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Abstract: Thermal Doppler broadening of spectral profiles for particle populations in the absence or presence of potential fields are described by kappa distributions. The kappa distribution provides a replacement for the Maxwell-Boltzmann distribution which can be considered as a generalization for describing systems characterized by local correlations among their particles as found in space and astrophysical plasmas. This paper presents all special cases of kappa distributions as members of a general pathway family of densities introduced by Mathai.

Keywords: astrophysical plasmas; kappa distributions; entropy; nonextensive statistical mechanics

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1. Introduction

Kappa distributions are used across the physics of astrophysical plasma processes, describing the velocities and energies of particles from solar wind and planetary magnetospheres to the heliosheath, and beyond to interstellar and intergalactic plasmas [1–3]. From the point of view of physics, the connection of kappa distributions with statistical mechanics and thermodynamics is important. The statistical origin of kappa distributions came from the maximization of Tsallis q-entropy [4]. From the point of view of astrophysics, kappa distributions were introduced in the 1960s by Binsack [5], Olbert [6], and Vasyliunas [7].

A particle is in thermal equilibrium when the exchange of heat or entropy in the system stops. The kappa index controls the exchange of entropy and it is also a measure of departure from the equilibrium state. The particle velocity distribution can be given in terms of a kappa distribution. In the case of a collisional plasma, where no local correlations among particles exist, the system is stabilized into a Maxwell-Boltzmann distribution. The kappa index is inversely proportional to the correlation between the energies of two particles. The d-D kappa distribution describes the particle velocity with dimensionality d. Livadiotis [8] and Livadiotis and McComas [9] take this particle velocity density as the following:

$$f(X; \theta, k_d) = c_d [1 + \frac{1}{k_d - \frac{d}{2}} \frac{(X - \mu)'(X - \mu)}{\theta^2}]^{-(k_d + 1)} \tag{1.1}$$

where X is a $d \times 1$ velocity vector, a prime denotes the transpose and $\mu = E[X] = <X>$ is the expected value of X, which is also a location parameter vector, where $E[(\cdot)]$ denotes the expected value with respect to the density $f(X; \theta, k_d)$, and the normalizing constant c_d is the following:

$$c_d = \frac{\Gamma(k_d + 1)}{\Gamma(k_d - \frac{d}{2} + 1)[\pi(k_d - \frac{d}{2})\theta^2]^{\frac{d}{2}}} \tag{1.2}$$

and $\theta = \sqrt{k_B T/m}$ is the thermal speed of the particle with mass m and temperature T expressed in speed units. If $k_d - \frac{d}{2}$ is invariant over the dimensionality d then let k_o be this constant value. Then, $k_d = k_o + \frac{d}{2}$. Then, the normalizing constant becomes

$$c_d = \frac{\Gamma(k_o + 1 + \frac{d}{2})}{\Gamma(k_o + 1)[\pi k_o \theta^2]^{\frac{d}{2}}} \tag{1.3}$$

and the functional part is

$$[1 + \frac{1}{k_d - \frac{d}{2}} \frac{(X-\mu)'(X-\mu)}{\theta^2}]^{-(k_d+1)} = [1 + \frac{1}{k_o} \frac{(X-\mu)'(X-\mu)}{\theta^2}]^{-(k_o+1+\frac{d}{2})} \tag{1.4}$$

Since $k_d - \frac{d}{2} = k_o$ is invariant $k_3 - \frac{3}{2} = k_d - \frac{d}{2}$, $k_3 = k_d - \frac{(d-3)}{2}$. If θ depends on the kappa index, then θ is of the form θ_k. Two forms of θ_k are usually taken. One is $\theta_k^2 = \frac{(k_3 - \frac{3}{2})}{k_3} \frac{(d-1)}{2} \theta^2$ and the other is $\theta_{k,d}^2 = \frac{k_o}{k_d} \theta^2 = \frac{(k_d - \frac{d}{2})}{k_d} \theta^2$. If θ_k is used then (1.4) becomes

$$[1 + \frac{1}{k_d} \frac{(X-\mu)'(X-\mu)}{\theta_k^2}]^{-(k_d + \frac{1}{2}(d-1))}. \tag{1.5}$$

On the other hand, if $\theta_{k,d}$ is used then (1.4) becomes

$$[1 + \frac{1}{k_d} \frac{(X-\mu)'(X-\mu)}{\theta_{k,d}^2}]^{-(k_d+1)}. \tag{1.6}$$

Thus, one can write the kappa density in (1.1) in a number of different ways. One set of representations will be for $d = 1, 2, 3$, another set will be with θ replaced by θ_k or $\theta_{k,d}$. When $k_o \to \infty$, (1.1) will go to the density

$$f_1(X; \theta, k_d) = c_* e^{-\frac{(X-\mu)'(X-\mu)}{\theta^2}} \tag{1.7}$$

where $c_* = (\pi\theta^2)^{-\frac{d}{2}}$. This density (1.6) is a multivariate real Gaussian and also becomes as $d - D$ Maxwellian distribution.

2. A Pathway Family of Densities

All the forms of the kappa distribution considered in Section 1 are special cases of a general pathway family of densities introduced in Mathai [10]. Let X be a $p \times 1$ real vector random variable with the covariance matrix $\Sigma = \text{Cov}(X) = E[(X - E(X))(X - E(X))']$ where a prime denotes the transpose and $E(\cdot)$ is the expected value of $(\cdot) =< (\cdot) >$ with respect to the density of X. The p components of X may be correlated among themselves. One can get rid of the effect of correlations by taking $Y = \Sigma^{-\frac{1}{2}}(X - \mu)$ where $\Sigma^{\frac{1}{2}}$ is the real positive definite square root of the positive definite matrix $\Sigma > O$. Then the covariance matrix in Y is the identity matrix, free of all correlations. The Euclidean distance of Y from the origin is also $(X - \mu)'\Sigma^{-1}(X - \mu)$, $\mu = E(X)$. Also, $(X - \mu)'\Sigma^{-1}(X - \mu) =$ a positive constant is an offset ellipsoid and it is also known in statistical literature as the ellipsoid of concentration. If the components of X are to be simply weighted, rather than removing the effect of correlations, then one can replace Σ^{-1} in the quadratic form by a real positive definite matrix $A > O$ and then we may consider $(X - \mu)'A(X - \mu)$. Consider the following density:

$$g(X)dX = c[(X-\mu)'\Sigma^{-1}(X-\mu)]^\gamma [1 + a((X-\mu)'\Sigma^{-1}(X-\mu))^\delta]^{-\eta} dX \tag{2.1}$$

for $a > 0, \delta > 0, \eta > 0, \gamma > 0$, all real scalar parameters, and c is the normalizing constant, which can be evaluated as

$$c = \frac{\delta\Gamma(\frac{p}{\delta})\Gamma(\eta)a^{\frac{\gamma}{\delta} + \frac{p}{2\delta}}}{|\Sigma|^{\frac{1}{2}}\pi^{\frac{p}{2}}\Gamma(\frac{\gamma}{\delta} + \frac{p}{2\delta})\Gamma(\eta - \frac{\gamma}{\delta} - \frac{p}{2\delta})}, \eta - \frac{\gamma}{\delta} - \frac{p}{2\delta} > 0 \tag{2.2}$$

see the derivation of c in the Appendix. Consider $\delta = 1, \Sigma = I$ (identity matrix), $a = \frac{1}{\theta^2(k_d - \frac{d}{2})}, \eta = k_d + 1, \gamma = 0$. Then (2.1) reduces to (1.1), the density given by Livadiotis (2018). The general representation

in (2.1) and (2.2) has many advantages. This (2.1) is a member of the pathway family of distributions defined in Mathai [10]. Let $a = \frac{b}{\alpha - a_o}, b > 0, \alpha > a_o$ and $\eta = \rho(\alpha - a_o), \rho > 0$. Then, (2.1) becomes

$$g_1(X)\mathrm{d}X = c_1[(X - \mu)'\Sigma^{-1}(X - \mu)]^\gamma[1 + \frac{b}{\alpha - a_o}((X - \mu)'\Sigma^{-1}(X - \mu))^\delta]^{-\rho(\alpha - a_0)}\mathrm{d}X. \qquad (2.3)$$

Note that when $\alpha \to a_o$ from the right, $g_1(X)$ goes to the density

$$g_2(X)\mathrm{d}X = c_2[(X - \mu)'\Sigma^{-1}(X - \mu)]^\gamma \mathrm{e}^{-b\rho[(X-\mu)'\Sigma^{-1}(X-\mu)]^\delta}\mathrm{d}X \qquad (2.4)$$

where

$$c_2 = \frac{\delta\Gamma(\frac{p}{2})(b\rho)^{\frac{\gamma}{\delta} + \frac{p}{2\delta}}}{\pi^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}\Gamma(\frac{\gamma}{\delta} + \frac{p}{2\delta})}, \delta > 0, b > 0, \rho > 0. \qquad (2.5)$$

The density in (2.4) can be taken as a power-transformed real multivariate Maxwell-Boltzmann density. For $\delta = 1$, (2.4) is a form of multivariate Maxwell-Boltzmann density. Hence, if (2.4) is the stable density in a physical system, then the unstable neighborhood and transitional stages are determined by (2.3) for various values of the pathway parameter α. We may also note that for $\alpha = k_d, a_o = \frac{d}{2}, b = \frac{1}{\theta^2}, \delta = 1, \gamma = 0, \rho = \frac{k_d+1}{k_d - \frac{d}{2}}$ one has (1.1) also. This density in (2.3) has another advantage. For $\alpha < a_o$ we can write $\alpha - a_o = -(a_o - \alpha), a_o - \alpha > 0$ so that the density in (2.3) switches into a type-1 beta form of the density given by

$$g_3(X)\mathrm{d}X = c_3[(X - \mu)'\Sigma^{-1}(X - \mu)]^\gamma[1 - \frac{b}{a_o - \alpha}((X - \mu)'\Sigma^{-1}(X_\mu))^\delta]^{\rho(a_o - \alpha)}\mathrm{d}X, \alpha < a_o \qquad (2.6)$$

where $[1 - \frac{b}{a_o - \alpha}((X - \mu)'\Sigma^{-1}(X - \mu))^\delta] > 0$ so that the density is defined within the ellipsoid

$$(X - \mu)'\Sigma^{-1}(X - \mu) = (\frac{a_o - \alpha}{b})^{\frac{1}{\delta}} \text{ for } \alpha < a_o, b > 0, \delta > 0$$

and

$$c_3 = \frac{\delta\Gamma(\frac{p}{\delta})\Gamma(1 + \rho(a_o - \alpha) + \frac{\gamma}{\delta} + \frac{p}{2\delta})(\frac{b}{a_o - \alpha})^{\frac{\gamma}{\delta} + \frac{p}{2\delta}}}{|\Sigma|^{\frac{1}{2}}\pi^{\frac{p}{2}}\Gamma(\frac{\gamma}{\delta} + \frac{p}{2\delta})\Gamma(1 + \rho(a_o - \alpha))}, \alpha < a_o. \qquad (2.7)$$

This normalizing constant is evaluated by going through the procedure in Appendix A1 and the final integral is evaluated by using a type-1 beta integral. Thus, g_1, g_2, g_3 belong to the pathway family of densities. If the power-transformed Maxwell-Boltzmann density in (2.4) is the stable density in a physical system then the unstable neighborhoods and the transitional stages are given by g_1 in (2.3) and g_3 in (2.6). One can switch among a generalized type-1 beta family, a type-2 beta family and a gamma family or Maxwell-Boltzmann family of distributions through the pathway parameter α. In the model in (2.3) one can identify α with k_d and a_o with $\frac{d}{2}$ if convenient.

3. Livadiotis' $d - D$ Density Through an Entropy Optimization

Let X be a $p \times 1$ real vector random variable. Let $f(X)$ be a density function, that is, $f(X) \geq 0$ for all X and $\int_X f(X)\mathrm{d}X = 1$ where $f(X)$ is a real-valued scalar function of X. Consider Mathai's entropy for the vector random variable, namely,

$$M_\alpha(f) = \frac{\int_X [f(X)]^{1 + \frac{a_o - \alpha}{\eta}}\mathrm{d}X - 1}{\alpha - a_o} \qquad (3.1)$$

for a_o a fixed quantity or anchoring point, α is the pathway parameter and the deviation of α from a_o is measured in $\eta > 0$ units. Then, when $\alpha \to a_o$, we can see that (3.1) reduces to Shannon's entropy $S(f) = -K \int_X f(X) \ln f(X)\mathrm{d}X$ where K is a constant. Shannon's entropy is for the scalar variable case and the corresponding real vector-variate form is denoted here as $S(f)$. Let $\Sigma > O$ be the covariance

matrix of X. Consider the ellipsoid of concentration $(X - \mu)'\Sigma^{-1}(X_\mu) = $ a positive constant, where μ is a $p \times 1$ location parameter vector. We will set moment-type constraints on the ellipsoid of concentration and then optimize (3.1). Consider the following constraints:

$$E[(X-\mu)'\Sigma^{-1}(X-\mu)]^{\gamma\left(\frac{a_o-\alpha}{\eta}\right)} = \text{ fixed}, E[(X-\mu)'\Sigma^{-1}(X-\mu)]^{\gamma\left(\frac{a_o-\alpha}{\eta}\right)+\delta} = \text{ fixed} \qquad (3.2)$$

for some parameters $\gamma > 0, \eta > 0, \delta > 0$. If we use calculus of variation for the optimization, then the Euler equation becomes the following where λ_1 and λ_2 are Lagrangian multipliers:

$$\frac{\partial}{\partial f}[f^{1+\frac{a_o-\alpha}{\eta}} - \lambda_1[(X-\mu)'\Sigma^{-1}(X-\mu)]^{\gamma\left(\frac{a_o-\alpha}{\eta}\right)}f + \lambda_2[(X-\mu)'\Sigma^{-1}(X-\mu)]^{\gamma\left(\frac{a_o-\alpha}{\eta}\right)+\delta}f] = 0.$$

This gives the solution for f as the following:

$$f = \nu_1[(X-\mu)'\Sigma^{-1}(X-\mu)]^\gamma[1 - \frac{\lambda_2}{\lambda_1}[(X-\mu)'\Sigma^{-1}(X-\mu)]^\delta]^{\frac{\eta}{a_o-\alpha}} \qquad (3.3)$$

where $\nu_1, \lambda_1, \lambda_2$ are constants. Let $\frac{\lambda_2}{\lambda_1} = b(a_o - \alpha), b > 0$ and let ν_1 be the normalizing constant to make (3.3) a density. Then, for $u = (X - \mu)'\Sigma^{-1}(X - \mu)$ we have the following densities from (3.3):

$$f_1(X) = \nu_1 u^\gamma[1 - b(a_o - \alpha)u^\delta]^{\frac{\eta}{a_o-\alpha}}, \alpha < a_o \qquad (3.4)$$

$$f_2(X) = \nu_2 u^\gamma[1 + b(\alpha - a_o)u^\delta]^{-\frac{\eta}{\alpha-a_o}}, \alpha > a_o \qquad (3.5)$$

$$f_3(X) = \nu_3 u^\gamma e^{-b\eta u^\delta}, \alpha \to a_o \qquad (3.6)$$

where in (3.4) we need an additional condition $1 - b(a_o - \alpha)u^\delta > 0$ in order to make (3.4) a density. Note that in the limiting form we have the following properties:

$$e^{-b\eta u^\delta} = \lim_{\alpha \to a_o}[1 - b(a_o - \alpha)u^\delta]^{\frac{\eta}{a_o-\alpha}} = \lim_{\alpha \to a_o}[1 + b(\alpha - a_o)u^\delta]^{-\frac{\eta}{\alpha-a_o}}$$

$$= \lim_{\alpha \to a_o}[1 - \frac{b}{a_o - \alpha}u^\delta]^{\eta(a_o-\alpha)} = \lim_{\alpha \to a_o}[1 + \frac{b}{\alpha - a_o}u^\delta]^{-\eta(\alpha-a_o)} \qquad (3.7)$$

Hence, one can take any one of the formats in (3.7) in the limiting case. Livadiotis' density in (1.1) is available from (3.5) by taking $b(\alpha - a_o) = \frac{1}{\theta(k_d - \frac{d}{2})}$ and $\frac{\eta}{\alpha - a_o} = k_d + 1$.

For $\gamma = 0, \delta = 1, \eta = 1, a_o = 1, b = 1$, equations (3.4),(3.5)(3.6) give a real multivariate version of Tsallis statistics of non-extensive statistical mechanics [4]. For $\delta = 1, \eta = 1, a_o = 1, b = 1, \delta = 1$, (3.5) and (3.6) can be taken as a multivariate extension of superstatistics of Beck and Cohen [12]. Matrix-variate versions of Tsallis statistics [4] and Beck and Cohen superstatistics [12] can also be defined.

4. A Matrix-Variate Generalization of Livadiotis' Density

Let $Y = (y_{ij})$ be a $p \times q, p \leq q$ and of rank p matrix with distinct real scalar variables y_{ij}'s as elements. Let $g(Y)$ be a real-valued scalar function of Y such that $g(Y) \geq 0$ for all Y and $\int_Y g(Y)dY = 1$ so that $g(Y)$ is a density function where $dY = \wedge_{i=1}^p \wedge_{j=1}^q dy_{ij} = $ the wedge product of all distinct differentials in Y. Let M be a $p \times q, p \leq q$ parameter matrix. Let $A > O$ be $p \times p$ and $B > O$ be $q \times q$ positive definite constant matrices. Let $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ be the positive definite square roots of A and B respectively. Let

$$U = A^{\frac{1}{2}}(Y - M)B^{\frac{1}{2}}, V = UU' = A^{\frac{1}{2}}(Y - M)B(Y - M)'A^{\frac{1}{2}} \qquad (4.1)$$

Then, the determinant of V, namely $|V|$, can be interpreted in different ways. $|V|$ is the product of the eigenvalues of the matrix V. Let $U_1, ..., U_p$ be the p linearly independent rows of U, the $p \times q, p \leq q$ and of rank p matrix U. Then, $U_j, j = 1, ..., p$ can be taken as p linearly independent points in a

q-dimensional Euclidean space with $p \leq q$. Then, $|V|^{\frac{1}{2}} = |UU'|^{\frac{1}{2}} =$ the volume of the p-parallelotope generated in the convex hull of the p linearly independent points, taken in the order. Thus, $|V|$ is also the square of the volume content of this parallelotope. Consider Mathis's entropy in (3.1) with $f(X)$ replaced by $g(Y)$ where Y is now $p \times q, p \leq q$ matrix of rank p. Consider the optimization of (3.1) with the real-valued scalar function $g(Y)$ under the following constraints:

$$E[|V|^{\gamma\left(\frac{a_o - \alpha}{\eta}\right)} = \text{ fixed}, E[|V|^{\gamma\left(\frac{a_o - \alpha}{\eta}\right)}[\text{tr}(V)]^\delta = \text{ fixed} \tag{4.2}$$

Then, proceeding as in Section 3, we can end up with the following pathway densities:

$$g_1(Y) = C_1 |V|^\gamma [1 - b(a_o - \alpha)(\text{tr}(V))]^{\frac{\eta}{a_o - \alpha}}, \alpha < a_o \tag{4.3}$$

$$g_{+}2(Y) = C_2 |V|^\gamma [1 + b(\alpha - a_o)(\text{tr}(V))]^{-\frac{\eta}{\alpha - a_o}}, \alpha > a_o \tag{4.4}$$

$$g_3(Y) = |V|^\gamma e^{-b\eta \text{tr}(V)} \tag{4.5}$$

where C_1, C_2, C_3 are the normalizing constants and in (4.3) the additional condition needed is $1 - b(a_o - \alpha)\text{tr}(V) > 0$ to make (4.3) a density, where

$$V = A^{\frac{1}{2}}(Y - M)B(Y - M)'A^{\frac{1}{2}}. \tag{4.6}$$

The normalizing constants are evaluated in Appendix A2 by using the following steps:

$$U = A^{\frac{1}{2}}(Y - M)B^{\frac{1}{2}} \Rightarrow dU = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}dY$$

and

$$V = UU' \Rightarrow dU = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p\left(\frac{q}{2}\right)}|V|^{\frac{q}{2} - \frac{p+1}{2}}dV \tag{i}$$

where $\Gamma_p(\alpha)$ is a real matrix-variate gamma function, associated with a real matrix-variate gamma integral, and it is the following:

$$\Gamma_p(\alpha) = \int_{S > O} |S|^{\alpha - \frac{p+1}{2}} e^{-\text{tr}(S)} dS, \Re(\alpha) > \frac{p-1}{2} \tag{ii}$$

$$= \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2} \tag{iii}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Note that (4.5) is a real rectangular matrix-variate Maxwell-Boltzmann density.

Note 4.1. Results parallel to all the results in Sections 3 and 4 are also available in the complex domain. Corresponding physics can also be dealt with in the complex domain. A real rectangular matrix-variate version of Tsallis statistics [4] is available from (4.3)-(4.5) for $\gamma + \frac{q}{2} = \frac{p+1}{2}, b = 1, \alpha = 1, a_o = 1, \eta = 1$. Corresponding results in the complex domain can also be worked out. For $b = 1, \alpha = 1, \eta = 1$, (4.4) and (4.5) give a real rectangular matrix-variate version of superstatistics [12]. Corresponding versions in the complex domain can also be worked out. Such rectangular or square matrix-variate versions may not be available in the literature.

5. Conclusion

The manuscript introduces a single pathway density that brings together all vector- and matrix-variate kappa distributions, including the Livadiotis's plasma models, Tsallis statistics, and their Maxwell-Boltzmann limit, by varying one parameter alpha. It is shown how each case follows from an entropy maximization principle. This unified approach based on Mathai's pathway covers type-2 beta (kappa), type-1 beta, and gamma/Maxwell-Boltzmann cases. All normalizing constants come from detailed multivariate integrals.

Appendix A. Derivation of the Normalizing Constant in (2.2)

The appendix is an optional section that can contain details and data supplemental to the main text—for example, explanations of experimental details that would disrupt the flow of the main text but nonetheless remain crucial to understanding and reproducing the research shown; figures of replicates for experiments of which representative data are shown in the main text can be added here if brief, or as Supplementary Data. Mathematical proofs of results not central to the paper can be added as an appendix.

Let X be a $p \times 1$ vector random variable and $\mu = E[X], \Sigma = \text{Cov}(X)$.

$$\frac{1}{c} = \int_X [(X-\mu)'\Sigma^{-1}(X-\mu)]^\gamma [1 + a((X-\mu)'\Sigma^{-1}(X-\mu))^\delta]^{-\eta} dX.$$

Consider the transformation

$$X = \Sigma^{-\frac{1}{2}}(X-\mu) \Rightarrow dZ = |\Sigma|^{-\frac{1}{2}} dX$$

see Mathai [11]. Let $u = ZZ'$. Then writing Z uniquely as a product of a unique lower triangular matrix and a unique semi-orthonormal matrix and then integrating out the differential element over a Stiefel manifold, we have a relation between du and dZ, that is

$$dZ = \frac{\pi^{\frac{q}{2}}}{\Gamma(\frac{p}{2})} u^{\frac{p}{2}-1} du, u > 0 \qquad (A_1)$$

see Mathai [11] for the details. Observe that u is real scalar whereas Z is a $p \times 1$ vector. Then

$$\int_X [(X-\mu)'\Sigma^{-1}(X-\mu)]^\gamma [1 + a((X-\mu)'\Sigma^{-1}(X-\mu))^\delta]^{-\eta} dX$$

$$= \frac{|\Sigma|^{\frac{1}{2}} \pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \int_{u=0}^\infty u^{\gamma + \frac{p}{2} - 1} (1 + au^\delta)^{-\eta} du.$$

Now, let $v = u^\delta, u > 0, \delta > 0 \Rightarrow du = \frac{1}{\delta} v^{\frac{1}{\delta} - 1} dv$. Integrating out by using a type-2 beta integral we have

$$\frac{1}{c} = \frac{|\Sigma|^{\frac{1}{2}} \pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \frac{\Gamma(\frac{\gamma}{\delta} + \frac{p}{2\delta})\Gamma(\eta - \frac{\gamma}{\delta} - \frac{p}{2\delta})}{\delta a^{\frac{\gamma}{\delta} + \frac{p}{2\delta}} \Gamma(\eta)}, \eta - \frac{\gamma}{\delta} - \frac{p}{2\delta} > 0 \qquad (A_2)$$

which gives the normalizing constant.

Appendix B. Derivation of the Normalizing Constant C in (4.4)

Let Y be $p \times q, p \le q$ and of rank p matrix of real scalar random variables as elements.

$$\frac{1}{C} = \int_Y [A^{\frac{1}{2}}(Y-M)B(Y-M)'A^{\frac{1}{2}}]^\gamma [1 + b(\alpha - a_o)\text{tr}((A^{\frac{1}{2}}(Y-M)B(Y-M)'A^{\frac{1}{2}}))]^{-\frac{\eta}{\alpha - a_o}} dY,$$

for $\alpha > a_o$. Let $U = A^{\frac{1}{2}}(Y-M)B^{\frac{1}{2}} \Rightarrow dU = |A|^{\frac{q}{2}} |B|^{\frac{p}{2}} dY$ where Y is $p \times q, p \le q$ and of rank p, M is a $p \times q, p \le q$ parameter matrix, $A > O, B > O$ are $p \times p$ and $q \times q$ constant positive definite matrices,

see Mathai [11] for the Jacobian in this transformation. Let $V = UU' \Rightarrow dU = \frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}|V|^{\frac{q}{2}-\frac{p+1}{2}}dV$, see Mathai [11] for details. Then

$$\frac{1}{C} = |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}\int_V |V|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}$$
$$\times [1 + b(\alpha - a)\text{tr}(V)]^{-\frac{\eta}{\alpha-a_o}}dV, \alpha > a_o$$

We can replace one factor by an equivalent integral, that is,

$$[1 + b(\alpha - a_o)\text{tr}(V)]^{-\frac{\eta}{\alpha-a_o}} \equiv \frac{1}{\Gamma(\frac{\eta}{\alpha-a_o})}\int_{z=0}^{\infty} z^{\frac{\eta}{\alpha-a_o}-1}e^{-z(1+b(\alpha-a_o)\text{tr}(V))}dz.$$

$$\frac{1}{C} = |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}e^{-z}[\int_V |V|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}e^{-z(\alpha-a_o)\text{tr}(V)}]dV$$

Evaluate the V-integral by using a real matrix-variate gamma integral. We have

$$\frac{1}{C} = |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}[b(\alpha-a_o)]^{-p(\gamma+\frac{q}{2})}\Gamma_p(\gamma+\frac{q}{2})$$
$$\times \int_{z=0}^{\infty} z^{\frac{\eta}{\alpha-a_o}-p(\gamma+\frac{q}{2})-1}e^{-z}dz$$
$$= |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{pq}{2}}}{\Gamma_p(\frac{q}{2})}\Gamma_p(\gamma+\frac{q}{2})[b(\alpha-a_o)]^{-p(\gamma+\frac{q}{2})}\Gamma(\frac{\eta}{\alpha-a_o}-p(\gamma+\frac{q}{2}))$$

for $\frac{\eta}{\alpha-a_o} - p(\gamma+\frac{q}{2}) > 0$. This completes the computations.

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