mathbb{N}$, where for all $r_k \geq N'$, there is either a $r_v \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_{\mathrm{H}}(G_{r_k}^{(k)})}(G_{r_k}^{(k)} \Delta G_{r_v}^{(v)}) \neq 0$$

or for all $r_v \geq N'$, there is a $r_k \in \mathbb{N}$, where

$$\mathcal{H}^{dim_{\mathrm{H}}(G_{r_v}^{(v)})}(G_{r_k}^{(k)} \Delta G_{r_v}^{(v)}) \neq 0$$

5.3.3. Explanation

We define $\overrightarrow{f_k}$ and $\overrightarrow{f_v}$ as equivalent, where $\overrightarrow{A_k}$ and $\overrightarrow{A_v}$ are constant in $f_{r_k}^{(k)} : A_{r_k}^{(k)} \to \mathbb{R}$ and $f_{r_v}^{(v)} : A_{r_v}^{(v)} \to \mathbb{R}$, such for all Borel measurable $f \in \mathbb{R}^A$, when either $\mathbb{E}[f_{r_k}^{(k)}]$ or $\mathbb{E}[f_{r_v}^{(v)}]$ exist:

$$\mathbb{E}[f_{r_k}^{(k)}] \neq \mathbb{E}[f_{r_v}^{(v)}] \quad (\S 5.2)$$

Hence, consider the following:

5.3.4. Example of Non-Equivalent Bounded Sequences of Functions

Suppose we define $\overrightarrow{f_1} = (f_{r_1}^{(1)})_{r_1 \in \mathbb{N}}$, where $f_{r_1}^{(1)} : A_{r_1}^{(1)} \to \mathbb{R}$ and $\overrightarrow{f_2} = (f_{r_2}^{(2)})_{r_2 \in \mathbb{N}}$, where $f_{r_2}^{(2)} : A_{r_2}^{(2)} \to \mathbb{R}$ such that:

(1) $f_{r_1}^{(1)}(x) = x$

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(2) $A_{r_1}^{(1)} = [-r_1, r_1]$
(3) $G_{r_1}^{(1)} = \{(x,x) : x \in [-r_1, r_1]\}$

and

(1) $f_{r_2}^{(2)}(x) = x + (1/r_2)\sin(x)$
(2) $A_{r_2}^{(1)} = [-2r_2, 2r_2]$
(3) $G_{r_2}^{(2)} = \{(x, x + (1/r_2)\sin(x)) : x \in [-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])\}$

Note, using def. 7, $S = \{1,2\}$, $k = 1$, and $v = 2$. In other words, $r_k = r_1$ and $r_v = r_2$. Now, suppose $N := 1$, where $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$. Then,

(1) For all $r_2 \geq N = 1$, there exists a $3 \leq 2r_2 + 1 =: r_2 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(G_{r_2}^{(2)})}(G_{r_1}^{(1)} \Delta G_{r_2}^{(2)}) \neq 0$$

We show this with the following:

Since $G_{r_2}^{(2)} = [-2r_2, 2r_2]$ is a 1-d interval, $\dim_{\mathrm{H}}(G_{r_1}^{(1)}) = 1$. Hence:

$$\mathcal{H}^1(\{(x,x) : x \in [-r_1, r_1]\}\Delta\{(x, x + (1/r_2)\sin(x)) : x \in [-2r_2, 2r_2]\}) = \quad (44)$$

$$\mathcal{H}^1(\{(x,x) : x \in [-(2r_2+1), (2r_2+1)]\}\Delta\{(x, x + (1/r_2)\sin(x)) : x \in [-2r_2, 2r_2]\}) = \quad (45)$$

$$\mathcal{H}^1(\{(x,x) : x \in [-2r_2 - 1, 2r_2 + 1]\}\Delta\{(x, x + (1/r_2)\sin(x)) : x \in [-2r_2, 2r_2]\}) = \quad (46)$$

$$\neq 0 \quad (47)$$

Which is true, since $x \neq (1/r_2)\sin(x)$ for $x \in ([-2r_2 - 1, -2r_2] \cup [2r_2, 2r_2 + 1]) \setminus \{\pi m : m \in \mathbb{N}, -r_2 \leq \pi m \leq r_2\}$.

Now, considering the examples in §5.3.2 and §5.3.2, here are some shortcuts to definitions of equivelant and non-equivelant sequences of bounded functions:

5.3.5. Shortcuts to Determining Equivelant and Non-equivelant Sequences of Bounded Functions

Suppose we define $\overrightarrow{f_1} = (f_{r_1}^{(1)})_{r_1 \in \mathbb{N}}$, where $f_{r_1}^{(1)} : A_{r_1}^{(1)} \to \mathbb{R}$ and $\overrightarrow{f_2} = (f_{r_2}^{(2)})_{r_2 \in \mathbb{N}}$, where $f_{r_2}^{(2)} : A_{r_2}^{(2)} \to \mathbb{R}$:

(1) $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$ cannot be equivalent, unless:

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_{r_1}^{(1)})}\left(\left\{(x_1, \cdots, x_n) : f_{r_1}^{(1)}(x_1, \cdots, x_n) \neq f_{r_2}^{(2)}(x_1, \cdots, x_n)\right\}\right) = \quad (48)$$

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_{r_2}^{(2)})}\left(\left\{(x_1, \cdots, x_n) : f_{r_1}^{(1)}(x_1, \cdots, x_n) \neq f_{r_2}^{(2)}(x_1, \cdots, x_n)\right\}\right) = 0 \quad (49)$$

(2) $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$ is non-equivelant, when either:

(a) $\mathcal{H}^{\dim_{\mathrm{H}}(A_{r_1}^{(1)})}\left(\left\{(x_1, \cdots, x_n) : f_{r_1}^{(1)}(x_1, \cdots, x_n) \neq f_{r_2}^{(2)}(x_1, \cdots, x_n)\right\}\right) > 0$
(b) $\mathcal{H}^{\dim_{\mathrm{H}}(A_{r_2}^{(2)})}\left(\left\{(x_1, \cdots, x_n) : f_{r_1}^{(1)}(x_1, \cdots, x_n) \neq f_{r_2}^{(2)}(x_1, \cdots, x_n)\right\}\right) > 0$

5.4. Defining the "Measure"

5.4.1. Preliminaries

We define the "**measure**" of $(f_r)_{r \in \mathbb{N}}$ in §5.4.2, where $(G_r)_{r \in \mathbb{N}}$ is a sequence of the graph of each f_r. To understand this "measure", continue reading.

(1) For every $r \in \mathbb{N}$, “over-cover" G_r with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_{\text{H}}(G_r)}$ measure. (We denote the equal measures ε, where the former sentence is defined $\mathbf{C}(\varepsilon, G_r, \omega)$: i.e., $\omega \in \Omega_{\varepsilon,r}$ enumerates all collections of these sets covering G_r. In case this step is unclear, see §8.1.)

(2) For every ε, r and ω, take a sample point from each set in $\mathbf{C}(\varepsilon, G_r, \omega)$. The set of these points is “the sample" which we define $\mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)$: i.e., $\psi \in \Psi_{\varepsilon,r,\omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon, G_r, \omega)$. (If this is unclear, see §8.2.)

(3) For every ε, r, ω and ψ,

  (a) Take a “pathway” of line segments: we start with a line segment from arbitrary point x_0 of $\mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)$ to the sample point with the smallest $(n+1)$-dimensional Euclidean distance to x_0 (i.e., when more than one sample point has the smallest $(n+1)$-dimensional Euclidean distance to x_0, take either of those points). Next, repeat this process until the “pathway” intersects with every sample point once. (In case this is unclear, see §8.3.1.)

  (b) Take the set of the length of all segments in ((3)a), except for lengths that are outliers (i.e., for any constant $C > 0$, the outliers are more than C times the interquartile range of the length of all line segments as $r \to \infty$). Define this $\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi))$. (If this is unclear, see 8.3.2.)

  (c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))$. (In case this is unclear, see §8.3.3)

  (d) Take the shannon entropy [14, p.61-95] of step ((3)c). We define this:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))) = \sum_{x \in \mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))} -x \log_2 x$$

  which will be *shortened* to $\mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))$. (If this is unclear, see §8.3.4.)

  (e) Maximize the entropy w.r.t all "pathways". This we will denote:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)} \mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))$$

  (In case this is unclear, see §8.3.5.)

(4) Therefore, the **maximum entropy**, using ((1)) and ((2)) is:

$$\mathrm{E}_{\max}(\varepsilon, r) = \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r, \omega), \psi)))$$

5.4.2. What Am I Measuring?

Suppose we define two sequences of the graph of the bounded functions converging to the graph of f: e.g., $(G_r^{\star})_{r \in \mathbb{N}}$ and $(G_j^{\star\star})_{j \in \mathbb{N}}$, where for **constant** ε and *cardinality* $|\cdot|$

(a) Using ((2)) and (3(3)e) of section 5.4.1, suppose:

$$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} =$$
$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \le \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\}$$

then (using $\underline{|\mathcal{S}(\mathbf{C}(\epsilon, G_r^{\star}, \omega), \psi)|}$) we get

$$\underline{\alpha}(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi))|$$

(b) Also, using ((2)) and (3(3)e) of section 5.4.1, suppose:

$$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} =$$
$$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \ge \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\}$$

then (using $\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|}$) we also get:

$$\overline{\alpha}(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi))|$$

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(1) If using $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ we have:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi), \liminf_{\varepsilon \to 0} = \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G_r^\star)_{r\in\mathbb{N}}$ increases at a rate **superlinear** to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$.

(2) If using equations $\overline{\alpha}(\varepsilon, j, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$ (swapping $r \in \mathbb{N}$ and $(G_r^\star)_{r\in\mathbb{N}}$, in $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$, with $j \in \mathbb{N}$ and $(G_j^{\star\star})_{j\in\mathbb{N}}$) we get:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{j \to \infty} \sup_{\omega \in \Omega_{\varepsilon,j}} \sup_{\psi \in \Psi_{\varepsilon,j,\omega}} \overline{\alpha}(\varepsilon, j, \omega, \psi), \liminf_{\varepsilon \to 0} \liminf_{j \to \infty} \inf_{\omega \in \Omega_{\varepsilon,j}} \inf_{\psi \in \Psi_{\varepsilon,j,\omega}} \underline{\alpha}(\varepsilon, j, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G_r^\star)_{r\in\mathbb{N}}$ increases at a rate **sublinear** to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$.

(3) If using equations $\overline{\alpha}(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}(\varepsilon, r, \omega, \psi)$, $\overline{\alpha}(\varepsilon, j, \omega, \psi)$, and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$, we **both** have:

(a) $\limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi)$ or $\liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi)$ are equal to zero, one or $+\infty$

(b) $\limsup_{\varepsilon \to 0} \limsup_{j \to \infty} \sup_{\omega \in \Omega_{\varepsilon,j}} \sup_{\psi \in \Psi_{\varepsilon,j,\omega}} \overline{\alpha}(\varepsilon, j, \omega, \psi)$ or $\liminf_{\varepsilon \to 0} \liminf_{j \to \infty} \inf_{\omega \in \Omega_{\varepsilon,j}} \inf_{\psi \in \Psi_{\varepsilon,j,\omega}} \underline{\alpha}(\varepsilon, j, \omega, \psi)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(G_r^\star)_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$.

5.4.3. Example of The "Measure" of $(G_r^\star)$ Increasing at Rate Super-Linear to that of $(G_j^{\star\star})$

Suppose, we have function $f : A \to \mathbb{R}$, where $A = \mathbb{Q} \cap [0, 1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{50}$$

such that:

$$(A_r^\star)_{r\in\mathbb{N}} = (\{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r\in\mathbb{N}}$$

and

$$(A_j^{\star\star})_{j\in\mathbb{N}} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq j, 0 \leq c \leq j\})_{j\in\mathbb{N}}$$

where for $f_r^\star : A_r^\star \to \mathbb{R}$,

$$f_r^\star(x) = f(x) \text{ for all } x \in A_r^\star \tag{51}$$

and $f_j^{\star\star} : A_j^{\star\star} \to \mathbb{R}$

$$f_j^{\star\star}(x) = f(x) \text{ for all } x \in A_j^{\star\star} \tag{52}$$

Hence, when $(G_r^\star)_{r\in\mathbb{N}}$ is:

$$(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\}\})_{r\in\mathbb{N}} \tag{53}$$

and $(G_j^{\star\star})_{j\in\mathbb{N}}$ is:

$$(G_j^{\star\star})_{j\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq j, 0 \leq c \leq j\}\})_{j\in\mathbb{N}} \tag{54}$$

Note, the following:

Since $\varepsilon > 0$ and $A = \mathbb{Q} \cap [0, 1]$ is countably infinite, there exists minimum ε which is 1. Therefore, we don't need $\varepsilon \to 0$. Also, we maximize $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$ by the following procedure:

(1) For every $r \in \mathbb{N}$, group $x \in G_r^\star$ into elements with an even numerator when simplified: i.e.,

$$x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\}$$

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which we call $S_{1,r}$, and group $x \in G_r^\star$ into elements with an odd denominator when simplified: i.e.,

$$x \in \mathbb{Q} \setminus \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\}$$

which we call $S_{2,r}$

(2) Arrange the points in $S_{1,r}$ from least to greatest and take the 2-d Euclidean distance between each pair of consecutive points in $S_{1,r}$. In this case, since all points lie on $y = 1$, take the absolute difference between the x-coordinates of $S_{1,r}$ then call this $\mathcal{D}_{1,r}$. (Note, this is similar to §5.4.1 step (3)(3)a).

(3) Repeat step ((2)) for $S_{2,r}$, then call this $\mathcal{D}_{2,r}$. (Note, all point of $S_{2,r}$ lie on $y = 0$.)

(4) Remove any outliers from $\mathcal{D}_r = \mathcal{D}_{1,r} \cup \mathcal{D}_{2,r} \cup \{d((\frac{n!-1}{n!}, 1), (1, 0))\}$ (i.e., d is the 2-d Euclidean distance between points $(\frac{n!-1}{n!}, 1)$ and $(1, 0)$). Note, in this case, $\mathcal{D}_{2,r}$ and $\{d((\frac{n!-1}{n!}, 1), (1, 0))\}$ should be outliers (i.e., for any $C > 0$, the lengths of $\mathcal{D}_{2,r}$ are more than C times the interquartile range of the lengths of $\mathcal{D}_r$) leaving us with $\mathcal{D}_{1,r}$.

(5) Multiply the remaining lengths in the pathway by a constant so they add up to one. (See `P[r]` of code 1 for an example)

(6) Take the entropy of the probability distribution. (See `entropy[r]` of code 1 for an example.)

We can illustrate this process with the following code:

Code 1: Illustration of step ((1))-((6))

```
Clear["*Global`*"]
A[r_] := A[r] = Range[0, r!]/(r!)

(*Below is step (1)*)
S1[r_] :=
 S1[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[r_] :=
 S2[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step (2)*)
Dist1[r_] := Dist1[r] = Differences[S1[r]]

(*Below is step (3)*)
Dist2[r_] := Dist2[r] = Differences[S2[r]]

(*Below is step (4)*)
NonOutliers[r_] :=
 NonOutliers[r] = Dist1[r] (*We exclude Dist2[r] since it's an outlier*)

(*Below is step (5)*)
P[r_] := P[r] = NonOutliers[r]/Total[NonOutliers[r]]

(*Below is step (6)*)
entropy[r_] := entropy[r] = Total[-P[r] Log[2, P[r]]]
```

Taking `Table[{r,entropy[r]},{r,3,8}]`, we get:

Code 2: Output of `Table[{r,entropy[r]},{r,3,8}]`

```
Clear["*Global`*"]
{{{3,1}, {4,(2 Log[11])/(11 Log[2]) + (9 Log[22])/(11 Log[2])},
 {5,(14 Log[59])/(59 Log[2]) + (45 Log[118])/(59 Log[2])},
 {6,(44 Log[359])/(359 Log[2]) + (315 Log[718])/(359 Log[2])},
 {7,(314 Log[2519])/(2519 Log[2]) + (2205 Log[5038])/(2519 Log[2])},
 {8,(314 Log[20159])/(20159 Log[2]) + (19845 Log[40318])/(20159 Log[2])}}}
```

and notice when:

(1) $c(r) = (r!)/2 - 1$

(2) $\{b(4) \mapsto 9,\ b(5) \mapsto 45,\ b(6) \mapsto 315,\ b(7) \mapsto 2205,\ b(8) \mapsto 19845\}$

(3) $a(r) + b(r) = c(r)$

the output of code 2 can be defined:

$$\frac{a(r)\log_2(c(r))}{c(r)} + \frac{b(r)\log(2c(r))}{c(r)} = \frac{a(r)\log_2(c(r)) + b(r)\log(2c(r))}{c(r)} \tag{55}$$

Hence, since $a(r) = c(r) - b(r) = (r!)/2 - 1 - b(r)$:

$$\frac{a(r)\log_2(c(r)) + b(r)\log(2c(r))}{c(r)} = \tag{56}$$

$$\frac{(r!/2 - 1 - b(r))\log_2(c(r)) + b(r)\log_2(2c(r))}{c(r)} = \tag{57}$$

$$\frac{(r!/2)\log_2(c(r)) - \log_2(c(r)) - b(r)\log_2(r) + b(r)\log_2(c(r)) + b(r)\log_2(2)}{c(r)} = \tag{58}$$

$$\frac{(r!/2)\log_2(c(r)) - \log_2(c(r)) + b(r)}{c(r)} = \tag{59}$$

$$\frac{(r!/2 - 1)\log_2(c(r)) + b(r)}{c(r)} = \tag{60}$$

$$\frac{(r!/2 - 1)\log_2(r!/2 - 1) + b(r)}{r!/2 - 1} = \tag{61}$$

$$\log_2(r!/2 - 1) + \frac{b(r)}{r!/2 - 1} = \tag{62}$$

and $\lim_{r\to\infty} b(r)/c(r) = 1$ (I need help proving this):

$$\log_2(r!/2 - 1) + \frac{b(r)}{r!/2 - 1} \sim \log_2(r!/2 - 1) + 1 \tag{63}$$

$$\log_2(r!/2 - 1) + \log_2(2) = \tag{64}$$

$$\log_2(2(r!/2 - 1)) \tag{65}$$

$$\log_2(r! - 2) \sim \log_2(r!) \tag{66}$$

Hence, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))) \sim \log_2(r!) \tag{67}$$

Now, repeat code 1 with:

$$(A_j^{\star\star})_{j\in\mathbb{N}} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \le j, 0 \le c \le j\})$$

Code 3: Illustration of step ((1))-((6)) on $(A_j^{\star\star})$

```
Clear["*Global`*"]
A[j_] := A[j] =
  DeleteDuplicates[Flatten[Table[Range[0, t]/t, {t, 1, j}]]]

(*Below is step 1*)
S1[j_] :=
 S1[j] = Sort[Select[A[j], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[j_] :=
 S2[j] = Sort[Select[A[j], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[j_] := Dist1[j] = Differences[S1[j]]

(*Below is step 3*)
Dist2[j_] := Dist2[j] = Differences[S2[j]]

(*Below is step 4*)
NonOutliers[j_] :=
 NonOutliers[j] = Join[Dist1[j], Dist2[j]] (*There are no outliers*)

(*Below is step 5*)
P[j_] := P[j] = NonOutliers[j]/Total[NonOutliers[j]]
```

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```
(*Below is step 6*)
entropy[j_] := entropy[j] = N[Total[-P[j] Log[2, P[j]]]]
```

Using this post [22], we assume an approximation of `Table[entropy[j],{j,3,Infinity}]` or $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_j^{\star\star},\omega'),\psi')))$ is:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_j^{\star\star},\omega'),\psi'))) \sim 2\log_2(j)+1-\log_2(3\pi) \tag{68}$$

Hence, using §5.4.2 ((0)a) and §5.4.2 ((1)), take $\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega'),\psi')\right| = \sum_{M=1}^{j}\phi(M) \approx \frac{3}{\pi^2}j^2$ (where ϕ is Euler's Totient function) computing the following:

$$\begin{aligned}
&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|} = \\
&\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega'),\psi')\right| : j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega'),\psi')))\le \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)))\right\} = \\
&\sup\left\{\frac{3}{\pi^2}j^2 : j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ 2\log_2(j)+1-\log_2(3\pi)\le\log_2(r!)\right\} =
\end{aligned} \tag{69}$$

where:

(1) For every $r\in\mathbb{N}$, we find a $j\in\mathbb{N}$, where $2\log_2(j)+1-\log_2(3\pi)\le\log_2(r!)$, but the absolute value of $\log_2(r!)-(2\log_2(j)+1-\log_2(3\pi))$ is minimized. In other words, for every $r\in\mathbb{N}$, we want $j\in\mathbb{N}$ where:

$$2\log_2(j)+1-\log_2(3\pi)\le\log_2(r!) \tag{70}$$
$$2\log_2(j)\le\log_2(r!)-1+\log_2(3\pi) \tag{71}$$
$$\left(2^{\log_2(j)}\right)^2\le 2^{\log_2(r!)-1+\log_2(3\pi)} \tag{72}$$
$$j^2\le\left(2^{\log_2(r!)}2^{\log_2(3\pi)}\right)/2 \tag{73}$$
$$j\le\sqrt{\frac{r!(3\pi)}{2}} \tag{74}$$
$$j=\left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor \tag{75}$$
$$\frac{3}{\pi^2}j^2=\frac{3}{\pi^2}\left(\left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor\right)^2\sim\underline{|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)|} \tag{76}$$

Finally, since $|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|=r!$, we wish to prove

$$1<\liminf_{\varepsilon\to0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)<+\infty$$

within §5.4.2 crit. (1):

$$\liminf_{\varepsilon\to0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)=\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\frac{|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^{\star},\omega),\psi)|}{\underline{|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi))|}} \tag{77}$$

$$=\lim_{r\to\infty}\frac{\frac{3}{\pi^2}\left(\left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor\right)^2}{r!} \tag{78}$$

where using mathematica, we get the limit is greater than one:

Code 4: Limit of eq. 78

```
Clear["*Global`*"]
N[Limit[((3/Pi^2) (Floor[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]

(*Output is 1.43239*)
```

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Also, using §5.4.2 ((0)b) and §5.4.2 ((1)), take $\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| = \sum_{M=1}^{j} \phi(M) \approx \frac{3}{\pi^2} j^2$ (where ϕ is Euler's Totient function) to compute the following:

$$\begin{aligned} &\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} = \\ &\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\} = \\ &\inf\left\{\frac{3}{\pi^2} j^2 : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ 2\log_2(j) + 1 - \log_2(3\pi) \geq \log_2(r!)\right\} = \end{aligned} \tag{79}$$

where:

(1) For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $2\log_2(j) + 1 - \log_2(3\pi) \geq \log_2(r!)$, but the absolute value of $(2\log_2(j) + 1 - \log_2(3\pi)) - \log_2(r!)$ is minimized. In other words,
for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$2\log_2(j) + 1 - \log_2(3\pi) \geq \log_2(r!) \tag{80}$$

$$2^{2\log_2(j)} \geq \log_2(r!) - 1 + \log_2(3\pi) \tag{81}$$

$$\left(2^{\log_2(j)}\right)^2 \geq 2^{\log_2(r!) - 1 + \log_2(3\pi)} \tag{82}$$

$$j^2 \geq \left(2^{\log_2(r!)} 2^{\log_2(3\pi)}\right)/2 \tag{83}$$

$$j \geq \sqrt{\frac{r!(3\pi)}{2}} \tag{84}$$

$$j = \left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil \tag{85}$$

$$\frac{3}{\pi^2} j^2 = \frac{3}{\pi^2}\left(\left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil\right)^2 \sim \overline{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)|} \tag{86}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)| = r!$, we wish to prove

$$1 < \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

within §5.4.2 crit. (1):

$$\limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi) = \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \frac{\overline{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)|}}{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))|} \tag{87}$$

$$= \lim_{r \to \infty} \frac{\frac{3}{\pi^2}\left(\left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil\right)^2}{r!} \tag{88}$$

where using mathematica, we get the limit is greater than one:

Code 5: Limit of eq. 88

```
N[Limit[((3/Pi^2) (Ceiling[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]
  (*The output is 1.43239*)
```

Hence, since the limits in eq. 78 and eq. 88 are greater than one and less than $+\infty$: i.e.,

$$1 < \liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) = \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty \tag{89}$$

what we're measuring from $(G_r^{\star})_{r \in \mathbb{N}}$ increases at a rate **superlinear** to that of $(G_j^{\star\star})_{j \in \mathbb{N}}$ (i.e., 5.4.2 crit. (1)).

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5.4.4. Example of The "Measure" from $(G_r^{\star})_{r\in\mathbb{N}}$ Increasing at a Rate Sub-Linear to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$

Using our previous example, we can use the following theorem:

Theorem 10. *If what we're measuring from $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate superlinear to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$, then what we're measuring from $(G_j^{\star\star})_{r\in\mathbb{N}}$ increases at a rate* ***sublinear*** *to that of $(G_r^{\star})_{r\in\mathbb{N}}$*

Hence, in our definition of super-linear (§5.4.2 crit. (1)), swap $G_r^{\star}$ and $r\in\mathbb{N}$ for $(G_j^{\star\star})$ and $j\in\mathbb{N}$ regarding $\overline{\alpha}(\epsilon,r,\omega,\psi)$ and $\underline{\alpha}(\epsilon,r,\omega,\psi)$ (i.e., $\overline{\alpha}(\epsilon,j,\omega,\psi)$ and $\underline{\alpha}(\epsilon,j,\omega,\psi)$) and notice thm. 10 is true when:

$$1<\limsup_{\varepsilon\to 0}\limsup_{j\to\infty}\sup_{\omega\in\Omega_{\varepsilon,j}}\sup_{\psi\in\Psi_{\varepsilon,j,\omega}}\overline{\alpha}(\varepsilon,j,\omega,\psi),\ \liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\inf_{\omega\in\Omega_{\varepsilon,j}}\inf_{\psi\in\Psi_{\varepsilon,j,\omega}}\underline{\alpha}(\varepsilon,j,\omega,\psi)<+\infty$$

5.4.5. Example of The "Measure" from $(G_r^{\star})_{r\in\mathbb{N}}$ Increasing at a Rate Linear to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$

Suppose, we have function $f:A\to\mathbb{R}$, where $A=\mathbb{Q}\cap[0,1]$, and:

$$f(x)=\begin{cases}1 & x\in\{(2s+1)/(2t):s\in\mathbb{Z},t\in\mathbb{N},t\neq 0\}\cap[0,1]\\ 0 & x\notin\{(2s+1)/(2t):s\in\mathbb{Z},t\in\mathbb{N},t\neq 0\}\cap[0,1]\end{cases} \tag{90}$$

such that:

$$(A_r^{\star})_{r\in\mathbb{N}}=(\{c/r!:c\in\mathbb{Z},0\le c\le r!\})_{r\in\mathbb{N}}$$

and

$$(A_j^{\star\star})_{j\in\mathbb{N}}=\left(\left\{c/((j!)^2):c\in\mathbb{N},1\le c\le (j!)^2\right\}\right)$$

where for $f_r^{\star}:A_r^{\star}\to\mathbb{R}$,

$$f_r^{\star}(x)=f(x)\text{ for all }x\in A_r^{\star} \tag{91}$$

and $f_j^{\star\star}:A_j^{\star\star}\to\mathbb{R}$

$$f_j^{\star\star}(x)=f(x)\text{ for all }x\in A_j^{\star\star} \tag{92}$$

Hence, when $(G_r^{\star})_{r\in\mathbb{N}}$ is:

$$(G_r^{\star})_{r\in\mathbb{N}}=(\{(x,f(x)):x\in\{c/r!:c\in\mathbb{Z},0\le c\le r!\}\})_{r\in\mathbb{N}} \tag{93}$$

and $(G_j^{\star\star})_{j\in\mathbb{N}}$ is:

$$(G_j^{\star\star})_{j\in\mathbb{N}}=\left(\left\{(x,f(x)):x\in\left\{c/((j!)^2):c\in\mathbb{Z},0\le c\le (j!)^2\right\}\right\}\right)_{j\in\mathbb{N}} \tag{94}$$

We already know, using §5.4.3:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)))\sim\log_2(r!-2)\sim\log_2(r!) \tag{95}$$

Also, using §5.4.3 steps (1)-(6) on $(A_j^{\star\star})_{j\in\mathbb{N}}$:

Code 6: Illustration of step ((1))-((6)) on $(A_j^{\star\star})$

```
Clear["*Global`*"]
A[j_] := A[j] =  Range[0, 7 (j!)]/(7 (j!))

(*Below is step 1*)
S1[j_] :=
 S1[j] = Sort[Select[A[j], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[j_] :=
 S2[j] = Sort[Select[A[j], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[j_] := Dist1[j] = Differences[S1[j]]
```

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```
(*Below is step 3*)
Dist2[j_] := Dist2[j] = Differences[S2[j]]

(*Below is step 4*)
NonOutliers[j_] :=
 NonOutliers[j] = Dist1[j] (*Dist2[j] is an outlier*)

(*Below is step 5*)
P[j_] := P[j] = NonOutliers[j]/Total[NonOutliers[j]]

(*Below is step 6*)

entropy[j_] := entropy[j] = N[Total[-P[j] Log[2, P[j]]]]

T = Table[{j,entropy[j]},{j,3,6}]
```

where the output is

Code 7: Output of Code 6

```
{{3,(8 Log[17])/(17 Log[2]) + (9 Log[34])/(17 Log[2])},
     {4,(8 Log[287])/(287 Log[2]) + (279 Log[574])/(287 Log[2])},
     {5,(224 Log[7199])/(7199 Log[2]) + (6975 Log[14398])/(7199 Log[2])},
     {6,(2024 Log[259199])/(259199 Log[2]) + (257175 Log[518398])/(259199 Log[2])}}
```

Notice when:

(1) $c(j) = (j!)^2/2 - 1$
(2) $\{b(4) \mapsto 9,\ b(5) \mapsto 279,\ b(6) \mapsto 6975,\ b(7) \mapsto 257175,\ b(8) \mapsto 19845\}$
(3) $a(j) + b(j) = c(j)$

the output of code 7 can be defined:

$$\frac{a(j)\log_2(c(j))}{c(j)} + \frac{b(j)\log(2c(j))}{c(j)} = \frac{a(j)\log_2(c(j)) + b(j)\log(2c(j))}{c(j)} \tag{96}$$

Hence, since $a(j) = c(j) - b(j) = (j!)^2/2 - 1 - b(j)$:

$$\frac{a(j)\log_2(c(j)) + b(j)\log(2c(j))}{c(j)} = \tag{97}$$

$$\frac{((j!)^2/2 - 1 - b(j))\log_2(c(j)) + b(j)\log_2(2c(j))}{c(j)} = \tag{98}$$

$$\frac{((j!)^2/2)\log_2(c(j)) - \log_2(c(j)) - b(j)\log_2(j) + b(j)\log_2(c(j)) + b(j)\log_2(2)}{c(j)} = \tag{99}$$

$$\frac{((j!)^2/2)\log_2(c(j)) - \log_2(c(j)) + b(j)}{c(j)} = \tag{100}$$

$$\frac{((j!)^2/2 - 1)\log_2(c(j)) + b(j)}{c(j)} = \tag{101}$$

$$\frac{((j!)^2/2 - 1)\log_2((j!)^2/2 - 1) + b(j)}{(j!)^2/2 - 1} = \tag{102}$$

$$\log_2((j!)^2/2 - 1) + \frac{b(j)}{(j!)^2/2 - 1} = \tag{103}$$

since $\lim_{r\to\infty} b(r)/c(r) = 1$ (this is proven in [23]):

$$\log_2((j!)^2/2-1) + \frac{b(j)}{(j!)^2/2-1} \sim \log_2((j!)^2/2-1)+1 \tag{104}$$
$$\log_2((j!)^2/2-1) + \log_2(2) = \tag{105}$$
$$\log_2((j!)^2-2)) \sim \tag{106}$$
$$\log_2((j!)^2) = \tag{107}$$
$$2\log_2(j!) \tag{108}$$

Hence, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))) \sim \tag{109}$$
$$2\log_2(j!) \tag{110}$$

Therefore, using §5.4.2 ((0)a) and §5.4.2 ((3)(3)a), take $\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| = (j!)^2$ to compute the following:

$$\begin{aligned}
&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} = \\
&\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \le \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\} = \\
&\sup\left\{(j!)^2 : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ 2\log_2(j!) \le \log_2(r!)\right\} =
\end{aligned} \tag{111}$$

where:

(1) For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $2\log_2(j!) \le \log_2(r!)$, but the absolute value of $\log_2(r!) - 2\log_2(j!)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$2\log_2(j!) \le \log_2(r!) \tag{112}$$
$$2^{2\log_2(j!)} \le 2^{\log_2(r!)} \tag{113}$$
$$(2^{\log_2(j!)})^2 \le r! \tag{114}$$
$$(j!)^2 \le r! \tag{115}$$
$$(j!)^2 = \lfloor r! \rfloor \tag{116}$$

To solve for j, we try the following code:

Code 8: Code for j in eq. 116

```
Clear["Global`*"]

T1 = Table[
   {sol[r_] := sol[r] = Reduce[j > 0 && ((j!)^2) <= r!, j, Integers],
    jsolve = Max[j /. Solve[sol[r], {j}, Integers]],
    (* Largest j that solves inequality (j!)^2<=r for every r *)
    , N[(jsolve!)^2/(r!)]}, {r, 3, 40}];

Tablejsolve =
 Table[{T1[[r - 3 + 1, 2]], r}, {r, 3,
   40}] (*Takes largest j-values for every r in r!*)

loweralphr =
 Table[{r, T1[[r - 3 + 1, 4]]}, {r, 3,
   40}] (* Takes largest largest j values and corresponding r value*)

ListPlot[loweralphr] (*Graph points of upperalph. Notice, the graph has
a lower bound of zero.*)
```

Note, the output is:

Code 9: Output for code 8

```
Clear["Global`*"]
(* Output of Tablejsolve*)
{{2, 3}, {2, 4}, {3, 5}, {4, 6}, {4, 7}, {5, 8}, {5, 9}, {6, 10}, {7, 11}, {7, 12},
 {8, 13}, {8, 14}, {9, 15}, {10, 16}, {10, 17}, {11, 18}, {11, 19}, {12, 20}, {13, 21},
 {13, 22}, {14, 23}, {14, 24}, {15, 25}, {15, 26}, {16, 27}, {17, 28}, {17, 29},
 {18, 30}, {18, 31}, {19, 32}, {20, 33}, {20, 34}, {21, 35}, {21, 36}, {22, 37}, {22, 38},
 {23, 39}, {24, 40}}

(*Output of loweralphr*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

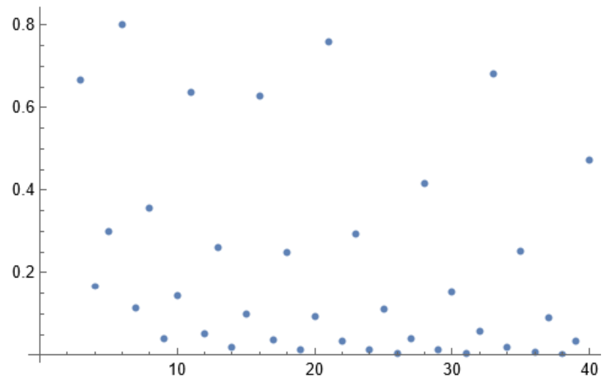

Figure 1. Plot of `loweralphr`.

Finally, since the lower bound of `loweralphr` is zero, we have shown:

$$\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)=0 \tag{117}$$

Next, using §5.4.2 ((0)b) and §5.4.2 ((3)(3)b), take $|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')|=r!$ and swap $r\in\mathbb{N}$ and $(G_r^{\star})_{r\in\mathbb{N}}$ with $j\in\mathbb{N}$ and $(G_j^{\star\star})_{j\in\mathbb{N}}$, to compute the following:

$$\begin{aligned}&\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega),\psi)\right|}=\\&\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')\right|:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,r,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')))\geq\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega),\psi)))\right\}=\\&\inf\{r!:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \log_2(r!)\geq 2\log_2(j!)\}=\end{aligned}\tag{118}$$

where:

(1) For every $j\in\mathbb{N}$, we find a $r\in\mathbb{N}$, where $\log_2(r!)\leq 2\log_2(j!)$, but the absolute value of $2\log_2(j!)-\log_2(r!)$ is minimized. In other words, for every $j\in\mathbb{N}$, we want $r\in\mathbb{N}$ where:

$$\log_2(r!)\leq 2\log_2(j!) \tag{119}$$
$$2^{\log_2(r!)}\leq 2^{2\log_2(j!)} \tag{120}$$
$$r!\leq(2^{\log_2(j!)})^2 \tag{121}$$
$$r!\leq(j!)^2 \tag{122}$$
$$r!=(j!)^2 \tag{123}$$

To solve r, we try the following code:

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Code 10: Code for r in eq. 123

```
Clear["Global`*"]

Clear["Global`*"]
T2 = Table[
   {sol[j_] := sol[j] = Reduce[j > 0 && r! <= (j!)^2, r, Integers],
    rsolve = Max[r /. Solve[sol[j], {r}, Integers]],
    (* Largest r that solves inequality (r!)<=(j!)^2 for every r *)
    , N[(rsolve!)/((j!)^2)]}, {j, 3, 40}];

Tablersolve =
 Table[{T2[[j - 3 + 1, 2]], j}, {j, 3,
   40}] (*Takes largest r-values for every j in j!*)

loweralphj =
 Table[{j, T2[[j - 3 + 1, 4]]}, {j, 3,
   40}] (* Takes largest largest r values and corresponding j value*)

ListPlot[loweralphj](*Graph points of upperalph. Notice, the graph
has a upperbound of Infinity *)
```

Note, the output is:

Code 11: Output for code 10

```
Clear["Global`*"]
(* Output of Tablersolve*)
{{4, 3}, {5, 4}, {7, 5}, {9, 6}, {10, 7}, {12, 8}, {14, 9}, {15, 10}, {17, 11}, {19, 12},
 {20, 13}, {22, 14}, {24, 15}, {26, 16}, {27, 17}, {29, 18}, {31, 19}, {32, 20}, {34, 21},
 {36, 22}, {38, 23}, {39, 24}, {41, 25}, {43, 26}, {44, 27}, {46, 28}, {48, 29}, {50, 30},
 {51, 31}, {53, 32}, {55, 33}, {57, 34}, {58, 35}, {60, 36}, {62, 37}, {64, 38}, {65, 39},
 {67, 40}}

(*Output of loweralphj*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

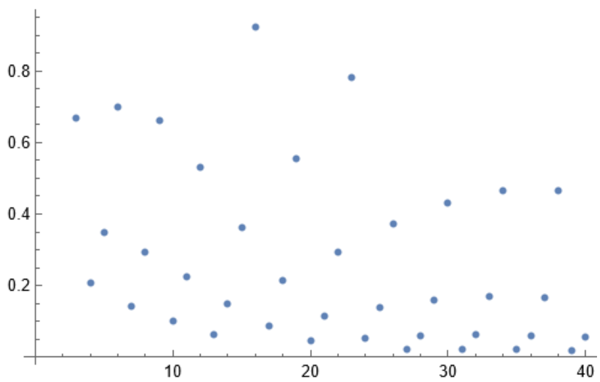

Figure 2. Plot of `loweralphj`.

since the lower bound of `loweralphj` is zero, we have shown:

$$\liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\inf_{\omega\in\Omega_{\varepsilon,j}}\inf_{\psi\in\Psi_{\varepsilon,j,\omega}}\underline{\alpha}(\varepsilon,j,\omega,\psi)=0 \tag{124}$$

Hence, using eq. 117 and 124, since **both**:

(1) $\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon,r,\omega,\psi)$ or $\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)$ are equal to zero, one or $+\infty$

(2) $\limsup_{\varepsilon\to 0}\limsup_{j\to\infty}\sup_{\omega\in\Omega_{\varepsilon,j}}\sup_{\psi\in\Psi_{\varepsilon,j,\omega}}\overline{\alpha}(\varepsilon,j,\omega,\psi)$ or $\liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\inf_{\omega\in\Omega_{\varepsilon,j}}\inf_{\psi\in\Psi_{\varepsilon,j,\omega}}\underline{\alpha}(\varepsilon,j,\omega,\psi)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(G_r^\star)_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$.

5.5. Defining The Actual Rate of Expansion of Sequence of Bounded Sets

5.5.1. Definition of Actual Rate of Expansion of Sequence of Bounded Sets

Suppose $(f_r)_{r\in\mathbb{N}}$ is a sequence of bounded functions converging to f, where $(G_r)_{r\in\mathbb{N}}$ is a sequence of the graph on each f_r, and $d(Q,R)$ is the Euclidean distance between points $Q,R\in\mathbb{R}^n$. Therefore, using the "chosen" center point $C\in\mathbb{R}^{n+1}$, when:

$$\mathcal{G}(C,G_r)=\sup\{d(C,y):y\in G_r\}$$

the **actual rate of expansion** is:

$$\mathcal{E}(C,G_r)=\mathcal{G}(C,G_{r+1})-\mathcal{G}(C,G_r)$$

Note, there are cases of $(G_r)_{r\in\mathbb{N}}$ when $\mathcal{E}$ isn't fixed and $\mathcal{E}\neq E$ (i.e., **the chosen, fixed rate of expansion**).

5.5.2. Example

Suppose, we have $f:A\to\mathbb{R}$, where $A=\mathbb{R}$ and $f(x)=x$, such that $(A_r^\star)_{r\in\mathbb{N}}=([-r,r])_{r\in\mathbb{N}}$ and for $f_r^\star:A_r^\star\to\mathbb{R}$:

$$f_r^\star(x)=f(x)\text{ for all }x\in A_r^\star$$

Hence, when $(G_r^\star)_{r\in\mathbb{N}}$ is:

$$(G_r^\star)_{r\in\mathbb{N}}=(\{(x,x):x\in[-r,r]\})_{r\in\mathbb{N}}$$

such that $C=(0,0)$, note the farthest point of $G_r^\star$ from C is either $(-r,-r)$ or (r,r). Hence, to compute $\mathcal{G}(C,G_r^\star)$, we can take $d((0,0),(r,r))$ or $d((0,0),(-r,r))$:

$$\begin{aligned}\mathcal{G}(C,G_r^\star)=&\sup\{d(C,y):y\in G_r^\star\}= &(125)\\&d((0,0),(r,r))= &(126)\\&\sqrt{(0-r)^2+(0-r)^2}= &(127)\\&\sqrt{r^2+r^2}= &(128)\\&\sqrt{2r^2}= &(129)\\&\sqrt{2r^2}= &(130)\\&\sqrt{2}|r| &(131)\\&\sqrt{2}r\text{ since }r>0 &(132)\end{aligned}$$

and the actual rate of expansion is:

$$\begin{aligned}\mathcal{E}(C,G_r^\star)=&\mathcal{G}(C,G_{r+1}^\star)-\mathcal{G}(C,G_r^\star)= &(133)\\&\sqrt{2}(r+1)-\sqrt{2}r= &(134)\\&\sqrt{2}(r+1)-\sqrt{2}r= &(135)\\&\sqrt{2}r+\sqrt{2}-\sqrt{2}r= &(136)\\&\sqrt{2} &(137)\end{aligned}$$

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5.6. Reminder

See if §3.2 is easier to understand.

6. My Attempt At Answering The Blockquote of §1.3.2

6.1. Choice Function

Suppose we define the following:

(1) $(f_k^{\star\star\star})_{k\in\mathbb{N}}$ is the sequence of bounded functions which satisfies ((1)), ((2)), ((3)), ((4)) and ((5)) of the **leading question** in §3.2
(2) $\mathbb{S}'(f)$ is all sequences of bounded functions satisfying ((1)) of the **leading question** where the expected values, defined in the papers of §2, is finite.
(3) $(f_j^{\star\star})_{j\in\mathbb{N}}$ is an element $\mathbb{S}'(f)$ but **not** an element in the set of equivelant sequences of bounded functions to that of $(f_k^{\star\star\star})_{k\in\mathbb{N}}$ (def. 7), where using the end of def. 7, we represent this criteria as:

$$(f_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(f) \setminus \sim (f_k^{\star\star\star})_{k\in\mathbb{N}}$$

Further note, from §5.4.2 ((0)b), if we take:

$$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi)|} = \tag{138}$$
$$\inf\left\{ \left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi))) \right\}$$

and from §5.4.2 ((0)a), we take:

$$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi)|} = \tag{139}$$
$$\sup\left\{ \left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi))) \right\}$$

Then, §5.4.1 ((2)), eq. 138, and eq. 139 is:

$$\sup_{\omega\in\Omega_{\varepsilon,k}} \sup_{\psi\in\Psi_{\varepsilon,k,\omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi)| = \left|\mathcal{S}'(\varepsilon, G_k^{\star\star\star})\right| = \left|\mathcal{S}'\right| \tag{140}$$

$$\sup_{\omega\in\Omega_{\varepsilon,k}} \sup_{\psi\in\Psi_{\varepsilon,k,\omega}} \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi)|} = \overline{\left|\mathcal{S}'(\varepsilon, G_k^{\star\star\star})\right|} = \overline{\left|\mathcal{S}'\right|} \tag{141}$$

$$\sup_{\omega\in\Omega_{\varepsilon,k}} \sup_{\psi\in\Psi_{\varepsilon,k,\omega}} \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_k^{\star\star\star}, \omega), \psi)|} = \underline{\left|\mathcal{S}'(\varepsilon, G_k^{\star\star\star})\right|} = \underline{\left|\mathcal{S}'\right|} \tag{142}$$

6.2. Approach

We manipulate the definitions of §5.4.2 ((0)a) and §5.4.2 ((0)b) to solve ((1)), ((2)), ((3)), ((4)) and ((5)) of the *leading question* in §3.2

6.3. Potential Answer

6.3.1. Preliminaries (Definition of T in Case of §3.2.1 ((5)))

When the difference of point $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ is:

$$X - Y = (x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n)$$

the average of $G_r^\star$ for every $r \in \mathbb{N}$ is:

$$\mathrm{Avg}(G_r^\star) = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)} \int_{G_r^\star} (x_1, \cdots, x_n)\, d\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)} \tag{143}$$

and $d(P, Q)$ is the n-d Euclidean distance between points $P, Q \in \mathbb{R}^n$, we define an *explicit* injective $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}$ such that:

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(1) If $d(\text{Avg}(G_r^{\star}), C) < d(\text{Avg}(G_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^{\star}) - C) < \mathscr{F}(\text{Avg}(G_j^{\star\star}) - C)$
(2) If $d(\text{Avg}(G_r^{\star}), C) > d(\text{Avg}(G_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^{\star}) - C) > \mathscr{F}(\text{Avg}(G_j^{\star\star}) - C)$
(3) If $d(\text{Avg}(G_r^{\star}), C) = d(\text{Avg}(F_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^{\star}) - C) \neq \mathscr{F}(\text{Avg}(G_j^{\star\star}) - C)$

where using "chosen" center point $C \in \mathbb{R}^n$:

$$T(C, G_r^{\star}) = \mathscr{F}(\text{Avg}(G_r^{\star}) - C) \tag{144}$$

6.3.2. Question

Does T exist? If so, how do we define it?
Hence, using $|\mathcal{S}'|$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, E, $\mathcal{E}(C, G_k^{\star\star\star})$ (§5.5), and $T(C, F_k^{\star\star\star})$, such that with the absolute value function $||\cdot||$, ceiling function $\lceil\cdot\rceil$, and nearest integer function $[\cdot]$, we define:

$$K(\varepsilon, G_k^{\star\star\star}) =$$
$$\left(1 + \left|\left|E - \mathcal{E}(C, G_k^{\star\star\star})\right|\right|\right)\left(\left|\left|\frac{|\mathcal{S}'|\left(1 + \left\lceil\frac{|\mathcal{S}'|\left(|\underline{\mathcal{S}'}| + 2|\mathcal{S}'|\right)}{\left(|\underline{\mathcal{S}'}| + |\mathcal{S}'|\right)\left(|\underline{\mathcal{S}'}| + |\mathcal{S}'| + |\overline{\mathcal{S}'}|\right)}\right\rceil\right)\left(1 + \left[|\underline{\mathcal{S}'}|/|\mathcal{S}'|\right]\right)}{\left(1 + \left[|\mathcal{S}'|/|\overline{\mathcal{S}'}|\right]\right)\left(1 + \left[|\underline{\mathcal{S}'}|/|\overline{\mathcal{S}'}|\right]\right)} - |\mathcal{S}'|\right|\right| + |\mathcal{S}'|\right) - T(C, G_k^{\star\star\star})\mathcal{E}(C, G_k^{\star\star\star}) \tag{145}$$

where $\mathcal{E}$, E, and T are "removed" when $\mathcal{E}, E = 0$, the choice function which answers the **leading question** in §3.2 could be the following, s.t. we explain the reason behind choosing the choice function in §6.4:

Theorem 11. *If we define:*

$$\mathcal{M}(\varepsilon, G_k^{\star\star\star}) = |\mathcal{S}'(\varepsilon, G_k^{\star\star\star})|\left(K(\varepsilon, G_k^{\star\star\star}) - |\mathcal{S}'(\varepsilon, G_k^{\star\star\star})|\right)$$
$$\mathcal{M}(\varepsilon, G_j^{\star\star}) = |\mathcal{S}'(\varepsilon, G_j^{\star\star})|\left(K(\varepsilon, G_j^{\star\star}) - |\mathcal{S}'(\varepsilon, G_j^{\star\star})|\right)$$

where for $\mathcal{M}(\varepsilon, G_k^{\star\star\star})$, we define $\mathcal{M}(\varepsilon, G_k^{\star\star\star})$ to be the same as $\mathcal{M}(\varepsilon, G_j^{\star\star})$ when swapping "$j \in \mathbb{N}$" with "$k \in \mathbb{N}$" (for eq. 138 & 139) and sets $G_k^{\star\star\star}$ with $G_j^{\star\star}$ (for eq. 138–145), then for constant $v > 0$ and variable $v^* > 0$, if:

$$\overline{\mathcal{S}}(\varepsilon, k, v^*, G_j^{\star\star}) = \inf\left(\left\{|\mathcal{S}'(\varepsilon, G_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, G_j^{\star\star}) \geq \mathcal{M}(\varepsilon, G_k^{\star\star\star}) \geq v^*\right\} \cup \{v^*\}\right) + v \tag{146}$$

and:

$$\underline{\mathcal{S}}(\varepsilon, k, v^*, G_j^{\star\star}) = \sup\left(\left\{|\mathcal{S}'(\varepsilon, G_j^{\star\star})| : j \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, G_j^{\star\star}) \leq \mathcal{M}(\varepsilon, G_k^{\star\star\star})\right\} \cup \{-v^*\}\right) + v \tag{147}$$

then for all $(f_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(f) \backslash \sim (f_k^{\star\star\star})_{k\in\mathbb{N}}$ (def. 7), if:

$$\inf\left\{||1 - c|| : \forall(\epsilon > 0)\exists(c > 0)\forall(k \in \mathbb{N})\exists(j \in \mathbb{N})\left(\left|\left|\frac{|\mathcal{S}'(\varepsilon, G_k^{\star\star\star})|}{|\mathcal{S}'(\varepsilon, G_j^{\star\star})|} - c\right|\right| < \varepsilon\right)\right\} \tag{148}$$

where $\lceil\cdot\rceil$ is the ceiling function, E is the fixed rate of expansion, Γ is the gamma function, n is the dimension of $\mathbb{R}^n$, $\dim_H(G_k^{\star\star\star})$ is the Hausdorff dimension of set $G_k^{\star\star\star} \subseteq \mathbb{R}^{n+1}$, and $\mathbf{A}_k$ is area of the smallest $(n+1)$-dimensional box that contains $A_k^{\star\star\star}$, then:

$$V(\varepsilon, G_k^{\star\star\star}, n) = \tag{149}$$
$$\left\lceil\left(\mathbf{A_k}^{1-\text{sign}(E)}(E - \text{sign}(E) + 1)\left(\frac{\exp(n\ln(\pi)/2)}{\Gamma(n/2+1)}\right)\left(k!^{(n-\dim_{\text{H}}(G_k^{\star\star\star}))}\right)\left(k^{\text{sign}(E)(\dim_{\text{H}}(G_k^{\star\star\star}) - \text{sign}(\dim_{\text{H}}(G_k^{\star\star\star}))+1)}\right) + (1 - \text{sign}(\dim_{\text{H}}(G_k^{\star\star\star})))\right)/\varepsilon\right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_k^{\star\star\star})|$$

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the choice function is:

$$\limsup_{\varepsilon\to 0}\lim_{v^*\to\infty}\limsup_{k\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}-c^{-V(\varepsilon,G_k^{\star\star\star},n)}\right)\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}-c^{-V(\varepsilon,G_k^{\star\star\star},n)}\right)= \tag{150}$$

$$\liminf_{\varepsilon\to 0}\lim_{v^*\to\infty}\liminf_{k\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}-c^{-V(\varepsilon,G_k^{\star\star\star},n)}\right)\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}-c^{-V(\varepsilon,G_k^{\star\star\star},n)}\right)=0 \tag{151}$$

such that $(G_k^{\star\star\star})_{k\in\mathbb{N}}$ *satisfies eq. 150 & eq. 151. (Note, we want* $\sup\emptyset=-\infty$, $\inf\emptyset=+\infty$*, and* $(f_k^{\star\star\star})_{k\in\mathbb{N}}$ *to answer the leading question of* §3.2*) where the answer to the blockquote of* §1.3.2 *is* $\mathbb{E}[f_k^{\star\star\star}]$ *(when it exists).*

6.4. Explaining The Choice Function and Evidence The Choice Function Is Credible

Notice, before reading the programming in code 12, without the "c"-terms in eq. 150 and eq. 151:

(1) The choice function in eq. 150 and eq. 151 is zero, when *what I'm measuring* from $(G_k^{\star\star\star})_{k\in\mathbb{N}}$ (§5.4.2 criteria (1)) increases at a rate superlinear to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$, where $\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))=0$.

(2) The choice function in eq. 150 and eq. 151 is zero, when for a given $(G_k^{\star\star\star})_{k\in\mathbb{N}}$ and $(G_j^{\star\star})_{j\in\mathbb{N}}$ there doesn't exist c where eq. 148 is satisfied or $c=0$.

(3) When c does exist, suppose:

$$\left\{\mathcal{J}(k):k\in\mathbb{N},\frac{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|}{|\mathcal{S}'(\varepsilon,G_{\mathcal{J}(k)}^{\star\star})|}\approx c\right\} \tag{152}$$

(a) When $|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|<|\mathcal{S}'(\varepsilon,G_{\mathcal{J}(k)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0}\lim_{v^*\to\infty}\limsup_{k\to\infty}\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}=c \tag{153}$$

$$\liminf_{\varepsilon\to 0}\lim_{v^*\to\infty}\liminf_{k\to\infty}\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}=0 \tag{154}$$

(b) When $|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|>|\mathcal{S}'(\varepsilon,G_{\mathcal{J}(k)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0}\lim_{v^*\to\infty}\limsup_{k\to\infty}\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}=+\infty \tag{155}$$

$$\liminf_{\varepsilon\to 0}\lim_{v^*\to\infty}\liminf_{k\to\infty}\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon,k,v^*,G_j^{\star\star})}{|\mathcal{S}'(\varepsilon,G_k^{\star\star\star})|+v}=1/c \tag{156}$$

Hence, for each sub-criteria under crit. ((3)), if we subtract one of their limits by their limit value, then eq. 150 and eq. 151 is zero. (We do this using the "c"-term in eq. 150 and 151). However, when the exponents of the "c"-terms aren't equal to -1, the limits of eq. 150 and 151 aren't equal to zero. We want this, infact, whenever we swap $\mathcal{S}'(\varepsilon,G_k^{\star\star})$ with $\mathcal{S}'(\varepsilon,G_j^{\star\star})$. Moreover, we define function $V(\varepsilon,G_k^{\star\star\star},n)$ (i.e., eq. 149), where:

(i) When $S'(\varepsilon,G_k^{\star\star\star})\gg\operatorname{Numerator}(V(\varepsilon,G_k^{\star\star\star},n))$, then eq. 150 and 151 without the "c"-terms are zero. (The "c"-terms approach zero and still allow eq. 150 and 151 to equal zero.)

(ii) When $S'(\varepsilon,G_k^{\star\star\star})\ll\operatorname{Numerator}(V(\varepsilon,G_k^{\star\star\star},n))$, then $\operatorname{sign}(\mathcal{M}(\varepsilon,G_k^{\star\star\star}))$ is zero which makes eq. 150 and 151 equal zero.

(iii) Here are some examples of the numerator of $V(\varepsilon, G_k^{\star\star\star}, n)$ (eq. 149):
   (A) When $E = 0$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, G_k^{\star\star\star}, n)$ is $\lceil(\mathbf{A}k! + 1)/\varepsilon\rceil$
   (B) When $E = z$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, G_k^{\star\star\star}, n)$ is $\lceil(2zk \cdot k! + 1)/\varepsilon\rceil$
   (C) When $E = 0$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, G_k^{\star\star\star}, n)$ is ceiling of constant $\mathbf{A}$ times the volume of an n-dimensional ball with finite radius: i.e.,
$$\left\lceil \frac{\mathbf{A}z_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2+1)} \Big/ \varepsilon \right\rceil$$
   (D) When $E = z_1$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, G_k^{\star\star\star}, n)$ is ceiling of the volume of the n-dimensional ball: i.e.,
$$\left\lceil \frac{z_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2+1)} k^{z_2} \Big/ \varepsilon \right\rceil$$

Now, consider the code for eq. 150 and eq. 151. (Note, the set theoretic limit of $G_k^{\star\star\star}$ is the graph of function $f : A \to \mathbb{R}$.) In this example, $A = \mathbb{Q} \cap [0,1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{157}$$

such that:
$$(A_k^{\star\star\star})_{r\in\mathbb{N}} = (\{c/k! : c \in \mathbb{Z}, 0 \leq c \leq k!\})_{k\in\mathbb{N}}$$
the ceiling function is $\lceil\cdot\rceil$, and:

$$(A_j^{\star\star})_{j\in\mathbb{N}} = (\{c/\lceil j!/3\rceil : c \in \mathbb{Z}, 0 \leq c \leq \lceil j!/3\rceil\})_{j\in\mathbb{N}}$$

such for $f_k^{\star} : A_k^{\star\star\star} \to \mathbb{R}$,
$$f_k^{\star\star\star}(x) = f(x) \text{ for all } x \in A_k^{\star\star\star} \tag{158}$$
and $f_j^{\star\star} : A_j^{\star\star} \to \mathbb{R}$
$$f_j^{\star\star}(x) = f(x) \text{ for all } x \in A_j^{\star\star} \tag{159}$$
Hence, when $(G_k^{\star\star\star})_{r\in\mathbb{N}}$ is:

$$(G_k^{\star\star\star})_{k\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/k! : c \in \mathbb{Z}, 0 \leq c \leq k!\}\})_{k\in\mathbb{N}} \tag{160}$$

and $(G_j^{\star\star})_{j\in\mathbb{N}}$ is:

$$(G_j^{\star\star})_{j\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/\lceil j!/3\rceil : c \in \mathbb{Z}, 0 \leq c \leq \lceil j!/3\rceil\}\})_{j\in\mathbb{N}} \tag{161}$$

Note, the following (we leave this to mathematicians to figure `LengthS1`, `LengthS2`, `Entropy1` and `Entropy2` for other A and f in code 12).

6.4.1. Evidence with Programming

Code 12: Code for eq. 150 and 151 to eq. 160 and eq. 161

```
Clear["Global`*"]

(*'A' is the domain of f*)
A=Intersect[Rationals,Interval[{0,1}]]

(*'f' is the function we are averaging over. In the case of f in §5.4.3, this can be
reperesented in mathematica using the following*)
f[x_] := f[x] =
```

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```
  Piecewise[{{1, Boole[IntegerQ[Denominator[x]/2]] == 1}, {0,
     Boole[IntegerQ[(Denominator[x] - 1)/2]] == 1}}]

eps=1 (*Since 'A' is rational, we set 'eps' or ε to 1*)

(*'LengthS1' is $|S'(\varepsilon,G_k^{\star\star\star})|$*)
LengthS1[k_] := LengthS1[k] = Ceiling[k!/3] + 1

(*'Entropy1' is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,k}} \sup_{\psi\in\Psi_{\varepsilon,k,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_k^{\star\star\star},\omega),\psi)))$ using asymptotic analysis *)
Entropy1[k_] := Entropy1[k] = Log2[k!/3]

(*'LengthS2' is $|S'(\varepsilon,G_j^{\star\star})|$*)
LengthS2[j_] := LengthS2[j] = j! + 1

(*'Entropy2' is the approximation $\sup_{\omega\in\Omega_{\varepsilon,j}} \sup_{\psi\in\Psi_{\varepsilon,j,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_j^{\star\star},\omega),\psi)))$ using asmyptotic analysis *)
Entropy2[j_] := Entropy2[j] = Log2[j!]

q = 35; (*We want q as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving 'TableLowAlphk' which is $\underline{|S'(\varepsilon,G_k^{\star\star\star})|}$*)
LowAlphValuesk = Table[
  {sol1[k_] :=
    sol1[k] = Reduce[j > 0 && Entropy2[j] <= Entropy1[k], j, Integers],
   LowSamplek = Max[j /. Solve[sol1[k], {j}, Integers]],
   LowAlphk = N[LengthS2[LowSamplek]]}, {k, 3, q}];
TableLowAlphk = Table[LowAlphValuesk[[k - 3 + 1, 3]], {k, 3, q}]

(*Below is the process of solving 'TableUpAlphk' which is $\overline{|S'(\varepsilon,G_k^{\star\star\star})|}$*)
UpAlphValuesk = Table[
   {sol11[k_] :=
     sol11[k] =
      Reduce[j < 5000 && Entropy2[j] >= Entropy1[k], j, Integers],
    UpSamplek = Min[j /. Solve[sol11[k], {j}, Integers]],
    UpAlphk = N[LengthS2[UpSamplek]]}, {k, 3, q}];
TableUpAlphk = Table[UpAlphValuesk[[k - 3 + 1, 3]], {k, 3, q}]

(*Below is the process of solving 'TableLowAlphj' which is $\underline{|S'(\varepsilon,G_j^{\star\star})|}$*)
LowAlphValuesj = Table[
   {sol2[j_] :=
     sol2[j] =
      Reduce[k > 0 && Entropy1[k] <= Entropy2[j], k, Integers],
    LowSamplej = Max[k /. Solve[sol2[j], {k}, Integers]],
    LowAlphj = N[LengthS1[LowSamplej]]}, {j, 3, q}];
TableLowAlphj = Table[LowAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

(*Below is the process of solving 'TableUpAlphj' which is $\overline{|S'(\varepsilon,G_j^{\star\star})|}$*)
UpAlphValuesj = Table[
   {sol21[j_] :=
     sol21[j] =
      Reduce[k < 5000 && Entropy1[k] >= Entropy2[j], k, Integers],
    UpSamplej = Min[k /. Solve[sol21[j], {k}, Integers]],
    UpAlphj = N[LengthS1[UpSamplej]]}, {j, 3, q}];
TableUpAlphj = Table[UpAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

a[k_] := a[k] = TableUpAlphk[[k - 3 + 1]] (*This is $\underline{|S'(\varepsilon,G_k^{\star\star\star})|}$*)
b[k_] := b[k] = LengthS1[k] (*This is $|S'(\varepsilon,G_k^{\star\star\star})|$*)
c[k_] := c[k] = TableLowAlphk[[k - 3 + 1]](*This is $\overline{|S'(\varepsilon,G_k^{\star\star\star})|}$*)

(*'K1' is $K(\varepsilon,G_k^{\star\star\star})$*)
K1[k_] :=
 K1[k] = N[
   RealAbs[(b[
          k] (1 + Ceiling[(b[
                k] (a[k] + 2 b[k]))/((a[k] + b[k]) (a[k] + b[k] +
                 c[k]))]) (1 + Round[a[k]/b[k]]))/((1 +
            Round[b[k]/c[k]]) (1 + Round[a[k]/c[k]])) - b[k]] + b[k]]

a1[j_] :=
 a1[j] = TableUpAlphj[[j - 3 + 1]] (*This is $\underline{|S'(\varepsilon,G_j^{\star\star})|}$*)
b1[j_] := b1[j] = LengthS2[j] (*This is $|S'(\varepsilon,G_j^{\star\star})|$*)
```

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```
c1[j_] := c1[j] = TableLowAlphj[[j - 3 + 1]](*This is |S'(ε,G_j^**)| *)

(*'K2' is K(ε,G_j^**)*)
K2[j_] :=
 K2[j] = N[
   RealAbs[(b1[
         j] (1 + Ceiling[(b1[
               j] (a1[j] + 2 b1[j]))/((a1[j] + b1[j]) (a1[j] +
                b1[j] + c1[j]))])) (1 + Round[a1[j]/b1[j]]))/((1 +
           Round[b1[j]/c1[j]]) (1 + Round[a1[j]/c1[j]])) - b1[j]] +
    b1[j]]

(*'Mk' is M'(ε,G_k^***)*)
Mk = Table[N[LengthS1[k] (K1[k] - LengthS1[k])], {k, 3, q - 1}]

(*'Mj' is M'(ε,G_j^**)*)
Mj = Table[N[LengthS2[j] (K2[j] - LengthS2[j])], {j, 3, q - 1}]

(*'DownS' is S_(ε,k,v*,G_j^**)*)
DownS = Table[
  LengthS2[Flatten[
      Position[Mj, Max[Select[Mj, # <= Mk[[k - 4 + 2]] &]]]][[1]] +
    4 - 2], {k, 4, q - 3}]

(*'UpS' is S̄(ε,k,v*,G_j^**)*)
UpS = Table[
  LengthS2[Flatten[
      Position[Mj, Min[Select[Mj, # >= Mk[[k - 4 + 2]] &]]]][[1]] +
    4 - 2], {k, 4, q - 3}]

E1 = 0 (*Constant rate of expansion*)
dimH = 0 (*Hausdorff Dimension of A*)
Ak = 1(*The smallest 1-dimensional box that covers A_k^*** is [0,1] which
has a length/'area' of one *)

(*'V' is V(ε,G_k^***) or eq. 149. Note, n is the dimension
of n-Euclidean Plane for which A is a subset*)
V[k_] := V[
   k] = V[k_, n_] :=
  V[k, n] =
   Ceiling[(Ak^(1 - Sign[E1])) (E1 + (1 - Sign[E1])) ((Pi^(n/2))/
         Gamma[n/2 + 1]) (k!^(n -
           dimH)) (k^(Sign[E1] (dimH - Sign[dimH] + 1))) + (1 -
          Sign[dimH])
       Simplify[V[k]]/eps]/LengthS1[k]

(* We couldn't add v, v* or convert this to a limit due to
limitations of the programming *)
ChoiceFunction =
 Table[N[((Sign[Mk[[k - 5 + 2]]] UpS[[k - 5 + 2]])/(LengthS1[
          k]) - (LengthS1[k]/LengthS2[k])^(-V[k,1]))*((Sign[Mk[[k - 5 + 2]]] DownS[[
          k - 5 + 2]])/(LengthS1[k]) - (LengthS1[k]/
          LengthS2[k])^(-V[k,1]))], {k, 5, q - 3}]
```

7. Questions

(1) Does §6 answer the **leading question** in §3.2

(2) Using thm. 11, when f is defined in §1.1, does $\mathbb{E}[f_k^{\star\star\star}]$ have a finite value?

(3) Using thm. 11, when f is defined in §1.2, does $\mathbb{E}[f_k^{\star\star\star}]$ have a finite value?

(4) If there's no time to check questions (1), (2) and (3), see §4.

8. Appendix of §5.4.1

8.1. Example of §5.4.1, step (1)

Suppose

(1) $A = \mathbb{R}$

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(2) When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{162}$$

(3) $(G_r^{\star})_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$

Then one example of $\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1)$, using §5.4.1 step (1), (where $G_1^{\star} = (\{(x, f(x)) : -1 \le x \le 1\})_{r\in\mathbb{N}}$) is:

$$\begin{aligned} &\left\{\left\{(x, f(x)) : -1 \le x \le \frac{\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{\sqrt{2}-6}{6} \le x \le \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{2\sqrt{2}-6}{6} \le x \le \frac{3\sqrt{2}-6}{6}\right\}\right. \\ &\left\{(x, f(x)) : \frac{3\sqrt{2}-6}{6} \le x \le \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{4\sqrt{2}-6}{6} \le x \le \frac{5\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{5\sqrt{2}-6}{6} \le x \le \frac{6\sqrt{2}-6}{6}\right\} \\ &\left.\left\{(x, f(x)) : \frac{6\sqrt{2}-6}{6} \le x \le \frac{7\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{7\sqrt{2}-6}{6} \le x \le \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \frac{8\sqrt{2}-6}{6} \le x \le \frac{9\sqrt{2}-6}{6}\right\}\right\} \end{aligned} \tag{163}$$

Note, the length of each partition is $\sqrt{2}/6$, where the borders could be approximated as:

$$\begin{aligned} &\{\{(x, f(x)) : -1 \le x \le -.764\}, \{(x, f(x)) : -.764 \le x \le -.528\}, \{(x, f(x)) : -.528 \le x \le -.293\} \\ &\{(x, f(x)) : -.293 \le x \le -.057\}, \{(x, f(x)) : -.057 \le x \le .178\}, \{(x, f(x)) : .178 \le x \le .414\} \\ &\{(x, f(x)) : .414 \le x \le .65\}, \{(x, f(x)) : .65 \le x \le .886\}, \{(x, f(x)) : .886 \le x \le 1.121\}\} \end{aligned} \tag{164}$$

which is illustrated using *alternating* orange/black lines of equal length covering $G_1^{\star}$ (i.e., the black vertical lines are the smallest and largest x-cooridinates of $G_1^{\star}$).

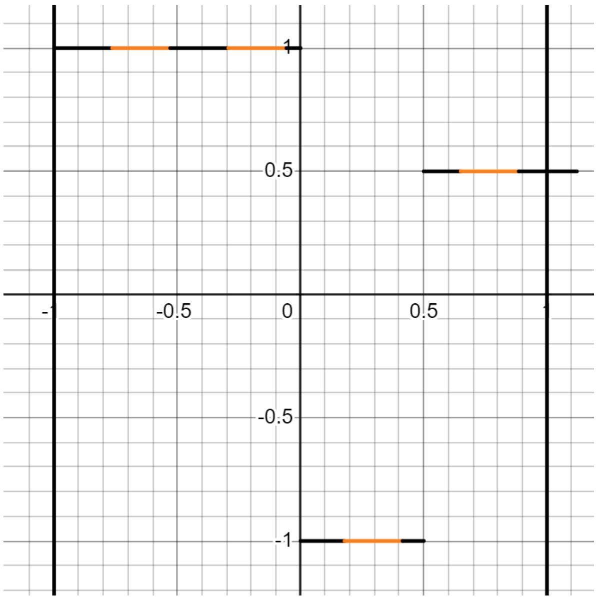

Figure 3. The alternating orange & black lines are the "covers" and the vertical lines are the boundaries of $G_1^{\star}$.

(Note, the alternating covers in Figure 3 satisfy step ((1)) of §5.4.1, because the Hausdorff measure *in its dimension* of the covers is $\sqrt{2}/6$ and there are 9 covers over-covering $G_1^{\star}$: i.e.,

Definition 12 (Minimum Covers of Measure $\varepsilon = \sqrt{2}/6$ covering $G_1^{\star}$). *We can compute the minimum covers of* $\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1)$*, using the formula:*

$$\lceil \mathcal{H}^{dim_H(G_1^{\star})}(G_1^{\star})/(\sqrt{2}/6) \rceil$$

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where $\lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_1^\star)}(G_1^\star)/(\sqrt{2}/6) \rceil = \lceil \text{Length}([-1,1])/(\sqrt{2}/6) \rceil = \lceil 2/(\sqrt{2}/6) \rceil = \lceil 6\sqrt{2} \rceil = \lceil 6(1.4) \rceil = \lceil 8 + .4 \rceil = 9$).

Note there are other examples of $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ for different ω. Here is another case:

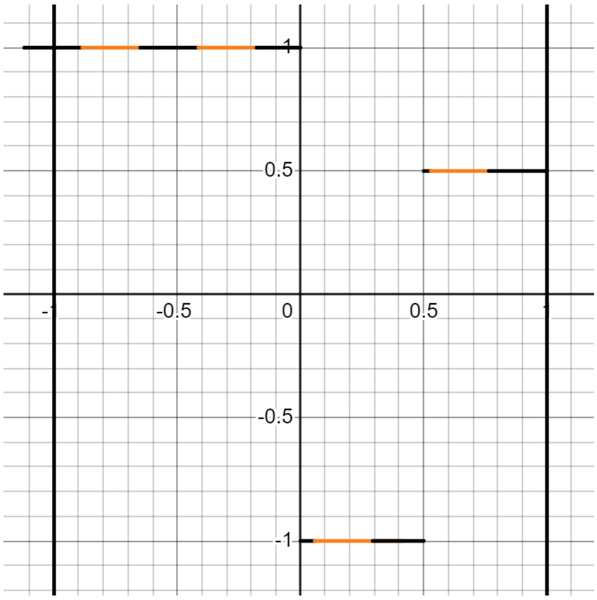

Figure 4. This is similar to Figure 3, except the start-points of the covers are shifted all the way to the left.

which can be defined (see eq. 163 for comparison):

$$\begin{aligned}&\left\{\left\{(x,f(x)) : \frac{6-9\sqrt{2}}{6} \le x \le \frac{6-8\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-8\sqrt{2}}{6} \le x \le \frac{6-7\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-7\sqrt{2}}{6} \le x \le \frac{6-6\sqrt{2}}{6}\right\}\right.\\ &\left\{(x,f(x)) : \frac{6-6\sqrt{2}}{6} \le x \le \frac{6-5\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-5\sqrt{2}}{6} \le x \le \frac{6-4\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-4\sqrt{2}}{6} \le x \le \frac{6-3\sqrt{2}}{6}\right\}\\ &\left.\left\{(x,f(x)) : \frac{6-3\sqrt{2}}{6} \le x \le \frac{6-2\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-2\sqrt{2}}{6} \le x \le \frac{6-\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-\sqrt{2}}{6} \le x \le 1\right\}\right\}\end{aligned} \tag{165}$$

In the case of $G_1^\star$, there are uncountable *different covers* $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ which can be used. For instance, when $0 \le \alpha \le (12 - 9\sqrt{2})/6$ (i.e., $\omega = \alpha - (12 - 9\sqrt{2})/6 + 1$) consider:

$$\begin{aligned}&\left\{\left\{(x,f(x)) : \alpha - 1 + \alpha \le x \le \alpha + \frac{\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{\sqrt{2}-6}{6} \le x \le \alpha + \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{2\sqrt{2}-6}{6} \le x \le \alpha + \frac{3\sqrt{2}-6}{6}\right\}\right.\\ &\left\{(x,f(x)) : \alpha + \frac{3\sqrt{2}-6}{6} \le x \le \alpha + \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{4\sqrt{2}-6}{6} \le x \le \alpha + \frac{5\sqrt{2}-6}{6}\right\},\\ &\left\{(x,f(x)) : \alpha + \frac{5\sqrt{2}-6}{6} \le x \le \alpha + \frac{6\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{6\sqrt{2}-6}{6} \le x \le \alpha + \frac{7\sqrt{2}-6}{6}\right\}\\ &\left.\left\{(x,f(x)) : \alpha + \frac{7\sqrt{2}-6}{6} \le x \le \alpha + \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{8\sqrt{2}-6}{6} \le x \le \alpha + \frac{9\sqrt{2}-6}{6}\right\}\right\}\end{aligned} \tag{166}$$

When $\alpha = 0$ and $\omega = (9\sqrt{2} - 6)/6$, we get Figure 4 and when $\alpha = (12 - 9\sqrt{2})/6$ and $\omega = 1$, we get Figure 3

8.2. *Example of §5.4.1, step (2)*

Suppose:

(1) $A = \mathbb{R}$

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(2) When defining $f : A \to \mathbb{R}$: i.e.,

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{167}$$

(3) $(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$
(4) $G_1^\star = \{(x, f(x)) : -1 \le x \le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using eq. 164 and Figure 3, which is *approximately*

$$\begin{aligned} &\{\{(x, f(x)) : -1 \le x \le -.764\}, \{(x, f(x)) : -.764 \le x \le -.528\}, \{(x, f(x)) : -.528 \le x \le -.293\} \\ &\{(x, f(x)) : -.293 \le x \le -.057\}, \{(x, f(x)) : -.057 \le x \le .178\}, \{(x, f(x)) : .178 \le x \le .414\} \\ &\{(x, f(x)) : .414 \le x \le .65\}, \{(x, f(x)) : .65 \le x \le .886\}, \{(x, f(x)) : .886 \le x \le 1.121\}\} \end{aligned} \tag{168}$$

Then, an example of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{169}$$

Below, we illustrate the sample: i.e., the set of all blue points *in each orange and black line of* $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$ covering $G_1^\star$:

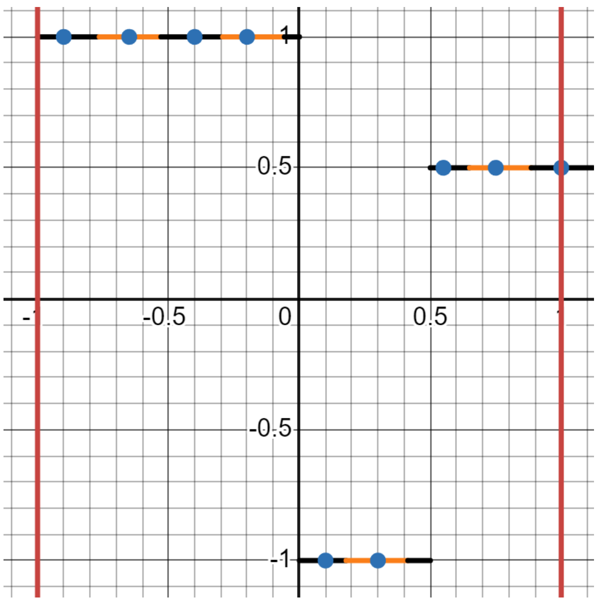

Figure 5. The blue points are the "sample points", the alternative black and orange lines are the "covers", and the red lines are the *smallest* & *largest* x-coordinates $G_1^\star$.

Note, there are multiple samples that can be taken, as long as one sample point is taken from each cover in $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$.

8.3. Example of §5.4.1, step (3)

Suppose

(1) $A = \mathbb{R}$
(2) When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{170}$$

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(3) $(G^{\star}_r)_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$
(4) $G^{\star}_1 = \{(x, f(x)) : -1 \le x \le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1)$, using eq. 164 and Figure 3, is approx.

$$\begin{aligned}&\{\{(x, f(x)) : -1 \le x \le -.764\}, \{(x, f(x)) : -.764 \le x \le -.528\}, \{(x, f(x)) : -.528 \le x \le -.293\}\\ &\{(x, f(x)) : -.293 \le x \le -.057\}, \{(x, f(x)) : -.057 \le x \le .178\}, \{(x, f(x)) : .178 \le x \le .414\}\\ &\{(x, f(x)) : .414 \le x \le .65\}, \{(x, f(x)) : .65 \le x \le .886\}, \{(x, f(x)) : .886 \le x \le 1.121\}\}\end{aligned} \tag{171}$$

(6) $\mathcal{S}(\mathbf{C}(13/6, G^{\star}_1, 1), 1)$, using eq. 169, is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{172}$$

Therefore, consider the following process:

8.3.1. Step (3)(3)a

If $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{173}$$

suppose $x_0 = (-.9, 1)$. Note, the following:

(1) $x_1 = (-.65, 1)$ is the next point in the "pathway" since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1), 1)$ with the smallest 2-d Euclidean distance to x_0 instead of x_0.
(2) $x_2 = (-.4, 1)$ is the third point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1), 1)$ with the smallest 2-d Euclidean distance to x_1 instead of x_0 and x_1.
(3) $x_3 = (-.2, 1)$ is the fourth point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1), 1)$ with the smallest 2-d Euclidean distance to x_2 instead of x_0, x_1, and x_2.
(4) we continue this process, where the "**pathway**" of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G^{\star}_1, 1), 1)$ is:

$$(-.9, 1) \to (-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1) \tag{174}$$

Note 13. *If more than one point has the minimum 2-d Euclidean distance from x_0, x_1, x_2, etc. take all potential pathways: e.g., using the sample in eq. 173, if $x_0 = (-.65, 1)$, then since $(-.9, 1)$ and $(-.4, 1)$ have the smallest Euclidean distance to $(-.65, 1)$, take **two** pathways:*

$$(-.65, 1) \to (-.9, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

and also:

$$(-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (-.9, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

8.3.2. Step (3)(3)b

Next, take the length of all line segments in each pathway. In other words, suppose $d(P, Q)$ is the n-th dim. Euclidean distance between points $P, Q \in \mathbb{R}^n$. Using the pathway in eq. 174, we want:

$$\begin{aligned}&\{d((-.9, 1), (-.65, 1)), d((-.65, 1), (-.4, 1)), d((-.4, 1), (-.2, 1)), d((-.2, 1), (.55, .5)),\\ &d((.55, .5), (.75, .5)), d((.75, .5), (1, .5)), d((1, .5), (.3, -1)), d((.3, -1), (.1, -1))\}\end{aligned} \tag{175}$$

Whose distances can be approximated as:

$$\{.25, .25, .2, .901389, .2, .25, 1.655295, .2\}$$

Also, we see the outliers [11] are .901389 and 1.655295 (i.e., notice that the outliers are more prominent for $\varepsilon \ll \sqrt{2}/6$). Therefore, remove .901389 and 1.655295 from our set of lengths:

$$\{.25, .25, .2, .2, .25, .2\}$$

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This is illustrated using:

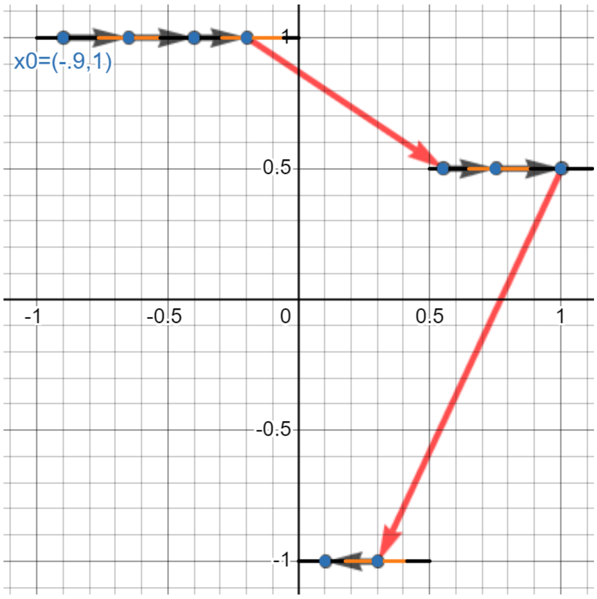

Figure 6. The black arrows are the "pathways" whose lengths aren't outliers. The length of the red arrows in the pathway are outliers.

Hence, when $x_0 = (-.9, 1)$, using §5.4.1 step (3)(3)b & eq. 173, we note:

$$\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)) = \{.25, .25, .2, .2, .25, .2\} \tag{176}$$

8.3.3. Step (3)(3)c

To convert the set of distances in eq. 176 into a probability distribution, we take:

$$\sum_{x \in \{.25,.25,.2,.2,.25,.2\}} x = .25 + .25 + .2 + .2 + .25 + .2 = 1.35 \tag{177}$$

Then divide each element in $\{.25, .25, .2, .2, .25, .2\}$ by 1.35

$$\{.25/(1.35), .25/(1.35), .2/(1.35), .2/(1.35), .25/(1.35), .2/(1.35)\}$$

which gives us the probability distribution:

$$\{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))) = \{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\} \tag{178}$$

8.3.4. Step (3)(3)d

Take the shannon entropy of eq. 178:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))) =$$
$$\sum_{x \in \mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))} -x \log_2 x = \sum_{x \in \{5/27,5/27,4/27,4/27,5/27,4/27\}} -x \log_2 x =$$
$$-(5/27)\log_2(5/27) - (5/27)\log_2(5/27) - (4/27)\log_2(4/27) - (4/27)\log_2(4/27) - (5/27)\log_2(5/27) - (4/27)\log_2(5/27) =$$
$$-(15/27)\log_2(5/27) - (12/27)\log_2(4/27) \approx 2.57604$$

We shorten $\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))))$ to $\mathrm{E}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))$, giving us:

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$$\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.57604 \tag{179}$$

8.3.5. Step (3)(3)e

Take the entropy, w.r.t all pathways, of the sample:

$$\{(-.9,1),(-.65,1),(-.4,1),(-.2,1),(.1,-1),(.3,-1),(.55,.5),(.75,.5),(1,.5)\} \tag{180}$$

In other words, we'll compute:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))=\sup_{x_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)}\mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))$$

We do this by repeating §8.3.1-§8.3.4 for different $x_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))$ (i.e., in the equation with multiple values, see note 13)

$$\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.57604 \tag{181}$$
$$\mathrm{E}(\mathcal{L}((-.65,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.3131, 2.377604 \tag{182}$$
$$\mathrm{E}(\mathcal{L}((-.4,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.3131 \tag{183}$$
$$\mathrm{E}(\mathcal{L}((-.2,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.57604 \tag{184}$$
$$\mathrm{E}(\mathcal{L}((-.1,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 1.86094 \tag{185}$$
$$\mathrm{E}(\mathcal{L}((-.3,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 1.85289 \tag{186}$$
$$\mathrm{E}(\mathcal{L}((.55,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.08327 \tag{187}$$
$$\mathrm{E}(\mathcal{L}((.75,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.31185 \tag{188}$$
$$\mathrm{E}(\mathcal{L}((1,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.2622 \tag{189}$$

Hence, since the largest value out of eq. 181-189 is 2.57604:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(13/6,G_1^\star,1),1)))=\sup_{x_0\in\mathcal{S}(\mathbf{C}(\varepsilon,G_1^\star,3),1)}\mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_1^\star,1),1)))\approx 2.57604$$

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