Article

Not peer-reviewed version

Fuzzy topological approaches via r-fuzzy γ-open sets in the sense of Sostak

Fahad Alsharari , Hind Y. Saleh , Osama M. Taha , Islam M. Taha *

Posted Date: 18 February 2025

doi: 10.20944/preprints202502.1193.v1

Keywords: F-topology; r-F-γ-open set; F-γ-closure operator; F-γ-continuity; F-γ-irresoluteness; r-F-γ-compact set; r-F-almost γ-compact set; r-F-γ-regular space; r-F-γ-normal space

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Article

Fuzzy Topological Approaches Via r-fuzzy γ-Open Sets in the Sense of Šostak

Fahad Alsharari [1], Hind Y. Saleh [2], Osama Taha[3] and Islam M. Taha[3,4,*]

[1] Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia
[2] Department of Mathematics, College of Basic Education, University of Duhok, Duhok, Iraq
[3] Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt
[4] Department of Basic Sciences, High Institute for Engineering and Technology, Sohag, Egypt
* Correspondence: itaha@ut.edu.sa

Abstract: In the present article, we define and investigate the notion of r-fuzzy γ-open (r-$\mathcal{F}$-γ-open) sets as a generalized novel class of fuzzy open ($\mathcal{F}$-open) sets on fuzzy topological spaces ($\mathcal{FTS}s$) in the sense of Šostak. This class is contained in the class of r-$\mathcal{F}$-β-open sets and contains all r-$\mathcal{F}$-pre-open and r-$\mathcal{F}$-semi-open sets. However, we introduce the interior and closure operators with respect to the classes of r-$\mathcal{F}$-γ-open and r-$\mathcal{F}$-γ-closed sets, and study some of their properties. After that, we define and discuss the notions of $\mathcal{F}$-γ-continuous (respectively (resp. for short) $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and (N, F). Also, we display and investigate the notions of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions. We also showed that $\mathcal{F}$-γ-continuity $\Longrightarrow$ $\mathcal{F}$-almost γ-continuity $\Longrightarrow$ $\mathcal{F}$-weakly γ-continuity, but the converse may not be true. Next, we present and characterize new $\mathcal{F}$-functions via r-$\mathcal{F}$-γ-open and r-$\mathcal{F}$-γ-closed sets, called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. The relationships between these classes of functions were discussed with the help of some examples. We also introduce some new types of $\mathcal{F}$-separation axioms, called r-$\mathcal{F}$-γ-regular (resp. r-$\mathcal{F}$-γ-normal) spaces via r-$\mathcal{F}$-γ-closed sets, and study some properties of them. Lastly, we explore and discuss some new types of $\mathcal{F}$-compactness, called r-$\mathcal{F}$-almost (resp. r-$\mathcal{F}$-nearly) γ-compact sets using r-$\mathcal{F}$-γ-open sets.

Keywords: $\mathcal{F}$-topology; r-$\mathcal{F}$-γ-open set; $\mathcal{F}$-γ-closure operator; $\mathcal{F}$-γ-continuity; $\mathcal{F}$-γ-irresoluteness; r-$\mathcal{F}$-γ-compact set; r-$\mathcal{F}$-almost γ-compact set; r-$\mathcal{F}$-γ-regular space; r-$\mathcal{F}$-γ-normal space

MSC: 54A05, 54A40, 54C05, 54C08, 54D15

---

1. Introduction

The concept of a fuzzy set ($\mathcal{F}$-set) of a nonempty set M is a mapping $\mathcal{D} : M \to I$ (where $I = [0,1]$), this concept was first defined in 1965 by Zadeh [1]. The theory of $\mathcal{F}$-sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. After the introduction of the concept of $\mathcal{F}$-sets, several research studies were conducted on the generalizations of $\mathcal{F}$-sets. The integration between $\mathcal{F}$-sets and some uncertainty approaches such as soft sets ($\mathcal{S}$-sets) and rough sets ($\mathcal{R}$-sets) has been discussed in [2-4]. The concept of a fuzzy topology ($\mathcal{F}$-topology) was presented in 1968 by Chang [5]. Several authors have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. According to Šostak [6], these notions, an $\mathcal{F}$-topology is a crisp subclass of the class of $\mathcal{F}$-sets and fuzziness in the notion of openness of an $\mathcal{F}$-set has not been considered, which seems to be a drawback in the process of fuzzification of the notion of a topological space. Therefore, Šostak [6] defined a novel definition of the notion of an $\mathcal{F}$-topology as the concept of openness of $\mathcal{F}$-sets. It is an extension of an $\mathcal{F}$-topology introduced by Chang [5]. Thereafter, many researchers (Ramadan [7], Chattopadhyay et. al. [8], El Gayyar et. al. [9], Höhle and Šostak [10], Ramadan et. al. [11], Kim et. al.

 doi:10.20944/preprints202502.1193.v1

[12], Abbas [13,14], Kim and Abbas [15], Aygun and Abbas [16,17], Li and Shi [18,19], Shi and Li [20], Fang and Guo [21], El-Dardery et. al. [22], Kalaivani and Roopkumar [23], Solovyov [24], Minana and Šostak [25]) have redefined the same notion and studied $\mathcal{FTS}s$ being unaware of Šostak work.

The generalizations of fuzzy open sets ($\mathcal{F}$-open sets) play an effective role in an $\mathcal{F}$-topology through their use to improve on many results, or to open the door to explore and discuss several of fuzzy topological notions such as $\mathcal{F}$-continuity [7,8], $\mathcal{F}$-connectedness [8], $\mathcal{F}$-compactness [8,9], $\mathcal{F}$-separation axioms [18], etc. Overall, the notions of r-$\mathcal{F}$-semi-open, r-$\mathcal{F}$-pre-open, r-$\mathcal{F}$-α-open, and r-$\mathcal{F}$-β-open sets were defined and studied by the authors of [12,14] on $\mathcal{FTS}s$ in the sense of Šostak [6]. Also, Kim et al. [12] defined and discussed some weaker forms of $\mathcal{F}$-continuity, called $\mathcal{F}$-semi-continuity (resp. $\mathcal{F}$-pre-continuity and $\mathcal{F}$-α-continuity) between $\mathcal{FTS}s$ in the sense of Šostak [6]. Furthermore, Abbas [14] explored and characterized the notions of $\mathcal{F}$-β-continuous (resp. $\mathcal{F}$-β-irresolute) functions between $\mathcal{FTS}s$ in the sense of Šostak [6]. Additionally, Kim and Abbas [15] introduced several types of r-$\mathcal{F}$-compactness on $\mathcal{FTS}s$ in the sense of Šostak [6].

The notion of fuzzy soft sets ($\mathcal{FS}$-sets) was first presented in 2001 by the author of [26], which combines $\mathcal{S}$-set [27] and $\mathcal{F}$-set [1]. Thereafter, the notion of an $\mathcal{FS}$-topology was defined and many of its properties such as $\mathcal{FS}$-continuity, $\mathcal{FS}$-closure operators, $\mathcal{FS}$-interior operators, and $\mathcal{FS}$-subspaces were introduced in [28,29]. Also, a novel approach to discussing $\mathcal{FS}$-separation and $\mathcal{FS}$-regularity axioms using $\mathcal{FS}$-sets was introduced by Taha [30,31] based on the approach developed by Aygünoğlu et al. [28]. Moreover, the notions of r-$\mathcal{FS}$-regularly-open, r-$\mathcal{FS}$-pre-open, r-$\mathcal{FS}$-semi-open, r-$\mathcal{FS}$-α-open, and r-$\mathcal{FS}$-β-open sets were introduced by the authors of [32-35] based on the approach developed by Aygünoğlu et al. [28]. Additionally, Alshammari et al. [36] defined and investigated the notions of r-$\mathcal{FS}$-δ-open sets and $\mathcal{FS}$-δ-continuous functions. Overall, Taha [37] introduced and discussed the notions of $\mathcal{FS}$-almost (resp. $\mathcal{FS}$-weakly) r-minimal continuity, which are weaker forms of $\mathcal{FS}$-r-minimal continuity [33] based on the approach developed by Aygünoğlu et al. [28].

We lay out the remainder of this article as follows. Section 2 contains some basic definitions and results that help in understanding the obtained results. In Section 3, we display a novel class of $\mathcal{F}$-open sets, called r-$\mathcal{F}$-γ-open sets on $\mathcal{FTS}s$ in the sense of Šostak [6]. The class of r-$\mathcal{F}$-γ-open sets is contained in the class of r-$\mathcal{F}$-β-open sets and contains all r-$\mathcal{F}$-α-open, r-$\mathcal{F}$-pre-open, and r-$\mathcal{F}$-semi-open sets. Some properties of r-$\mathcal{F}$-γ-open sets along with their mutual relationships have been specified with the help of some illustrative examples. After that, we define the concepts of $\mathcal{F}$-γ-closure and $\mathcal{F}$-γ-interior operators, and study some of their properties. In Section 4, we explore and investigate the concepts of $\mathcal{F}$-γ-continuous (resp. $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and (N, F). Moreover, we define and study the concepts of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions. We also showed that $\mathcal{F}$-γ-continuity $\Longrightarrow$ $\mathcal{F}$-almost γ-continuity $\Longrightarrow$ $\mathcal{F}$-weakly γ-continuity, but the converse may not be true. In Section 5, we introduce and discuss some novel $\mathcal{F}$-functions using r-$\mathcal{F}$-γ-open and r-$\mathcal{F}$-γ-closed sets, called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Furthermore, we define some new types of $\mathcal{F}$-separation axioms, called r-$\mathcal{F}$-γ-regular (resp. r-$\mathcal{F}$-γ-normal) spaces, and study some properties of them. Also, we explore and discuss some new types of $\mathcal{F}$-compactness, called r-$\mathcal{F}$-almost (resp. r-$\mathcal{F}$-nearly) γ-compact sets using r-$\mathcal{F}$-γ-open sets. In the last section, we close this article with conclusions and proposed future papers.

2. Preliminaries

In this manuscript, nonempty sets will be denoted by M, N, W, etc. On M, I^M is the class of all $\mathcal{F}$-sets. For $\mathcal{D} \in I^M$, $\mathcal{D}^c(m) = 1 - \mathcal{D}(m)$, for each $m \in M$. Also, for $\sigma \in I$, $\underline{\sigma}(m) = \sigma$, for each $m \in M$.

 doi:10.20944/preprints202502.1193.v1

An $\mathcal{F}$-point m_σ on M is an $\mathcal{F}$-set, defined as follows: $m_\sigma(u) = \sigma$ if $u = m$, and $m_\sigma(u) = 0$ for any $u \in M - \{m\}$. Moreover, we say that m_σ belong to $\mathcal{D} \in I^M$ ($m_\sigma \in \mathcal{D}$), if $\sigma \leq \mathcal{D}(m)$. On M, $P_\sigma(M)$ is the class of all $\mathcal{F}$-points.

On M, an $\mathcal{F}$-set $\mathcal{D} \in I^M$ is a quasi-coincident with $\mathcal{P} \in I^M$ ($\mathcal{D}\ q\ \mathcal{P}$), if there is $m \in M$, with $\mathcal{D}(m) + \mathcal{P}(m) > 1$. Otherwise, $\mathcal{D}$ is not quasi-coincident with $\mathcal{P}$ ($\mathcal{D}\ \overline{q}\ \mathcal{P}$).

Lemma 2.1. [38] Let $\mathcal{D}, \mathcal{P} \in I^M$. Thus,

(1) $\mathcal{D}\ q\ \mathcal{P}$ iff there is $m_\sigma \in \mathcal{D}$ such that $m_\sigma\ q\ \mathcal{P}$,

(2) if $\mathcal{D}\ q\ \mathcal{P}$, then $\mathcal{D} \wedge \mathcal{P} \neq \underline{0}$,

(3) $\mathcal{D}\ \overline{q}\ \mathcal{P}$ iff $\mathcal{D} \leq \mathcal{P}^c$,

(4) $\mathcal{D} \leq \mathcal{P}$ iff $m_\sigma \in \mathcal{D}$ implies $m_\sigma \in \mathcal{P}$ iff $m_\sigma\ q\ \mathcal{D}$ implies $m_\sigma\ q\ \mathcal{P}$ iff $m_\sigma\ \overline{q}\ \mathcal{P}$ implies $m_\sigma\ \overline{q}\ \mathcal{D}$,

(5) $m_\sigma\ \overline{q}\ \bigvee_{j\in\Omega} \mathcal{P}_j$ iff there is $j_\circ \in \Omega$ such that $m_\sigma\ \overline{q}\ \mathcal{P}_{j_\circ}$.

Definition 2.1. [6,7] A mapping $\Im : I^M \longrightarrow I$ is said to be a fuzzy topology on M if it satisfies the following conditions:

(1) $\Im(\underline{1}) = \Im(\underline{0}) = 1$.

(2) $\Im(\mathcal{D} \wedge \mathcal{P}) \geq \Im(\mathcal{D}) \wedge \Im(\mathcal{P})$, for each $\mathcal{D}, \mathcal{P} \in I^M$.

(3) $\Im(\bigvee_{j\in\Omega} \mathcal{D}_j) \geq \bigwedge_{j\in\Omega} \Im(\mathcal{D}_j)$, for each $\mathcal{D}_j \in I^M$.

Thus, $(M, \Im)$ is said to be a fuzzy topological space ($\mathcal{FTS}$) in the sense of Šostak.

Definition 2.2. [8,11] In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$ (where $I_\circ = (0, 1]$), we define $\mathcal{F}$-operators $C_\Im$ and $I_\Im : I^M \times I_\circ \to I^M$ as follows:

$$C_\Im(\mathcal{D}, r) = \bigwedge \{\mathcal{P} \in I^M : \mathcal{D} \leq \mathcal{P},\ \Im(\mathcal{P}^c) \geq r\}.$$

$$I_\Im(\mathcal{D}, r) = \bigvee \{\mathcal{P} \in I^M : \mathcal{P} \leq \mathcal{D},\ \Im(\mathcal{P}) \geq r\}.$$

Definition 2.3. [12,14] Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is said to be r-$\mathcal{F}$-regularly-open (resp. r-$\mathcal{F}$-pre-open, r-$\mathcal{F}$-β-open, r-$\mathcal{F}$-semi-open, r-$\mathcal{F}$-α-open, and r-$\mathcal{F}$-open) if $\mathcal{D} = I_\Im(C_\Im(\mathcal{D}, r), r)$ (resp. $\mathcal{D} \leq I_\Im(C_\Im(\mathcal{D}, r), r)$, $\mathcal{D} \leq C_\Im(I_\Im(C_\Im(\mathcal{D}, r), r), r)$, $\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, r), r)$, $\mathcal{D} \leq I_\Im(C_\Im(I_\Im(\mathcal{D}, r), r), r)$, and $\mathcal{D} \leq I_\Im(\mathcal{D}, r)$).

Definition 2.4. [9,15] Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is said to be r-$\mathcal{F}$-compact (resp. r-$\mathcal{F}$-nearly compact and r-$\mathcal{F}$-almost compact) iff for every family $\{\mathcal{P}_j \in I^M \mid \Im(\mathcal{P}_j) \geq r\}_{j\in\Omega}$, with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{P}_j$, there is a finite sub-set $\Omega_\circ$ of Ω, with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} \mathcal{P}_j$ (resp. $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} I_\Im(C_\Im(\mathcal{P}_j, r), r)$ and $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} C_\Im(\mathcal{P}_j, r)$).

Definition 2.5. [7,12] Let $(M, \Im)$ and $(N, \digamma)$ be $\mathcal{FTS}s$. An $\mathcal{F}$-function $h: I^M \longrightarrow I^N$ is said to be

(1) $\mathcal{F}$-continuous if $\Im(h^{-1}(\mathcal{P})) \geq \digamma(\mathcal{P})$, for every $\mathcal{P} \in I^N$;

(2) $\mathcal{F}$-open if $\digamma(h(\mathcal{D})) \geq \Im(\mathcal{D})$, for every $\mathcal{D} \in I^M$;

(3) $\mathcal{F}$-closed if $\digamma((h(\mathcal{D}))^c) \geq \Im(\mathcal{D}^c)$, for every $\mathcal{D} \in I^M$.

Definition 2.6. [12,14] Let $(M, \Im)$ and $(N, \digamma)$ be $\mathcal{FTS}s$ and $r \in I_\circ$. An $\mathcal{F}$-function $h: I^M \longrightarrow I^N$ is said to be $\mathcal{F}$-α-continuous (resp. $\mathcal{F}$-pre-continuous, $\mathcal{F}$-semi-continuous, and $\mathcal{F}$-β-continuous) if $h^{-1}(\mathcal{P})$ is an r-$\mathcal{F}$-α-open (resp. r-$\mathcal{F}$-pre-open, r-$\mathcal{F}$-semi-open, and r-$\mathcal{F}$-β-open) set, for every $\mathcal{P} \in I^N$ with $\digamma(\mathcal{P}) \geq r$.

Some basic notations and results that we need in the sequel are found in [7-15].

3. On r-Fuzzy γ-Open Sets

Here, we define and study a new class of $\mathcal{F}$-open sets, called r-$\mathcal{F}$-γ-open sets on $\mathcal{FTS}s$ in the sense of Šostak [6]. The class of r-$\mathcal{F}$-γ-open sets is contained in the class of r-$\mathcal{F}$-β-open sets and contains all r-$\mathcal{F}$-α-open, r-$\mathcal{F}$-pre-open, and r-$\mathcal{F}$-semi-open sets. Also, we explore the concepts of $\mathcal{F}$-γ-closure and $\mathcal{F}$-γ-interior operators, and investigate some of their properties.

Definition 3.1. Let $(M, \Im)$ be an $\mathcal{FTS}$ and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D} \in I^M$ is said to be

(1) r-$\mathcal{F}$-γ-open set if $\mathcal{D} \leq C_\Im(I_\Im(\mathcal{D}, r), r) \vee I_\Im(C_\Im(\mathcal{D}, r), r)$;

(2) r-$\mathcal{F}$-γ-closed set if $\mathcal{D} \geq C_\Im(I_\Im(\mathcal{D}, r), r) \wedge I_\Im(C_\Im(\mathcal{D}, r), r)$.

Remark 3.1. The complement of r-$\mathcal{F}$-γ-open set (resp., r-$\mathcal{F}$-γ-closed set) is r-$\mathcal{F}$-γ-closed set (resp., r-$\mathcal{F}$-γ-open set).

Proposition 3.1. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$,

(1) every r-$\mathcal{F}$-pre-open set is r-$\mathcal{F}$-γ-open;

(2) every r-$\mathcal{F}$-γ-open set is r-$\mathcal{F}$-β-open;

(3) every r-$\mathcal{F}$-semi-open set is r-$\mathcal{F}$-γ-open.

Proof. (1) If $\mathcal{D}$ is an r-$\mathcal{F}$-pre-open set,

$$\mathcal{D} \leq I_\Im(C_\Im(\mathcal{D}, r), r) \leq I_\Im(C_\Im(\mathcal{D}, r), r) \vee I_\Im(\mathcal{D}, r) \leq I_\Im(C_\Im(\mathcal{D}, r), r) \vee C_\Im(I_\Im(\mathcal{D}, r), r).$$

Thus, $\mathcal{D}$ is r-$\mathcal{F}$-γ-open set.

(2) If $\mathcal{D}$ is an r-$\mathcal{F}$-γ-open set,

$$\mathcal{D} \leq C_{\Im}(I_{\Im}(\mathcal{D},r),r) \vee I_{\Im}(C_{\Im}(\mathcal{D},r),r) \leq C_{\Im}(I_{\Im}(C_{\Im}(\mathcal{D},r),r),r) \vee I_{\Im}(C_{\Im}(\mathcal{D},r),r)$$
$$\leq C_{\Im}(I_{\Im}(C_{\Im}(\mathcal{D},r),r),r).$$

Thus, $\mathcal{D}$ is r-$\mathcal{F}$-β-open set.

(3) If $\mathcal{D}$ is an r-$\mathcal{F}$-semi-open set,

$$\mathcal{D} \leq C_{\Im}(I_{\Im}(\mathcal{D},r),r) \leq C_{\Im}(I_{\Im}(\mathcal{D},r),r) \vee I_{\Im}(\mathcal{D},r) \leq C_{\Im}(I_{\Im}(\mathcal{D},r),r) \vee I_{\Im}(C_{\Im}(\mathcal{D},r),r).$$

Thus, $\mathcal{D}$ is r-$\mathcal{F}$-γ-open set.
□

Remark 3.2. From the previous discussions and definitions, we have the following diagram.

r-$\mathcal{F}$-**pre-open set**

↗ ↘

r-$\mathcal{F}$-**open set** $\longrightarrow$ r-$\mathcal{F}$-**ff-open set** $\longrightarrow$ r-$\mathcal{F}$-**fl-open set** $\longrightarrow$ r-$\mathcal{F}$-**fi-open set**

↘ ↗

r-$\mathcal{F}$-**semi-open set**

Remark 3.3. The converse of the above diagram fails as Examples 3.1, 3.2, and 3.3.

Example 3.1. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.3}\}$, $\mathcal{P} = \{\frac{m_1}{0.2}, \frac{m_2}{0.6}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.7}\}$. Define $\Im : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} = \mathcal{P} \wedge \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P} \vee \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{V}$ is $\frac{1}{3}$-$\mathcal{F}$-γ-open set, but it is neither $\frac{1}{3}$-$\mathcal{F}$-pre-open nor $\frac{1}{3}$-$\mathcal{F}$-α-open.

Example 3.2. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}\}$, $\mathcal{P} = \{\frac{m_1}{0.7}, \frac{m_2}{0.8}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\Im : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Thus, $\mathcal{V}$ is $\frac{1}{2}$-$\mathcal{F}$-γ-open set, but it is not $\frac{1}{2}$-$\mathcal{F}$-semi-open.

Example 3.3. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.4}, \frac{m_2}{0.5}\}$. Define $\Im : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{V}$ is $\frac{1}{4}$-$\mathcal{F}$-β-open set, but it is not $\frac{1}{4}$-$\mathcal{F}$-γ-open.

Corollary 3.1. In an $\mathcal{FTS}$ $(M, \Im)$ and $r \in I_\circ$,

(1) the union of r-$\mathcal{F}$-γ-open sets is r-$\mathcal{F}$-γ-open;

(2) the intersection of r-$\mathcal{F}$-γ-closed sets is r-$\mathcal{F}$-γ-closed.

Proof. Easily proved by Definition 3.1. □

Corollary 3.2. In an $\mathcal{FTS}$ $(M, \Im)$, for each r-$\mathcal{F}$-γ-closed set $\mathcal{D} \in I^M$:

(1) If $\mathcal{D}$ is an r-$\mathcal{F}$-regularly-open set, then $\mathcal{D}$ is r-$\mathcal{F}$-pre-closed.

(2) If $\mathcal{D}$ is an r-$\mathcal{F}$-regularly-closed set, then $\mathcal{D}$ is r-$\mathcal{F}$-semi-closed.

(3) If $I_\Im(\mathcal{D}, r) = \underline{0}$, then $\mathcal{D}$ is r-$\mathcal{F}$-semi-closed.

(4) If $C_\Im(\mathcal{D}, r) = \underline{0}$, then $\mathcal{D}$ is r-$\mathcal{F}$-pre-closed.

Proof. The proof follows by Definitions 2.3 and 3.1. □

Corollary 3.3. In an $\mathcal{FTS}$ $(M, \Im)$, for each r-$\mathcal{F}$-γ-open set $\mathcal{P} \in I^M$:

(1) If $\mathcal{P}$ is an r-$\mathcal{F}$-regularly-open set, then $\mathcal{P}$ is r-$\mathcal{F}$-semi-open.

(2) If $\mathcal{P}$ is an r-$\mathcal{F}$-regularly-closed set, then $\mathcal{P}$ is r-$\mathcal{F}$-pre-open.

(3) If $I_\Im(\mathcal{P}, r) = \underline{0}$, then $\mathcal{P}$ is r-$\mathcal{F}$-pre-open.

(4) If $C_\Im(\mathcal{P}, r) = \underline{0}$, then $\mathcal{P}$ is r-$\mathcal{F}$-semi-open.

Proof. The proof follows by Definitions 2.3 and 3.1. □

Definition 3.2. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$, we define an $\mathcal{F}$-γ-closure operator $\gamma C_\Im : I^M \times I_\circ \longrightarrow I^M$ as follows: $\gamma C_\Im(\mathcal{D}, r) = \bigwedge \{\mathcal{P} \in I^M : \mathcal{D} \leq \mathcal{P},\ \mathcal{P} \text{ is } r\text{-}\mathcal{F}\text{-}\gamma\text{-closed}\}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Proposition 3.2. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is r-$\mathcal{F}$-γ-closed iff $\gamma C_\Im(\mathcal{D}, r) = \mathcal{D}$.

Proof. Easily proved from Definition 3.2. □

Theorem 3.1. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D}, \mathcal{P} \in I^M$ and $r \in I_\circ$. An $\mathcal{F}$-operator $\gamma C_\Im : I^M \times I_\circ \longrightarrow I^M$ satisfies the following properties.

(1) $\gamma C_\Im(\underline{0}, r) = \underline{0}$.

(2) $\mathcal{D} \leq \gamma C_\Im(\mathcal{D}, r) \leq C_\Im(\mathcal{D}, r)$.

(3) $\gamma C_\Im(\mathcal{D}, r) \leq \gamma C_\Im(\mathcal{P}, r)$ if $\mathcal{D} \leq \mathcal{P}$.

(4) $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r) = \gamma C_\Im(\mathcal{D}, r)$.

(5) $\gamma C_\Im(\mathcal{D} \vee \mathcal{P}, r) \geq \gamma C_\Im(\mathcal{D}, r) \vee \gamma C_\Im(\mathcal{P}, r)$.

(6) $\gamma C_\Im(C_\Im(\mathcal{D}, r), r) = C_\Im(\mathcal{D}, r)$.

Proof. (1), (2), and (3) are easily proved by Definition 3.2.

(4) From (2) and (3), $\gamma C_\Im(\mathcal{D}, r) \leq \gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)$. Now, we show $\gamma C_\Im(\mathcal{D}, r) \geq \gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)$. If $\gamma C_\Im(\mathcal{D}, r)$ does not contain $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)$, there is $m \in M$ and $\sigma \in (0, 1)$ with $\gamma C_\Im(\mathcal{D}, r)(m) < \sigma < \gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)(m)$. (Z)

Since $\gamma C_\Im(\mathcal{D}, r)(m) < \sigma$, by Definition 3.2, there is $\mathcal{V} \in I^M$ as an r-$\mathcal{F}$-γ-closed set and $\mathcal{D} \leq \mathcal{V}$ with $\gamma C_\Im(\mathcal{D}, r)(m) \leq \mathcal{V}(m) < \sigma$. Since $\mathcal{D} \leq \mathcal{V}$, then $\gamma C_\Im(\mathcal{D}, r) \leq \mathcal{V}$. Again, by the definition of $\gamma C_\Im$, $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r) \leq \mathcal{V}$.

Hence, $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)(m) \leq \mathcal{V}(m) < \sigma$, which is a contradiction for (Z). Thus, $\gamma C_\Im(\mathcal{D}, r) \geq \gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r)$. Therefore, $\gamma C_\Im(\gamma C_\Im(\mathcal{D}, r), r) = \gamma C_\Im(\mathcal{D}, r)$.

(5) Since $\mathcal{D} \leq \mathcal{D} \vee \mathcal{P}$ and $\mathcal{P} \leq \mathcal{D} \vee \mathcal{P}$, hence by (3), $\gamma C_\Im(\mathcal{D}, r) \leq \gamma C_\Im(\mathcal{D} \vee \mathcal{P}, r)$ and $\gamma C_\Im(\mathcal{P}, r) \leq \gamma C_\Im(\mathcal{D} \vee \mathcal{P}, r)$. Thus, $\gamma C_\Im(\mathcal{D} \vee \mathcal{P}, r) \geq \gamma C_\Im(\mathcal{D}, r) \vee \gamma C_\Im(\mathcal{P}, r)$.

(6) From Proposition 3.2 and $C_\Im(\mathcal{D}, r)$ is an r-$\mathcal{F}$-γ-closed set, then $\gamma C_\Im(C_\Im(\mathcal{D}, r), r) = C_\Im(\mathcal{D}, r)$. □

Definition 3.3. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$, we define an $\mathcal{F}$-γ-interior operator $\gamma I_\Im : I^M \times I_\circ \longrightarrow I^M$ as follows: $\gamma I_\Im(\mathcal{D}, r) = \bigvee \{\mathcal{P} \in I^M : \mathcal{P} \leq \mathcal{D},\ \mathcal{P}$ is r-$\mathcal{F}$-γ-open$\}$.

Proposition 3.3. Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. Then

(1) $\gamma C_\Im(\mathcal{D}^c, r) = (\gamma I_\Im(\mathcal{D}, r))^c$;

(2) $\gamma I_\Im(\mathcal{D}^c, r) = (\gamma C_\Im(\mathcal{D}, r))^c$.

Proof. (1) For each $\mathcal{D} \in I^M$ and $r \in I_\circ$, we have $\gamma C_\Im(\mathcal{D}^c, r) = \bigwedge\{\mathcal{P} \in I^M : \mathcal{D}^c \leq \mathcal{P},\ \mathcal{P}$ is r-$\mathcal{F}$-γ-closed$\} = [\bigvee\{\mathcal{P}^c \in I^M : \mathcal{P}^c \leq \mathcal{D},\ \mathcal{P}^c$ is r-$\mathcal{F}$-γ-open$\}]^c = (\gamma I_\Im(\mathcal{D}, r))^c$.

(2) Similar to that of (1). □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Proposition 3.4. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D} \in I^M$ and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is r-$\mathcal{F}$-γ-open iff $\gamma I_\Im(\mathcal{D}, r) = \mathcal{D}$.

Proof. Easily proved from Definition 3.3. □

Theorem 3.2. In an $\mathcal{FTS}$ $(M, \Im)$, for each $\mathcal{D}, \mathcal{P} \in I^M$ and $r \in I_\circ$. An $\mathcal{F}$-operator $\gamma I_\Im : I^M \times I_\circ \longrightarrow I^M$ satisfies the following properties.

(1) $\gamma I_\Im(\underline{1}, r) = \underline{1}$.

(2) $I_\Im(\mathcal{D}, r) \leq \gamma I_\Im(\mathcal{D}, r) \leq \mathcal{D}$.

(3) $\gamma I_\Im(\mathcal{D}, r) \leq \gamma I_\Im(\mathcal{P}, r)$ if $\mathcal{D} \leq \mathcal{P}$.

(4) $\gamma I_\Im(\gamma I_\Im(\mathcal{D}, r), r) = \gamma I_\Im(\mathcal{D}, r)$.

(5) $\gamma I_\Im(\mathcal{D}, r) \wedge \gamma I_\Im(\mathcal{P}, r) \geq \gamma I_\Im(\mathcal{D} \wedge \mathcal{P}, r)$.

Proof. The proof is similar to that of Theorem 3.1.
□

4. On Fuzzy γ-Continuity

Here, we define and discuss the concepts of $\mathcal{F}$-γ-continuous and $\mathcal{F}$-γ-irresolute functions between $\mathcal{FTS}$s $(M, \Im)$ and $(N, \digamma)$. We also define and study the concepts of $\mathcal{F}$-almost and $\mathcal{F}$-weakly γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions. We showed that $\mathcal{F}$-γ-continuity $\Longrightarrow$ $\mathcal{F}$-almost γ-continuity $\Longrightarrow$ $\mathcal{F}$-weakly γ-continuity, but the converse may not be true.

Definition 4.1. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, \digamma)$ is called

(1) $\mathcal{F}$-γ-continuous if $h^{-1}(\mathcal{D})$ is an r-$\mathcal{F}$-γ-open set, for every $\mathcal{D} \in I^N$ with $\digamma(\mathcal{D}) \geq r$;

(2) $\mathcal{F}$-γ-irresolute if $h^{-1}(\mathcal{D})$ is an r-$\mathcal{F}$-γ-open set, for every r-$\mathcal{F}$-γ-open set $\mathcal{D} \in I^N$.

Remark 4.1. From the previous definitions, we have the following diagram.

$\mathcal{F}$-**pre-continuity**

↗ ↘

$\mathcal{F}$-**continuity** ⟶ $\mathcal{F}$-**ff-continuity** ⟶ $\mathcal{F}$-**fl-continuity** ⟶ $\mathcal{F}$-**fi-continuity**

↘ ↗

$\mathcal{F}$-**semi-continuity**

Remark 4.2. The converse of the above diagram fails as Examples 4.1, 4.2, and 4.3.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Example 4.1. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.3}\}$, $\mathcal{P} = \{\frac{m_1}{0.2}, \frac{m_2}{0.6}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.7}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{C} = \mathcal{P}, \\ \frac{1}{4}, & \text{if } \mathcal{C} = \mathcal{D}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{P} \wedge \mathcal{D}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{P} \vee \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if } \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is neither $\mathcal{F}$-pre-continuous nor $\mathcal{F}$-α-continuous.

Example 4.2. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}\}$, $\mathcal{P} = \{\frac{m_1}{0.7}, \frac{m_2}{0.8}\}$, $\mathcal{V} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{P}, \\ \frac{1}{4}, & \text{if } \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{6}, & \text{if } \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is not $\mathcal{F}$-semi-continuous.

Example 4.3. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.4}, \frac{m_2}{0.5}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if } \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-β-continuous, but it is not $\mathcal{F}$-γ-continuous.

Theorem 4.1. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, \digamma)$ is $\mathcal{F}$-γ-continuous iff for any $m_\sigma \in P_\sigma(M)$ and any $\mathcal{D} \in I^N$ with $\digamma(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq \mathcal{D}$.

Proof. ($\Rightarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $\digamma(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, then $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), r)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then we obtain $m_\sigma \in \gamma I_\Im(h^{-1}(\mathcal{D}), r) = \mathcal{A}$ (say). Hence, $\mathcal{A} \in I^M$ is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq \mathcal{D}$.

($\Leftarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $\digamma(\mathcal{D}) \geq r$ and $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq \mathcal{D}$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(\mathcal{D})$ and $m_\sigma \in \gamma I_\Im(h^{-1}(\mathcal{D}), r)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), r)$, so $h^{-1}(\mathcal{D})$ is an r-$\mathcal{F}$-γ-open set. Then, h is $\mathcal{F}$-γ-continuous.

□

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Theorem 4.2. Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-function and $r \in I_\circ$, the following statements are equivalent for every $\mathcal{P} \in I^M$ and $\mathcal{D} \in I^N$:

(1) h is $\mathcal{F}$-γ-continuous.

(2) $h^{-1}(\mathcal{D})$ is r-$\mathcal{F}$-γ-closed, for every $\mathcal{D} \in I^N$ with $F(\mathcal{D}^c) \geq r$.

(3) $h(\gamma C_\Im(\mathcal{P}, r)) \leq C_F(h(\mathcal{P}), r)$.

(4) $\gamma C_\Im(h^{-1}(\mathcal{D}), r) \leq h^{-1}(C_F(\mathcal{D}, r))$.

(5) $h^{-1}(I_F(\mathcal{D}, r)) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), r)$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by $h^{-1}(\mathcal{D}^c) = (h^{-1}(\mathcal{D}))^c$ and Definition 4.1.

(2) $\Rightarrow$ (3) Let $\mathcal{P} \in I^M$. By (2), we have $h^{-1}(C_F(h(\mathcal{P}), r))$ is r-$\mathcal{F}$-γ-closed. Thus,

$$\gamma C_\Im(\mathcal{P}, r) \leq \gamma C_\Im(h^{-1}(h(\mathcal{P})), r) \leq \gamma C_\Im(h^{-1}(C_F(h(\mathcal{P}), r)), r) = h^{-1}(C_F(h(\mathcal{P}), r)).$$

Therefore, $h(\gamma C_\Im(\mathcal{P}, r)) \leq C_F(h(\mathcal{P}), r)$.

(3) $\Rightarrow$ (4) Let $\mathcal{D} \in I^N$. By (3), $h(\gamma C_\Im(h^{-1}(\mathcal{D}), r)) \leq C_F(h(h^{-1}(\mathcal{D})), r) \leq C_F(\mathcal{D}, r)$. Thus, $\gamma C_\Im(h^{-1}(\mathcal{D}), r) \leq h^{-1}(h(\gamma C_\Im(h^{-1}(\mathcal{D}), r))) \leq h^{-1}(C_F(\mathcal{D}, r))$.

(4) $\Leftrightarrow$ (5) The proof follows by $h^{-1}(\mathcal{D}^c) = (h^{-1}(\mathcal{D}))^c$ and Proposition 3.3.

(5) $\Rightarrow$ (1) Let $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$. By (5), we obtain $h^{-1}(\mathcal{D}) = h^{-1}(I_F(\mathcal{D}, r)) \leq \gamma I_\Im(h^{-1}(\mathcal{D}), r) \leq h^{-1}(\mathcal{D})$. Then, $\gamma I_\Im(h^{-1}(\mathcal{D}), r) = h^{-1}(\mathcal{D})$. Thus, $h^{-1}(\mathcal{D})$ is r-$\mathcal{F}$-γ-open, so h is $\mathcal{F}$-γ-continuous. □

Lemma 4.1. Every $\mathcal{F}$-γ-irresolute function is $\mathcal{F}$-γ-continuous.

Proof. The proof follows by Definition 4.1. □

Remark 4.3. The converse of Lemma 4.1 fails as Example 4.4.

Example 4.4. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-continuous, but it is not $\mathcal{F}$-γ-irresolute.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Theorem 4.3. Let $h: (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function and $r \in I_\circ$, the following statements are equivalent for every $\mathcal{D} \in I^M$ and $\mathcal{P} \in I^N$:

(1) h is $\mathcal{F}$-γ-irresolute.

(2) $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-closed, for every r-$\mathcal{F}$-γ-closed set $\mathcal{P}$.

(3) $h(\gamma C_\Im(\mathcal{D},r)) \leq \gamma C_F(h(\mathcal{D}),r)$.

(4) $\gamma C_\Im(h^{-1}(\mathcal{P}),r) \leq h^{-1}(\gamma C_F(\mathcal{P},r))$.

(5) $h^{-1}(\gamma I_F(\mathcal{P},r)) \leq \gamma I_\Im(h^{-1}(\mathcal{P}),r)$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ and Definition 4.1.

(2) $\Rightarrow$ (3) Let $\mathcal{D} \in I^M$. By (2), we have $h^{-1}(\gamma C_F(h(\mathcal{D}),r))$ is r-$\mathcal{F}$-γ-closed. Thus,

$$\gamma C_\Im(\mathcal{D},r) \leq \gamma C_\Im(h^{-1}(h(\mathcal{D})),r) \leq \gamma C_\Im(h^{-1}(\gamma C_F(h(\mathcal{D}),r)),r) = h^{-1}(\gamma C_F(h(\mathcal{D}),r)).$$

Therefore, $h(\gamma C_\Im(\mathcal{D},r)) \leq \gamma C_F(h(\mathcal{D}),r)$.

(3) $\Rightarrow$ (4) Let $\mathcal{P} \in I^N$. By (3), $h(\gamma C_\Im(h^{-1}(\mathcal{P}),r)) \leq \gamma C_F(h(h^{-1}(\mathcal{P})),r) \leq \gamma C_F(\mathcal{P},r)$. Thus, $\gamma C_\Im(h^{-1}(\mathcal{P}),r) \leq h^{-1}(h(\gamma C_\Im(h^{-1}(\mathcal{P}),r))) \leq h^{-1}(\gamma C_F(\mathcal{P},r))$.

(4) $\Leftrightarrow$ (5) The proof follows by $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ and Proposition 3.3.

(5) $\Rightarrow$ (1) Let $\mathcal{P} \in I^N$ be an r-$\mathcal{F}$-γ-open set. By (5),

$$h^{-1}(\mathcal{P}) = h^{-1}(\gamma I_F(\mathcal{P},r)) \leq \gamma I_\Im(h^{-1}(\mathcal{P}),r) \leq h^{-1}(\mathcal{P}).$$

Thus, $\gamma I_\Im(h^{-1}(\mathcal{P}),r) = h^{-1}(\mathcal{P})$. Therefore, $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-open, so h is $\mathcal{F}$-γ-irresolute. □

Proposition 4.1. Let $(M,\Im)$, (W,η) and (N,F) be $\mathcal{FTS}s$, and $h: (M,\Im) \longrightarrow (W,\eta)$, $f: (W,\eta) \longrightarrow (N,F)$ be two $\mathcal{F}$-functions. The composition $f \circ h$ is $\mathcal{F}$-γ-irresolute (resp., $\mathcal{F}$-γ-continuous) if, h is $\mathcal{F}$-γ-irresolute and f is $\mathcal{F}$-γ-irresolute (resp., $\mathcal{F}$-γ-continuous).

Proof. The proof follows from Definition 4.1. □

Definition 4.2. An $\mathcal{F}$-function $h: (M,\Im) \longrightarrow (N,F)$ is called $\mathcal{F}$-almost γ-continuous if $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(I_F(C_F(\mathcal{D},r),r)),r)$, for every $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$.

Lemma 4.2. Every $\mathcal{F}$-γ-continuous function is $\mathcal{F}$-almost γ-continuous.

Proof. The proof follows by Definitions 4.1 and 4.2. □

Remark 4.4. The converse of Lemma 4.2 fails as Example 4.5.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Example 4.5. Let $M = \{m_1, m_2, m_3\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}, \frac{m_3}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.6}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if } \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & \text{if } \mathcal{U} = \mathcal{P}, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mathcal{U} = \mathcal{V}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-almost γ-continuous, but it is not $\mathcal{F}$-γ-continuous.

Theorem 4.4. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is $\mathcal{F}$-almost γ-continuous iff for any $m_\sigma \in P_\sigma(M)$ and any $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq I_F(C_F(\mathcal{D}, r), r)$.

Proof. ($\Rightarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, then $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(I_F(C_F(\mathcal{D}, r), r)), r)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then $m_\sigma \in \gamma I_\Im(h^{-1}(I_F(C_F(\mathcal{D}, r), r)), r) = \mathcal{A}$ (say). Therefore, $\mathcal{A} \in I^M$ is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq I_F(C_F(\mathcal{D}, r), r)$.

($\Leftarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ such that $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq I_F(C_F(\mathcal{D}, r), r)$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(I_F(C_F(\mathcal{D}, r), r))$ and $m_\sigma \in \gamma I_\Im(h^{-1}(I_F(C_F(\mathcal{D}, r), r)), r)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_\Im(h^{-1}(I_F(C_F(\mathcal{D}, r), r)), r)$. Therefore, h is $\mathcal{F}$-almost γ-continuous.
□

Theorem 4.5. Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-function, $\mathcal{P} \in I^N$, and $r \in I_\circ$, the following statements are equivalent:

(1) h is $\mathcal{F}$-almost γ-continuous.

(2) $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-open, for every r-$\mathcal{F}$-regularly open set $\mathcal{P}$.

(3) $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-closed, for every r-$\mathcal{F}$-regularly closed set $\mathcal{P}$.

(4) $\gamma C_\Im(h^{-1}(\mathcal{P}), r) \leq h^{-1}(C_F(\mathcal{P}, r))$, for every r-$\mathcal{F}$-γ-open set $\mathcal{P}$.

(5) $\gamma C_\Im(h^{-1}(\mathcal{P}), r) \leq h^{-1}(C_F(\mathcal{P}, r))$, for every r-$\mathcal{F}$-semi-open set $\mathcal{P}$.

Proof. (1) $\Rightarrow$ (2) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{P}$ be an r-$\mathcal{F}$-regularly open set with $m_\sigma \in h^{-1}(\mathcal{P})$. Hence, by (1), there is $\mathcal{A} \in I^M$ that is an r-$\mathcal{F}$-γ-open with $m_\sigma \in \mathcal{A}$ and $h(\mathcal{A}) \leq I_F(C_F(\mathcal{P}, r), r)$. Thus, $\mathcal{A} \leq h^{-1}(I_F(C_F(\mathcal{P}, r), r)) = h^{-1}(\mathcal{P})$ and $m_\sigma \in \gamma I_\Im(h^{-1}(\mathcal{P}), r)$. Therefore, $h^{-1}(\mathcal{P}) \leq \gamma I_\Im(h^{-1}(\mathcal{P}), r)$, so $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-open.

(2) $\Rightarrow$ (3) If $\mathcal{P} \in I^N$ is r-$\mathcal{F}$-regularly closed, then by (2), $h^{-1}(\mathcal{P}^c) = (h^{-1}(\mathcal{P}))^c$ is r-$\mathcal{F}$-γ-open. Thus, $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-γ-closed.

(3) $\Rightarrow$ (4) If $\mathcal{P} \in I^N$ is r-$\mathcal{F}$-b-open and since $C_F(\mathcal{P}, r)$ is r-$\mathcal{F}$-regularly closed, then by (3), $h^{-1}(C_F(\mathcal{P}, r))$ is r-$\mathcal{F}$-γ-closed. Since $h^{-1}(\mathcal{P}) \leq h^{-1}(C_F(\mathcal{P}, r))$, hence

$$\gamma C_\Im(h^{-1}(\mathcal{P}), r) \leq h^{-1}(C_F(\mathcal{P}, r)).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

(4) $\Rightarrow$ (5) The proof follows from the fact that any r-$\mathcal{F}$-semi-open set is r-$\mathcal{F}$-γ-open set.

(5) $\Rightarrow$ (3) If $\mathcal{P} \in I^N$ is r-$\mathcal{F}$-regularly closed, then $\mathcal{P}$ is r-$\mathcal{F}$-semi-open. By (5), $\gamma C_{\Im}(h^{-1}(\mathcal{P}), r) \leq h^{-1}(C_F(\mathcal{P}, r)) = h^{-1}(\mathcal{P})$. Hence, $h^{-1}(\mathcal{P})$ is r-$\mathcal{F}$-b-closed.

(3) $\Rightarrow$ (1) If $m_\sigma \in P_\sigma(M)$ and $\mathcal{P} \in I^N$ with $F(\mathcal{P}) \geq r$ such that $m_\sigma \in h^{-1}(\mathcal{P})$, then $m_\sigma \in h^{-1}(I_F(C_F(\mathcal{P}, r), r))$. Since $[I_F(C_F(\mathcal{P}, r), r)]^c$ is r-$\mathcal{F}$-regularly closed, then by (3), we have $h^{-1}([I_F(C_F(\mathcal{P}, r), r)]^c)$ is r-$\mathcal{F}$-γ-closed. Hence, $h^{-1}(I_F(C_F(\mathcal{P}), r))$ is r-$\mathcal{F}$-γ-open and

$$m_\sigma \in \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{P}, r), r)), r).$$

Thus, $h^{-1}(\mathcal{P}) \leq \gamma I_{\Im}(h^{-1}(I_F(C_F(\mathcal{P}, r), r)), r)$. Therefore, h is $\mathcal{F}$-almost γ-continuous.
□

Definition 4.3. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is called $\mathcal{F}$-weakly γ-continuous if $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D}, r)), r)$, for every $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$.

Lemma 4.3. Every $\mathcal{F}$-γ-continuous function is $\mathcal{F}$-weakly γ-continuous.

Proof. The proof follows by Definitions 4.1 and 4.3. □

Remark 4.5. The converse of Lemma 4.3 fails as Example 4.6.

Example 4.6. Let $M = \{m_1, m_2, m_3\}$ and define $\mathcal{D}, \mathcal{P}, \mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.4}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}, \frac{m_3}{0.4}\}$, $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.6}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & \text{if } \mathcal{U} = \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if } \mathcal{U} = \mathcal{V}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-weakly γ-continuous, but it is not $\mathcal{F}$-γ-continuous.

Theorem 4.6. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is $\mathcal{F}$-weakly γ-continuous iff for any $m_\sigma \in P_\sigma(M)$ and any $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq C_F(\mathcal{D}, r)$.

Proof. ($\Rightarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ containing $h(m_\sigma)$, then $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D}, r)), r)$. Since $m_\sigma \in h^{-1}(\mathcal{D})$, then $m_\sigma \in \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D}, r)), r) = \mathcal{A}$ (say). Hence, $\mathcal{A} \in I^M$ is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq C_F(\mathcal{D}, r)$.

($\Leftarrow$) Let $m_\sigma \in P_\sigma(M)$ and $\mathcal{D} \in I^N$ with $F(\mathcal{D}) \geq r$ such that $m_\sigma \in h^{-1}(\mathcal{D})$. According to the assumption there is $\mathcal{A} \in I^M$ that is r-$\mathcal{F}$-γ-open containing m_σ with $h(\mathcal{A}) \leq C_F(\mathcal{D}, r)$. Hence, $m_\sigma \in \mathcal{A} \leq h^{-1}(C_F(\mathcal{D}, r))$ and $m_\sigma \in \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D}, r)), r)$. Thus, $h^{-1}(\mathcal{D}) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{D}, r)), r)$. Therefore, h is $\mathcal{F}$-weakly γ-continuous.
□

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Theorem 4.7. Let $h: (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, the following statements are equivalent:

(1) h is $\mathcal{F}$-weakly γ-continuous.

(2) $h^{-1}(\mathcal{P}) \geq \gamma C_{\Im}(h^{-1}(I_F(\mathcal{P},r)),r)$, if $\mathcal{P} \in I^N$ with $F(\mathcal{P}^c) \geq r$.

(3) $\gamma I_{\Im}(h^{-1}(C_F(\mathcal{P},r)),r) \geq h^{-1}(I_F(\mathcal{P},r))$.

(4) $\gamma C_{\Im}(h^{-1}(I_F(\mathcal{P},r)),r) \leq h^{-1}(C_F(\mathcal{P},r))$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by Proposition 3.3 and Definition 4.3.

(2) $\Rightarrow$ (3) Let $\mathcal{P} \in I^N$. Hence by (2),

$$\gamma C_{\Im}(h^{-1}(I_F(C_F(\mathcal{P}^c,r),r)),r) \leq h^{-1}(C_F(\mathcal{P}^c,r)).$$

Thus, $h^{-1}(I_F(\mathcal{P},r)) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{P},r)),r)$.

(3) $\Leftrightarrow$ (4) The proof follows from Proposition 3.3.

(4) $\Rightarrow$ (1) Let $\mathcal{P} \in I^N$ with $F(\mathcal{P}) \geq r$. Hence by (4), $\gamma C_{\Im}(h^{-1}(I_F(\mathcal{P}^c,r)),r) \leq h^{-1}(C_F(\mathcal{P}^c,r)) = h^{-1}(\mathcal{P}^c)$. Thus, $h^{-1}(\mathcal{P}) \leq \gamma I_{\Im}(h^{-1}(C_F(\mathcal{P},r)),r)$, so h is $\mathcal{F}$-weakly γ-continuous. □

Lemma 4.4. Every $\mathcal{F}$-almost γ-continuous function is $\mathcal{F}$-weakly γ-continuous.

Proof. The proof follows by Definitions 4.2 and 4.3. □

Remark 4.6. The converse of Lemma 4.4 fails as Example 4.7.

Example 4.7. Let $M = \{m_1, m_2, m_3\}$ and define $\mathcal{D},\mathcal{P},\mathcal{V} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.6}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$, $\mathcal{P} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.5}\}$, $\mathcal{V} = \{\frac{m_1}{0.3}, \frac{m_2}{0.2}, \frac{m_3}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im, \eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if } \mathcal{U} = \mathcal{D}, \\ \frac{1}{2}, & \text{if } \mathcal{U} = \mathcal{V}, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if } \mathcal{U} = \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M,\Im) \longrightarrow (M,\eta)$ is $\mathcal{F}$-weakly γ-continuous, but it is not $\mathcal{F}$-almost γ-continuous.

Remark 4.7. From the previous discussions and definitions, we have the following diagram.

$\mathcal{F}$**-fl-continuity** $\longrightarrow$ $\mathcal{F}$**-almost fl-continuity** $\longrightarrow$ $\mathcal{F}$**-weakly fl-continuity**

Proposition 4.2. Let $(M,\Im)$, (W,η) and (N,F) be $\mathcal{FTS}$s, and $h : (M,\Im) \longrightarrow (W,\eta)$, $g : (W,\eta) \longrightarrow (N,F)$ be two $\mathcal{F}$-functions. The composition $g \circ h$ is $\mathcal{F}$-almost γ-continuous if, h is $\mathcal{F}$-γ-irresolute (resp., $\mathcal{F}$-γ-continuous) and g is $\mathcal{F}$-almost γ-continuous (resp., $\mathcal{F}$-continuous).

Proof. The proof follows by the previous definitions. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

5. Further Selected Topics

Here, we introduce and establish some new $\mathcal{F}$-functions using r-$\mathcal{F}$-γ-open and r-$\mathcal{F}$-γ-closed sets, called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Furthermore, we define some new types of $\mathcal{F}$-separation axioms, called r-$\mathcal{F}$-γ-regular and r-$\mathcal{F}$-γ-normal spaces, and study some properties of them. Also, we explore and discuss some new types of $\mathcal{F}$-compactness, called r-$\mathcal{F}$-almost and r-$\mathcal{F}$-nearly γ-compact sets using r-$\mathcal{F}$-γ-open sets.

- **Some new fuzzy functions:**

Definition 5.1. An $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, \digamma)$ is called

(1) $\mathcal{F}$-γ-open if $h(\mathcal{D})$ is an r-$\mathcal{F}$-γ-open set, for every $\mathcal{D} \in I^M$ with $\Im(\mathcal{D}) \geq r$;

(2) $\mathcal{F}$-γ-closed if $h(\mathcal{D})$ is an r-$\mathcal{F}$-γ-closed set, for every $\mathcal{D} \in I^M$ with $\Im(\mathcal{D}^c) \geq r$;

(3) $\mathcal{F}$-γ-irresolute open if $h(\mathcal{D})$ is an r-$\mathcal{F}$-γ-open set, for every r-$\mathcal{F}$-γ-open set $\mathcal{D} \in I^M$;

(4) $\mathcal{F}$-γ-irresolute closed if $h(\mathcal{D})$ is an r-$\mathcal{F}$-γ-closed set, for every r-$\mathcal{F}$-γ-closed set $\mathcal{D} \in I^M$.

Lemma 5.1. (1) Each $\mathcal{F}$-γ-irresolute open function is $\mathcal{F}$-γ-open.

(2) Each $\mathcal{F}$-γ-irresolute closed function is $\mathcal{F}$-γ-closed.

Proof. The proof follows from Definition 5.1. □

Remark 5.1. The converse of Lemma 5.1 fails as Example 5.1.

Example 5.1. Let $M = \{m_1, m_2\}$ and define $\mathcal{D}, \mathcal{P} \in I^M$ as follows: $\mathcal{D} = \{\frac{m_1}{0.5}, \frac{m_2}{0.5}\}$, $\mathcal{P} = \{\frac{m_1}{0.5}, \frac{m_2}{0.4}\}$. Define $\mathcal{F}$-topologies $\Im$, $\eta : I^M \longrightarrow I$ as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{C} = \mathcal{D}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \mathcal{C} = \mathcal{P}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity $\mathcal{F}$-function $f : (M, \Im) \longrightarrow (M, \eta)$ is $\mathcal{F}$-γ-open, but it is not $\mathcal{F}$-γ-irresolute open.

Theorem 5.1. Let $h : (M, \Im) \longrightarrow (N, \digamma)$ be an $\mathcal{F}$-function, the following statements are equivalent for every $\mathcal{A} \in I^M$ and $\mathcal{D} \in I^N$:

(1) h is $\mathcal{F}$-γ-open.

(2) $h(I_{\Im}(\mathcal{A}, r)) \leq \gamma I_{\digamma}(h(\mathcal{A}), r)$.

(3) $I_{\Im}(h^{-1}(\mathcal{D}), r) \leq h^{-1}(\gamma I_{\digamma}(\mathcal{D}, r))$.

(4) For every $\mathcal{D}$ and every $\mathcal{A}$ with $\Im(\mathcal{A}^c) \geq r$ and $h^{-1}(\mathcal{D}) \leq \mathcal{A}$, there is $\mathcal{P} \in I^N$ is r-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{A}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Proof. (1) $\Rightarrow$ (2) Since $h(I_{\Im}(\mathcal{A},r)) \leq h(\mathcal{A})$, hence by (1), $h(I_{\Im}(\mathcal{A},r))$ is r-$\mathcal{F}$-γ-open. Thus,

$$h(I_{\Im}(\mathcal{A},r)) \leq \gamma I_{F}(h(\mathcal{A}),r).$$

(2) $\Rightarrow$ (3) Put $\mathcal{A} = h^{-1}(\mathcal{D})$, hence by (2), $h(I_{\Im}(h^{-1}(\mathcal{D}),r)) \leq \gamma I_{F}(h(h^{-1}(\mathcal{D})),r) \leq \gamma I_{F}(\mathcal{D},r)$. Thus, $I_{\Im}(h^{-1}(\mathcal{D}),r) \leq h^{-1}(\gamma I_{F}(\mathcal{D},r))$.

(3) $\Rightarrow$ (4) Let $\mathcal{D} \in I^{N}$ and $\mathcal{A} \in I^{M}$ with $\Im(\mathcal{A}^{c}) \geq r$ such that $h^{-1}(\mathcal{D}) \leq \mathcal{A}$. Since $\mathcal{A}^{c} \leq h^{-1}(\mathcal{D}^{c})$, $\mathcal{A}^{c} = I_{\Im}(\mathcal{A}^{c},r) \leq I_{\Im}(h^{-1}(\mathcal{D}^{c}),r)$. Hence by (3), $\mathcal{A}^{c} \leq I_{\Im}(h^{-1}(\mathcal{D}^{c}),r) \leq h^{-1}(\gamma I_{F}(\mathcal{D}^{c},r))$. Then, we have $\mathcal{A} \geq (h^{-1}(\gamma I_{F}(\mathcal{D}^{c},r)))^{c} = h^{-1}(\gamma C_{F}(\mathcal{D},r))$. Thus, there is $\gamma C_{F}(\mathcal{D},r) \in I^{N}$ is r-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \gamma C_{F}(\mathcal{D},r)$ and $h^{-1}(\gamma C_{F}(\mathcal{D},r)) \leq \mathcal{A}$.

(4) $\Rightarrow$ (1) Let $\mathcal{B} \in I^{M}$ with $\Im(\mathcal{B}) \geq r$. Put $\mathcal{D} = (h(\mathcal{B}))^{c}$ and $\mathcal{A} = \mathcal{B}^{c}$, then $h^{-1}(\mathcal{D}) = h^{-1}((h(\mathcal{B}))^{c}) \leq \mathcal{A}$. Hence by (4), there is $\mathcal{P} \in I^{N}$ is r-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{A} = \mathcal{B}^{c}$. Thus, $h(\mathcal{B}) \leq h(h^{-1}(\mathcal{P}^{c})) \leq \mathcal{P}^{c}$. On the other hand, since $\mathcal{D} \leq \mathcal{P}$, $h(\mathcal{B}) = \mathcal{D}^{c} \geq \mathcal{P}^{c}$. Hence, $h(\mathcal{B}) = \mathcal{P}^{c}$, so $h(\mathcal{B})$ is an r-$\mathcal{F}$-γ-open set. Therefore, h is $\mathcal{F}$-γ-open.
□

Theorem 5.2. Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, the following statements are equivalent for every $\mathcal{B} \in I^{M}$ and $\mathcal{D} \in I^{N}$:

(1) h is $\mathcal{F}$-γ-closed.

(2) $\gamma C_{F}(h(\mathcal{B}),r) \leq h(C_{\Im}(\mathcal{B},r))$.

(3) $h^{-1}(\gamma C_{F}(\mathcal{D},r)) \leq C_{\Im}(h^{-1}(\mathcal{D}),r)$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ with $\Im(\mathcal{B}) \geq r$ and $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^{N}$ is r-$\mathcal{F}$-γ-open with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 5.1.
□

Theorem 5.3. Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, the following statements are equivalent for every $\mathcal{B} \in I^{M}$ and $\mathcal{D} \in I^{N}$:

(1) h is $\mathcal{F}$-γ-irresolute open.

(2) $h(\gamma I_{\Im}(\mathcal{B},r)) \leq \gamma I_{F}(h(\mathcal{B}),r)$.

(3) $\gamma I_{\Im}(h^{-1}(\mathcal{D}),r) \leq h^{-1}(\gamma I_{F}(\mathcal{D},r))$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ is an r-$\mathcal{F}$-γ-closed set with $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^{N}$ is r-$\mathcal{F}$-γ-closed with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 5.1.
□

Theorem 5.4. Let $h : (M,\Im) \longrightarrow (N,F)$ be an $\mathcal{F}$-function, the following statements are equivalent for every $\mathcal{B} \in I^{M}$ and $\mathcal{D} \in I^{N}$:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

(1) h is $\mathcal{F}$-γ-irresolute closed.

(2) $\gamma C_{F}(h(\mathcal{B}), r) \leq h(\gamma C_{\Im}(\mathcal{B}, r))$.

(3) $h^{-1}(\gamma C_{F}(\mathcal{D}, r)) \leq \gamma C_{\Im}(h^{-1}(\mathcal{D}), r)$.

(4) For every $\mathcal{D}$ and every $\mathcal{B}$ is an r-$\mathcal{F}$-γ-open set with $h^{-1}(\mathcal{D}) \leq \mathcal{B}$, there is $\mathcal{P} \in I^N$ is r-$\mathcal{F}$-γ-open with $\mathcal{D} \leq \mathcal{P}$ and $h^{-1}(\mathcal{P}) \leq \mathcal{B}$.

Proof. The proof is similar to that of Theorem 5.1.
□

Proposition 5.1. Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-function, then h is $\mathcal{F}$-γ-irresolute open iff h is $\mathcal{F}$-γ-irresolute closed.

Proof. The proof follows from;

$$h^{-1}(\gamma C_{F}(\mathcal{V}, r)) \leq \gamma C_{\Im}(h^{-1}(\mathcal{V}), r) \iff h^{-1}(\gamma I_{F}(\mathcal{V}^c, r)) \leq \gamma I_{\Im}(h^{-1}(\mathcal{V}^c), r).$$

□

Definition 5.2. A bijective $\mathcal{F}$-function $h : (M, \Im) \longrightarrow (N, F)$ is called $\mathcal{F}$-γ-irresolute homeomorphism if h^{-1} and h are $\mathcal{F}$-γ-irresolute.

The proof of the following corollary is easy.

Corollary 5.1. Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-function, the following statements are equivalent for every $\mathcal{P} \in I^M$ and $\mathcal{V} \in I^N$:

(1) h is $\mathcal{F}$-γ-irresolute homeomorphism.

(2) h is $\mathcal{F}$-γ-irresolute closed and $\mathcal{F}$-γ-irresolute.

(3) h is $\mathcal{F}$-γ-irresolute open and $\mathcal{F}$-γ-irresolute.

(4) $h(\gamma I_{\Im}(\mathcal{P}, r)) = \gamma I_{F}(h(\mathcal{P}), r)$.

(5) $h(\gamma C_{\Im}(\mathcal{P}, r)) = \gamma C_{F}(h(\mathcal{P}), r)$.

(6) $\gamma I_{\Im}(h^{-1}(\mathcal{V}), r) = h^{-1}(\gamma I_{F}(\mathcal{V}, r))$.

(7) $\gamma C_{\Im}(h^{-1}(\mathcal{V}), r) = h^{-1}(\gamma C_{F}(\mathcal{V}, r))$.

- **r-fuzzy γ-regular and γ-normal spaces:**

Definition 5.3. Let $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{B} \in I^M$, and $r \in I_\circ$. An $\mathcal{FTS}$ $(M, \Im)$ is called

(1) r-$\mathcal{F}$-γ-regular space if $m_\sigma \, \overline{q} \, \mathcal{A}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{A}$, there is $\mathcal{C}_j \in I^M$ with $\Im(\mathcal{C}_j) \geq r$ for $j \in \{1, 2\}$, such that $m_\sigma \in \mathcal{C}_1$, $\mathcal{A} \leq \mathcal{C}_2$, and $\mathcal{C}_1 \, \overline{q} \, \mathcal{C}_2$.

(2) r-$\mathcal{F}$-γ-normal space if $\mathcal{A} \, \overline{q} \, \mathcal{B}$ for each r-$\mathcal{F}$-γ-closed sets $\mathcal{A}$ and $\mathcal{B}$, there is $\mathcal{C}_j \in I^M$ with $\Im(\mathcal{C}_j) \geq r$ for $j \in \{1, 2\}$, such that $\mathcal{A} \leq \mathcal{C}_1$, $\mathcal{B} \leq \mathcal{C}_2$, and $\mathcal{C}_1 \, \overline{q} \, \mathcal{C}_2$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Theorem 5.5. Let $(M, \Im)$ be an $\mathcal{FTS}$, $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{P} \in I^M$, the following statements are equivalent:

(1) $(M, \Im)$ is r-$\mathcal{F}$-γ-regular space.

(2) If $m_\sigma \in \mathcal{A}$ for each r-$\mathcal{F}$-γ-open set $\mathcal{A}$, there is $\mathcal{P}$ with $\Im(\mathcal{P}) \geq r$ and $m_\sigma \in \mathcal{P} \leq C_\Im(\mathcal{P}, r) \leq \mathcal{A}$.

(3) If $m_\sigma \,\overline{q}\, \mathcal{A}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{A}$, there is $\mathcal{D}_j \in I^M$ with $\Im(\mathcal{D}_j) \geq r$ for $j \in \{1, 2\}$, such that $m_\sigma \in \mathcal{D}_1$, $\mathcal{A} \leq \mathcal{D}_2$, and $C_\Im(\mathcal{D}_1, r) \,\overline{q}\, C_\Im(\mathcal{D}_2, r)$.

Proof. (1) $\Rightarrow$ (2) Let $m_\sigma \in \mathcal{A}$ for each r-$\mathcal{F}$-γ-open set $\mathcal{A}$, then $m_\sigma \,\overline{q}\, \mathcal{A}^c$. Since $(M, \Im)$ is r-$\mathcal{F}$-γ-regular, then there is $\mathcal{P}, \mathcal{D} \in I^M$ with $\Im(\mathcal{P}) \geq r$ and $\Im(\mathcal{D}) \geq r$, such that $m_\sigma \in \mathcal{P}$, $\mathcal{A}^c \leq \mathcal{D}$, and $\mathcal{P} \,\overline{q}\, \mathcal{D}$. Thus, $m_\sigma \in \mathcal{P} \leq \mathcal{D}^c \leq \mathcal{A}$, so $m_\sigma \in \mathcal{P} \leq C_\Im(\mathcal{P}, r) \leq \mathcal{A}$.

(2) $\Rightarrow$ (3) Let $m_\sigma \,\overline{q}\, \mathcal{A}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{A}$, then $m_\sigma \in \mathcal{A}^c$. By (2), there is $\mathcal{D}$ with $\Im(\mathcal{D}) \geq r$ and $m_\sigma \in \mathcal{D} \leq C_\Im(\mathcal{D}, r) \leq \mathcal{A}^c$. Since $\Im(\mathcal{D}) \geq r$, then $\mathcal{D}$ is an r-$\mathcal{F}$-γ-open set and $m_\sigma \in \mathcal{D}$. Again, by (2), there is $\mathcal{U}$ with $\Im(\mathcal{U}) \geq r$ and $m_\sigma \in \mathcal{U} \leq C_\Im(\mathcal{U}, r) \leq \mathcal{D} \leq C_\Im(\mathcal{D}, r) \leq \mathcal{A}^c$. Hence, $\mathcal{A} \leq (C_\Im(\mathcal{D}, r))^c = I_\Im(\mathcal{D}^c, r) \leq \mathcal{D}^c$. Put $\mathcal{V} = I_\Im(\mathcal{D}^c, r)$, thus $\Im(\mathcal{V}) \geq r$. Then, $C_\Im(\mathcal{V}, r) \leq \mathcal{D}^c \leq (C_\Im(\mathcal{U}, r))^c$. Therefore, $C_\Im(\mathcal{V}, r) \,\overline{q}\, C_\Im(\mathcal{U}, r)$.

(3) $\Rightarrow$ (1) Easily proved by Definition 5.3. □

Theorem 5.6. Let $(M, \Im)$ be an $\mathcal{FTS}$, $m_\sigma \in P_\sigma(M)$, $\mathcal{A}, \mathcal{B} \in I^M$, the following statements are equivalent:

(1) $(M, \Im)$ is r-$\mathcal{F}$-γ-normal space.

(2) If $\mathcal{B} \leq \mathcal{A}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{B}$ and r-$\mathcal{F}$-γ-open set $\mathcal{A}$, there is $\mathcal{D}$ with $\Im(\mathcal{D}) \geq r$ and $\mathcal{B} \leq \mathcal{D} \leq C_\Im(\mathcal{D}, r) \leq \mathcal{A}$.

(3) If $\mathcal{A} \,\overline{q}\, \mathcal{B}$ for each r-$\mathcal{F}$-γ-closed sets $\mathcal{A}$ and $\mathcal{B}$, there is $\mathcal{D}_j \in I^M$ with $\Im(\mathcal{D}_j) \geq r$ for $j \in \{1, 2\}$, such that $\mathcal{A} \leq \mathcal{D}_1$, $\mathcal{B} \leq \mathcal{D}_2$, and $C_\Im(\mathcal{D}_1, r) \,\overline{q}\, C_\Im(\mathcal{D}_2, r)$.

Proof. The proof is similar to that of Theorem 5.5.
□

Theorem 5.7. Let $h : (M, \Im) \longrightarrow (N, F)$ be a bijective $\mathcal{F}$-γ-irresolute and $\mathcal{F}$-open function. If $(M, \Im)$ is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space), then (N, F) is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space).

Proof. If $n_\sigma \,\overline{q}\, \mathcal{B}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{B} \in I^N$ and $\mathcal{F}$-γ-irresolute function h, then $h^{-1}(\mathcal{B})$ is an r-$\mathcal{F}$-γ-closed set. Put $n_\sigma = h(m_\sigma)$, then $m_\sigma \,\overline{q}\, h^{-1}(\mathcal{B})$. Since $(M, \Im)$ is r-$\mathcal{F}$-γ-regular, there is $\mathcal{D}_1, \mathcal{D}_2 \in I^M$ with $\Im(\mathcal{D}_1) \geq r$ and $\Im(\mathcal{D}_2) \geq r$ such that $m_\sigma \in \mathcal{D}_1$, $h^{-1}(\mathcal{B}) \leq \mathcal{D}_2$, and $\mathcal{D}_1 \,\overline{q}\, \mathcal{D}_2$. Since h is a bijective $\mathcal{F}$-open, hence $n_\sigma \in h(\mathcal{D}_1)$, $\mathcal{B} = h(h^{-1}(\mathcal{B})) \leq h(\mathcal{D}_2)$, and $h(\mathcal{D}_1) \,\overline{q}\, h(\mathcal{D}_2)$. Therefore, (N, F) is an r-$\mathcal{F}$-γ-regular space. The other case also follows similar lines. □

Theorem 5.8. Let $h : (M, \Im) \longrightarrow (N, F)$ be an injective $\mathcal{F}$-continuous and $\mathcal{F}$-γ-irresolute closed function. If (N, F) is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space), then $(M, \Im)$ is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Proof. If $m_\sigma \,\bar{q}\, \mathcal{B}$ for each r-$\mathcal{F}$-γ-closed set $\mathcal{B} \in I^M$ and injective $\mathcal{F}$-γ-irresolute closed function h, hence $h(\mathcal{B})$ is an r-$\mathcal{F}$-γ-closed set and $h(m_\sigma) \,\bar{q}\, h(\mathcal{B})$. Since $(N, \digamma)$ is r-$\mathcal{F}$-γ-regular, there is $\mathcal{D}_1, \mathcal{D}_2 \in I^N$ with $\digamma(\mathcal{D}_1) \geq r$ and $\digamma(\mathcal{D}_2) \geq r$ such that $h(m_\sigma) \in \mathcal{D}_1$, $h(\mathcal{B}) \leq \mathcal{D}_2$, and $\mathcal{D}_1 \,\bar{q}\, \mathcal{D}_2$. Since h is $\mathcal{F}$-continuous, then $m_\sigma \in h^{-1}(\mathcal{D}_1)$, $\mathcal{B} \leq h^{-1}(\mathcal{D}_2)$ with $\Im(h^{-1}(\mathcal{D}_1)) \geq r$, $\Im(h^{-1}(\mathcal{D}_2)) \geq r$, and $h^{-1}(\mathcal{D}_1) \,\bar{q}\, h^{-1}(\mathcal{D}_2)$. Hence, $(M, \Im)$ is an r-$\mathcal{F}$-γ-regular space. The other case also follows similar lines.
□

Theorem 5.9. Let $h : (M, \Im) \longrightarrow (N, \digamma)$ be a surjective $\mathcal{F}$-γ-irresolute, $\mathcal{F}$-open, and $\mathcal{F}$-closed function. If $(M, \Im)$ is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space), then $(N, \digamma)$ is an r-$\mathcal{F}$-γ-regular space (resp., r-$\mathcal{F}$-γ-normal space).

Proof. The proof is similar to that of Theorem 5.7.
□

- **Several new types of fuzzy compactness:**

Definition 5.4. Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called r-$\mathcal{F}$-γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j \text{ is } r\text{-}\mathcal{F}\text{-}\gamma\text{-open}\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite sub-set $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} \mathcal{B}_j$.

Lemma 5.2. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-γ-compact set is r-$\mathcal{F}$-compact.

Proof. The proof follows from Definitions 2.4 and 5.4.
□

Theorem 5.10. Let $h : (M, \Im) \longrightarrow (N, \digamma)$ be an $\mathcal{F}$-γ-continuous function, then $h(\mathcal{D})$ is an r-$\mathcal{F}$-compact set if $\mathcal{D} \in I^M$ is an r-$\mathcal{F}$-γ-compact set.

Proof. Let $\{\mathcal{B}_j \in I^N \mid \digamma(\mathcal{B}_j) \geq r\}_{j\in\Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, then $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j) \text{ is } r\text{-}\mathcal{F}\text{-}\gamma\text{-open}\}$ (by h is $\mathcal{F}$-γ-continuous) with $\mathcal{D} \leq \bigvee_{j\in\Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is r-$\mathcal{F}$-γ-compact, there is a finite sub-set $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} h^{-1}(\mathcal{B}_j)$. Hence, $h(\mathcal{D}) \leq \bigvee_{j\in\Omega_\circ} \mathcal{B}_j$. Therefore, $h(\mathcal{D})$ is r-$\mathcal{F}$-compact. □

Definition 5.5. Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called r-$\mathcal{F}$-almost γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j \text{ is } r\text{-}\mathcal{F}\text{-}\gamma\text{-open}\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite sub-set $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} C_\Im(\mathcal{B}_j, r)$.

Lemma 5.3. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-almost γ-compact set is r-$\mathcal{F}$-almost compact.

Proof. The proof follows from Definitions 2.4 and 5.5.
□

Lemma 5.4. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-γ-compact set is r-$\mathcal{F}$-almost γ-compact.

Proof. The proof follows from Definitions 5.4 and 5.5.
□

Remark 5.2. The converse of Lemma 5.4 fails as Example 5.2.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

Example 5.2. Let $W = [0,1]$, $t \in \mathbb{N} - \{1\}$, and $\mathcal{A}, \mathcal{B}_t \in I^W$ defined as follows:

$$\mathcal{A}(w) = \begin{cases} 1, & \text{if} \quad w = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases} \qquad \mathcal{B}_t(w) = \begin{cases} 0.8, & \text{if} \quad w = 0, \\ tw, & \text{if} \quad 0 < w \leq \frac{1}{t}, \\ 1\,, & \text{if} \quad \frac{1}{t} < w \leq 1. \end{cases}$$

Also, $\Im$ defined on W as follows:

$$\Im(\mathcal{C}) = \begin{cases} 1, & \text{if} \quad \mathcal{C} \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \mathcal{C} \leq \mathcal{A}, \\ \frac{t}{t+1}, & \text{if} \quad \mathcal{C} \leq \mathcal{B}_t, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, W is $\frac{1}{2}$-$\mathcal{F}$-almost γ-compact, but it is not $\frac{1}{2}$-$\mathcal{F}$-γ-compact.

Theorem 5.11. Let $h : (M, \Im) \longrightarrow (N, F)$ be an $\mathcal{F}$-continuous function, then $h(\mathcal{D})$ is an r-$\mathcal{F}$-almost compact set if $\mathcal{D} \in I^M$ is an r-$\mathcal{F}$-almost γ-compact set.

Proof. Let $\{\mathcal{B}_j \in I^N \mid F(\mathcal{B}_j) \geq r\}_{j\in\Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, then $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j)$ is r-$\mathcal{F}$-γ-open$\}$ (by h is $\mathcal{F}$-γ-continuous), such that $\mathcal{D} \leq \bigvee_{j\in\Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is r-$\mathcal{F}$-almost γ-compact, there is a finite sub-set $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} C_\Im(h^{-1}(\mathcal{B}_j), r)$. Since h is $\mathcal{F}$-continuous function,

$$\begin{aligned} \mathcal{D} &\leq \bigvee_{j\in\Omega_\circ} C_\Im(h^{-1}(\mathcal{B}_j), r) \\ &\leq \bigvee_{j\in\Omega_\circ} h^{-1}(C_F(\mathcal{B}_j, r)) \\ &= h^{-1}(\bigvee_{j\in\Omega_\circ} C_F(\mathcal{B}_j, r)). \end{aligned}$$

Hence, $h(\mathcal{D}) \leq \bigvee_{j\in\Omega_\circ} C_F(\mathcal{B}_j, r)$. Therefore, $h(\mathcal{D})$ is r-$\mathcal{F}$-almost compact. □

Definition 5.6. Let $(M, \Im)$ be an $\mathcal{FTS}$, $\mathcal{D} \in I^M$, and $r \in I_\circ$. An $\mathcal{F}$-set $\mathcal{D}$ is called r-$\mathcal{F}$-nearly γ-compact if for each family $\{\mathcal{B}_j \in I^M \mid \mathcal{B}_j$ is r-$\mathcal{F}$-γ-open$\}_{j\in\Omega}$ with $\mathcal{D} \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, there is a finite sub-set $\Omega_\circ$ of Ω with $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} I_\Im(C_\Im(\mathcal{B}_j, r), r)$.

Lemma 5.5. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-nearly γ-compact set is r-$\mathcal{F}$-nearly compact.

Proof. The proof follows from Definitions 2.4 and 5.6.
□

Lemma 5.6. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-γ-compact set is r-$\mathcal{F}$-nearly γ-compact.

Proof. The proof follows from Definitions 5.4 and 5.6.
□

Remark 5.3. The converse of Lemma 5.6 fails as Example 5.3.

Example 5.3. Let $W = [0,1]$, $0 < t < 1$, and $\mathcal{A}, \mathcal{B}, \mathcal{D}_t \in I^W$ defined as follows:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

$$\mathcal{A}(w) = \begin{cases} \frac{1}{2}, & \text{if} \quad 0 \leq w < 1, \\ 1, & \text{if} \quad w = 1, \end{cases} \qquad \mathcal{B}(w) = \begin{cases} 1, & \text{if} \quad w = 0, \\ \frac{1}{2}, & \text{if} \quad 0 < w \leq 1, \end{cases}$$

$$\mathcal{D}_t(w) = \begin{cases} \frac{w}{t}, & \text{if} \quad 0 \leq w < t, \\ \frac{1-w}{1-t}, & \text{if} \quad t < w \leq 1. \end{cases}$$

Also, $\Im$ defined on W as follows:

$$\Im(\mathcal{P}) = \begin{cases} 1, & \text{if} \quad \mathcal{P} \in \{\mathcal{A}, \mathcal{B}, \underline{1}, \underline{0}\}, \\ \max(\{1-t, t\}), & \text{if} \quad \mathcal{P} = \mathcal{D}_t, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, W is $\frac{1}{2}$-$\mathcal{F}$-nearly γ-compact, but it is not $\frac{1}{2}$-$\mathcal{F}$-γ-compact.

Theorem 5.12. Let $h : (M, \Im) \longrightarrow (N, \digamma)$ be an $\mathcal{F}$-continuous and $\mathcal{F}$-open, then $h(\mathcal{D})$ is an r-$\mathcal{F}$-nearly compact set if $\mathcal{D} \in I^M$ is an r-$\mathcal{F}$-nearly γ-compact set.

Proof. Let $\{\mathcal{B}_j \in I^N \mid \digamma(\mathcal{B}_j) \geq r\}_{j\in\Omega}$ with $h(\mathcal{D}) \leq \bigvee_{j\in\Omega} \mathcal{B}_j$, then $\{h^{-1}(\mathcal{B}_j) \in I^M \mid h^{-1}(\mathcal{B}_j)$ is r-$\mathcal{F}$-γ-open$\}$ (by h is $\mathcal{F}$-γ-continuous), such that $\mathcal{D} \leq \bigvee_{j\in\Omega} h^{-1}(\mathcal{B}_j)$. Since $\mathcal{D}$ is r-$\mathcal{F}$-nearly γ-compact, there is a finite sub-set $\Omega_\circ$ of Ω, such that $\mathcal{D} \leq \bigvee_{j\in\Omega_\circ} I_\Im(C_\Im(h^{-1}(\mathcal{B}_j), r), r)$. Since h is $\mathcal{F}$-continuous and $\mathcal{F}$-open,

$$\begin{aligned} h(\mathcal{D}) &\leq \bigvee_{j\in\Omega_\circ} h(I_\Im(C_\Im(h^{-1}(\mathcal{B}_j), r), r)) \\ &\leq \bigvee_{j\in\Omega_\circ} I_\digamma(h(C_\Im(h^{-1}(\mathcal{B}_j), r)), r) \\ &\leq \bigvee_{j\in\Omega_\circ} I_\digamma(h(h^{-1}(C_\digamma(\mathcal{B}_j, r))), r) \\ &\leq \bigvee_{j\in\Omega_\circ} I_\digamma(C_\digamma(\mathcal{B}_j, r), r). \end{aligned}$$

Therefore, $h(\mathcal{D})$ is r-$\mathcal{F}$-nearly compact. □

Lemma 5.7. In an $\mathcal{FTS}$ $(M, \Im)$, every r-$\mathcal{F}$-nearly γ-compact set is r-$\mathcal{F}$-almost γ-compact.

Proof. The proof follows from Definitions 5.5 and 5.6.
□

Remark 5.4. From the previous discussions and definitions, we have the following diagram.

r-$\mathcal{F}$-γ-compact set	$\rightarrow$	**r-$\mathcal{F}$-compact set**
$\downarrow$		$\downarrow$
r-$\mathcal{F}$-nearly γ-compact set	$\rightarrow$	**r-$\mathcal{F}$-nearly compact set**
$\downarrow$		$\downarrow$
r-$\mathcal{F}$-almost γ-compact set	$\rightarrow$	**r-$\mathcal{F}$-almost compact set**

6. Conclusions and Future Works

In the present manuscript, a novel class of $\mathcal{F}$-open sets, called r-$\mathcal{F}$-γ-open sets has been introduced on $\mathcal{FTS}s$ in Šostak's sense [6]. Some characterizations of r-$\mathcal{F}$-γ-open sets along with their mutual relationships have been discussed with the help of some illustrative examples. Furthermore, the notions of $\mathcal{F}$-γ-interior and $\mathcal{F}$-γ-closure operators have been defined and investigated. After that, the notions of $\mathcal{F}$-γ-continuous (resp. $\mathcal{F}$-γ-irresolute) functions between $\mathcal{FTS}s$ $(M, \Im)$ and (N, F) has been explored and discussed. Moreover, the notions of $\mathcal{F}$-almost (resp. $\mathcal{F}$-weakly) γ-continuous functions, which are weaker forms of $\mathcal{F}$-γ-continuous functions have been defined and characterized. We also showed that $\mathcal{F}$-γ-continuity $\Longrightarrow$ $\mathcal{F}$-almost γ-continuity $\Longrightarrow$ $\mathcal{F}$-weakly γ-continuity, but the converse may not be true. Thereafter, we defined and studied some new $\mathcal{F}$-functions using r-$\mathcal{F}$-γ-open and r-$\mathcal{F}$-γ-closed sets, called $\mathcal{F}$-γ-open (resp. $\mathcal{F}$-γ-irresolute open, $\mathcal{F}$-γ-closed, $\mathcal{F}$-γ-irresolute closed, and $\mathcal{F}$-γ-irresolute homeomorphism) functions. Also, we introduced and studied some new types of $\mathcal{F}$-separation axioms, called r-$\mathcal{F}$-γ-regular (resp. r-$\mathcal{F}$-γ-normal) spaces using r-$\mathcal{F}$-γ-closed sets. Finally, some new types of $\mathcal{F}$-compactness, called r-$\mathcal{F}$-almost (resp. r-$\mathcal{F}$-nearly) γ-compact sets have been defined and discussed via r-$\mathcal{F}$-γ-open sets. In the next works, we intend to the following topics:

• Defining upper (lower) γ-continuous $\mathcal{F}$-multifunctions and r-$\mathcal{F}$-γ-connected sets.

• Extending these notions given here to include fuzzy soft topological (r-minimal) spaces [28,33,37].

• Finding a use for these notions given here in the frame of fuzzy ideals as defined in [39-41].

• Introducing the notions as defined in [42-44] by using r-$\mathcal{F}$-γ-open sets.

References

1. Zadeh, L.A. Fuzzy Sets. *Inform. Control* **1965**, 8, 338-353.
2. Ahmad, B.; Kharal, A. On fuzzy soft sets. *Adva. Fuzzy Syst.* **2009**, 2009, 586507.
3. Cagman, N.; Enginoglu, S.; Citak, F. Fuzzy soft set theory and its application. *Iran. J. Fuzzy Syst.* **2011**, 8(3), 137-147.
4. Atef, M.; Ali, M.I.; Al-shami, T.M. Fuzzy soft covering based multi-granulation fuzzy rough sets and their applications. *Comput. Appl. Math.* **2021**, 40(4), 115.
5. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, 24, 182-190.
6. Šostak, A.P. On a fuzzy topological structure. *In: Proceedings of the 13th winter school on abstract analysis, Section of topology, Palermo: Circolo Matematico di Palermo* **1985**, 89-103.
7. Ramadan, A.A. Smooth topological spaces. *Fuzzy Set. Syst.* **1992**, 48, 371-375.
8. Chattopadhyay, K.C.; Samanta, S.K. Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness. *Fuzzy Set. Syst.* **1993**, 54(2), 207-212.
9. El-Gayyar, M.K.; Kerre, E.E.; Ramadan, A.A. Almost compactness and near compactness in smooth topological spaces. *Fuzzy Set. Syst.* **1994**, 62(2), 193-202.
10. Höhle, U.; Šostak, A.P. A general theory of fuzzy topological spaces. *Fuzzy Set. Syst.* **1995**, 73, 131-149.
11. Ramadan, A.A.; Abbas, S.E.; Kim, Y.C. Fuzzy irresolute mappings in smooth fuzzy topological spaces. *J. Fuzzy Math.* **2001**, 9(4), 865-877.
12. Kim, Y.C.; Ramadan A.A.; Abbas, S.E. Weaker forms of continuity in Šostaks fuzzy topology. *Indian J. Pure Appl. Math.* **2003**, 34(2), 311-333.
13. Abbas, S.E. Fuzzy super irresolute functions. *Inter. J. Math. Mathematical Sci.* **2003**, 42, 2689-2700.
14. Abbas, S.E. Fuzzy β-irresolute functions. *Appl. Math. Comp.* **2004**, 157, 369-380.
15. Kim, Y.C.; Abbas, S.E. On several types of r-fuzzy compactness. *J. Fuzzy Math.* **2004**, 12(4), 827-844.
16. Aygün, H.; Abbas, S.E. On characterization of some covering properties in L-fuzzy topological spaces in Šostak sense. *Inform. Sciences* **2004**, 165, 221-233,

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 February 2025 doi:10.20944/preprints202502.1193.v1

17. Aygün,, H.; Abbas, S.E. Some good extensions of compactness in Šostaks L-fuzzy topology. *Hacett. J. Math. Stat.* **2007**, 36(2), 115-125,
18. Li, H.Y.; Shi., F.G Some separation axioms in I-fuzzy topological spaces. *Fuzzy Set. Syst.* **2008**, 159, 573-587.
19. Li, H.Y.; Shi, F.G. Measures of fuzzy compactness in L-fuzzy topological spaces. *Comput. Math. Appl.* **2010**, 59, 941-947.
20. Shi, F.G.; Li, R.X. Compactness in L-fuzzy topological spaces. *Hacet. J. Math. Stat.* **2011**, 40(6), 767-774,
21. Fang, J.; Guo, Y. Quasi-coincident neighborhood structure of relative I-fuzzy topology and its applications. *Fuzzy Set. Syst.* **2012**, 190, 105-117.
22. El-Dardery, M.; Ramadan, A.A.; Kim, Y.C. L-fuzzy topogenous orders and L-fuzzy topologies. *J. Intell. Fuzzy Syst.* **2013**, 24(4), 685-691.
23. Kalaivani, C.; Roopkumar, R. Fuzzy Perfect Mappings and Q-Compactness in Smooth Fuzzy Topological Spaces. *Fuzzy Inform. Eng.* **2014**, 6(1), 115-131.
24. Solovyov, S.A. On fuzzification of topological categories. *Fuzzy Set. Syst.* **2014**, 238, 1-25.
25. Minana, J.J.; Šostak, A.P. Fuzzifying topology induced by a strong fuzzy metric. *Fuzzy Set. Syst.* **2016**, 300, 24-39,
26. Maji, P.K.; Biswas, R.; Roy, A.R. Fuzzy soft sets. *J. Fuzzy Math.* **2001**, 9, 589-602.
27. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.* **1999**, 37, 19-31.
28. Aygünoğlu, A.; Çetkin, V.; Aygün, H. An introduction to fuzzy soft topological spaces. *Hacet. J. Math. Stat.* **2014**, 43, 193-208.
29. Çetkin, V.; Aygünoğlu, A.; Aygün, H. On soft fuzzy closure and interior operators. *Util. Math.* **2016**, 99, 341-367.
30. Taha, I.M. A new approach to separation and regularity axioms via fuzzy soft sets. *Ann. Fuzzy Math. Inform.* **2020**, 20, 115-123.
31. Taha, I.M. Some new separation axioms in fuzzy soft topological spaces. *Filomat* **2021**, 35, 1775-1783.
32. Çetkin, V.; Aygün, H. Fuzzy soft semiregularization spaces. *Ann. Fuzzy Math. Inform.* **2014**, 7, 687-697.
33. Taha, I.M. Compactness on fuzzy soft r-minimal spaces. *Int. J. Fuzzy Logic Intell. Syst.* **2021**, 21, 251-258.
34. Alqurashi, W.; Taha, I.M. On fuzzy soft α-open sets, α-continuity, and α-compactness: some novel results. *Eur. J. Pure Appl. Math.* **2024**, 17, 4112-4134.
35. Alshammari, I.; Taha, I.M. On fuzzy soft β-continuity and β-irresoluteness: some new results. *AIMS Math.* **2024**, 9, 11304-11319.
36. Alshammari, I.; Taha, O.; El-Bably, M.K.; Taha, I.M. On r-fuzzy soft δ-open sets with applications in fuzzy soft topological spaces. *Eur. J. Pure Appl. Math.* **2025**, 18(1), 1-21.
37. Taha, I.M. Some new results on fuzzy soft r-minimal spaces. *AIMS Math.* **2022**, 7, 12458-12470.
38. Kandil, A.; El-Shafei, M.E. Regularity axioms in fuzzy topological spaces and FRi-proximities. *Fuzzy Set. Syst.* **1988**, 27, 217-231.
39. Taha, I.M. On r-fuzzy ℓ-open sets and continuity of fuzzy multifunctions via fuzzy ideals. *J. Math. Comput. Sci.* **2020**, 10(6), 2613-2633.
40. Taha, I.M. On r-generalized fuzzy ℓ-closed sets: properties and applications. *J. Math.* **2021**, 2021, 1-8.
41. Taha, I.M. r-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓ-continuity via fuzzy idealization. *J. Math. Comput. Sci.* **2022**, 25(1), 1-9.
42. Saleh, H.Y.; Asaad, B.A.; Mohammed, R.A. Connectedness, local connectedness, and components on bipolar soft generalized topological spaces. *J. Math. Comput. Sci.* **2023**, 30(4), 302-321.
43. Saleh, H.Y.; Salih, A.A.; Asaad, B.A.; Mohammed, R.A. Binary bipolar soft points and topology on binary bipolar soft sets with their symmetric properties. *Symmetry* **2024**, 16(1), 1-18.
44. Saleh, H.Y.; Asaad, B.A.; Mohammed, R.A. Novel classes of bipolar soft generalized topological structures: compactness and homeomorphisms. *Fuzzy Inform. Eng.* **2024**, 16(1), 49-73.