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Characterization of Pomonoids by Properties of I-Regular S-Posets

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Abstract

In 2005, Shi defined I-regular S-posets and used this concept to characterize PP-pomonoids and po-cancellable pomonoids. In this paper, we continue to develop the homological classification of pomonoids by using the I-regularity of S-posets. First, we characterize pomonoids over which all I-regular S-posets have one of the properties around projectivity or injectivity, and many known results are generalized. Moreover, some possible conditions on pomonoids that describe when their diagonal posets are I-regular are found. Finally, some characterizations of pomonoids by the I-regularity of their Rees factor posets are given.

Keywords: I-regular; S-poset; Rees factor; regular pomonoid

MSC: 20M30; 20M50

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1. Preliminaries

In this paper, unless otherwise specified, S will be a partially ordered monoid (or simply, a pomonoid). A nonempty poset A is called a *left S-poset* if there exists a mapping $S \times A \to A$, $(s, a) \longmapsto sa$ which satisfies the following conditions: (i) The action is monotonic in each variable; (ii) $t(sa) = (ts)a$ and $1a = a$ for all $a \in A$ and all $s, t \in S$. Right S-posets B_S are defined analogously, and $\Theta = \{\theta\}$ denotes the one-element S-poset. In this paper, a left (right) ideal of S refers to a nonempty subset I of S satisfying $SI \subseteq I$ ($IS \subseteq I$).

A *morphism of left S-posets* is a monotonic mapping $f : A \to B$ which satisfies $f(sa) = sf(a)$ for every $a \in A$ and $s \in S$. Morphisms of right S-posets are defined similarly, and morphisms of posets are just monotonic mappings. In this way, the categories ${}_S\mathrm{Pos}$ (left S-posets), Pos_S (right S-posets) and Pos (posets) are obtained. In these categories, the monomorphisms are the injective morphisms, whereas the regular monomorphisms are the order-embeddings¡ªi.e., morphisms $f : A \to B$ for which $f(a) \leq f(a')$ implies $a \leq a'$ for all $a, a' \in A$ (see [2]). Research on flatness properties of S-posets was initiated in the mid-1980s by S. Fakhruddin in [5], and this work has recently been continued in the articles [1–3,7].

An S-subposet ${}_SB$ of an S-poset ${}_SA$ is called *convex* if, for any $a \in A$ and $b, b' \in B$, $b' \leq a \leq b$ implies $a \in B$. An element $c \in S$ is called *right (left) po-cancellable* if for all $s, s' \in S$, $sc \leq s'c$ ($cs \leq cs'$) implies $s \leq s'$. A pomonoid is called *left (right) collapsible* if for all $s, t \in S$, there exists $u \in S$ such that $us = ut$ ($su = tu$). A pomonoid is called *weakly right (left) reversible* if for all $s, t \in S$, there exist $u, v \in S$ such that $us \leq vt$ ($su \leq tv$). A left S-poset is called *simple* if it has no proper subposets, and *completely reducible* if it is a coproduct of simple posets.

An *order congruence* on an S-poset ${}_SA$ is an S-act congruence ρ such that the factor act A/ρ can be equipped with a compatible order, making the natural map $A \to A/\rho$ an S-poset morphism. A left S-poset ${}_SA$ is called *cyclic* if $A = Sa = \{sa \mid s \in S\}$ for some $a \in A$. In [12], an S-poset A is cyclic if and only if there exists an order congruence λ on S such that $A \cong S/\lambda$. If K is a convex left ideal of a pomonoid S, then there exists an S-poset congruence where one of its classes is K and all the others are singletons. Moreover, the factor S-poset by this congruence is called the *Rees factor S-poset of S by K* and denoted S/K.

Various flatness properties of S-posets are defined in terms of tensor products. To define the tensor product $A \otimes_S B$ of a right S-poset A_S and a left S-poset ${}_SB$ (see [13]), we consider a preorder θ on the set $A \times B$, defined by $(a,b)\theta(a',b')$ if and only if

$$\begin{array}{ll} a \leq a_1 s_1, & \\ a_1 t_1 \leq a_2 s_2, & s_1 b \leq t_1 b_2, \\ a_2 t_2 \leq a_3 s_3, & s_2 b_2 \leq t_2 b_3, \\ \vdots & \\ a_n t_n \leq a', & s_n b_n \leq t_n b', \end{array}$$

for some $n \in \mathbb{N}$, $a_1, \cdots, a_n \in A$, $b_2, \cdots, b_n \in B$ and $s_1, t_1, \cdots, s_n, t_n \in S$. Then $\theta \cap \theta^{-1}$ is an equivalence relation on $A \times B$, and we denote the equivalence class of (a,b) by $a \otimes b$. The quotient set

$$A \otimes_S B := (A \times B)/(\theta \cap \theta^{-1}) = \{a \otimes b \mid a \in A, b \in B\}$$

is a poset with respect to the order

$$a \otimes b \leq a' \otimes b' \Leftrightarrow (a,b)\theta(a',b').$$

This poset $A \otimes_S B$ is called the tensor product of A_S and ${}_SB$. Note that $as \otimes b = a \otimes sb$ for every $a \in A$, $b \in B$ and $s \in S$. In a natural way, one obtains a functor $A_S \otimes -$ of tensor multiplication from ${}_S\mathrm{Pos}$ to Pos.

In [3,12], the definitions of flatness, weak flatness, principally weak flatness and weak torsion freeness are formulated as follows:

- A left S-poset ${}_SB$ is called *flat* if, for every right S-poset A_S and all pairs $(a,b),(a',b')$ in $A \times B$, $a \otimes b = a' \otimes b'$ in $A \otimes_S B$ implies the same equality holds in $(aS \cup a'S) \otimes_S B$. Equivalently, the functor $- \otimes_S B$ takes embeddings in Pos_S to monomorphisms in Pos.
- A left S-poset ${}_SB$ is called *(principally) weakly flat* if the functor $- \otimes_S B$ maps embeddings of (principal) right ideals $I \subseteq S$ in Pos_S to monomorphisms in Pos.
- A left S-poset ${}_SB$ is called *weakly torsion free* if $cb = cb'$ implies $b = b'$ whenever $b, b' \in B$ and c is a left po-cancellable element.

For a more complete discussion of flatness properties of posets over pomonoids, the reader is referred to [3], [8], [9]. The following relations exist among flatness properties of S-posets:

$$\begin{array}{ccccccccccccccc}
\text{free} & \Rightarrow & \text{projective} & \Rightarrow & \text{strongly flat} & \Rightarrow & (P) & \Rightarrow & (WP) & \Rightarrow & (PWP) & & & & \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & & & \\
 & & & & & & (P_w) & \Rightarrow & (WP)_w & \Rightarrow & (PWP)_w & & & & \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & & & \\
 & & & & & & \text{po-flat} & \Rightarrow & \text{w.po-f.} & \Rightarrow & \text{p.w.po-f.} & \Rightarrow & \text{GP-po-f.} & \Rightarrow & \text{po-t.f.} \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \text{flat} & \Rightarrow & \text{w.f.} & \Rightarrow & \text{p.w.f.} & \Rightarrow & \text{GP-f.} & \Rightarrow & \text{w.t.f.}
\end{array}$$

Let A be an S-poset. An element $a \in A$ is called *I-regular* if there exists an S-morphism $f : Sa \to S$ such that $f(a)a = a$. An S-poset A is called *I-regular* if all elements of A are I-regular.

It is clear that a regular pomonoid S is I-regular as a left S-poset, but the converse is not true. For example, if S is a right po-cancellable pomonoid, then S is an I-regular left S-poset without being a regular pomonoid. In the special case where S is an ordered group, S is not only a regular pomonoid but also an I-regular left S-poset.

In [13], Shi introduced the concept of I-regular S-posets and gave characterizations of two classes of pomonoids (left PP-pomonoid and right po-cancellable pomonoid) by the I-regularity of S-posets.

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In this paper, we continue to study I-regular S-posets. In Section 2, we characterize pomonoids over which all I-regular S-posets have one of the properties around projectivity or injectivity. In [13], characterizations of pomonoids over which all free (projective) S-posets are I-regular are given; in [12], the authors characterized pomonoids over which all strongly flat S-posets are I-regular. Consequently, we continue to investigate pomonoids over which all left S-posets with one of the properties are I-regular in Section 3. In Section 4, we investigate the direct product of I-regular S-posets. Finally, we study the classification of pomonoids by the I-regularity property of right Rees factor S-posets and tabulate the results.

2. All I-Regular S-Posets Are $\cdots$

In this section, we investigate pomonoids over which all I-regular left S-posets have one of the properties introduced in Section 1. To achieve this goal, we need the following lemmas.

Lemma 1 ([13], Proposition 4.2)**.** *Let A be an S-poset and $a \in A$. The following assertions are equivalent:*
(1) *a is I-regular.*
(2) *There exists an element $e \in E(S)$ such that $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$.*
(3) *$Sa \cong Se$ in ${}_S\mathbf{Pos}$ for some $e \in E(S)$.*
(4) *Sa is projective.*

If $a \in A, e^2 = e \in S$ as in Lemma 1 (2), then we call $\{a, e\}$ an I-regular pair (in A).

From [13] a pomonoid S is called *left PP pomonoid* if the S-subposet Sx is projective for all $x \in S$.(Note, however, that Sx may be an ideal of S in the ordered sense.) By Lemma 1, an S-poset is I-regular if and only if all cyclic S-subposets of A are projective. Thus we have

Lemma 2 ([13], Lemma 4.7)**.** *A pomonoid S is I-regular if and only if S is a left PP pomonoid.*

Lemma 3. *${}_S\Theta$ is I-regular if and only if S contains a right zero element.*

Proof. This follows from Theorem 1 of [3] and Lemma 1. □

Lemma 4 ([13], Lemma 4.5)**.** *All S-subposets of I-regular S-poset are I-regular, and coproducts of I-regular S-posets are I-regular.*

Lemma 5. *([3])Let S be a pomonoid, $a, a' \in A_S, b, b' \in {}_SB$. Then $a \otimes b \leq a' \otimes b'$ in $A_S \otimes_S B$ if and only if there exist $a_1, \cdots, a_m \in A_S, b_2, \cdots, b_m \in {}_SB, s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{rlrl} a & \leq a_1 s_1, & & \\ a_1 t_1 & \leq a_2 s_2, & s_1 b & \leq t_1 b_2, \\ a_2 t_2 & \leq a_3 s_3, & s_2 b_2 & \leq t_2 b_3, \\ & \vdots & & \vdots \\ a_m t_m & \leq a', & s_m b_m & \leq t_m b'. \end{array}$$

Similar to ([6], Lemma 3.2), we have

Lemma 6. *Let S be a pomonoid. If there exists an I-regular left S-poset, then there exist a largest I-regular left ideal of S.*

In the following, $T(S)$ is always used to represent the largest I-regular left ideal of S.

Theorem 1. *For any pomonoid S, all I-regular left S-posets are principally weakly flat if and only if for every idempotent $e \in T(S)$ and every element $s \in S$ the product se is a regular element in S.*

Proof. Let $s \in S$ and $e^2 = e \in T$. If $Sse = Se$, then there exists $t \in S$ such that $tse = e$, and then it follows that $setse = se$, hence se is a regular element. In other case we have $Sse \neq Se$. We can construct an S-act M as follows:

$$M = \{(x, te) \mid te \in Se \backslash Sse\} \cup \{(y, te) \mid te \in Se \backslash Sse\} \cup \{(z, te) \mid te \in Sse\}$$

where x, y, z are three elements not belonging to S, and define a left S-action on M by

$$r(w, te) = \begin{cases} (w, rte), & if\ rte \in Se \setminus Sse, \quad w \in \{x, y\}, \\ (z, rte), & if\ rte \in Sse, \end{cases}$$

$$r(z, te) = (z, rte).$$

The order on M is defined as:

$$(w_1, s) \leq (w_2, t) \Leftrightarrow (w_1 = w_2 \text{ and } s \leq t) \text{ or } (w_1 \neq w_2, s \leq i \leq t \text{ for some } i \in Sse).$$

Then M is an S-poset according to the above definition. It is clear that there have isomorphisms $S(x, e) \simeq Se \simeq S(y, e)$. Since $Se \leq T(S)$, Se is I-regular by Lemma 4, thus $S(x, e)$ and $S(y, e)$ are also I-regular. From Lemma 4, it follows that $M = S(x, e) \cup S(y, e)$ is I-regular. By assumption, M is principally weakly flat. Clearly $s(x, e) = (z, se) = s(y, e)$, then $se \otimes (x, e) = se \otimes (y, e)$ in $seS \otimes M$. By Lemma 5, there exist $s_1, \cdots, s_n, u_1, v_1, \cdots u_n, v_n \in S, b_2, b_3, \cdots, b_n \in seS, w_1, \cdots, w_n \in \{x, y, z\}$ such that

$$\begin{aligned} & & (x, e) &\leq u_1(w_1, s_1), \\ seu_1 &\leq b_2 v_1, & v_1(w_1, s_1) &\leq u_2(w_2, s_2), \\ b_2 u_2 &\leq b_3 v_2, & v_2(w_2, s_2) &\leq u_3(w_3, s_3), \\ &\cdots & &\cdots \\ b_n u_n &\leq sev_n, & v_n(w_n, s_n) &\leq (y, e). \end{aligned}$$

Denote 1 by $u_0 s_0$ and $v_{n+1} s_{n+1}$, by the definition of M, there exist $k \in \{0, 1, \cdots, n, n+1\}$ and $j \in Sse$ such that $v_k s_k \leq j \leq u_{k+1} s_{k+1}$. So we have $se \leq see \leq seu_1 s_1 \leq b_2 v_1 s_1 \leq b_2 u_2 s_2 \leq \cdots \leq b_{k+1} v_k s_k \leq b_{k+1} j \leq b_{k+1} u_{k+1} s_{k+1} \leq \cdots \leq b_n u_n s_n \leq b_n v_n s_n \leq sev_n s_n \leq se$, which implies that $se = b_{k+1} j$, hence $se \in seSse$. Now the result follows.

Conversely, suppose ${}_S A$ is an I-regular left S-poset and for $a, a' \in A, s \in S, s \otimes a = s \otimes a'$ in $S \otimes A$. Then there exist $a_2, a_3, \cdots, a_m, b_2, b_3, \cdots, b_k \in A$, $u_1, v_1, \cdots, u_m, v_m \in S$, $p_1, q_1, \cdots, p_k, q_k \in S$, $s_1, \cdots, s_m, r_1, \cdots, r_k \in S$ such that

$$\begin{aligned} s &\leq s_1 u_1, & & \\ s_1 v_1 &\leq s_2 u_2, & u_1 a &\leq v_1 a_2, \\ s_2 v_2 &\leq s_3 u_3, & u_2 a_2 &\leq v_2 a_3, \\ &\cdots & &\cdots \\ s_m v_m &\leq s, & u_m a_m &\leq v_m a', \\ s &\leq r_1 p_1, & & \\ r_1 q_1 &\leq r_2 p_2, & p_1 a' &\leq q_1 b_2, \\ r_2 q_2 &\leq r_3 p_3, & p_2 b_2 &\leq q_2 b_3, \\ &\cdots & &\cdots \\ r_k q_k &\leq s, & p_k b_k &\leq q_k a. \end{aligned}$$

Because ${}_SA$ is I-regular, there exist $e, f \in E(S)$ such that $\{a, e\}, \{a', f\}$ are I-regular pairs and Se, Sf are I-regular left ideals, thus $e, f \in T(S)$. By hypothesis there exist $x, y \in S$ such that $se = sexse$ and $sf = sfysf$. From $s \otimes a = s \otimes a'$, we obtain that $sa = sa'$. We can now canculate

$$\begin{aligned} sexsa' &\leq sexr_1p_1a' \leq sexr_1q_1b_2 \leq sexr_2p_2b_2 \\ &\leq \cdots \leq sexr_kp_kb_k \leq sexr_kq_ka \\ &\leq sexsa = sexsea = sea = sa = sa'. \end{aligned}$$

Thus $sexsf \leq sf$(using I-regular pair). Hence

$$\begin{aligned} s \otimes a &= s \otimes ea = se \otimes a = sexse \otimes a \\ &= sex \otimes sa = sex \otimes sa' = sex \otimes sfa' \\ &= sexsf \otimes a' \leq sf \otimes a' = s \otimes fa' \\ &= s \otimes a' \end{aligned}$$

in $sS \otimes A$. Similarly, using I-regular pair, we can obtain $sfyse \leq se$. Therefore

$$\begin{aligned} s \otimes a' &= s \otimes fa' = sf \otimes a' = sfysf \otimes a' \\ &= sfy \otimes sfa' = sfy \otimes sa' = sfy \otimes sa \\ &= sfyse \otimes a \leq se \otimes a = s \otimes ea \\ &= s \otimes a \end{aligned}$$

in $sS \otimes A$. So A is principally weakly flat. □

Corollary 1. *For any pomonoid S, the following statements are equivalent:*
(1) *S is a regular pomonoid.*
(2) *S is a left PP pomonoid and all I-regular left S-posets are principally weakly po-flat.*
(3) *S is a left PP pomonoid and all I-regular left S-posets are principally weakly flat.*

Proof. (1) $\Rightarrow$ (2). Let S be a regular pomonoid. Then S is left PP and all left S-posets are principally weakly po-flat by ([11], Theorem 2.3).

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). From Proposition 4.6 of [13], it follows that S is a left PP pomonoid if and only if S is an I-regular S-poset if and only if $T(S) = S$. So by Theorem 1, S is a regular pomonoid. □

Theorem 2. *Let S be a pomonoid. If all I-regular left S-posets are GP-flat, then for every idempotent $e \in T(S)$ and every element $s \in S$, there exist $n \in N, x \in S$ such that $(se)^n = (se)^n xse$.*

Proof. It is similar to that of Theorem 1. □

Theorem 3. *For any pomonoid S, the following statements are equivalent:*
(1) *All I-regular left S-posets are weakly torsion free.*
(2) *For every left po-cancellable element r and for every idempotent $e \in T(S)$, $re\mathcal{L}e$.*

Proof. (1) $\Rightarrow$ (2). For left po-cancellable element $r \in S$ and $e^2 = e \in T(S)$, if $Sre \neq Se$, then M is an I-regular S-poset constructed in Theorem 1. By assumption M must be weakly torsion free. But now from $r(x, e) = (z, re) = r(y, e)$, we get $(x, e) = (y, e)$, a contradiction. Hence $Sre = Se$, which means that $re\mathcal{L}e$.

(2) $\Rightarrow$ (1). Let A be an I-regular S-poset and $ra = rb$ for any left po-cancellable element r, any $a, b \in A$. Since A is I-regular, there exist $e, f \in E(S)$ such that $\{a, e\}, \{b, f\}$ are I-regular pairs. Thus we have $rea = rfb$. Since $e \in T(S)$, we have $re\mathcal{L}e$, which implies there exists $t \in S$ such that $tre = e$.

Therefore $ea = trea = trfb$ and $rb = ra = rtrfb$, which imply $rf = rtrf$. Since r is a left po-cancellable element, then $f = trf$ and so $a = ea = trfb = fb = b$. Therefore A is weakly torsion free. □

Theorem 4. *For any pomonoid S, the following statements are equivalent:*
(1) *All I-regular left S-posets are projective.*
(2) *All I-regular left S-posets are strongly flat.*
(3) *All I-regular left S-posets satisfy Condition (P).*
(4) *Every idempotent of $T(S)$ generates a minimal left ideal.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

(3)⇒(4). Let $e^2 = e \in T(S)$. Then Se is I-regular by Lemma 4. Suppose I is a left ideal of S such that $I \subseteq Se, I \neq Se$. Let M be the I-regular S-poset constructed in Theorem 1. By assumption, M satisfies Condition (P). By ([12], Proposition 2.11), M must be a coproduct of cyclic S-subposets which is impossible because $S(x,e) \cap S(y,e) = \{(z,se)\}$. Hence Se is a minimal left ideal.

(4)⇒(1). Let A be an I-regular left S-poset. For any $a \in A$, the cyclic subposet Sa is, by Lemma 1, isomorphic to some left ideal $Se, e \in T(S)$. By assumption, all such ideals Se are simple. Hence, A is a coproduct of simple subposets each of which is isomorphic to a left ideal generated by an idempotent. By Theorem 3.4 of [13], A is projective. □

Recall from [4] and [14], a left S-poset A is called *regular-injective* if for any regular monomorphism $h : B \to C$ and morphism $f : B \to A$ there exists a morphism $g : C \to A$ such that $f = gh$. A left S-poset A is called *regular-(principally) weakly injective* if for any regular monomorphism $i : I \to S$ where I is a (principally) left ideal of S and for any S-poset morphism $f : I \to A$ there exists an S-poset morphism $g : S \to A$ such that $f = gi$. A left S-poset A is called *regular-divisible* if $dA = A$ for every right po-cancellable element d of S.

Proposition 1. *For any pomonoid S, all regular-principally weakly injective S-posets are regular-divisible.*

Proof. Let M be a regular-principally weakly injective S-poset and let d be any right po-cancellable element of S. Let $m \in M$. Since d is right po-cancellable, there exists an S-poset morphism $f : Sd \to M$ defined by $f(sd) = sm$ for all $s \in S$. Since M is regular-principally weakly injective, there exists an S-poset morphism $g : S \to M$ such that $f = gi$ where i is the regular monomorphism of Sd into S. Now

$$m = f(d) = (gi)(d) = g(d) = d(g(1)) \in dM.$$

Hence $M \subseteq dM$ and thus $dM = M$ which means that M is regular-divisible. □

From the previous definitions and proposition, we have following implications:

regular-injective ⇒ regular-weakly injective ⇒

regular-principally weakly injective ⇒ regular-divisible.

Theorem 5. *Let S be a pomonoid. All I-regular left S-posets are regular-divisible if and only if all left ideals Se, $e^2 = e \in T(S)$, are regular-divisible.*

Proof. Necessity is obvious because Se is I-regular by Lemma 4.

Sufficiency. Let A be an arbitrary I-regular S-poset and let $a \in A$. Then by Lemma 1, Sa is isomorphic to Se, $e \in T(S)$. Since Se is regular-divisible, then for any right po-cancellable $d \in S$, we have $dSe = Se$ and thus $dSa = Sa$. But then

$$dA = d(\bigcup_{a\in A} Sa) = \bigcup_{a\in A} dSa = \bigcup_{a\in A} Sa = A$$

which shows that A is regular-divisible. □

For any $q \in S$, an element $p \in S$ is said to be *q-po-cancellable*, if for any $s, t \in S$, $sp \leq tp$ always implies $sq \leq tq$.

Theorem 6. *Let S be a pomonoid. If all I-regular left S-posets are regular-principally weakly injective, then the largest left ideal $T(S) \subseteq S$ is regular and if $p \in S \setminus T(S)$ is e-po-cancellable for $e^2 = e \in T(S)$, then $e \in pS$.*

Proof. For any $t \in T(S)$, we have $St \subseteq T(S)$ and St is I-regular. By assumption, St is regular-principally weakly injective, and there exists a morphism $g : S \to St$ such that $gi = 1_{St}$ where $i : St \to S$ is the inclusion morphism. So $st = g(1)$ for some $s \in S$. Now $tst = tg(1) = g(t) = t$. Thus t is regular.

Let $p \in S \setminus T(S)$ be e-po-cancellable for $e^2 = e \in T(S)$. Then by setting $f(p) = e$ we get an S-poset morphism f from Sp into Se. Since Se is I-regular, it is regular-principally weakly injective and there exists a morphism $g : S \to Se$ such that $f = gi$ where $i : Sp \to S$ is the inclusion homomorphism. Now $e = f(p) = g(p) = pg(1) \in pS$. □

Lemma 7. *S is a regular pomonoid if and only if all left S-posets are regular-principally weakly injective.*

Proof. Let A be a left S-poset and $f : Ss \to A, s \in S$ be an S-homomorphism. If S is regular, then there exists $s' \in S$ such that $s = ss's$. Set $f(s's) = a$ and define a mapping $g : S \to A$ by $g(1) = a$. Then g is well defined and $g(s) = sg(1) = sa = sf(s's) = f(s)$. Since g is the extension of f to S, A is regular-principally weakly injective.

Conversely, suppose that all left S-posets are regular-principally weakly injective. Then for every $s \in S$, the principal left ideal Ss of S is a regular-principally weakly injective left S-poset. Hence the identity map i of Ss to Ss can extended to the S-homomorphism g of S onto Ss. Set $g(1) = s's$ for some $s' \in S$. Then $s = i(s) = g(s) = sg(1) = ss's$, hence s is a regular element. □

Theorem 7. *A pomonoid S is an I-regular left S-poset and all I-regular left S-posets are regular-principally weakly injective if and only if S is a regular pomonoid.*

Proof. Necessity. Suppose that S is an I-regular left S-poset and all I-regular left S-posets are regular-principally weakly injective. Then $T(S) \subseteq S$ is regular by Theorem 6. Let $p \in S \setminus T(S)$. If S is an I-regular S-poset, then there exists an idempotent $e \in S$ such that $h : Sp \to Se$ is an isomorphism. Since Se is I-regular, it is regular-principally weakly injective and then there exists an S-homomorphism $g : S \to Se$ such that g is an extension of h. Hence $e = h(p) = g(p) = pg(1)$, and then $p = ep = pg(1)p$, so p is also regular. Thus S is a regular pomonoid.

Sufficiency. If S is a regular pomonoid, then S is a left PP pomonoid and so S an I-regular left S-poset by Lemma 2. Using Lemma 7, we obtain all left S-posets are regular-principally weakly injective. So the result follows. □

From [13], a right S-poset A is called *faithful (strongly faithful)* if from $sa \leq ta, s, t \in S$, for all (some) $a \in A$ it follows that $s \leq t$.

Theorem 8. *For any pomonoid S, the following statements are equivalent:*
(1) *All I-regular left S-posets are faithful.*
(2) *For any $e^2 = e \in T(S)$, Se is faithful.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$. Let A be an I-regular left S-poset and $ux \leq vx$, $u, v \in S, x \in A$. Take $a \in A$, there exists $e^2 = e \in T(S)$ such that $\{a, e\}$ is an I-regular pair. For any $s \in S$, we have $usa \leq vsa$, it follows that $use \leq vse$. Since Se is faithful, $u \leq v$. □

Theorem 9. *let S be a pomonoid. Then all I-regular left S-posets are strongly faithful if and only if S is right po-cancellable.*

Proof. Let all I-regular left S-posets are strongly faithful and for any $s,t,z \in S$, $sz \leq tz$. For $a \in A$, we have $s(za) \leq t(za)$. Since A is strongly faithful, $s \leq t$ and so S is right po-cancellable.

Conversely, suppose that A is an I-regular left S-poset and for $a \in A, s,t \in S$, $sa \leq ta$. Then there exists $e \in E(S)$ such that $ea = a$ and $se \leq te$. Since S is right po-cancellative, we have $s \leq t$ and so A is strongly faithful as required. □

3. All $\cdots$ S-Posets Are I-Regular

In [13], characterizations of pomonoid over which all free (projective) S-posets are I-regular have been given. In [12], the authors characterized pomonoids over which all strongly flat S-posets are I-regular. In this section, we continue to investigate pomonoids over which all left S-posets with one of the properties are I-regular.

Proposition 2. *For any pomonoid S, all strongly faithful S-posets are I-regular.*

Proof. Let A be a strongly faithful S-poset. Then for any $a \in A$, there exists a morphism $f : Sa \to S$ defined by $f(sa) = s$ which satisfies $f(a)a = a$. Therefore A is I-regular. □

Theorem 10. *For any pomonoid S, all completely reducible left S-posets are I-regular if and only if S contains a right zero element.*

Proof. Necessity. The one element left S-poset ${}_S\Theta$ is obviously completely reducible. Hence by assumption ${}_S\Theta$ is I-regular. By Lemma 1, ${}_S\Theta$ is projective, which implies that S contains a right zero element from ([3], Theorem 1).

Sufficiency. From the existence of a right zero element, it follows that the only simple left S-poset is one-element S-poset. Obviously the one-element poset is projective and I-regular by Lemma 1. But then, by Lemma 3, every completely reducible left S-poset is I-regular. □

Lemma 8. *Let S be a left zero semigroup with 1 adjoined and A a weakly po-flat left S-poset. Suppose that $a \in A$ and $s,t \in S\backslash\{1\}$ are such that $sa \leq ta$. Then $s \leq t$.*

Proof. From $sa \leq ta$, it follows that $s \otimes a \leq t \otimes a$ in $S \otimes A$. Thus we have $s \otimes a \leq t \otimes a$ in $(sS \cup tS) \otimes A$, since A is weakly po-flat. Therefore, by Lemma 5, there exist $u_1, v_1, \cdots, u_n, v_n \in S, s_1, \cdots, s_n \in (sS \cup tS), a_2, \cdots, a_n \in A$ such that

$$\begin{array}{ll} s \leq s_1u_1, & \\ s_1v_1 \leq s_2u_2, & u_1a \leq v_1a_2, \\ s_2v_2 \leq s_3u_3, & u_2a_2 \leq v_2a_3, \\ \cdots & \cdots \\ s_nv_n \leq t, & u_na_n \leq v_na. \end{array}$$

Since $s_1, \cdots, s_n \in (sS \cup tS)$, where s,t are left zero elements, it is easy to show that $s_1, s_2, \cdots, s_n$ are also left zero elements. Thus $s \leq t$ and the result follows. □

Theorem 11. *Let S be a left zero semigroup with 1 adjoined. Then all weakly po-flat left S-posets are I-regular.*

Proof. If $S = \{1\}$, then the result is clear. Now let S be a left zero semigroup with 1 adjoined. Suppose that A is a weakly po-flat left S-poset and $a \in A$. By Lemma 1, we will show that Sa is a projective left S-poset.

Suppose that $sa \neq a$ for any $s \in S \setminus \{1\}$. Define a mapping $f : Sa \to S$ as follows: $f(ta) = t, t \in S$. Suppose $ta = t'a$. If $t, t' \in S \setminus \{1\}$, then $t = t'$ by Lemma 8. If $t \in S \setminus \{1\}$ and $t' = 1$, then $ta = a$, it is a contradiction. If $t' \in S \setminus \{1\}$ and $t = 1$, the result is similar. This means that f is well-defined. It is clear that f is an isomorphism of left S-posets. Thus Sa is projective.

Now suppose that there exists an element $s \in S \setminus \{1\}$ such that $sa = a$. Define a mapping $f : Sa \to Ss$ as following:

$$f(a) = s,$$

$$f(ta) = t, \ t \in S \setminus \{1\}.$$

By Lemma 8, it is easy to see that f is well-defined. Clearly $f : Sa \to Ss$ is an isomorphism of left S-posets. Thus Sa is projective since s is an idempotent of S. □

4. Direct Product of I-Regular S-Posets

In the following, we first investigate I-regularity of $D(S)$. First, we remind the reader of some preliminaries.

First notice that for $a \in A, \rho_a : {}_SS \to {}_SA$ denotes the right translation map defined by $\rho_a(s) = sa$ and $\overrightarrow{ker}\rho_a = \{(s,t) | sa \leq ta\}$. Let ${}_SA$ and ${}_SB$ be left S-posets over a pomonoid S. It is known that ${}_SA$ is I-regular if and only if for every $a \in A$ there exists an idempotent $e \in S$ such that $\overrightarrow{ker}\rho_a = \overrightarrow{ker}\rho_e$. It is also known that a pomonoid S is left PP if and only if for each $s \in S$, $\overrightarrow{ker}\rho_s = \overrightarrow{ker}\rho_e$ for some idempotent $e \in S$. Therefore, each left PP pomonoid as a left S-poset is I-regular. Furthermore, if we denote by $Con_S A$ the set of all congruences on the poset ${}_SA$, the order relation on $Con_S A$ is defined by $\rho \leq \lambda$ if and only if $\rho \cap \lambda = \rho$. Then clearly $(\overrightarrow{Con_S A}, \cap)$ is a pomonoid with identity $({}_SA \times {}_SA)$. It can be routinely verified that for $(a,b) \in A \times B$, $\overrightarrow{ker}\rho_{(a,b)} = \overrightarrow{ker}\rho_a \cap \overrightarrow{ker}\rho_b$.

The next theorem gives a characterization of pomonoids over which $D(S)$ is I-regular.

Theorem 12. *Let S be a pomonoid. The diagonal S-poset $D(S)$ is I-regular if and only if*
(1) *S is a left PP pomonoid.*
(2) *The set $R = \{\overrightarrow{ker}\rho_e | e \in E(S)\} \cup (S \times S)$ is a subpomonoid of $T = (Con_S S, \cap)$.*

Proof. Necessity. Take $s \in S$. Since $D(S)$ is I-regular, $\overrightarrow{ker}\rho_s = \overrightarrow{ker}\rho_{(s,s)} = \overrightarrow{ker}\rho_e$ for some idempotent $e \in S$. Thus S is a left PP pomonoid. On the other hand, by assumption for each pair of idempotents $e, f \in S$, $\overrightarrow{ker}\rho_e \cap \overrightarrow{ker}\rho_f = \overrightarrow{ker}\rho_{(e,f)} = \overrightarrow{ker}\rho_h$ for some idempotent $h \in S$ which complete the proof of necessity.

Sufficiency. Let $(s,t) \in D(S)$ for $s, t \in S$. Since S is a left PP pomonoid, S is I-regular by Lemma 2. Thus there exist idempotents e, f in S such that $\overrightarrow{ker}\rho_s = \overrightarrow{ker}\rho_e$ and $\overrightarrow{ker}\rho_t = \overrightarrow{ker}\rho_f$. Since R is a subpomonoid of T, there exists an idempotent $h \in S$ such that $\overrightarrow{ker}\rho_e \cap \overrightarrow{ker}\rho_f = \overrightarrow{ker}\rho_h$. Now we get $\overrightarrow{ker}\rho_{(s,t)} = \overrightarrow{ker}\rho_s \cap \overrightarrow{ker}\rho_t = \overrightarrow{ker}\rho_e \cap \overrightarrow{ker}\rho_f$
$= \overrightarrow{ker}\rho_h$. Hence $D(S)$ is I-regular. □

Theorem 13. *The following are equivalent for a left PP pomonoid S:*
(1) *Every finite product of I-regular S-posets is I-regular.*
(2) *S^n is I-regular for every $n \in N$.*
(3) *The diagonal S-poset $D(S)$ is I-regular.*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (1). Let ${}_SA$ and ${}_SB$ be two I-regular posets. Take $(a,b) \in A \times B$. Suppose that $\overrightarrow{ker}\rho_a = \overrightarrow{ker}\rho_e$ and $\overrightarrow{ker}\rho_b = \overrightarrow{ker}\rho_f$ for some idempotents $e, f \in S$. By Theorem 12, we have $\overrightarrow{ker}\rho_{(a,b)} = \overrightarrow{ker}\rho_a \cap \overrightarrow{ker}\rho_b = \overrightarrow{ker}\rho_e \cap \overrightarrow{ker}\rho_f = \overrightarrow{ker}\rho_h$ for some idempotent $h \in S$. Now by induction, we obtain the desired result. □

Theorem 14. *Let S be a pomonoid, $E(S)$ the set of idempotents of S. Then the following conditions are equivalent:*
(1) *The diagonal S-poset $D(S)$ is I-regular and $|E(S)| = 1$.*
(2) *S is right po-cancellable.*

Proof. (1)⇒(2). Let $su \leq tu$, for $u, s, t \in S$. Then $(s,t) \in \overrightarrow{ker}\rho_u = \overrightarrow{ker}\rho_{(u,u)}$. Since $D(S)$ is I-regular, there exists $e \in E(S)$ such that $\overrightarrow{ker}\rho_{(u,u)} = \overrightarrow{ker}\rho_e$. But $|E(S)| = 1$, that is $e = 1$, then $s \leq t$. Hence S is right po-cancellable.

(2)⇒(1). If S is right po-cancellable, then $|E(S)| = 1$ and S is a left PP pomonoid. For $(s,t) \in D(S)$, it is clear that $\overrightarrow{ker}\rho_s = \overrightarrow{ker}\rho_1$ and $\overrightarrow{ker}\rho_t = \overrightarrow{ker}\rho_1$. So $\overrightarrow{ker}\rho_{(s,t)} = \overrightarrow{ker}\rho_s \cap \overrightarrow{ker}\rho_t = \overrightarrow{ker}\rho_1 \cap \overrightarrow{ker}\rho_1 = \overrightarrow{ker}\rho_1$. Hence (s,t) is an I-regular element of $D(S)$ and it is I-regular. □

Proposition 3. *For a right collapsible pomonoid S, if $\prod_{i \in I} A_i$ is I-regular, then for every $i \in I$, A_i is I-regular.*

Proof. Let $sa_i \leq ta_i$, for $a_i \in A_i$. Since S is right collapsible, there exists $r \in S$ such that $sr = tr$. Consider the fixed element $a_j \in A_j$, for $j \neq i$, and take

$$d_k = \begin{cases} a_i, & k = i \\ ra_j, & k \neq i. \end{cases}$$

Then $s(d_k)_I \leq t(d_k)_I$. Since $\prod_{i \in I} A_i$ is I-regular, by Lemma 1 there exists $e \in E(S)$ such that $(d_k)_I = e(d_k)_I$, $se \leq te$. So $a_i = ea_i$ and A_i is I-regular. □

Proposition 4. *For a left PP pomonoid S, the following are equivalent:*
(1) *If $\prod_{i \in I} A_i$ is I-regular, then A_i is I-regular.*
(2) *${}_S\Theta$ is I-regular.*
(3) *S has a right zero element.*

Proof. (1) ⇒ (2). Let S be a left PP pomonoid. From Lemma 2 it follows that S is I-regular as a left S-poset. Since $S \cong S \times {}_S\Theta$, we have by assumption ${}_S\Theta$ is I-regular.

(2) ⇒ (3). By Lemma 3, it is obvious.

(3) ⇒ (1). If S has a right zero element, then S is right collapsible. By Proposition 3, the result follows. □

5. Classification of Pomonoids by I-Regularity Property of Right Rees Factor S-Posets

In this section we give a classification of pomonoids by I-regularity property of their right Rees factor S-posets.

Lemma 9. *([10], Lemma 1.8) Let S be a pomonoid and K a convex, proper right ideal of S. The following assertions are equivalent:*
(1) *S/K is free.*
(2) *S/K is projective.*
(3) *S/K is strongly flat.*
(4) *S/K satisfies condition (P).*
(5) $|K| = 1$.

Lemma 10. *([3], Theorem 1) Let S be a pomonoid. Then:*
(1) *Θ_S is free if and only if $|S| = 1$.*
(2) *Θ_S is projective if and only if S has a left zero element.*
(3) *Θ_S satisfies condition (E) if and only if S is left collapsible.*

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(4) *The following assertions are equivalent:*

(a) Θ_S *satisfies condition* (P)*;*

(b) Θ_S *satisfies condition* (P_w)*;*

(c) Θ_S *is po-flat;*

(d) Θ_S *is flat;*

(e) Θ_S *is weakly po-flat;*

(f) Θ_S *is weakly flat;*

(g) S *is weakly right reversible.*

(5) Θ_S *is (always) principally weakly (po-)flat and (po-)torsion free.*

Theorem 15. *Let* S *be a pomonoid and* K *a convex, right ideal of* S*. Then* S/K *is I-regular if and only if* $|K| = 1$ *and* S *is right* PP*, or* $K_S = S$ *and* S *contains a left zero element.*

Proof. Suppose that S/K is I-regular for the convex right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is I-regular and so by Lemma 3, S contains a left zero element.

Case 2. K_S is a convex proper right ideal of S. Since S/K is I-regular, S/K is projective. Thus by Lemma 9, $|K| = 1$ and so $S/K \cong S_S$. Since S/K is I-regular, S_S is I-regular and so by Lemma 2, S is right PP as required.

Conversely, suppose $|K| = 1$ and S is right PP. Then $S/K \cong S_S$ and so by Lemma 2, S/K is I-regular.

If $K_S = S$ and S contains a left zero element, then $S/K \cong \Theta_S$ and by Lemma 3, S/K is I-regular. □

Theorem 16. *Let* S *be a pomonoid. The following statements are equivalent:*

(1) *All projective right Rees factor* S*-posets are I-regular.*

(2) *All free right Rees factor* S*-posets are I-regular.*

(3) *If* S *has a left zero element, then* S *is right* PP*.*

Proof. Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$. Suppose that S contains a left zero element z. If $K_S = zS = \{z\}$, then $S/K_S \cong S_S$ and so S/K_S is free, since S_S is free. Thus by assumption S/K_S is I-regular and by Lemma 2, S is right PP.

$(3) \Rightarrow (1)$. Suppose that S/K is projective for the convex right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is projective and so by Lemma 3, S/K is I-regular.

Case 2. K_S is a convex proper right ideal of S. Since S/K is projective, by Lemma 9 $|K| = 1$ and so $K_S = zS = \{z\}$ for some $z \in S$. Thus z is a left zero element and so by assumption S is right PP. From Lemma 2, it follows that $S/K \cong S_S$ is I-regular. □

Theorem 17. *Let* S *be a pomonoid. The following statements are equivalent:*

(1) *All strongly flat right Rees factor* S*-posets are I-regular.*

(2) *If* S *is left collapsible, then* S *contains a left zero element and* S *is right* PP*.*

Proof. $(1) \Rightarrow (2)$. If S is left collapsible, then by Lemma 10, Θ_S satisfies Condition (E) and so it is strongly flat. Thus by assumption Θ_S is I-regular and so by Lemma 3, S contains a left zero element. Thus S is right PP by Theorem 16.

$(2) \Rightarrow (1)$. Suppose that S/K is strongly flat for the convex right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is strongly flat and so by Lemma 10, S is left collapsible. By assumption S contains a left zero element, thus by Lemma 3, S/K is I-regular.

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Case 2. K_S is a convex proper right ideal of S. Since S/K is strongly flat, by Lemma 9, $|K| = 1$ and so $K_S = zS = \{z\}$ for some $z \in S$. Thus z is a left zero element and so S is left collapsible. By assumption S is right PP. Hence $S/K \cong S_S$ is I-regular by Lemma 2. □

Theorem 18. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All right Rees factor S-posets satisfying Condition* (P) *are I-regular.*
(2) *If S is weakly right reversible, then S contains a left zero element and S is right* PP.

Proof. It is similar to that of Theorem 17. □

Recall from [10], let K be convex, proper right ideal of pomnoid S, S/K is principally weakly flat if, and only if K is left stabilizing. S/K is principally po-flat if, and only if K is strongly left stabilizing.

Theorem 19. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All weakly flat right Rees factor S-posets are I-regular.*
(2) *If S is weakly right reversible, then S contains a left zero element and S is right* PP, *and S has no proper, left stabilizing convex right ideal K with* $|K| > 1$.

Proof. $(1) \Rightarrow (2)$. If S is weakly right reversible, then S contains a left zero element and also S is right PP by Theorem 18. If S has a proper, left stabilizing convex right ideal K with $|K| > 1$, then S/K is weakly flat and by assumption $|K| = 1$, a contradiction is obtained.

$(2) \Rightarrow (1)$. Let K be a convex right ideal of pomonoid S and S/K is weakly flat. If K is convex, proper right ideal of S, by ([10], Lemma 1.7), S is weakly right reversible and K is a proper, left stabilizing convex right ideal of S. By assumption $|K| = 1$ and S is right PP, then $S/K \cong S$ is I-regular. But if $K = S$, $S/K \cong \Theta$ is weakly flat and by Lemma 10, S is weakly right reversible. By assumption S has a left zero element and by Lemma 3, S/K is I-regular. □

The following theorem can be proved by a similar argument of the proof of Theorem 19.

Theorem 20. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All weakly po-flat right Rees factor S-posets are I-regular.*
(2) *If S is weakly right reversible, then S contains a left zero and S is right* PP, *and S has no proper, strongly left stabilizing convex right ideal K with* $|K| > 1$.

Theorem 21. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All principally weakly flat right Rees factor S-posets are I-regular.*
(2) *S has a left zero element and S is right* PP, *and S has no proper, left stabilizing convex right ideal K with* $|K| > 1$.

Proof. $(1) \Rightarrow (2)$. Since Θ is principally weakly flat, by assumption Θ is I-regular. Using Lemma 3 we obtain S has a left zero element and also S is right PP by Theorem 16. Suppose that S has proper, left stabilizing convex right ideal K with $|K| > 1$. Then S/K is principally weakly flat and by assumption $|K| = 1$, a contradiction is obtained.

$(2) \Rightarrow (1)$. Let K be a convex right ideal of pomonoid S and S/K be principally weakly flat. If K is convex, proper right ideal of S, then by ([10], Lemma 1.7), K is a proper, left stabilizing convex right ideal of S. By assumption $|K| = 1$ and S is right PP, hence $S/K \cong S$ is I-regular. But if $K = S$, then $S/K \cong \Theta$ is principally weakly flat. Since S has a left zero element and by Lemma 3, S/K is I-regular. □

Similarly, one can prove the following theorem.

Theorem 22. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All principally weakly po-flat right Rees factor S-posets are I-regular.*

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(2) *S has a left zero element and S is right PP, and S has no proper, strongly left stabilizing convex right ideal K with $|K| > 1$.*

Theorem 23. *Let S be a pomonoid. The following statements are equivalent:*
(1) *All I-regular right Rees factor S-posets are free.*
(2) *If S has a left zero element, then $S = \{1\}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that S has a left zero element. From Lemma 3, it follows that Θ is I-regular. By assumption Θ_S is free, we obtain that $S = \{1\}$ by Lemma 10.

$(2) \Rightarrow (1)$. Suppose that S/K is I-regular for the convex right ideal K of S. Then there are two cases as follows:

Case 1. $K = S$. Then $S/K \cong \Theta$ and so by Lemma 3, S contains a left zero element. Hence by assumption $S = \{1\}$ and so $S/K \cong \Theta$ is free by Lemma 10.

Case 2. K is a proper, convex right ideal of S. Since S/K is projective, by Lemma 9 we have $|K| = 1$. Thus $S/K \cong S_S$ is free, since S_S is free. □

Theorem 24. *Let S be a pomonoid. Then all I-regular right Rees factor S-posets are projective.*

Proof. It follows from Lemma 1. □

Remark 1. *If the order of S is discrete(as an S-act), then by the main results in this paper, we can easily obtain all the characterization of monoids by properties of regular (Rees factor) S-acts.*

Below we tabulate the results.

$\Rightarrow$	I-regualr
free	$\exists$ l.zero $\Rightarrow$ rpp
projective	$\exists$ l.zero $\Rightarrow$ rpp
strongly flat	l.c. $\Rightarrow$ $\exists$ l.zero + rpp
(P)	r.r. $\Rightarrow$ $\exists$ l.zero + rpp
w.f	w.r.r.$\Rightarrow$$\exists$ l.zero + rpp + no proper l.s. convex right ideal K with $\|K\| > 1$
w.po-f.	w.r.r.$\Rightarrow$ $\exists$ l.zero + rpp + no proper s.l.s. concex right ideal K with $\|K\| > 1$
p.w.f.	$\exists$ l.zero + rpp + no proper l.s. convex right ideal K with $\|K\| > 1$
p.w.po-f.	$\exists$ l.zero + rpp + no proper s.l.s. convex right ideal K with $\|K\| > 1$

Abbreviations: l.c.=left collapsible; l.zero=left zero; w.r.r.=weakly right reversible; r.r.=right reversible; rpp=right PP; s.l.s.=strongly left stabilizing; l.s.=left stabilizing.

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