Preprints.org

A Few Explicit Properties Involving Integral Transforms and Fractional Calculus of Srivastava's Triple Hypergeometric Function Hλ, IC, P,α

Mohammad Kamarujjama , Ajija Yasmin [*] , Showkat Ahmad Dar

Article

A Few Explicit Properties Involving Integral Transforms and Fractional Calculus of Srivastava's Triple Hypergeometric Function $H_{C,p,\alpha}^{\lambda,l}$

M. Kamarujjama, Ajija Yasmin * and Showkat Ahmad Dar

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India
* Correspondence: ajijayasmin001@gmail.com

Abstract: In this paper, we obtain an extension of Srivastava's triple hypergeometric function $\mathbf{H}_C(\cdot)$ by employing the extended Beta function $B_{p,\alpha}^{\lambda,l}(x_1, x_2)$ introduced in Oraby *et al.* [12]. We give some of the main properties of this extended function, which include several integral representations, derivative formulas, and a few integral transforms, namely, Euler-Beta transform, Mellin transform, Laplace transform, and Whittaker transform. Further, we establish some results based on the consequences of Riemann-Liouville fractional integral and differential operators on $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$. Lastly, we discuss some recursion formulas.

Keywords: Srivastava's triple hypergeometric functions; Beta and Gamma functions; generalized Mittag-Leffler function; integral transforms and Riemann-Liouville fractional integrals and derivatives

MSC: 33C60; 33C70; 33B15; 33C05; 33E12; 26A33

---

1. Introduction and Preliminaries

There is a long history of hypergeometric functions of a single variable in many areas of physics, statistics, and mathematics. The Gauss hypergeometric function is defined as

$$_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array}; x\right) = \sum_{i=0}^{\infty} \frac{(a_1)_i (a_2)_i}{(b_1)_i} \frac{x^i}{i!} \qquad (a_1, a_2 \in \mathsf{C}, b_1 \in \mathsf{C}\backslash \mathsf{Z}_0^-, |x| < 1), \tag{1.1}$$

where $(a_1)_i$ denotes the Pochhammer symbol defined by

$$(a_1)_i = \frac{\Gamma(a_1 + i)}{\Gamma(a_1)} = \left\{ \begin{array}{ll} 1, & (i = 0, \ a_1 \in \mathsf{C} \setminus \{0\}), \\ \\ a_1(a_1 + 1)...(a_1 + i - 1), & (i \in \mathsf{N}, a_1 \in \mathsf{C}) \end{array} \right\}. \tag{1.2}$$

Here, the classical Gamma function [13] is defined as

$$\Gamma(a_1) = \int_0^\infty e^{-t} t^{a_1 - 1} \, dt, \qquad (R(a_1) > 0). \tag{1.3}$$

Extensions of the function (1.1) to include p numerator parameters a_i ($1 \leq i \leq p$) and q denominator parameters b_i ($1 \leq i \leq q$) also find wide application; see [17]. Triple hypergeometric functions have been introduced and studied by Srivastava and Karlsson [23, Chapter 3], who provide a table of 205 distinct such functions. In [19,20], Srivastava introduced the triple hypergeometric functions H_A, H_B, and H_C of the second order. It is known that H_C and H_B are generalizations of Appell's hypergeometric functions F_1 and F_2, while H_A is the generalization of both F_1 and F_2.

The focus of this study will be on Srivastava's triple hypergeometric function H_C, which is given by [23, p. 43, 1.5(11) to 1.5(13)] (see also [19] and [22, p. 68])

$$H_C(a_1, a_2, a_3; b_1; x_1, x_2, x_3) := \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{i+k}(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_{i+j+k}} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j} \frac{B(a_1+i+k, b_1+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}. \tag{1.4}$$

In [6, p.243], the convergence region for the hypergeometric series $H_C(\cdot)$ is provided as $|x_1| < \alpha_1$, $|x_2| < \alpha_2$, $|x_3| < \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ satisfy the relation

$$\alpha_1 + \alpha_2 + \alpha_3 - 2\sqrt{(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)} < 2. \tag{1.5}$$

Now, we find it suitable to introduce a new parameter c into $H_C(\cdot)$ as follows

$$H_C^{(c)}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) :=$$

$$\sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j} \frac{B(a_1+c+i+k, b_1+c+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}, \tag{1.6}$$

which reduces to (1.4) when $c = 0$.

Here, $B(x_1, x_2)$ is the classical Beta function defined as [11, (5.12.1)]

$$B(x_1, x_2) = \begin{cases} \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt, & (R(x_1) > 0, R(x_2) > 0) \\ \dfrac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2)}, & ((x_1, x_2) \in \mathsf{C}\backslash\mathsf{Z}_0^-). \end{cases} \tag{1.7}$$

In 1997, Chaudhry *et al.* [2, Eq.(1.7)] introduced a *p*-extension of $B(x_1, x_2)$ given by

$$B_p(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \exp\left[\frac{-p}{t(1-t)}\right]dt, \qquad (R(p) \geq 0). \tag{1.8}$$

In 2018, a further extension of the Beta function was given, as shown by Shadab *et al.* [15]

$$B_p^\lambda(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_\lambda\left[\frac{-p}{t(1-t)}\right]dt, \qquad (R(p) \geq 0), \tag{1.9}$$

which reduces to (1.8) when $\lambda = 1$ and $E_\lambda(x)$ denotes the Mittag-Leffler function defined by [10],

$$E_\lambda(x) = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(\lambda i + 1)}, \qquad (x \in \mathsf{C}; \lambda > 0).$$

In 2020, Oraby *et al.* [12] gave a generalized Beta function in the form

$$B_{p,\alpha}^{\lambda,l}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{\lambda,\alpha}\left[\frac{-p}{t^l(1-t)^l}\right]dt \tag{1.10}$$

$$(R(x_1) > 0, R(x_2) > 0, R(p) \geq 0, R(l) > 0),$$

which reduces to (1.9) when $\alpha = l = 1$.

Where $E_{\lambda,l}(x)$ is the generalized Mittag-Leffler function introduced in 1905 by Wiman [24] defined as

$$E_{\lambda,\alpha}(x) = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(\lambda i + \alpha)} \ , \qquad (x, \alpha \in \mathsf{C}; \lambda > 0, R(\alpha) > 0).$$

In 2021, Abubakar [1] introduced and studied a generalized Beta function as

$$^{\Psi}B_{\tau}^l(x_1, x_2) = {}^{\Psi}B_{\tau}^l \left[\begin{array}{c} (\rho_m, \acute{\rho}_m)_{1,\eta_1} \\ (\zeta_n, \acute{\zeta}_n)_{1,\eta_2} \end{array} \middle| x_1, x_2 \right]$$

$$= \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \, {}_{\eta_1}\Psi_{\eta_2} \left(-\frac{\tau}{t^l(1-t)^l} \right) dt, \qquad (1.11)$$

($x_1, x_2, \tau, l, \rho_m, \zeta_n \in \mathsf{C}$; $\acute{\rho}_m, \acute{\zeta}_n \in \mathsf{R}$, $R(x_1) > 0, R(x_2) > 0, R(\tau) > 0, R(l) > 0$ and $m = 1, ..., \eta_1, n = 1, ..., \eta_2$)

here ${}_{\eta_1}\Psi_{\eta_2}$ is the Fox-Wright function ([7],[8]) defined as

$$_{\eta_1}\Psi_{\eta_2} \left[\begin{array}{c} (\rho_m, \acute{\rho}_m)_{1,\eta_1} \\ (\zeta_n, \acute{\zeta}_n)_{1,\eta_2} \end{array} \middle| z \right] = {}_{\eta_1}\Psi_{\eta_2} \left[\begin{array}{c} (\rho_1, \acute{\rho}_1), ..., (\rho_{\eta_1}, \acute{\rho}_{\eta_1}); \\ (\zeta_1, \acute{\zeta}_1), ..., (\zeta_{\eta_2}, \acute{\zeta}_{\eta_2}); \end{array} z \right]$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{\eta_1} \Gamma(\rho_m + \acute{\rho}_m k)}{\prod_{n=1}^{\eta_2} \Gamma(\zeta_n + \acute{\zeta}_n k)} \frac{z^k}{k!}, \qquad (1.12)$$

where $z, \rho_m, \zeta_n \in \mathsf{C}$, and the coefficients $\acute{\rho}_m \geq 0, \acute{\zeta}_n \geq 0$ ($m = 1, ..., \eta_1; n = 1, ..., \eta_2$). If we put

$$\Delta = \sum_{n=1}^{\eta_2} \acute{\zeta}_n - \sum_{m=1}^{\eta_1} \acute{\rho}_m, \quad \delta = \prod_{m=1}^{\eta_1} \acute{\rho}_m^{-\acute{\rho}_m} \prod_{n=1}^{\eta_2} \acute{\zeta}_n^{\acute{\zeta}_n}, \quad \mu^* = \sum_{n=1}^{\eta_2} \zeta_n - \sum_{m=1}^{\eta_1} \rho_m + \frac{1}{2}(\eta_1 - \eta_2),$$

(1.12) converges for $|z| < \infty$ when $\Delta > -1$, for $|z| < \delta$ when $\Delta = -1$ and for $|z| = \delta$ if, in addition, $R(\mu^*) > \frac{1}{2}$.

For $\acute{\rho}_1 = ... = \acute{\rho}_{\eta_1} = \acute{\zeta}_1 = ... = \acute{\zeta}_{\eta_2} = 1$ eq. (1.12) reduces to the generalized hypergeometric function ${}_{\eta_1}F_{\eta_2}$ [26]

$$_{\eta_1}\Psi_{\eta_2} \left[\begin{array}{c} (\rho_1, 1), ..., (\rho_{\eta_1}, 1); \\ (\zeta_1, 1), ..., (\zeta_{\eta_2}, 1); \end{array} z \right] = \frac{\prod_{m=1}^{\eta_1} \Gamma(\rho_m)}{\prod_{n=1}^{\eta_2} \Gamma(\zeta_n)} \, {}_{\eta_1}F_{\eta_2} \left[\begin{array}{c} \rho_1, ..., \rho_{\eta_1}; \\ \zeta_1, ..., \zeta_{\eta_2}; \end{array} z \right]. \qquad (1.13)$$

The plan of this paper is as follows. The extended function $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ is defined in section 2 based on the extended Beta function in (1.10), and some integral representations are presented involving the generalized Mittag-Leffler function and the Gauss hypergeometric function ${}_2F_1$. Further, we discuss some derivative formulas and integral trasforms, namely, the Euler-Beta transform, Laplace transform, Mellin transform, and Whittaker transform. Moreover, some recursion formulas are established. Also, our work on Srivastava triple hypergeometric function is motivated by the work given in ([3],[4]).

2. The Extended Srivastava Triple Hypergeometric Function $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_C(\cdot)$, together with its integral representations, in ([19], [22]). Here, we define the extended Srivastava triple $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ by means of the extended Beta function defined in (1.10)

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+j-a_1)}{B(a_1,b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}, \tag{2.1}$$

where, $a_1,a_2,a_3 \in \mathbb{C}$ and $b_1 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and the region of convergence is $|x_1| < \alpha_1$, $|x_2| < \alpha_2$, $|x_3| < \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ satisfy (1.5).

Now, using the extended beta function given in (1.10) in (2.1), we get $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(.)$ in terms of extended Mittag-Leffler function as follows:

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) = \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1,b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$\times \int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1} E_{\lambda,\alpha}\left[-\frac{p}{t^l(1-t)^l}\right] dt. \tag{2.2}$$

We use of the above eq. (2.2) throughout the paper.

3. A Few Explicit Properties

In this section, we calculate two explicit properties, namely, integral representations and derivative properties.

3.1. Integral Representations

Here, we get the following integral representations for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ function involving the product of Mittag-Leffler function of two parameter and the Gauss hypergeometric function $_2F_1$.

Theorem 1. *For $R(p) \geq 0$, $R(a_n) > 0$ ($n = 1,2,3$) and $R(b_1 - a_1) > 0$, the integral representation of $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ is as follows :*

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)$$

$$\times (1-x_1 t)^{-a_2}(1-x_3 t)^{-a_3} \, _2F_1\left[\begin{matrix} a_2, a_3 \\ b_1-a_1 \end{matrix}; \frac{(1-t)x_2}{(1-x_1 t)(1-x_3 t)}\right] dt, \tag{3.1}$$

where $|x_n| < 1$ ($n = 1,2,3$).

Proof: By changing the order of integration and summation (with uniform convergence of the integral) in (2.2) and using the relation (1.7), we obtain

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)$$

$$\times \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1-a_1)_j} \frac{(x_1 t)^i}{i!} \frac{(x_2(1-t))^j}{j!} \frac{(x_3 t)^k}{k!} \, dt. \tag{3.2}$$

Applying the result $(\lambda)_{i+j} = (\lambda)_j (\lambda + j)_i$ in the treble sum in (3.2), yields

$$\sum_{j=0}^{\infty} \frac{(a_2)_j (a_3)_j}{(b_1 - a_1)_j} \frac{(x_2(1-t))^j}{j!} \sum_{i=0}^{\infty} (a_2 + j)_i \frac{(x_1 t)^i}{i!} \sum_{k=0}^{\infty} (a_3 + j)_k \frac{(x_3 t)^k}{k!}.$$

Now, emploeying the binomial theorem

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m z^m}{m!} = (1-z)^{-\lambda}, \qquad (\lambda \in \mathsf{C}, \ |z| < 1)$$

in the sum to evaluate the sums over i and k, we get

$$= (1 - x_1 t)^{-a_2} (1 - x_3 t)^{-a_3} \sum_{j=0}^{\infty} \frac{(a_2)_j (a_3)_j}{(b_1 - a_1)_j} \left[\frac{x_2(1-t)}{(1 - x_1 t)(1 - x_3 t)} \right]^j \frac{1}{j!},$$

identify the sum over j as (1.1), we get the representaion (3.1).

Remark 1. Here, we discuss some special cases by making suitable transformations of the integration variable. If we put

$$
\begin{array}{llll}
(i) & t & = & \dfrac{w}{1+w}, \quad \dfrac{dt}{dw} = \dfrac{1}{(1+w)^2}, \\[2ex]
(ii) & t & = & \dfrac{(\theta_2 - \theta_3)(w - \theta_1)}{(\theta_2 - \theta_1)(w - \theta_3)}, \quad \dfrac{dt}{dw} = \dfrac{(\theta_2 - \theta_3)(\theta_1 - \theta_3)}{(\theta_2 - \theta_1)(w - \theta_3)^2}, \\[2ex]
(iii) & t & = & \sin^2 w, \quad \dfrac{dt}{dw} = 2 \sin w \cos w, \\[2ex]
(iv) & t & = & \dfrac{(1+\rho)w}{1+\rho w}, \quad \dfrac{dt}{dw} = \dfrac{(1+\rho)}{(1+\rho w)^2},
\end{array}
$$

we get the integral representations as follows:

$$(i) \qquad \frac{\Gamma(a_1) \Gamma(b_1 - a_1)}{\Gamma(b_1)} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$= \int_0^{\infty} w^{a_1 - 1} (1+w)^{a_2 + a_3 - b_1} \Lambda_1^{-a_2} \Lambda_2^{-a_3} E_{\lambda,\alpha}\left(-\frac{p}{\delta_1^l \delta_2^l}\right) {}_2F_1\left(\begin{array}{c} a_2, a_3 \\ b_1 - a_1 \end{array}; \delta_3 x_2 \right) dw, \qquad (3.3)$$

where

$$\delta_1 = \frac{w}{1+w}, \quad \delta_2 = \frac{1}{1+w}, \quad \delta_3 = \frac{1+w}{\Lambda_1 \Lambda_2}, \quad \Lambda_1 = 1 + (1 - x_1)w, \quad \Lambda_2 = 1 + (1 - x_3)w;$$

$$(ii) \qquad \frac{\Gamma(a_1) \Gamma(b_1 - a_1)}{\Gamma(b_1)} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) = \frac{(\theta_2 - \theta_3)^{a_1} (\theta_1 - \theta_3)^{b_1 - a_1}}{(\theta_2 - \theta_1)^{b_1 - a_2 - a_3 - 1}}$$

$$\times \int_{\theta_1}^{\theta_2} \frac{(w - \theta_1)^{a_1 - 1} (\theta_2 - w)^{b_1 - a_1 - 1}}{(w - \theta_3)^{b_1 - a_2 - a_3}} \Lambda_1^{-a_2} \Lambda_2^{-a_3} E_{\lambda,\alpha}\left(-\frac{p}{\delta_1^l \delta_2^l}\right) {}_2F_1\left(\begin{array}{c} a_2, a_3 \\ b_1 - a_1 \end{array}; \delta_3 x_2 \right) dw, \qquad (3.4)$$

where, with $\theta_3 < \theta_1 < \theta_2$,

$$\delta_1 = \frac{(\theta_2 - \theta_3)(w - \theta_1)}{(\theta_2 - \theta_1)(w - \theta_3)}, \quad \delta_2 = \frac{(\theta_1 - \theta_3)(\theta_2 - w)}{(\theta_2 - \theta_1)(w - \theta_3)}, \quad \delta_3 = \frac{(\theta_1 - \theta_3)(\theta_2 - \theta_1)(\theta_2 - w)(w - \theta_3)}{\Lambda_1 \Lambda_2},$$

$$\Lambda_1 = (\theta_2 - \theta_1)(w - \theta_3) - x_1(\theta_2 - \theta_3)(w - \theta_1), \quad \Lambda_2 = (\theta_2 - \theta_1)(w - \theta_3) - x_3(\theta_2 - \theta_3)(w - \theta_1);$$

$$(iii) \qquad \frac{\Gamma(a_1)\Gamma(b_1-a_1)}{\Gamma(b_1)} \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)$$

$$= 2\int_0^{\pi/2} (\sin^2 w)^{a_1-\frac{1}{2}} (\cos^2 w)^{b_1-a_1-\frac{1}{2}} \Lambda_1^{-a_2}\Lambda_2^{-a_3} E_{\lambda,\alpha}\left(-\frac{p}{\delta_1^l\delta_2^l}\right) {}_2F_1\left(\begin{array}{c} a_2,a_3 \\ b_1-a_1 \end{array};\delta_3 x_2\right)dw, \qquad (3.5)$$

where

$$\delta_1 = \sin^2 w, \quad \delta_2 = \cos^2 w, \quad \delta_3 = \frac{\cos^2 w}{\Lambda_1\Lambda_2}, \quad \Lambda_1 = 1 - x_1\sin^2 w, \quad \Lambda_2 = 1 - x_3\sin^2 w;$$

and

$$(iv) \qquad \frac{\Gamma(a_1)\Gamma(b_1-a_1)}{\Gamma(b_1)} \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)$$

$$= (1+\rho)^{a_1}\int_0^1 \frac{w^{a_1-1}(1-w)^{b_1-a_1-1}}{(1+\rho w)^{b_1-a_2-a_3}} \Lambda_1^{-a_2}\Lambda_2^{-a_3} E_{\lambda,\alpha}\left(-\frac{p}{\delta_1^l\delta_2^l}\right) {}_2F_1\left(\begin{array}{c} a_2,a_3 \\ b_1-a_1 \end{array};\delta_3 x_2\right)dw, \qquad (3.6)$$

where, with $\rho > -1$,

$$\delta_1 = \frac{(1+\rho)w}{1+\rho w}, \quad \delta_2 = \frac{1-w}{1+\rho w}, \quad \delta_3 = \frac{(1-w)(1+\rho w)}{\Lambda_1\Lambda_2},$$

$$\Lambda_1 = 1 + \rho w - (1+\rho)x_1 w, \quad \Lambda_2 = 1 + \rho w - (1+\rho)x_3 w.$$

3.2. Derivative Properties

In this part, we discuss some interesting derivative properties of $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$.

Theorem 2. *The following derivative formula holds for* $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$

$$\left(\frac{d}{dp}\right)^m\left[p^{\alpha-1}\mathbf{H}_{C,p^\lambda,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]$$

$$= p^{\alpha-m-1}\mathbf{H}_{C,p^\lambda,\alpha-m}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3), \qquad (3.7)$$

where $\alpha, a_n \in \mathsf{C}, b_1 \in \mathsf{C}\backslash\mathsf{Z}_0^-; m \in \mathsf{N}$ *and* $R(\alpha-m) > 0, R(\alpha) > 0, \lambda > 0, R(a_n) > 0; n = 1,2,3.$

Proof: Using (2.2) and employing term-wise differentiation m times in L.H.S of (3.7), we obtain

$$\left(\frac{d}{dp}\right)^m\left[p^{\alpha-1}\mathbf{H}_{C,p^\lambda,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]$$

$$= \left(\frac{d}{dp}\right)^m\left[p^{\alpha-1}\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\right.$$

$$\left. \times \int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}E_{\lambda,\alpha}\left\{-\frac{p^\lambda}{t^l(1-t)^l}\right\}dt\right]$$

$$= \sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}$$

$$\times \sum_{n=0}^{\infty}\frac{1}{\Gamma(\lambda n+\alpha)}\left(-\frac{1}{t^l(1-t)^l}\right)^n\left(\frac{d}{dp}\right)^m\left[p^{\lambda n+\alpha-1}\right]dt$$

$$= p^{\alpha-m-1}\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\int_0^1 t^{a_1+i+k-1}$$

$$\times (1-t)^{b_1+j-a_1-1} \sum_{n=0}^{\infty} \left(-\frac{p^{\lambda}}{t^l(1-t)^l} \right)^n \frac{1}{\Gamma(\lambda n + \alpha - m)} \, dt$$

$$= p^{\alpha-m-1} \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j \, B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!} \int_0^1 t^{a_1+i+k-1}$$

$$\times (1-t)^{b_1+j-a_1-1} E_{\lambda,\alpha-m} \left(-\frac{p^{\lambda}}{t^l(1-t)^l} \right) dt$$

$$= p^{\alpha-m-1} \mathbf{H}_{C,p^{\lambda},\alpha-m}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3).$$

Theorem 3. *We have the following differentiation formula as follows:*

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$= \alpha \mathbf{H}_{C,p,\alpha+1}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) + \lambda p \frac{d}{dp} \mathbf{H}_{C,p,\alpha+1}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3), \tag{3.8}$$

where $\alpha \in \mathsf{C}, b_1 \in \mathsf{C}\backslash \mathsf{Z}_0^-$, and $\lambda > 0$, $R(\alpha) > 0$.

Proof: Using (2.2) in the R.H.S of (3.8), yields

$$\alpha \mathbf{H}_{C,p,\alpha+1}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) + \lambda p \frac{d}{dp} \mathbf{H}_{C,p,\alpha+1}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$= \alpha \mathbf{H}_{C,p,\alpha+1}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) + \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j \, B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!} \int_0^1 t^{a_1+i+k-1}$$

$$\times (1-t)^{b_1+j-a_1-1} \sum_{n=0}^{\infty} \left(-\frac{1}{t^l(1-t)^l} \right)^n \frac{\lambda n p^n}{\Gamma(\lambda n + \alpha + 1)} \, dt \tag{3.9}$$

Now employing the relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ in (3.9), we get

$$= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j \, B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!} \int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}$$

$$\times \sum_{n=0}^{\infty} \left(-\frac{p}{t^l(1-t)^l} \right)^n \frac{1}{\Gamma(\lambda n + \alpha)} \, dt$$

$$= \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3).$$

4. Integral Transform of $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$

Here, we explore some integral transforms, namely, the Euler-Beta transform, Laplace transform, Mellin transform, and Whittaker transform of $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$.

4.1. Euler-Beta Transform

The Euler-Beta transform of the function $f(p)$ is defined as [18]

$$\mathbf{B}\{f(p); a, b\} = \int_0^1 p^{a-1}(1-p)^{b-1} f(p) \, dp. \tag{4.1}$$

Theorem 4. *The Euler-Beta transform of* $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ *is as follows:*

$$\mathbf{B}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);a,b\right\} = \Gamma(b)\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1,b_1+j-a_1)}$$

$$\times\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\;{}^{\Psi}B_1^l\left[\begin{array}{c}(1,1),(a,1)\\(\alpha,\lambda),(a+b,1)\end{array}\middle|(a_1+i+k),(b_1+j-a_1)\right], \tag{4.2}$$

where, $\alpha,a,b,l,(a_1+i+k),(b_1+j-a_1)\in\mathbb{C}; \lambda>0, R(\alpha)>0, R(a)>0, R(b)>0, R(a_1+i+k)>0, R(b_1+j-a_1)>0.$

Proof: Using (4.1) in (2.2), we obtain

$$\mathbf{B}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);a,b\right\}$$

$$= \int_0^1 p^{a-1}(1-p)^{b-1}\,\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\,dp$$

$$= \sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$\times\left(\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}\left\{\int_0^1 p^{a-1}(1-p)^{b-1}E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)dp\right\}\right)dt. \tag{4.3}$$

By evaluating the integral as

$$\int_0^1 p^{a-1}(1-p)^{b-1}E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)dp$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{t^l(1-t)^l}\right)^n\frac{1}{\Gamma(\lambda n+\alpha)}\int_0^1 p^{a+n-1}(1-p)^{b-1}\,dp$$

$$= \Gamma(b)\sum_{n=0}^{\infty}\left(-\frac{1}{t^l(1-t)^l}\right)^n\frac{\Gamma(a+n)\Gamma(n+1)}{\Gamma(\lambda n+\alpha)\Gamma(a+b+n)\,n!},$$

employing the Fox-Wright function (1.12) in the above sum and using in (4.3), we get,

$$\mathbf{B}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);a,b\right\} = \sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$\times\Gamma(b)\left(\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}{}_2\Psi_2\left[\begin{array}{c}(1,1),(a,1)\\(\alpha,\lambda),(a+b,1)\end{array};-\frac{1}{t^l(1-t)^l}\right]\right)dt.$$

Upon evaluating the integral using the generalized Beta function defined in (1.11), we get the required result (4.2).

4.2. Laplace Transform

The Laplace transform of a function $f(p)$ is defined as [18]

$$\mathbf{L}\{f(p)\} = \int_0^{\infty}e^{-sp}f(p)\,dp \tag{4.4}$$

Theorem 5. *The Laplace transform of* $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ *is as follows:*

$$\int_0^\infty p^{a-1} e^{-sp} \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) \, dp = \frac{1}{s^a} \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1, b_1 + j - a_1)}$$

$$\times \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!} \, \Psi B_{(\frac{1}{s})}^l \left[\begin{array}{c} (1,1),(a,1) \\ (\alpha,\lambda) \end{array} \middle| (a_1 + i + k), (b_1 + j - a_1) \right], \tag{4.5}$$

where, $\lambda, \alpha, a, (a_1 + i + k), (b_1 + j - a_1) \in \mathsf{C}; R(\lambda) > 0, R(\alpha) > 0, R(a) > 0, R(s) > 0, R(a_1 + i + k) > 0, R(b_1 + j - a_1) > 0$ *and* $\left| -\frac{1}{st^l(1-t)^l} \right| < 1.$

Proof: Using (2.2) in the integral on the left-hand side of (4.5), we get

$$\int_0^\infty p^{a-1} e^{-sp} \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) \, dp$$

$$= \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j \, B(a_1, b_1 + j - a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$\times \left(\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1} \left\{ \int_0^\infty e^{-sp} \, p^{a-1} E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l} \right) dp \right\} \right) dt. \tag{4.6}$$

By evaluating the integral as

$$\int_0^\infty e^{-sp} \, p^{a-1} E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l} \right) dp$$

$$= \sum_{n=0}^\infty \left(-\frac{1}{t^l(1-t)^l} \right)^n \frac{1}{\Gamma(\lambda n + \alpha)} \int_0^\infty e^{-sp} \, p^{a+n-1} \, dp$$

$$= \frac{1}{s^a} \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(n+1)}{\Gamma(\lambda n + \alpha) \, n!} \left(-\frac{1}{s \, t^l(1-t)^l} \right)^n,$$

employing the Fox-Wright function (1.12) in the above sum and using in (4.6), we get,

$$\int_0^\infty p^{a-1} e^{-sp} \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) \, dp = \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j \, B(a_1, b_1 + j - a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$\times \frac{1}{s^a} \left(\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1} {}_2\Psi_1 \left[\begin{array}{c} (1,1),(a,1) \\ (\alpha,\lambda) \end{array}; -\frac{1}{s \, t^l(1-t)^l} \right] \right) dt.$$

Upon evaluating the integral using the generalized beta function defined in (1.11), we reach the result (4.5).

4.3. Mellin Transform

The Mellin transform of the function $f(p)$ is defined as [18],

$$f^*(s) = \mathbf{M}\{f(p); s\} = \int_0^\infty p^{s-1} f(p) \, dp, \qquad R(s) > 0, \tag{4.7}$$

and the inverse Mellin transform is given by

$$f(p) = \mathbf{M}^{-1}\{f^*(s); p\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-s} f^*(s) \, ds, \qquad c \in \mathsf{R}. \tag{4.8}$$

Theorem 6. *The Mellin transform of* $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ *is as follows:*

$$\mathbf{M}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);s\right\} = \int_0^\infty p^{s-1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\,dp$$

$$= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}H_C^{(ls)}(a_1,a_2,a_3;b_1;x_1,x_2,x_3), \tag{4.9}$$

where $\alpha, s \in \mathsf{C}; R(s) > \alpha > 0, R(\alpha) > 0, \lambda > 0, b_1 \in \mathsf{C}\backslash\mathsf{Z}_0^-$ *and* $H_C^{(ls)}(\cdot)$ *is defined in* (1.6).

Proof: Using (2.2) in the integral on the left-hand side of (4.9), we get

$$\mathbf{M}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);s\right\} = \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\, B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$\times \int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}\left\{\int_0^\infty p^{s-1}E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)dp\right\}dt. \tag{4.10}$$

Now, if we put $\gamma = q = 1$ and $\xi = -p/t^l(1-t)^l$ in the following result ([16], Th. 4.1)

$$E_{\lambda,\alpha}^{\gamma,q}(\xi)\,d\xi = \frac{1}{2\pi i\,\Gamma(\gamma)}\int_L \frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\alpha-\lambda s)}(-\xi)^{-s}\,ds,$$

where L is the contour of integration that begins at $c - i\infty$ and ends at $c + i\infty$, $c \in \mathsf{R}$,
  we get,

$$E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}\left(\frac{p}{t^l(1-t)^l}\right)^{-s}ds$$

$$= \frac{1}{2\pi i}\int_L f^*(s)\,p^{-s}\,ds,$$

where

$$f^*(s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}\left[t^l(1-t)^l\right]^s. \tag{4.11}$$

Using (4.7), (4.8), and (4.11), which leads to

$$\int_0^\infty p^{s-1}E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l}\right)dp = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}\left[t^l(1-t)^l\right]^s \tag{4.12}$$

Finally, put (4.12) in (4.10), yields

$$\mathbf{M}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);s\right\} = \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\, B(a_1,b_1+j-a_1)}$$

$$\times \frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}\int_0^1 t^{a_1+i+k+ls-1}(1-t)^{b_1+j+ls-a_1-1}dt.$$

Evaluate the integral using the classical Beta function, yields

$$\mathbf{M}\left\{\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3);s\right\}$$

$$= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha-\lambda s)}\sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j}\frac{B(a_1+ls+i+k,b_1+ls+j-a_1)}{B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}.$$

Identifying the above sum as $H_C^{(ls)}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)$ defined in (1.6), we get the result (4.9).

Corollary 1: The inverse Mellin formula for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ is as follows:

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) = \mathbf{M}^{-1}\{f^*(s); p\}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-s} \frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(\alpha - \lambda s)} H_C^{(ls)}(a_1, a_2, a_3; b_1; x_1, x_2, x_3) ds, \tag{4.13}$$

where $c > \alpha$

4.4. Whittaker Transform

Theorem 7. *The Whittaker transform $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ is as follows:*

$$\int_0^\infty p^{a-1}\, e^{-\frac{1}{2}bp}\, \mathbf{W}_{\eta,\mu}(bp)\, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)\, dp$$

$$= \frac{1}{b^a} \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1, b_1 + j - a_1)} \frac{x_1^i\, x_2^j\, x_3^k}{i!\, j!\, k!}$$

$$\times {}^{\Psi}B_{\left(\frac{1}{b}\right)}^l \left[\begin{array}{c} (1,1), (\frac{1}{2} \pm \mu + a, 1) \\ (\alpha, \lambda),\, (1 - \eta + a, 1) \end{array} \middle| (a_1 + i + k), (b_1 + j - a_1) \right] \tag{4.14}$$

where, $\alpha, a, (a_1 + i + k), (b_1 + j - a_1) \in \mathbb{C}; \lambda > 0, R(\alpha) > 0, R(a) > 0, R(a_1 + i + k) > 0, R(b_1 + j - a_1) > 0$.

Proof: Using (4.14) in (2.2), we get

$$\int_0^\infty p^{a-1}\, e^{-\frac{1}{2}bp}\, \mathbf{W}_{\eta,\mu}(bp)\, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)\, dp$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1, b_1 + j - a_1)} \frac{x_1^i\, x_2^j\, x_3^k}{i!\, j!\, k!}$$

$$\times \left(\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1} \left\{ \int_0^\infty p^{a-1}\, e^{-\frac{1}{2}bp}\, \mathbf{W}_{\eta,\mu}(bp)\, E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l} \right) dp \right\} \right) dt. \tag{4.15}$$

Now consider the right-most integral of (4.15) as I and put $bp = v$, yields

$$I = \int_0^\infty p^{a-1}\, e^{-\frac{1}{2}bp}\, \mathbf{W}_{\eta,\mu}(bp)\, E_{\lambda,\alpha}\left(-\frac{p}{t^l(1-t)^l} \right) dp$$

$$= \int_0^\infty \left(\frac{v}{b}\right)^{a-1} e^{\frac{-v}{2}}\, \mathbf{W}_{\eta,\mu}(v) \sum_{n=0}^{\infty} \left(-\frac{v}{b\, t^l(1-t)^l} \right)^n \frac{1}{\Gamma(\lambda n + \alpha)} \frac{1}{b}\, dv$$

$$= \frac{1}{b^a} \sum_{n=0}^{\infty} \left(-\frac{1}{b\, t^l(1-t)^l} \right)^n \frac{1}{\Gamma(\lambda n + \alpha)} \int_0^\infty v^{n+a-1}\, e^{-\frac{v}{2}}\, \mathbf{W}_{\eta,\mu}(v)\, dv. \tag{4.16}$$

Now employing the result

$$\int_0^\infty v^{a-1}\, e^{-\frac{v}{2}}\, \mathbf{W}_{\eta,\mu}(v)\, dv = \frac{\Gamma(\frac{1}{2} + \mu + a)\,\Gamma(\frac{1}{2} - \mu + a)}{\Gamma(1 - \eta + a)}\,, \qquad \left(R(\mu \pm a) > \frac{-1}{2} \right),$$

where $\mathbf{W}_{\eta,\mu}(v)$ is the Whittaker function [25], in (4.16), we get

$$I = \frac{1}{b^a} \sum_{n=0}^{\infty} \left(-\frac{1}{b\, t^l(1-t)^l} \right)^n \frac{\Gamma(\frac{1}{2} + \mu + a + n)\,\Gamma(\frac{1}{2} - \mu + a + n)\,\Gamma(n+1)}{\Gamma(\lambda n + \alpha)\,\Gamma(1 - \eta + a + n)\, n!}$$

Using (1.12), we get

$$I = \frac{1}{b^a} \, {}_3\Psi_2 \left[\begin{array}{c} (1,1), (\frac{1}{2}+\mu+a,1), (\frac{1}{2}-\mu+a,1) \\ (\alpha,\lambda), (1-\eta+a,1) \end{array} ; - \frac{1}{b\,t^l(1-t)^l} \right]. \tag{4.17}$$

Applying (4.17) in (4.15), we get

$$\int_0^\infty p^{a-1} e^{-\frac{1}{2}bp} \, \mathbf{W}_{\eta,\mu}(bp) \, \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\,dp = \sum_{i,j,k=0}^\infty \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$\times \int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1} \frac{1}{b^a} \, {}_3\Psi_2 \left[\begin{array}{c} (1,1), (\frac{1}{2}\pm\mu+a,1) \\ (\alpha,\lambda), (1-\eta+a,1) \end{array} ; - \frac{1}{b\,t^l(1-t)^l} \right] dt.$$

Upon evaluating the integral in terms of the generalized Beta function defined in (1.11), we obtain the desired result (4.14).

5. Fractional Calculus of $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$

Here, we investigate some results associated with the right-sided Riemann-Liouville fractional integral operator I_{a+}^ξ and the derivative operator D_{a+}^ξ, which are defined respectively as follows [9,14]:

$$(I_{a+}^\xi f)(z) = \frac{1}{\Gamma(\xi)} \int_a^z \frac{f(p)}{(z-p)^{1-\xi}}\,dp, \quad (R(\xi) > 0,\ \xi \in \mathsf{C}), \tag{5.1}$$

and

$$(D_{a+}^\xi f)(z) = \left(\frac{d}{dz}\right)^m (I_{a+}^{m-\xi} f)(z), \quad (m=[R(\xi)]+1;\ R(\xi) > 0, \alpha \in \mathsf{C}), \tag{5.2}$$

where $[t]$ is the greatest integer.

Hilfer [5] defined a generalized Riemann-Liouville fractional derivative operator $D_{a+}^{\xi,\gamma}$ of order $0 < \xi < 1$ and type $0 \le \gamma \le 1$ with respect to z as follows:

$$(D_{a+}^{\xi,\gamma} f)(z) = \left(I_{a+}^{\gamma(1-\xi)} \frac{d}{dz}\right)\left(I_{a+}^{(1-\gamma)(1-\xi)} f\right)(z), \qquad (m=[R(\xi)]+1;\ R(\xi) > 0, \xi \in \mathsf{C}). \tag{5.3}$$

Theorem 8. *Let $a \in \mathsf{R}_+;\ \alpha, \xi \in \mathsf{C};\ \lambda > 0, R(\alpha) > 0, R(\xi) > 0$, and $b_1 \in \mathsf{C}\backslash\mathsf{Z}_0^-$, then for $z > a$ the following relations hold for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$*

$$(i) \left(I_{a+}^\xi \left[(p-a)^{\alpha-1} \mathbf{H}_{C,(p-a)^\lambda,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) \right]\right)(z)$$

$$= (z-a)^{\alpha+\xi-1} \mathbf{H}_{C,(z-a)^\lambda,\alpha+\xi}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) \tag{5.4}$$

$$(ii) \left(D_{a+}^\xi \left[(p-a)^{\alpha-1} \mathbf{H}_{C,(p-a)^\lambda,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) \right]\right)(z)$$

$$= (z-a)^{\alpha-\xi-1} \mathbf{H}_{C,(z-a)^\lambda,\alpha-\xi}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) \tag{5.5}$$

and

$$(iii) \left(D_{a+}^{\xi,\gamma} \left[(p-a)^{\alpha-1} \mathbf{H}_{C,(p-a)^\lambda,\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3) \right]\right)(z)$$

$$= (z-a)^{\alpha-\xi-1} \mathbf{H}_{C,(z-a)^\lambda,\alpha-\xi}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3). \tag{5.6}$$

Proof. Use (2.2) in (5.1), and with the help of the relation [14]

$$\left(I_{a+}^{\xi}\left[(p-a)^{\mu-1}\right]\right)(z) = \frac{\Gamma(\mu)}{\Gamma(\mu+\xi)}(z-a)^{\xi+\mu-1},$$

for $z > a$ yields,

$$\left(I_{a+}^{\xi}\left[(p-a)^{\alpha-1}\mathbf{H}_{C,(p-a)^{\lambda},\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]\right)(z)$$

$$= \left(I_{a+}^{\xi}\left[(p-a)^{\alpha-1}\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\right.\right.$$

$$\left.\left.\times\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}E_{\lambda,\alpha}\left\{-\frac{(p-a)^{\lambda}}{t^l(1-t)^l}\right\}dt\right]\right)(z)$$

$$= \sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}$$

$$\times\sum_{n=0}^{\infty}\left(-\frac{1}{t^l(1-t)^l}\right)^n\frac{1}{\Gamma(\lambda n+\alpha)}\left(I_{a+}^{\xi}\left[(p-a)^{\lambda n+\alpha-1}\right]dt\right)(z)$$

$$= (z-a)^{\alpha+\xi-1}\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$\times\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}E_{\lambda,\alpha+\xi}\left\{-\frac{(z-a)^{\lambda}}{t^l(1-t)^l}\right\}dt$$

$$= (z-a)^{\alpha+\xi-1}\mathbf{H}_{C,(z-a)^{\lambda},\alpha+\xi}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3).$$

Next, employing (2.2) in (5.2), we get

$$\left(D_{a+}^{\xi}\left[(p-a)^{\alpha-1}\mathbf{H}_{C,(p-a)^{\lambda},\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]\right)(z)$$

$$= \left(\frac{d}{dz}\right)^m\left(I_{a+}^{m-\xi}\left[(p-a)^{\alpha-1}\mathbf{H}_{C,(p-a)^{\lambda},\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]\right)(z)$$

$$= \left(\frac{d}{dz}\right)^m\left[(z-a)^{\alpha+m-\xi-1}\mathbf{H}_{C,(z-a)^{\lambda},\alpha+m-\xi}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right].$$

Using (3.7), we get the desired result (5.5).

Finally, with the help of eqs. (2.2) and (5.3), we obtain

$$\left(D_{a+}^{\xi,\gamma}\left[(p-a)^{\alpha-1}\mathbf{H}_{C,(p-a)^{\lambda},\alpha}^{\lambda,l}(a_1,a_2,a_3;b_1;x_1,x_2,x_3)\right]\right)(z)$$

$$= \left(D_{a+}^{\xi,\gamma}\left[(p-a)^{\alpha-1}\sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\right.\right.$$

$$\left.\left.\times\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}E_{\lambda,\alpha}\left\{-\frac{(p-a)^{\lambda}}{t^l(1-t)^l}\right\}dt\right]\right)(z)$$

$$= \sum_{i,j,k=0}^{\infty}\frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j\,B(a_1,b_1+j-a_1)}\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\int_0^1 t^{a_1+i+k-1}(1-t)^{b_1+j-a_1-1}$$

$$\times\sum_{n=0}^{\infty}\left(-\frac{1}{t^l(1-t)^l}\right)^n\frac{1}{\Gamma(\lambda n+\alpha)}\left(D_{a+}^{\xi,\gamma}\left[(p-a)^{\lambda n+\alpha-1}\right]dt\right)(z). \tag{5.7}$$

Applying the relation of Srivastava and Tomovski ([21], p.203, eq.(2.18))

$$\left(D_{a+}^{\xi,\gamma}\left[(p-a)^{\beta-1}\right]\right)(z) = \frac{\Gamma(\beta)}{\Gamma(\beta-\xi)}(z-a)^{\beta-\xi-1}$$

$$(z > a;\, 0 < \xi < 1;\, 0 \le \gamma \le 1;\, R(\beta) > 0),$$

in (5.7), we get the relation (5.6).

6. Recursion Formulas for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$

In this section, we investigate two recursion formulas for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$. The first formula gives recursions with respect to the numerator parameters a_2 and a_3, and the second a recursion with respect to the denominator parameter b_1.

Theorem 9. *We have the following recursions for* $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ *with respect to the numerator parameters* a_2 *and* a_3 *as follows:*

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$+\frac{x_1 a_1}{b_1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2+1, a_3; b_1+1; x_1, x_2, x_3) + \frac{x_2 a_3}{b_1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3+1; b_1+1; x_1, x_2, x_3). \quad (6.1)$$

and

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3+1; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$+\frac{x_2 a_2}{b_1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3+1; b_1+1; x_1, x_2, x_3) + \frac{x_3 a_1}{b_1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2, a_3+1; b_1+1; x_1, x_2, x_3). \quad (6.2)$$

Proof. From (2.1) and using the identity $(a_3+1)_{j+k} = (a_3)_{j+k}(1 + j/a_3 + k/a_3)$, we obtain

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3+1; b_1; x_1, x_2, x_3)$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3+1)_{j+k}}{(b_1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$= \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$+\frac{x_2}{a_3}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!}\frac{x_2^{j-1}}{(j-1)!}\frac{x_3^k}{k!}$$

$$+\frac{x_3}{a_3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} \frac{(a_2)_{i+j}(a_3)_{j+k}}{(b_1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^{k-1}}{(k-1)!}. \quad (6.3)$$

Consider the first sum in (6.3), which we denote by S_1. Put $j \to j+1$ and apply the identity $(a)_{j+1} = a(a+1)_j$, we find

$$S_1 = \frac{x_2}{a_3}\sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j+1}(a_3)_{j+k+1}}{(b_1)_{j+1}} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+j+1-a_1)}{B(a_1, b_1+j+1-a_1)} \frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}$$

$$= \frac{x_2 a_2}{b_1}\sum_{i,j,k=0}^{\infty} \frac{(a_2+1)_{i+j}(a_3+1)_{j+k}}{(b_1+1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+i+k, b_1+1+j-a_1)}{B(a_1, b_1+j+1-a_1)} \frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}.$$

$$= \frac{x_2 a_2}{b_1}\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3+1; b_1+1; x_1, x_2, x_3). \quad (6.4)$$

Similarly, take the second sum in (6.3) as S_2 with $k \to k+1$, we obtain

$$S_2 = x_3 \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3+1)_{j+k}}{(b_1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+1+i+k, b_1+j-a_1)}{B(a_1, b_1+j-a_1)} \frac{x^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}.$$

Now, applying the result

$$B(p+1, q) = \frac{p}{p+q} B(p, q),$$

yields

$$S_2 = \frac{x_3 a_1}{b_1} \sum_{i,j,k=0}^{\infty} \frac{(a_2)_{i+j}(a_3+1)_{j+k}}{(b_1+1)_j} \frac{B_{p,\alpha}^{\lambda,l}(a_1+1+i+k, b_1+j-a_1)}{B(a_1+1, b_1+j-a_1)} \frac{x^i}{i!} \frac{x_2^j}{j!} \frac{x_3^k}{k!}$$

$$= \frac{x_3 a_1}{b_1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2, a_3+1; b_1+1; x_1, x_2, x_3). \tag{6.5}$$

Combining (6.4) and (6.5) with (6.3), we get the required result (6.2). Similarly, we can establish the proof of the result (6.1).

Corollary 2: From (6.1) and (6.2), the following recursions for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ hold:

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+M, a_3; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$+ \frac{x_1 a_1}{b_1} \sum_{k=1}^{M} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2+k, a_3; b_1+1; x_1, x_2, x_3) + \frac{x_2 a_3}{b_1} \sum_{k=1}^{M} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+k, a_3+1; b_1+1; x_1, x_2, x_3) \tag{6.6}$$

and

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3+M; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$+ \frac{x_2 a_2}{b_1} \sum_{k=1}^{M} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3+k; b_1+1; x_1, x_2, x_3) + \frac{x_3 a_1}{b_1} \sum_{k=1}^{M} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2, a_3+k; b_1+1; x_1, x_2, x_3), \tag{6.7}$$

for positive integer M.

Theorem 10. *The function $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ satisfies the following recursion formulas:*

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2-1, a_3; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$- \frac{x_1 a_1}{b_1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2, a_3; b_1+1; x_1, x_2, x_3) - \frac{x_2 a_3}{b_1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3+1; b_1+1; x_1, x_2, x_3). \tag{6.8}$$

and

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3-1; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$- \frac{x_2 a_2}{b_1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2+1, a_3; b_1+1; x_1, x_2, x_3) - \frac{x_3 a_1}{b_1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2, a_3; b_1+1; x_1, x_2, x_3). \tag{6.9}$$

Proof. Replace a_2 and a_3 with $a_2 - 1$ and $a_3 - 1$ in (6.1) and (6.2), we get the results (6.8) and (6.9) respectively.

Corollary 3: From (6.1) and (6.2), the following recursions for $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ hold:

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2-M, a_3; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$- \frac{x_1 a_1}{b_1} \sum_{k=0}^{M-1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1+1, a_2-k, a_3; b_1+1; x_1, x_2, x_3) - \frac{x_2 a_3}{b_1} \sum_{k=0}^{M-1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2-k, a_3+1; b_1+1; x_1, x_2, x_3) \tag{6.10}$$

and

$$\mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3 - M; b_1; x_1, x_2, x_3) = \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2, a_3; b_1; x_1, x_2, x_3)$$

$$- \frac{x_2 a_2}{b_1} \sum_{k=0}^{M-1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1, a_2 + 1, a_3 - k; b_1 + 1; x_1, x_2, x_3) - \frac{x_3 a_1}{b_1} \sum_{k=0}^{M-1} \mathbf{H}_{C,p,\alpha}^{\lambda,l}(a_1 + 1, a_2, a_3 - k; b_1 + 1; x_1, x_2, x_3), \qquad (6.11)$$

for positive integer M.

7. Conclusion

In this paper, we have introduced the extended Srivastava's triple hypergeometric function defined by $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$ in (2.1). We have given some integral representations of this function that involve the generalized Mittag-Leffler function and Gauss hypergeometric function. We have also explored some derivative properties and integral transforms of the function $\mathbf{H}_{C,p,\alpha}^{\lambda,l}(\cdot)$, namely the Euler-Beta transform, Laplace Transform, Mellin transform, and Whittaker transform and some recursion relations.

Conflicts of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethical Approval: This article does not contain any studies with human participants or animals performed by any of the author.

References

1. Abubakar, U.M., New generalized beta function associated with the Fox-Wright function, Journal of Fractional Calculus and Application **12(2)** (2021), 204-227.
2. Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M., Extension of Euler's beta function, J. Comput. Appl. Math. **78(1)** (1997), 19-32.
3. Dar, S.A., Paris, R.B., A (p, q)-extension of Srivastava's triple hypergeometric function HB(x,y,z) and its properties, Journal of Computational and Applied Mathematics 348(2019), 237-245.
4. Dar, S.A., Paris, R.B., A (p, v)-extension of Srivastava's triple hypergeometric function HC and its properties, Publicationsde L'Institut Mathesmatique **108(122)** (2020), 33-45.
5. Hilfer, R., Application of fractional calculus in physics, World Scientific Publishing Co. Pte. Ltd, Singapore, (2000).
6. Karlsson, P.W., Regions of convergences for hypergeometric series in three variables, Math. Scand, **48** (1974), 241-248.
7. Kilbas, A.A., Saigo, M., H-Transforms: Theory and Applications (Analytical Methods and Special Functions), CRC Press Company Boca Raton, New York, 2004.
8. Kilbas, A.A., Saigo, M., Trujillo, J.J., On the generalized Wright function, Frac. Calc. Appl. Anal. **5(4)** (2002), 437–460.
9. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., Theory and applications of fractional differential equations, elsevier Vol. **204**, (2006).
10. Mittag-Leffler, G. M., Une généralisation de l'intégrale de Laplace-Abel, CR Acad. Sci. Paris (Ser. II) **137** (1903), 537-539 .
11. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark C.W. (eds.), *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge, (2010).
12. Oraby, K. M., Rizq, A., Abdelhak, E., Taema, M., Magdy, M., Khaled, M., Generalization of Beta functions in terms of Mittag-Leffler function, Frontiers in Scientific Research and Technology **1(1)** (2020), 81-88.
13. Rainville, E. D., *Special functions*, The Macmillan Company, Newyork, (1960).
14. Samko, S. G., Kilbas, A. A., Marichev, O. I., *Fractional integrals and derivatives*, Vol. 1. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon (1993).
15. Shadab, M., Jabee, S., Choi, J., An extended beta function and its applications, Far East Journal of Mathematical Sciences. **103(1)** (2018), 235-251.

16. Shukla, A.K., Prajapati, J.C., On a generalization of Mittag-Leffler function and its properties, Journal of mathematical analysis and applications, **336(2)** (2007), 797-811.
17. Slater, L.J., *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge (1966).
18. Sneddon, I.N., *The Use of Integral Transforms*, New Delhi,Tata McGraw Hill, (1979).
19. Srivastava, H.M., Hypergeometric functions of three variables, Ganita **15** (1964), 97-108.
20. Srivastava, H.M., Some integrals representing triple hypergeometric functions, Rend. Circ. Mat. Palermo (Ser. 2) **16** (1967), 99-115.
21. Srivastava, H.M., Tomovski, ˆ Z. Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernal, Appl. Math. Comput.**211** (2009), 198-210.
22. Srivastava, H.M., Manocha, H.L., *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1984).
23. Srivastava, H.M., Karlsson, P.W., *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1985).
24. Wiman, A., Über den Fundamentalsatz in der Teorie der Funktionen $E_\alpha(x)$, (1905), 191-201.
25. Whittaker, E.T., Watson, G.N., *A course of Modern Analysis*, vol **I**, **II**, (reprint of 1927 ed.) Cambridge Univ. Press, New York, (1962).
26. Yang, G., Shiri, B., Kong, H., Wu, G. C., Intermediate value problems for fractional differential equations, Computational and Applied Mathematics, **40(6)** (2021), 1-20.