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Composable Conditions for Constructing Knowledge Structure Based on Variable Precision *FT*-Rough Set Model

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Abstract: Constructing a knowledge structure using the variable precision *FT*-rough set model is an effective approach. Since directly constructing a knowledge structure for a subject or field is challenging, synthesizing global information from local information becomes a viable solution. However, local information often overlaps (partially), making it crucial to ensure consistency between global and local information, which is an urgent issue to address. Therefore, based on the variable precision *FT*-rough set model and the knowledge structure constructed from it, this paper investigates the conditions for the composability of knowledge structures constructed using the lower (upper) inverse operator of the variable precision *FT*-rough set. Under these conditions, the knowledge structure constructed from the local fuzzy approximation space can be integrated into the knowledge structure constructed from the global fuzzy approximation space.

Keywords: knowledge space theory; variable precision *FT*-rough set; fuzzy approximation space; knowledge structure; composability condition

1. Introduction

Knowledge Space Theory (KST) [1] is a mathematical theoretical framework grounded in pedagogy and psychology, providing an effective methodology for studying educational principles and enabling scientific educational assessment and learning guidance. KST has since evolved into various research branches, including competency-based knowledge space theory [2,3] and polytomous knowledge space theory [4,5]. A systematic overview of KST's theoretical advances and applications was provided by Li et al. [6]. KST has been successfully applied in fields such as assisted learning and adaptive testing. For instance, the learning platform ALEKS (Assessment and Learning in Knowledge Spaces) [7] is developed based on this theory.

Construction of knowledge structures is one of the key research topics in KST. A knowledge structure encompasses all possible knowledge states, where each knowledge state represents a subset of problems in a specific domain or subject. Thus these knowledge states reflect an individual's cognitive level in the corresponding field. Doignon et al. [2] introduced skill maps and skill multimaps, establishing relationships between problems and skills, and proposed methods for constructing knowledge structures based on these maps. In recent years, integrating Rough Set Theory (RST) [8] and Fuzzy Set Theory (FST) [9] into knowledge spaces has emerged as a significant research direction. Notable contributions include: Yao et al. [10] pioneered the introduction of rough set approximation concepts into KST to construct knowledge structures. Liu [11] established

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connections between rough set-based upper and lower approximation operators and skill maps and skill multimaps, proposing new methods for knowledge structure construction. Sun et al. [12] incorporated Fuzzy Set Theory into KST, proposing a novel theoretical framework for constructing knowledge structures using fuzzy skill maps and fuzzy skill multimaps. Xu et al. [13] further advanced this direction by introducing variable precision models based on fuzzy skill maps and variable precision competency models based on fuzzy skill multimaps.

In the study of rough sets, several distinct rough set models have been developed, including: the variable precision *T*-rough set proposed by Zhu et al. [14]; the double-universe *T*-rough fuzzy set model under general fuzzy binary relations introduced by Thao et al. [15]; the fuzzy *T*-rough set (*FT*-rough set) investigated by Zhang et al. [16], who examined the properties of its upper and lower inverse operators under union and intersection operations. The *FT*-rough set model offers the advantage of handling continuous data while preserving data integrity. However, the conditions for knowledge state induction by its upper and lower inverse operators can be either too strict or too lenient. In contrast, the variable precision rough set model permits a certain misclassification rate during the classification process. Therefore, employing the upper and lower inverse operators of variable precision *FT*-rough sets to induce knowledge states allows for threshold adjustment of the required skill mastery ratio when solving problems. This approach effectively mitigates the issues of overly strict or lenient conditions mentioned above.

When constructing a knowledge structure for a specific domain or subject, it is necessary to first define a problem domain Q. One or more experts are then required to identify all valid knowledge states. However, since the number of possible subsets (knowledge states) grows exponentially with the size of Q ($|Q|$), and $|Q|$ is typically large in practical applications, it becomes clearly infeasible to have experts directly determine all possible states for the entire problem domain [1]. A viable alternative approach involves constructing large knowledge structures by combining smaller ones. Specifically, this method decomposes Q into sufficiently small subdomains that can be completely covered, with different experts independently identifying and assigning relevant problem-solving skills to each subdomain. However, inconsistencies may arise in skill assignments among different experts, leading to two key challenges: How to ensure global consistency when aggregating local information? Is it possible to properly distribute global information to local contexts? To address these issues, Heller et al. [17] introduced the concept of distributed skill functions, which can represent the integration of a finite number of skill functions. This approach enables the synthesis of local information provided by multiple experts into a globally consistent framework, leveraging the composability property of knowledge structures to resolve the aforementioned problems.

This paper first introduces relevant basic concepts, then addresses the aforementioned problems by investigating the composability conditions of knowledge structures constructed based on the upper (lower) inverse operators of variable precision *FT*-rough sets.

2. Preliminaries

In this section, the core concepts of fuzzy sets, *FT*-rough sets, and KST are briefly reviewed to establish the theoretical framework.

The following is a brief overview of fuzzy sets, which proposed by Zadeh [9] in 1965.

A fuzzy set over the universe S is a map from S to the real interval $[0,1]$, i.e., $Y:S\to[0,1]$. $\forall s\in S$, $Y(s)\in[0,1]$ denotes the membership grade of s with respect to Y. Generally, the family of all fuzzy sets on S is denoted as: $\mathcal{F}(S)=\{Y\mid Y:S\to[0,1]\}$. For convenience, we denote $Y:S\to[0,1]$ by $Y=\{(s,Y(s)):s\in S\}$. Additionally, we suppress $(s,Y(s))$ if $Y(s)=0$ for any $s\in S$. The definitions of equality relations, inclusion relations, and the operations of union, and intersection on $\mathcal{F}(S)$ are as follows [9,12]:

$Y_1=Y_2 \Leftrightarrow Y_1(s)=Y_2(s)$, $\forall s\in S$;

$Y_1\subseteq Y_2 \Leftrightarrow Y_1(s)\le Y_2(s)$, $\forall s\in S$;

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$(Y_1 \bigcup Y_2)(s) \Leftrightarrow Y_1(s) \vee Y_2(s)$, $\forall s \in S$;

$(Y_1 \bigcap Y_2)(s) \Leftrightarrow Y_1(s) \wedge Y_2(s)$, $\forall s \in S$.

Let Q and S be two nonempty finite sets, if for each $q \in Q$, there corresponds a nonempty fuzzy set on S, then $T: Q \to \mathcal{F}(S) \setminus \{\varnothing\}$ is called a fuzzy set-valued map [9], where $T(q) \in \mathcal{F}(S) \setminus \{\varnothing\}$. For convenience, we denote $T(q)$ by T_q. And we call the triple (Q,S,T) a fuzzy approximation space.

Based on the fuzzy set-valued map, and for $Y \in \mathcal{F}(S)$, the upper inverse($T^{+}(Y)$) and lower inverse($T^{-1}(Y)$) approximations of Y are respectively defined as:

$T^{+}(Y) = \{q \in Q \mid \forall s \in S, T_q(s) \le Y(s)\}$,

$T^{-1}(Y) = \{q \in Q \mid \exists s \in S, 0 < T_q(s) \le Y(s)\}$.

Then the ordered pair $(T^{+}(Y), T^{-1}(Y))$ is called an *FT*-rough set.

Knowledge states, knowledge structures, etc. are fundamental concepts in KST. Let Q be a nonempty finite set of items and $\mathcal{K}$ be a family of subsets of Q, then a knowledge structure is the pair $(Q,\mathcal{K})$, where $\mathcal{K}$ contains at least the empty set $\varnothing$ and Q. And each element K in $\mathcal{K}$ is referred to a knowledge state. If for any $M, N \in \mathcal{K}$, their union $M \bigcup N$ is also a knowledge state of $\mathcal{K}$, then $(Q,\mathcal{K})$ is called a knowledge space. If for any $M, N \in \mathcal{K}$, their intersection $M \bigcap N$ is also a knowledge state of $\mathcal{K}$, then $(Q,\mathcal{K})$ is called a simple closure space. And if $(Q,\mathcal{K})$ is both a knowledge space and a simple closure space, then it is called a quasi-ordinal space.

Let $(Q,\mathcal{K})$ be a knowledge structure, for $Q' \subset Q$ and $Q' \ne \varnothing$, the projection (or trace) of $\mathcal{K}$ on Q' is defined as $\mathcal{K}|_{Q'} = \{K \bigcap Q' \mid K \in \mathcal{K}\}$. Here $\mathcal{K}|_{Q'}$ is a substructure of $\mathcal{K}$, and $\mathcal{K}$ is the parent structure of $\mathcal{K}|_{Q'}$ [1]. Based on this, the composition of knowledge structures is defined as follows.

Definition 1 [1]. Let $(Q_i,\mathcal{K}_i)$, $i \in I$ and $(Q,\mathcal{K})$ be knowledge structures. $(Q,\mathcal{K})$ is said to be a composition of the family of knowledge structures $\{(Q_i,\mathcal{K}_i) \mid i \in I\}$, if the following conditions are satisfied:

(1) $Q = \bigcup_{i \in I} Q_i$;

(2) $\forall i \in I, \mathcal{K}_i = \mathcal{K}|_{Q_i}$.

Definition 2 [17]. Let (Q_i, S_i, T_i), $i \in I$ be fuzzy approximation spaces, their composition (Q,S,T) is defined by:

(1) $Q = \bigcup_{i \in I} Q_i$;

(2) $S = \bigcup_{i \in I} S_i$;

(3) $\forall q \in Q$, $T_q = \bigcup_{i \in I} (T_i^{*})_q$, where $(T_i^{*})_q = \begin{cases} (T_i)_q, & q \in Q_i \\ \varnothing, & q \notin Q_i \end{cases}$.

It can be easily proved that (Q,S,T) is also a fuzzy approximation space.

In [12], the concepts from fuzzy sets theory are extended to knowledge spaces, leading to the definition of a fuzzy skill map. Let (Q,S,T) be a fuzzy approximation space, and regarding Q as a nonempty finite set of items(or questions) and S as a nonempty finite set of skills. Then T represents a map from the item(or questions) set Q to the fuzzy skill set family $\mathcal{F}(S) \setminus \{\varnothing\}$. In KST, it means that to solve a question, an individual should first master the skills related to solving the question to a certain extent, i.e. $T_q \ne \varnothing$. The triple (Q,S,T) is also referred to as fuzzy skill map in KST.

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3. Variable Precision *FT*-Rough Sets and the Knowledge Structure Con-tructed

This section introduces the variable precision *FT*-rough set model and the properties of its operators, and proposes a knowledge structure constructed based on this model.

Definition 3. Let (Q,S,T) be a fuzzy approximation space. For any $B\in\mathcal{F}(S)$, $\beta\in(0.5,1]$, the β-lower inverse and β-upper inverse of B with respect to T are defined as

$$T_\beta^{-1}(B)=\{q\in Q\mid\frac{|\{s\in S\mid 0<T_q(s)\le B(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta\},$$

$$T_\beta^{+}(B)=\{q\in Q\mid\frac{|\{s\in S\mid 0<T_q(s)\le B(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}\ge\beta\}.$$

Then the ordered pair $(T_\beta^{+}(B),T_\beta^{-1}(B))$ is called a β-variable precision *FT*-rough set.

The upper and lower inverse operators of the variable precision *FT*-rough sets possess the following properties:

Proposition 1. Let (Q,S,T) be a fuzzy approximation space. For $A,B\in\mathcal{F}(S)$, $\beta\in(0.5,1]$, then

(1) $T_\beta^{-1}(\varnothing)=T_\beta^{+}(\varnothing)=\varnothing$;

(2) $T_\beta^{-1}(S)=T_\beta^{+}(S)=Q$;

(3) $T_\beta^{+}(B)\subseteq T_\beta^{-1}(B)$;

(4) If $A\subseteq B$, then $T_\beta^{+}(A)\subseteq T_\beta^{+}(B)$, $T_\beta^{-1}(A)\subseteq T_\beta^{-1}(B)$;

(5) $T_\beta^{+}(A\cap B)\subseteq T_\beta^{+}(A)\cap T_\beta^{+}(B)$;

(6) $T_\beta^{+}(A)\cup T_\beta^{+}(B)\subseteq T_\beta^{+}(A\cup B)$;

(7) $T_\beta^{-1}(A)\cup T_\beta^{-1}(B)\subseteq T_\beta^{-1}(A\cup B)$;

(8) $T_\beta^{-1}(A\cap B)\subseteq T_\beta^{-1}(A)\cap T_\beta^{-1}(B)$;

(9) If $0.5<\beta_1<\beta_2\le 1$, then $T_{\beta_2}^{+}(B)\subseteq T_{\beta_1}^{+}(B)\subseteq T_{\beta_1}^{-1}(B)\subseteq T_{\beta_2}^{-1}(B)$.

The proof of Proposition 1 is similar to the proof of the properties of *FT*-rough sets in [16], and it is not listed here.

From Definition 3, it follows that when the threshold β takes the value of 1, the variable precision -rough sets will degenerate into the *FT*-rough sets. Therefore, the variable precision *FT*-rough set has wider applicability than the *FT*-rough sets.

As shown in Proposition 1, the upper and lower inverse operators of variable precision *FT*-rough sets can induce knowledge states and derive knowledge structures.

Definition 4 [13]. Let (Q,S,T) be a fuzzy approximation space. For any $q\in Q$, $B\in\mathcal{F}(S)$, we call

$$D(B/T_q)=\frac{|\{s\in S\mid 0<T_q(s)\le B(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}$$

a fuzzy skill inclusion degree with respect to T_q and B.

Definition 5. Given a fuzzy approximation space (Q,S,T), the knowledge state delineated by the lower inverse model of the variable precision *FT*-rough set with respect to B ($B\in\mathcal{F}(S)$) is defined by

$$K_{B,\beta}^{-}=T_\beta^{-1}(B)=\{q\in Q\mid D(B/T_q)>1-\beta\},$$

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where $\beta \in (0.5,1]$.

The family of all the knowledge states delineated by the lower inverse model is denoted by $\mathcal{K}_\beta^- = \{K_{B,\beta}^- \mid B \in \mathcal{F}(S)\}$.

Definition 6. Given a fuzzy approximation space (Q,S,T), the knowledge state delineated by the upper inverse model of the variable precision *FT*-rough set with respect to B ($B \in \mathcal{F}(S)$) is defined by

$$K_{B,\beta}^+ = T_\beta^+(B) = \{q \in Q \mid D(B/T_q) \geq \beta\},$$

where $\beta \in (0.5,1]$.

The family of all the knowledge states delineated by the lower inverse model is denoted by $\mathcal{K}_\beta^+ = \{K_{B,\beta}^+ \mid B \in \mathcal{F}(S)\}$.

Theorem 1. *Given a fuzzy approximation space* (Q,S,T)*, and the families of the knowledge states delineated by the lower and upper inverse models are* $\mathcal{K}_\beta^- = \{K_{B,\beta}^- \mid B \in \mathcal{F}(S)\}$ *and* $\mathcal{K}_\beta^+ = \{K_{B,\beta}^+ \mid B \in \mathcal{F}(S)\}$*, respectively. Then both* $\mathcal{K}_\beta^-$ *and* $\mathcal{K}_\beta^+$ *are knowledge structures.*

Proof of Theorem 1. It can be easily deduced from (1) and (2) of Proposition 1 that when $B = \varnothing$, we have $T_\beta^{-1}(B) = T_\beta^{-1}(\varnothing) = \varnothing$, and $T_\beta^+(B) = T_\beta^+(\varnothing) = \varnothing$; when $B = S$, we have $T_\beta^{-1}(B) = T_\beta^{-1}(S) = Q$, and $T_\beta^+(B) = T_\beta^+(S) = Q$. Then

$$\mathcal{K}_\beta^- = \{T_\beta^{-1}(B) \mid B \in \mathcal{F}(S)\} = \{K_{B,\beta}^- \mid B \in \mathcal{F}(S)\},$$

$$\mathcal{K}_\beta^+ = \{T_\beta^+(B) \mid B \in \mathcal{F}(S)\} = \{K_{B,\beta}^+ \mid B \in \mathcal{F}(S)\},$$

are the knowledge structures delineated by the lower and upper inverse models of the variable precision *FT*-rough set respectively.

4. Composability of Knowledge Structure

Let the fuzzy approximation space (Q,S,T) be composed of the family of fuzzy approximation spaces $\{(Q_i,S_i,T_i) \mid i \in I\}$. The variable precision threshold is β ($\beta \in (0.5,1]$), the knowledge structures delineated by the lower inverse and upper inverse models of the variable precision *FT*-rough set in the fuzzy approximation space (Q_i,S_i,T_i), $i \in I$ are denoted as $(\mathcal{K}_i)_\beta^-$ and $(\mathcal{K}_i)_\beta^+$ respectively. The knowledge structures delineated by the lower inverse and upper inverse models of the variable precision *FT*-rough set in the fuzzy approximation space (Q,S,T) are denoted as $\mathcal{K}_\beta^-$ and $\mathcal{K}_\beta^+$ respectively. This section studies the conditions under which $\mathcal{K}_\beta^-$ and $\mathcal{K}_\beta^+$ can be composed of the knowledge structure families $\{(\mathcal{K}_i)_\beta^- \mid i \in I\}$ and $\{(\mathcal{K}_i)_\beta^+ \mid i \in I\}$ respectively.

For convenience, the following notation is given:

Notation 1. Let $Y \in \mathcal{F}(S)$, $S_j \subseteq S$, denote

$$Y|_{S_j}(s) = \begin{cases} Y(s), & s \in S_j \\ 0, & s \in S - S_j \end{cases}.$$

Then the knowledge structure $\mathcal{K}_\beta^-$ delineated by the lower inverse model of the variable precision *FT*-rough set in the fuzzy approximation space (Q,S,T) is not necessarily the composition of the knowledge structure family $\{(\mathcal{K}_i)_\beta^- \mid i \in I\}$. The knowledge structure $\mathcal{K}_\beta^+$ delineated by the upper inverse model is not necessarily the composition of $\{(\mathcal{K}_i)_\beta^+ \mid i \in I\}$. Example 1 below illustrates this problem.

Example 1. Let (Q_1,S_1,T_1) and (Q_2,S_2,T_2) be two fuzzy approximation spaces, where

$Q_1 = \{q_1,q_2,q_3\}$, $S_1 = \{s_1,s_2,s_3\}$, $(T_1)_{q_1} = \{(s_1,0.7),(s_2,0.5)\}$,

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$(T_1)_{q_2} = \{(s_1,0.8),(s_2,0.6),(s_3,0.6)\}$, $(T_1)_{q_3} = \{(s_2,0.7),(s_3,0.6)\}$;

$Q_2 = \{q_1,q_3,q_4\}$, $S_2 = \{s_2,s_3,s_4\}$, $(T_2)_{q_1} = \{(s_2,0.6),(s_3,0.8)\}$,

$(T_2)_{q_3} = \{(s_2,0.8),(s_3,0.7)\}$, $(T_2)_{q_4} = \{(s_3,0.7),(s_4,0.6)\}$.

Let (Q,S,T) is composed of (Q_1,S_1,T_1) and (Q_2,S_2,T_2), then

$Q = \{q_1,q_2,q_3,q_4\}$, $S = \{s_1,s_2,s_3,s_4\}$,

$(T)_{q_1} = \{(s_1,0.7),(s_2,0.6),(s_3,0.8)\}$, $(T)_{q_2} = \{(s_1,0.8),(s_2,0.6),(s_3,0.6)\}$,

$(T)_{q_3} = \{(s_2,0.8),(s_3,0.7)\}$, $(T)_{q_4} = \{(s_3,0.7),(s_4,0.6)\}$.

In the following, taking $\beta \in (\frac{2}{3},1]$ as an example, calculate the knowledge structures delineated by the lower and upper inverse models under the variable precision *FT*-rough set in (Q_1,S_1,T_1), (Q_2,S_2,T_2) and (Q,S,T), respectively.

(1) In (Q_1,S_1,T_1):

$(\mathcal{K}_1)^-_\beta = \{\varnothing,\{q_1\},\{q_1,q_2\},\{q_2,q_3\},Q_1\}$,

$(\mathcal{K}_1)^+_\beta = \{\varnothing,\{q_1\},\{q_3\},\{q_1,q_3\},\{q_2,q_3\},Q_1\}$;

(2) In (Q_2,S_2,T_2):

$(\mathcal{K}_2)^-_\beta = \{\varnothing,\{q_1\},\{q_4\},\{q_1,q_3\},\{q_1,q_4\},\{q_3,q_4\},Q_2\}$,

$(\mathcal{K}_2)^+_\beta = \{\varnothing,\{q_1\},\{q_3\},\{q_4\},\{q_1,q_3\},\{q_1,q_4\},\{q_3,q_4\},Q_2\}$;

(3) In (Q,S,T):

$$\mathcal{K}^-_\beta = \{\varnothing,\{q_1\},\{q_2\},\{q_4\},\{q_1,q_2\},\{q_1,q_4\},\{q_2,q_4\},\{q_1,q_2,q_3\},\{q_1,q_2,q_4\},\{q_2,q_3,q_4\},Q\}$$

$$\mathcal{K}^+_\beta = \{\varnothing,\{q_1\},\{q_2\},\{q_3\},\{q_4\},\{q_1,q_2\},\{q_1,q_3\},\{q_1,q_4\},\{q_2,q_3\},\{q_2,q_4\},\{q_3,q_4\},\{q_1,q_2,q_3\},\{q_1,q_2,q_4\},\{q_1,q_3,q_4\},\{q_2,q_3,q_4\},Q\}$$

Then the projections of $\mathcal{K}^-_\beta$ on Q_1 and Q_2 are respectively:

$\mathcal{K}^-_\beta|_{Q_1} = \{\varnothing,\{q_1\},\{q_2\},\{q_1,q_2\},\{q_2,q_3\},Q_1\}$,

$\mathcal{K}^-_\beta|_{Q_2} = \{\varnothing,\{q_1\},\{q_4\},\{q_1,q_3\},\{q_1,q_4\},\{q_3,q_4\},Q_2\}$.

The projections of $\mathcal{K}^+_\beta$ on Q_1 and Q_2 are respectively:

$\mathcal{K}^+_\beta|_{Q_1} = \{\varnothing,\{q_1\},\{q_2\},\{q_3\},\{q_1,q_2\},\{q_1,q_3\},\{q_2,q_3\},Q_1\}$,

$\mathcal{K}^+_\beta|_{Q_2} = \{\varnothing,\{q_1\},\{q_3\},\{q_4\},\{q_1,q_3\},\{q_1,q_4\},\{q_3,q_4\},Q_2\}$.

It can be seen that $(\mathcal{K}_1)^-_\beta \neq \mathcal{K}^-_\beta|_{Q_1}$, $(\mathcal{K}_2)^-_\beta = \mathcal{K}^-_\beta|_{Q_2}$, $(\mathcal{K}_1)^+_\beta \neq \mathcal{K}^+_\beta|_{Q_1}$, $(\mathcal{K}_2)^+_\beta = \mathcal{K}^+_\beta|_{Q_2}$. Therefore, when $\beta \in (\frac{2}{3},1]$, $\mathcal{K}^-_\beta$ is not the composition of $(\mathcal{K}_1)^-_\beta$ and $(\mathcal{K}_2)^-_\beta$, and $\mathcal{K}^+_\beta$ is not the composition of $(\mathcal{K}_1)^+_\beta$ and $(\mathcal{K}_2)^+_\beta$.

4.1. Composability of the Knowledge Structure Delineated by the Lower Inverse Model

Let the fuzzy approximation space (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i) \mid i \in I\}$. For (Q_j,S_j,T_j), ($j \in I$), and any $Y_j \in \mathcal{F}(S_j)$, then the knowledge state delineated via the lower inverse model of the variable precision *FT*-rough set by Y_j is

$$(T_j)^{-1}_\beta(Y_j) = \{q \in Q_j \mid D(Y_j/(T_j)_q) > 1-\beta\},$$

where

$$\beta \in (0.5,1] \text{ and } D(Y_j/(T_j)_q) = \frac{|\{s \in S_j \mid 0 < (T_j)_q(s) \le Y_j(s)\}|}{|\{s \in S_j \mid (T_j)_q(s) > 0\}|}.$$

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For any $Y \in \mathcal{F}(S)$, the knowledge state delineated via the lower inverse model of the variable precision *FT*-rough set by Y is

$$T_\beta^{-1}(Y) = \{q \in Q \mid D(Y / T_q) > 1-\beta\},$$

where

$$D(Y / T_q) = \frac{|\{s \in S \mid 0 < T_q(s) \le Y(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|}.$$

For $j \in I$, there is

$$T_\beta^{-1}(Y)|_{Q_j} = T_\beta^{-1}(Y) \cap Q_j = \{q \in Q_j \mid D(Y / T_q) > 1-\beta\}.$$

Theorem 2. *Let (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i) \mid i \in I\}$. For any $q \in Q_j$ and $s \in S_j$, $j \in I$, when $(T_j)_q(s) = 0$, it satisfies $T_q(s) = 0$, then for any $Y \in \mathcal{F}(S)$, $\beta \in (0.5,1]$, there is $T_\beta^{-1}(Y|_{S_j}) \cap Q_j \subseteq (T_j)_\beta^{-1}(Y|_{S_j})$.*

Proof of Theorem 2.

$$T_\beta^{-1}(Y|_{S_j}) \cap Q_j = \{q \in Q \mid D(Y|_{S_j} / T_q) > 1-\beta\} \cap Q_j = \{q \in Q_j \mid D(Y|_{S_j} / T_q) > 1-\beta\}.$$

If there is $q' \in T_\beta^{-1}(Y|_{S_j}) \cap Q_j$, then

$$D(Y|_{S_j} / T_{q'}) = \frac{|\{s \in S \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}|}{|\{s \in S \mid T_{q'}(s) > 0\}|} > 1-\beta.$$

For any $Y \in \mathcal{F}(S)$ and $q \in Q_j$, by Definition 2 and when $(T_j)_q(s) = 0$, satisfying $T_q(s) = 0$, we have

$$\left|\{s \in S \mid T_{q'}(s) > 0\}\right| \ge |\{s \in S_j \mid T_{q'}(s) > 0\}| = \left|\{s \in S_j \mid (T_j)_{q'}(s) > 0\}\right|$$

and

$$\begin{aligned} & |\{s \in S \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}| \\ =& |\{s \in S_j \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}| + |\{s \in S - S_j \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}| \\ =& |\{s \in S_j \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}| \\ \le& |\{s \in S_j \mid 0 < (T_j)_{q'}(s) \le Y|_{S_j}(s)\}|. \end{aligned}$$

Then

$$\begin{aligned} D(Y|_{S_j} / (T_j)_{q'}) &= \frac{|\{s \in S_j \mid 0 < (T_j)_{q'}(s) \le Y|_{S_j}(s)\}|}{|\{s \in S_j \mid (T_j)_{q'}(s) > 0\}|} \\ &\ge \frac{|\{s \in S \mid 0 < T_{q'}(s) \le Y|_{S_j}(s)\}|}{|\{s \in S \mid T_{q'}(s) > 0\}|} = D(Y|_{S_j} / T_{q'}) > 1-\beta. \end{aligned}$$

Then $q' \in (T_j)_\beta^{-1}(Y|_{S_j})$. Therefore $T_\beta^{-1}(Y|_{S_j}) \cap Q_j \subseteq (T_j)_\beta^{-1}(Y|_{S_j})$. □

Theorem 3. *Let (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i) \mid i \in I\}$. For any $q \in Q_j$, $j \in I$, $\beta \in (0.5,1]$, the following conclusions hold:*

(1) If

i) $(T_j)_q = T_q|_{S_j}$ holds for $s \in S_j$,

ii) $T_q(s) = 0$ holds for $s \in S - S_j$,

then for any $Y \in \mathcal{F}(S)$, there is $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$.

(2) If for any $Y \in \mathcal{F}(S)$, there is $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$, then $(T_j)_q = T_q|_{S_j}$.

(3) Let $(\mathcal{K}_j)_\beta^-$ and $\mathcal{K}_\beta^-$ are knowledge structures delineated by the lower inverse model of the variable precision FT-rough set in the fuzzy approximation space (Q_j,S_j,T_j) and (Q,S,T) respectively. If $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$, then $(\mathcal{K}_j)_\beta^- \subseteq \mathcal{K}_\beta^-|_{Q_j}$.

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Proof of Theorem 3. (1) Since when $s \in S_j$, $(T_j)_q = T_q|_{S_j}$, then when $(T_j)_q(s) = 0$, we have $T_q(s) = 0$. It is easy to know from Theorem 2 that for any $Y \in \mathcal{F}(S)$, $T_\beta^{-1}(Y|_{S_j}) \cap Q_j \subseteq (T_j)_\beta^{-1}(Y|_{S_j})$.

For any $Y \in \mathcal{F}(S)$, if there is $q \in (T_j)_\beta^{-1}(Y|_{S_j})$, then $D(Y|_{S_j}/(T_j)_q) > 1-\beta$ holds, i.e.

$$\frac{|\{s \in S_j \mid 0 < (T_j)_q(s) \le Y|_{S_j}(s)\}|}{|\{s \in S_j \mid (T_j)_q(s) > 0\}|} > 1-\beta;$$

and since when $s \in S_j$, $(T_j)_q = T_q|_{S_j}$, then

$$\frac{|\{s \in S_j \mid 0 < T_q|_{S_j}(s) \le Y|_{S_j}(s)\}|}{|\{s \in S_j \mid T_q|_{S_j}(s) > 0\}|} > 1-\beta.$$

Also, when $s \in S - S_j$, $T_q(s) = 0$, then

$$\frac{|\{s \in S \mid 0 < T_q(s) \le Y|_{S_j}(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|} > 1-\beta,$$

then there is $D(Y|_{S_j}/T_q) > 1-\beta$, therefore $q \in T_\beta^{-1}(Y|_{S_j}) \cap Q_j$, then $(T_j)_\beta^{-1}(Y|_{S_j}) \subseteq T_\beta^{-1}(Y|_{S_j}) \cap Q_j$.

Therefore, $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$ holds.

(2) Use the proof by contradiction. For $q \in Q_j$, there exists $s \in S_j$, such that $(T_j)_q(s) \ne T_q|_{S_j}(s)$, that is, it satisfies $0 \le (T_j)_q(s) < T_q(s)$. Let $n_j = |\{s \in S_j \mid (T_j)_q(s) > 0\}|$, $n = |\{s \in S \mid T_q(s) > 0\}|$,

i) When $(T_j)_q(s) \ne 0$, let

$$Y^*(u) = \begin{cases} \dfrac{(T_j)_q(s) + T_q(s)}{2}, & u = s \\ 0, & u \ne s, u \in S \end{cases},$$

then we have $Y^* \in \mathcal{F}(S)$, then there exists $\beta \in (0.5,1]$ such that $D(Y^*|_{S_j}/(T_j)_q) = \frac{1}{n_j} > 1-\beta$. Therefore, $q \in (T_j)_\beta^{-1}(Y^*|_{S_j})$. And if for any $\beta \in (0.5,1]$, $D(Y^*|_{S_j}/T_q) = 0 \le 1-\beta$, then there must be $q \notin T_\beta^{-1}(Y^*|_{S_j}) \cap Q_j$. This contradicts $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$.

ii) When $(T_j)_q(s) = 0$, let

$$Y^*(u) = \begin{cases} T_q(s), & u = s \\ 0, & u \ne s, u \in S \end{cases},$$

then $Y^* \in \mathcal{F}(S)$, and for any $\beta \in (0.5,1]$, $D(Y^*|_{S_j}/(T_j)_q) = 0 \le 1-\beta$, then there must be $q \notin (T_j)_\beta^{-1}(Y^*|_{S_j})$, and there exists $\beta \in (0.5,1]$ such that $D(Y^*|_{S_j}/T_q) = \frac{1}{n} > 1-\beta$, then $q \in T_\beta^{-1}(Y^*|_{S_j}) \cap Q_j$, which also contradicts $T_\beta^{-1}(Y|_{S_j}) \cap Q_j = (T_j)_\beta^{-1}(Y|_{S_j})$. Therefore, (2) holds.

(3) For any $B \in \mathcal{F}(S_j)$, $K_{B,\beta}^- = (T_j)_\beta^{-1}(B) \in (\mathcal{K}_j)_\beta^-$ be the knowledge state delineated by B via the lower inverse model. Let

$$Y^*(s) = \begin{cases} B'(s), & s \in S_j \\ 0, & s \in S - S_j \end{cases},$$

then $Y^* \in \mathcal{F}(S)$, we have

$$(T_j)_\beta^{-1}(B') = (T_j)_\beta^{-1}(Y^*|_{S_j}) = T_\beta^{-1}(Y^*|_{S_j}) \cap Q_j \in \mathcal{K}_\beta^-|_{Q_j},$$

then $(\mathcal{K}_j)_\beta^- \subseteq \mathcal{K}_\beta^-|_{Q_j}$. □

However, it can be seen from Example 1, when $\beta \in (\frac{2}{3},1]$, there is $(\mathcal{K}_1)_\beta^- \subseteq \mathcal{K}_\beta^-|_{Q_1}$, $(\mathcal{K}_2)_\beta^- \subseteq \mathcal{K}_\beta^-|_{Q_2}$, but $(T_1)_{q_1} \ne T_{q_1}|_{S_1}$, $(T_1)_{q_2} = T_{q_2}|_{S_1}$, $(T_1)_{q_3} \ne T_{q_3}|_{S_1}$. Therefore, Theorem 3(1) is a sufficient but not necessary condition for $(\mathcal{K}_j)_\beta^- \subseteq \mathcal{K}_\beta^-|_{Q_j}$.

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Corollary 1. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i)\mid i\in I\}$. *For any* $j\in I$, $\beta=1$, *the following statements are equivalent.*

(1) *For any* $q\in Q_j$ *and* $s\in S_j$, $(T_j)_q=T_q|_{S_j}$.

(2) *For any* $Y\in\mathcal{F}(S)$, $T_1^{-1}(Y|_{S_j})\cap Q_j=(T_j)_1^{-1}(Y|_{S_j})$.

Similar to Theorem 3, $(\mathcal{K}_j)_1^{-}\subseteq\mathcal{K}_1^{-}|_{Q_j}$ can be deduced from Corollary1 (1) or (2).

Theorem 4. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i)\mid i\in I\}$. *If* $T_q=T_q|_{S_j}$ *holds for any* $q\in Q_j$, $j\in I$ *and* $\beta\in(0.5,1]$, *then* $T_\beta^{-1}(Y|_{S_j})\cap Q_j=T_\beta^{-1}(Y)\cap Q_j$ *holds for any* $Y\in\mathcal{F}(S)$.

Proof of Theorem 4. i) For any $Y\in\mathcal{F}(S)$, let $q\in T_\beta^{-1}(Y|_{S_j})\cap Q_j$, then $q\in Q_j$, $j\in I$ and $D(Y|_{S_j}/T_q)>1-\beta$, i.e. $\frac{|\{s\in S\mid 0<T_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta$. Since for any $q\in Q_j$, $T_q=T_q|_{S_j}$, then $T_q(s)=0$ holds for $s\in S-S_j$. Then

$$\{s\in S\mid 0<T_q(s)\le Y|_{S_j}(s)\}=\{s\in S\mid 0<T_q(s)\le Y(s)\},$$

Then there is

$$\frac{|\{s\in S\mid 0<T_q(s)\le Y(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta,$$

That is $D(Y/T_q)>1-\beta$. Therefore $q\in T_\beta^{-1}(Y)\cap Q_j$, then $T_\beta^{-1}(Y|_{S_j})\cap Q_j\subseteq T_\beta^{-1}(Y)\cap Q_j$.

ii) For any $Y\in\mathcal{F}(S)$, let $q\in T_\beta^{-1}(Y)\cap Q_j$, then $q\in Q_j$, $j\in I$ and $D(Y/T_q)>1-\beta$, that is $\frac{|\{s\in S\mid 0<T_q(s)\le Y(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta$. Since for any $q\in Q_j$, $T_q=T_q|_{S_j}$, then $T_q(s)=0$ holds for $s\in S-S_j$. Then

$$\frac{|\{s\in S\mid 0<T_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta,$$

that is $D(Y|_{S_j}/T_q)>1-\beta$, then there is $q\in T_\beta^{-1}(Y|_{S_j})\cap Q_j$.

Then $T_\beta^{-1}(Y)\cap Q_j\subseteq T_\beta^{-1}(Y|_{S_j})\cap Q_j$.

Therefore, $T_\beta^{-1}(Y|_{S_j})\cap Q_j=T_\beta^{-1}(Y)\cap Q_j$ holds. □

However, if $T_\beta^{-1}(Y|_{S_j})\cap Q_j=T_\beta^{-1}(Y)\cap Q_j$ holds for any $Y\in\mathcal{F}(S)$ and $\beta\in(0.5,1]$, $T_q=T_q|_{S_j}$ does not necessarily holds. The following uses the proof by contradiction to show that if $T_\beta^{-1}(Y|_{S_j})\cap Q_j=T_\beta^{-1}(Y)\cap Q_j$ holds for any $Y\in\mathcal{F}(S)$ and $\beta=1$, then there is $T_q=T_q|_{S_j}$ for any $q\in Q_j$, $j\in I$.

For $q\in Q_j$, if there is $T_q\ne T_q|_{S_j}$, then there exists $s\in S-S_j$, such that $T_q(s)>0$. Let $Y^*(u)=\begin{cases}T_q(s), & u=s\\ 0, & u\ne s, u\in S\end{cases}$, then $Y^*\in\mathcal{F}(S)$, $Y^*|_{S_j}=\varnothing$, then $T_\beta^{-1}(Y^*|_{S_j})=\varnothing$. Therefore, for any $\beta\in(0.5,1]$, there is $T_\beta^{-1}(Y^*|_{S_j})\cap Q_j=\varnothing$. And obviously $|\{s\in S\mid 0<T_q(s)\le Y^*(s)\}|>0$, then $D(Y^*/T_q)=\frac{|\{s\in S\mid 0<T_q(s)\le Y^*(s)\}|}{|\{s\in S\mid T_q(s)>0\}|}>1-\beta=0$.

Then there must be $q\in\{q\in Q_j\mid D(Y^*/T_q)>0\}=T_\beta^{-1}(Y^*)\cap Q_j$, which contradicts $T_1^{-1}(Y|_{S_j})\cap Q_j=T_1^{-1}(Y)\cap Q_j$ for any $Y\in\mathcal{F}(S)$. Therefore, $T_q=T_q|_{S_j}$ holds for $\beta=1$.

Based on Theorems 3 and Theorems 4, we derive the conditions for the composability of the knowledge structure delineated by the lower inverse model of variable precision FT-rough sets.

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Theorem 5. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i)\mid i\in I\}$. *For any* $q\in Q_j$, $j\in I$ *and* $\beta\in(0.5,1]$, *if* $T_q=(T_j)_q$ *holds for* $s\in S_j$, *and* $T_q(s)=0$ *holds for* $s\in S-S_j$, *then*

(1) $T_\beta^{-1}(Y)\bigcap Q_j=(T_j)_\beta^{-1}(Y|_{S_j})$ *for any* $Y\in\mathcal{F}(S)$,

(2) $(\mathcal{K}_j)_\beta^-=\mathcal{K}_\beta^-|_{Q_j}$.

Proof of Theorem 5. (1) For any $q\in Q_j$, $(T_j)_q\subseteq T_q|_{S_j}\subseteq T_q$. If $T_q=(T_j)_q$ holds for $s\in S_j$, and $T_q(s)=0$ holds for $s\in S-S_j$, then $T_q=T_q|_{S_j}$ holds for $s\in S_j$. Then by Theorem 3, for any $Y\in\mathcal{F}(S)$, we have $T_\beta^{-1}(Y|_{S_j})\bigcap Q_j=(T_j)_\beta^{-1}(Y|_{S_j})$. And by Theorem 4, we have $T_\beta^{-1}(Y|_{S_j})\bigcap Q_j=T_\beta^{-1}(Y)\bigcap Q_j$. Therefore, $T_\beta^{-1}(Y)\bigcap Q_j=(T_j)_\beta^{-1}(Y|_{S_j})$ holds.

(2) By (1), for any $K_{Y,\beta}^-\in\mathcal{K}_\beta^-$, $\beta\in(0.5,1]$, there exists $Y\in\mathcal{F}(S)$ such that $K_{Y,\beta}^-|_{Q_j}=T_\beta^{-1}(Y)\bigcap Q_j=(T_j)_\beta^{-1}(Y|_{S_j})\in(\mathcal{K}_j)_\beta^-$. Then $\mathcal{K}_\beta^-|_{Q_j}\subseteq(\mathcal{K}_j)_\beta^-$. And according to Theorem 3, $(\mathcal{K}_j)_\beta^-\subseteq\mathcal{K}_\beta^-|_{Q_j}$, so $(\mathcal{K}_j)_\beta^-=\mathcal{K}_\beta^-|_{Q_j}$ holds. □

Theorem 5 is a sufficient condition for $(\mathcal{K}_j)_\beta^-=\mathcal{K}_\beta^-|_{Q_j}$, but not necessary. The following Example 2 shows this.

Example 2. Let (Q_1,S_1,T_1) and (Q_2,S_2,T_2) be fuzzy approximation spaces, where

$Q_1=\{q_1,q_2,q_3\}$, $S_1=\{s_1,s_2\}$, $(T_1)_{q_1}=\{(s_1,0.7),(s_2,0.5)\}$,

$(T_1)_{q_2}=\{(s_1,0.8),(s_2,0.7)\}$, $(T_1)_{q_3}=\{(s_2,0.9)\}$;

$Q_2=\{q_2,q_4\}$, $S_2=\{s_2,s_3\}$, $(T_2)_{q_2}=\{(s_2,0.7)\}$, $(T_2)_{q_4}=\{(s_2,0.8),(s_3,0.9)\}$.

(Q,S,T) is composed of (Q_1,S_1,T_1) and (Q_2,S_2,T_2).

Then

$Q=\{q_1,q_2,q_3,q_4\}$, $S=\{s_1,s_2,s_3\}$, $(T)_{q_1}=\{(s_1,0.7),(s_2,0.5)\}$,

$(T)_{q_2}=\{(s_1,0.8),(s_2,0.7)\}$, $(T)_{q_3}=\{(s_2,0.9)\}$, $(T)_{q_4}=\{(s_2,0.8),(s_3,0.9)\}$.

Then for $\beta\in(\frac{1}{2},1]$, the knowledge structures delineated by the lower inverse model under the variable precision *FT*-rough set in (Q_1,S_1,T_1), (Q_2,S_2,T_2) and (Q,S,T) are respectively:

$(\mathcal{K}_1)_\beta^-=\{\varnothing,\{q_1\},\{q_1,q_2\},Q_1\}$, $(\mathcal{K}_2)_\beta^-=\{\varnothing,\{q_2\},\{q_4\},Q_2\}$,

$\mathcal{K}_\beta^-=\{\varnothing,\{q_1\},\{q_4\},\{q_1,q_2\},\{q_1,q_4\},\{q_1,q_2,q_4\},Q\}$.

Then

$\mathcal{K}_\beta^-|_{Q_1}=\{\varnothing,\{q_1\},\{q_1,q_2\},Q_1\}=(\mathcal{K}_1)_\beta^-$,

$\mathcal{K}_\beta^-|_{Q_2}=\{\varnothing,\{q_2\},\{q_4\},Q_2\}=(\mathcal{K}_2)_\beta^-$.

So for $\beta\in(\frac{1}{2},1]$, $\mathcal{K}_\beta^-$ is composed of $(\mathcal{K}_1)_\beta^-$ and $(\mathcal{K}_2)_\beta^-$. However, $T_{q_2}\neq(T_2)_{q_2}$. Therefore, for any $q\in Q_j$, $s\in S_j$, $j\in I$, $T_q=(T_j)_q$ is not a necessary condition for the composabilily of the knowledge structure.

Corollary 2. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i)\mid i\in I\}$. *Then we have:*

(1) If Q_i, $i\in I$ *are pairwise disjoint, then for* $\beta\in(0.5,1]$, $\mathcal{K}_\beta^-$ *is composed of the family of knowledge structures* $\{(\mathcal{K}_j)_\beta^-\mid j\in I\}$.

(2) If S_i, $i\in I$ *are pairwise disjoint, then for* $\beta=1$ *and any* $j\in I$, *there is* $(\mathcal{K}_j)_1^-\subseteq\mathcal{K}_1^-|_{Q_j}$.

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Proof of Corollary 2. (1) If Q_i, $i \in I$ are pairwise disjoint, then for any $q \in Q_j$, $j \in I$, when $s \in S_j$, there is $T_q = (T_j)_q$; when $s \in S - S_j$, there is $T_q(s) = 0$. Then by Theorem 5, there is $(\mathcal{K}_j)_\beta^- = \mathcal{K}_\beta^- |_{Q_j}$. Therefore, $\mathcal{K}_\beta^-$ is composed of the knowledge structure family $\{(\mathcal{K}_j)_\beta^- \mid j \in I\}$.
(2) If S_i, $i \in I$ are pairwise disjoint, then for any $q \in Q_j$, $j \in I$, when $s \in S_j$, there is $(T_j)_q = T_q |_{S_j}$. Then by corollary 1, there is $(\mathcal{K}_j)_1^- \subseteq \mathcal{K}_1^- |_{Q_j}$. □

4.2. Composability of the Knowledge Structure Delineated by the Upper Inverse Model

Let the fuzzy approximation space (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i) \mid i \in I\}$. For (Q_j,S_j,T_j), ($j \in I$), and any $Y_j \in \mathcal{F}(S_j)$, the knowledge state delineated via the upper inverse model of the variable precision *FT*-rough set by Y_j is

$$(T_j)_\beta^+(Y_j) = \{q \in Q_j \mid D(Y_j / (T_j)_q) \geq \beta\},$$

where

$$\beta \in (0.5,1] \text{ and } D(Y_j / (T_j)_q) = \frac{|\{s \in S_j \mid 0 < (T_j)_q(s) \leq Y_j(s)\}|}{|\{s \in S_j \mid (T_j)_q(s) > 0\}|}.$$

For any $Y \in \mathcal{F}(S)$, the knowledge state delineated via the upper inverse model on the variable precision *FT*-rough set by Y is

$$T_\beta^+(Y) = \{q \in Q \mid D(Y / T_q) \geq \beta\},$$

where

$$\beta \in (0.5,1] \text{ and } D(Y / T_q) = \frac{|\{s \in S \mid 0 < T_q(s) \leq Y(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|}.$$

For $j \in I$, there is

$$T_\beta^+(Y)|_{Q_j} = T_\beta^+(Y) \cap Q_j = \{q \in Q_j \mid D(Y / T_q) \geq \beta\}.$$

Theorem 6. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i) \mid i \in I\}$. *For any* $q \in Q_j$ *and* $s \in S_j$, $j \in I$, *when* $(T_j)_q(s) = 0$, *it satisfies* $T_q(s) = 0$, *then for any* $Y \in \mathcal{F}(S)$, $\beta \in (0.5,1]$, *there is* $T_\beta^+(Y|_{S_j}) \cap Q_j \subseteq (T_j)_\beta^+(Y|_{S_j})$.

Proof of Theorem 6.

$$T_\beta^+(Y|_{S_j}) \cap Q_j = \{q \in Q \mid D(Y|_{S_j} / T_q) \geq \beta\} \cap Q_j = \{q \in Q_j \mid D(Y|_{S_j} / T_q) \geq \beta\}.$$

If there is $q \in T_\beta^+(Y|_{S_j}) \cap Q_j$, then

$$D(Y|_{S_j} / T_q) = \frac{|\{s \in S \mid 0 < T_q(s) \leq Y|_{S_j}(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|} \geq \beta.$$

For any $Y \in \mathcal{F}(S)$ and $q \in Q_j$, by Definition 2 and when $(T_j)_q(s) = 0$, satisfying $T_q(s) = 0$, we have

$$\left|\{s \in S \mid T_q(s) > 0\}\right| \geq |\{s \in S_j \mid T_q(s) > 0\}| = \left|\{s \in S_j \mid (T_j)_q(s) > 0\}\right|,$$

and

$$\begin{aligned} & |\{s \in S \mid 0 < T_q(s) \leq Y|_{S_j}(s)\}| \\ &= |\{s \in S_j \mid 0 < T_q(s) \leq Y|_{S_j}(s)\}| + |\{s \in S - S_j \mid 0 < T_q(s) \leq Y|_{S_j}(s)\}| \\ &= |\{s \in S_j \mid 0 < T_{q'}(s) \leq Y|_{S_j}(s)\}| \\ &\leq |\{s \in S_j \mid 0 < (T_j)_q(s) \leq Y|_{S_j}(s)\}|. \end{aligned}$$

Then

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$$D(Y|_{S_j}/(T_j)_q)=\frac{|\{s\in S_j\,|\,0<(T_j)_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S_j\,|\,(T_j)_q(s)>0\}|}$$
$$\ge\frac{|\{s\in S\,|\,0<T_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S\,|\,T_q(s)>0\}|}=D(Y|_{S_j}/T_q)\ge\beta.$$

Then $q\in(T_j)^+_\beta(Y|_{S_j})$. Therefore $T^+_\beta(Y|_{S_j})\bigcap Q_j\subseteq(T_j)^+_\beta(Y|_{S_j})$. □

Theorem 7. *Let (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i)\,|\,i\in I\}$. For any $q\in Q_j$, $j\in I$, $\beta\in(0.5,1]$, the following conclusions hold:*

(1) If

i) $(T_j)_q=T_q|_{S_j}$ holds for any $s\in S_j$,

ii) $T_q(s)=0$ holds for any $s\in S-S_j$,

then for any $Y\in\mathcal{F}(S)$, there is $T^+_\beta(Y|_{S_j})\bigcap Q_j=(T_j)^+_\beta(Y|_{S_j})$.

(2) If for any $Y\in\mathcal{F}(S)$, there is $T^+_\beta(Y|_{S_j})\bigcap Q_j=(T_j)^+_\beta(Y|_{S_j})$, then $(T_j)_q=T_q|_{S_j}$.

(3) Let $(\mathcal{K}_j)^+_\beta$ and $\mathcal{K}^+_\beta$ are knowledge structures delineated by the upper inverse model of the variable precision FT-rough set in the fuzzy approximation space (Q_j,S_j,T_j) and (Q,S,T) respectively. If $T^+_\beta(Y|_{S_j})\bigcap Q_j=(T_j)^+_\beta(Y|_{S_j})$, then $(\mathcal{K}_j)^+_\beta\subseteq\mathcal{K}^+_\beta|_{Q_j}$.

Proof of Theorem 7. (1) Since when $s\in S_j$, $(T_j)_q=T_q|_{S_j}$, then when $(T_j)_q(s)=0$, we have $T_q(s)=0$. It is easy to know from Theorem 6 that for any $Y\in\mathcal{F}(S)$, $T^+_\beta(Y|_{S_j})\bigcap Q_j\subseteq(T_j)^+_\beta(Y|_{S_j})$.

For any $Y\in\mathcal{F}(S)$, if there is $q\in(T_j)^+_\beta(Y|_{S_j})$, then $D(Y|_{S_j}/(T_j)_q)\ge\beta$ holds, i.e.

$$\frac{|\{s\in S_j\,|\,0<(T_j)_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S_j\,|\,(T_j)_q(s)>0\}|}\ge\beta.$$

And since when $s\in S_j$, $(T_j)_q=T_q|_{S_j}$, then

$$\frac{|\{s\in S_j\,|\,0<T_q|_{S_j}(s)\le Y|_{S_j}(s)\}|}{|\{s\in S_j\,|\,T_q|_{S_j}(s)>0\}|}\ge\beta.$$

Also, since when $s\in S-S_j$, $T_q(s)=0$, then

$$\frac{|\{s\in S\,|\,0<T_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S\,|\,T_q(s)>0\}|}\ge\beta.$$

Then there is $D(Y|_{S_j}/T_q)\ge\beta$, and then $q\in T^+_\beta(Y|_{S_j})\bigcap Q_j$. So, there is $(T_j)^+_\beta(Y|_{S_j})\subseteq T^+_\beta(Y|_{S_j})\bigcap Q_j$.

Therefore, $T^+_\beta(Y|_{S_j})\bigcap Q_j=(T_j)^+_\beta(Y|_{S_j})$ holds.

(2) Use the proof by contradiction. For $q\in Q_j$, there exists $s\in S_j$, such that $(T_j)_q(s)\ne T_q|_{S_j}(s)$, that is, it satisfies $0\le(T_j)_q(s)<T_q(s)$. Let $n_j=|\{s\in S_j\,|\,(T_j)_q(s)>0\}|$, $n=|\{s\in S\,|\,T_q(s)>0\}|$, then $n_j\le n$.

i) When $(T_j)_q(s)\ne0$, let

$$Y^*(u)=\begin{cases}\dfrac{(T_j)_q(s)+T_q(s)}{2}, & u=s\\ (T_j)_q(u), & u\ne s,u\in S_j\\ 0, & u\in S-S_j\end{cases},$$

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then $Y^* \in \mathcal{F}(S)$. Then there exists $\beta \in (0.5,1]$ such that $D(Y^*|_{S_j}/T_q) = \frac{n_j - 1}{n} < \beta$, and then $q \notin T_\beta^+(Y^*|_{S_j}) \cap Q_j$. And for any $\beta \in (0.5,1]$, $D(Y^*|_{S_j}/(T_j)_q) = \frac{n_j}{n_j} = 1 \ge \beta$, then there must be $q \in (T_j)_\beta^+(Y^*|_{S_j})$. This contradicts $T_\beta^+(Y|_{S_j}) \cap Q_j = (T_j)_\beta^+(Y|_{S_j})$.

ii) When $(T_j)_q(s) = 0$, let

$$Y^*(u) = \begin{cases} 0, & u = s \\ (T_j)_q(u), & u \ne s, u \in S_j \\ 0, & u \in S - S_j \end{cases},$$

then $Y^* \in \mathcal{F}(S)$, and for any $\beta \in (0.5,1]$, $D(Y^*|_{S_j}/(T_j)_q) = \frac{n_j}{n_j} = 1 \ge \beta$, then there must be $q \in (T_j)_\beta^+(Y^*|_{S_j})$, and there exists $\beta' \in (0.5,1]$ such that $D(Y^*|_{S_j}/T_q) = \frac{n_j}{n} < \beta'$, then $q \notin T_{\beta'}^+(Y^*|_{S_j}) \cap Q_j$, which also contradicts $T_\beta^+(Y|_{S_j}) \cap Q_j = (T_j)_\beta^+(Y|_{S_j})$. Therefore, (2) holds.

(3) For any $B \in \mathcal{F}(S_j)$, $K_{B,\beta}^+ = (T_j)_\beta^+(B) \in (\mathcal{K}_j)_\beta^+$ be the knowledge state delineated by B via the upper inverse model. Let

$$Y^*(s) = \begin{cases} B(s), & s \in S_j \\ 0, & s \in S - S_j \end{cases},$$

then $Y^* \in \mathcal{F}(S)$, we have

$$(T_j)_\beta^+(B) = (T_j)_\beta^+(Y^*|_{S_j}) = T_\beta^+(Y^*|_{S_j}) \cap Q_j \in \mathcal{K}_\beta^+|_{Q_j},$$

then $(\mathcal{K}_j)_\beta^+ \subseteq \mathcal{K}_\beta^+|_{Q_j}$. □

Similar to Theorem 3, Theorem 7(1) is a sufficient but not necessary condition for $(\mathcal{K}_j)_\beta^+ \subseteq \mathcal{K}_\beta^+|_{Q_j}$.

Theorem 8. *Let* (Q,S,T) *be composed of a family of fuzzy approximation spaces* $\{(Q_i,S_i,T_i) \mid i \in I\}$. *If* $T_q = T_q|_{S_j}$ *holds for any* $q \in Q_j$, $j \in I$ *and* $\beta \in (0.5,1]$, *then* $T_\beta^+(Y|_{S_j}) \cap Q_j = T_\beta^+(Y) \cap Q_j$ *holds for any* $Y \in \mathcal{F}(S)$.

Proof of Theorem 8. i) For any $Y \in \mathcal{F}(S)$, let $q \in T_\beta^+(Y|_{S_j}) \cap Q_j$, then $q \in Q_j$, $j \in I$ and $D(Y|_{S_j}/T_q) \ge \beta$, i.e.

$$\frac{|\{s \in S \mid 0 < T_q(s) \le Y|_{S_j}(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|} \ge \beta.$$

Since for any $q \in Q_j$, there is $T_q = T_q|_{S_j}$, then $T_q(s) = 0$ holds for $s \in S - S_j$.

Then

$$\{s \in S \mid 0 < T_q(s) \le Y|_{S_j}(s)\} = \{s \in S \mid 0 < T_q(s) \le Y(s)\}.$$

Then there is

$$\frac{|\{s \in S \mid 0 < T_q(s) \le Y(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|} \ge \beta,$$

that is $D(Y/T_q) \ge \beta$, and then $q \in T_\beta^+(Y) \cap Q_j$. Then $T_\beta^+(Y|_{S_j}) \cap Q_j \subseteq T_\beta^+(Y) \cap Q_j$.

ii) For any $Y \in \mathcal{F}(S)$, let $q \in T_\beta^+(Y) \cap Q_j$, then $q \in Q_j$, $j \in I$ and $D(Y/T_q) \ge \beta$, that is

$$\frac{|\{s \in S \mid 0 < T_q(s) \le Y(s)\}|}{|\{s \in S \mid T_q(s) > 0\}|} \ge \beta.$$

Since for any $q \in Q_j$, $j \in I$, there is $T_q = T_q|_{S_j}$, then $T_q(s) = 0$ holds for $s \in S - S_j$. Then

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$$\frac{|\{s\in S\,|\,0<T_q(s)\le Y|_{S_j}(s)\}|}{|\{s\in S\,|\,T_q(s)>0\}|}\ge\beta,$$

that is $D(Y|_{S_j}/T_q)\ge\beta$, then there is $q\in T_\beta^+(Y|_{S_j})\cap Q_j$.

Then $T_\beta^+(Y)\cap Q_j\subseteq T_\beta^+(Y|_{S_j})\cap Q_j$.

Therefore, $T_\beta^+(Y|_{S_j})\cap Q_j=T_\beta^+(Y)\cap Q_j$ holds. □

However, if $T_\beta^+(Y|_{S_j})\cap Q_j=T_\beta^+(Y)\cap Q_j$ holds for any $Y\in\mathcal{F}(S)$ and any $\beta\in(0.5,1]$, $T_q=T_q|_{S_j}$ does not necessarily holds. The following uses the proof by contradiction to show that if $T_\beta^+(Y|_{S_j})\cap Q_j=T_\beta^+(Y)\cap Q_j$ holds for any $Y\in\mathcal{F}(S)$ and $\beta=1$, then there is $T_q=T_q|_{S_j}$ for any $q\in Q_j$, $j\in I$.

For $q\in Q_j$, if there is $T_q\ne T_q|_{S_j}$, then there exists $s\in S-S_j$, such that $T_q(s)>0$. Let

$$Y^*(u)=\begin{cases}1, & u\in S_j\\ T_q(s), & u=s\\ 0, & u\ne s, u\in S-S_j\end{cases},$$

then $Y^*\in\mathcal{F}(S)$.

Let $n=|\{s\in S\,|\,T_q(s)>0\}|$, then

$$D(Y^*/T_q)=\frac{|\{s\in S\,|\,0<T_q(s)\le Y^*(s)\}|}{|\{s\in S\,|\,T_q(s)>0\}|}=\frac{n}{n}=1\ge\beta.$$

Then for any $\beta\in(0.5,1]$, there must be $q\in T_\beta^+(Y^*)\cap Q_j$. And

$$D(Y^*|_{S_j}/T_q)=\frac{|\{s\in S\,|\,0<T_q(s)\le Y^*|_{S_j}(s)\}|}{|\{s\in S\,|\,T_q(s)>0\}|}=\frac{n-1}{n}<1,$$

then if $\beta=1$, there is $q\notin T_{\beta'}^+(Y^*|_{S_j})\cap Q_j$, which contradicts $T_1^+(Y|_{S_j})\cap Q_j=T_1^+(Y)\cap Q_j$ for any $Y\in\mathcal{F}(S)$. Therefore, $T_q=T_q|_{S_j}$ holds for $\beta=1$.

Based on Theorems 7 and 8, we derive the conditions for the composability of the knowledge structure delineated by the upper inverse model of variable precision FT-rough sets.

Theorem 9. *Let (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i)\,|\,i\in I\}$. For any $q\in Q_j$, $j\in I$ and $\beta\in(0.5,1]$, if $T_q=(T_j)_q$ holds for $s\in S_j$, and $T_q(s)=0$ holds for $s\in S-S_j$, then*

(1) For any $Y\in\mathcal{F}(S)$, we have $T_\beta^+(Y)\cap Q_j=(T_j)_\beta^+(Y|_{S_j})$.

(2) $(\mathcal{K}_j)_\beta^+=\mathcal{K}_\beta^+|_{Q_j}$.

Proof of Theorem 9. (1) For any $q\in Q_j$, $(T_j)_q\subseteq T_q|_{S_j}\subseteq T_q$. Since $T_q=(T_j)_q$ holds for $s\in S_j$, then $(T_j)_q=T_q|_{S_j}$. And $T_q(s)=0$ holds for $s\in S-S_j$, then $T_q=T_q|_{S_j}$ holds for $s\in S$. Then by Theorem 7, for any $Y\in\mathcal{F}(S)$ and $\beta\in(0.5,1]$, we have $T_\beta^+(Y|_{S_j})\cap Q_j=(T_j)_\beta^+(Y|_{S_j})$. And by Theorem 8, we have $T_\beta^+(Y|_{S_j})\cap Q_j=T_\beta^+(Y)\cap Q_j$. Therefore, $T_\beta^+(Y)\cap Q_j=(T_j)_\beta^+(Y|_{S_j})$ holds.

(2) By (1), for any $K_{Y,\beta}^+\in\mathcal{K}_\beta^+$, $\beta\in(0.5,1]$, there exists $Y\in\mathcal{F}(S)$ such that $K_{Y,\beta}^+|_{Q_j}=T_\beta^+(Y)\cap Q_j=(T_j)_\beta^+(Y|_{S_j})\in(\mathcal{K}_j)_\beta^+$. Then $\mathcal{K}_\beta^+|_{Q_j}\subseteq(\mathcal{K}_j)_\beta^+$. And according to Theorem 7, there is $(\mathcal{K}_j)_\beta^+\subseteq\mathcal{K}_\beta^+|_{Q_j}$, so $(\mathcal{K}_j)_\beta^+=\mathcal{K}_\beta^+|_{Q_j}$ holds. □

Similar to Theorem 5, Theorem 9 is a sufficient condition for $(\mathcal{K}_j)_\beta^+=\mathcal{K}_\beta^+|_{Q_j}$, but not necessary.

Corollary 3. *Let (Q,S,T) be composed of a family of fuzzy approximation spaces $\{(Q_i,S_i,T_i)\,|\,i\in I\}$. If Q_i, $i\in I$ are pairwise disjoint, then for $\beta\in(0.5,1]$, $\mathcal{K}_\beta^+$ is composed of the family of knowledge structures $\{(\mathcal{K}_j)_\beta^+\,|\,j\in I\}$.*

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Proof of Corollary 3. If Q_i, $i \in I$ are pairwise disjoint, then for any $q \in Q_j$, $j \in I$, when $s \in S_j$, there is $T_q = (T_j)_q$; when $s \in S - S_j$, there is $T_q(s) = 0$. Then by Theorem 9, there is $(\mathcal{K}_j)^+_\beta = \mathcal{K}^+_\beta |_{Q_j}$. Therefore, $\mathcal{K}^+_\beta$ is composed of the knowledge structure family $\{(\mathcal{K}_j)^+_\beta \mid j \in I\}$.

Corollary 2 and Corollary 3 provide a method to ensure that the global information is a consistent aggregation of local information. Specifically, if the problem domains in all local fuzzy approximation spaces are pairwise disjoint, then the global knowledge structure delineated via the lower inverse (or upper inverse) model of variable precision *FT*-rough sets in the global fuzzy approximation space is the composition of the local knowledge structures.

5. Conclusions

This paper introduces the variable precision *FT*- rough set model and its properties, as well as the knowledge structure delineated by the lower (upper) inverse operator based on this model. On this basis, the conditions for the composability of the knowledge structure delineated by the lower (upper) inverse operator of the variable precision *FT*- rough sets are studied. Meanwhile, the conditions under which knowledge structures constructed from local fuzzy approximation spaces can be composed into those built from global fuzzy approximation spaces are investigated. Future work explores the necessary and sufficient conditions for composing knowledge structures into a well-graded one, along with the construction of learning paths. Additionally, the composability of dynamic knowledge structures is a promising research direction.

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Abbreviations

The following abbreviations are used in this manuscript:

KST	Knowledge Space Theory
ALEKS	Assessment and Learning in Knowledge Spaces
RST	Rough Set Theory
FST	Fuzzy Set Theory
FT-rough set	Fuzzy T-rough set

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