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Stability of an Additive-Quadratic-Cubic Functional Equation

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Abstract

We will prove the generalized stability of an additive-quadratic-cubic functional equation in the sprit of Găvruţa.

Keywords: an additive-quadratic-cubic mapping; stability of an additive-quadratic-cubic functional equation

MSC: 39A30;39B82; 39B52.

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1. Introduction

Throughout this paper, let V and W be real vector spaces. The study of the stability of functional equations began with Hyers' study of the additive functional equation

$$f(x_1 + x_2) - f(x_1) - f(x_2) = 0$$

for all $x_1, x_2 \in V$(see [6,15]). Subsequently, Rassias [14] and Găvruţa [5] generalized Hyers' result on the stability of the additive functional equation, and many mathematicians have since applied the methods of Rassias and Găvruţa to the study of the stability of various functional equations.

Some mathematicians have studied the stability of the additive-quadratic functional equation

$$\begin{aligned} &f(x_1 + x_2 + x_3) - f(x_1 + x_2) - f(x_1 + x_3) \\ &- f(x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) = 0 \end{aligned}$$

for all $x_1, x_2, x_3 \in V$(see [2,4,7,10,12,13,16]).

For a given mapping $f : V \to W$, we use the following abbreviation:

$$\begin{aligned} Ef(x_1, x_2, x_3, x_4) :=& f(x_1 + x_2 + x_3 + x_4) - f(x_1 + x_2 + x_3) - f(x_1 + x_2 + x_4) \\ &- f(x_1 + x_3 + x_4) - f(x_2 + x_3 + x_4) + f(x_1 + x_2) + f(x_1 + x_3) \\ &+ f(x_1 + x_4) + f(x_2 + x_3) + f(x_2 + x_4) + f(x_3 + x_4) \\ &- f(x_1) - f(x_2) - f(x_3) - f(x_4) \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in V$. We consider the functional equation

$$Ef(x_1, x_2, x_3, x_4) = 0 \tag{1}$$

for all $x_1, x_2, x_3, x_4 \in V$. The functional equation (1) is called *an additive-quadratic-cubic functional equation* and its solution is called *an additive-quadratic-cubic mapping*. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum_{s=1}^{3} a_s x^s$ is a particular solution of the functional equation (1), where a_s are real constants and $\mathbb{R}$ is the set of real numbers.

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In this paper, we will prove the stability theorem of the additive quadratic-cubic functional equation (1) in the sense of Găvruţa. Furthermore, as corollaries of this theorem, we will show stability theorems of functional equation (1) in the sense of Hyers and Rassias.

2. Stability of a General Undecic Functional Equation

Throughout this section, we use the following definitions:

For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned}
\underset{y}{\Delta} f(x) &:= f(x+y) - f(x), \\
f_1(x) &:= \frac{1}{12}\big(f(4x) - 12f(2x) + 32f(x)\big), \\
f_2(x) &:= -\frac{1}{8}\big(f(4x) - 10f(2x) + 16f(x)\big), \\
f_3(x) &:= \frac{1}{24}\big(f(4x) - 6f(2x) + 8f(x)\big), \\
\Gamma f(x) &:= f(8x) - 14f(4x) + 56f(2x) - 64f(x)
\end{aligned}$$

for all $x, y \in V$.

Note that

$$\begin{aligned}
Ef(x_1,x_2,x_3,x_4) &:= \sum_{m=1}^{4}(-1)^m\left(\sum_{1\le i_1<\cdots<i_m\le 4} f(x_{i_1}+x_{i_2}+\cdots+x_{i_m})\right), \\
Ef(x_1,x_2,x_3,x_4) &:= \underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(0) - f(0)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in V$, where $m, i_1, i_2, \ldots, i_m$ are positive integers..

The following theorem introduced by Albert and Baker [1] can be obtained by Corollary 1, Theorem 3 and Corollary 3 in Djoković's paper [3].

Theorem 1. ([1, Theorem C]) *For a given mapping $f : V \to W$, the followings are equivalent:*

(i) *A mapping $f : V \to W$ satisfies the functional equation $\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ for all $x, x_1, x_2, x_3, x_4 \in V$;*

(ii) *There exist mappings $\hat{f}_1, \hat{f}_2, \hat{f}_3 : V \to W$ that satisfy $f(x) = \sum_{k=1}^{3} \hat{f}_k(x) + f(0)$ and*

$$\sum_{s=1}^{k}\binom{k}{s}(-1)^{k-s}\hat{f}_k(x+sy) - k!\hat{f}_k(y) = 0 \tag{2}$$

for all $x, y \in V$ and each $k \in \{1,2,3\}$.

When $k \in \{1,2,3\}$, if a mapping $f : V \to W$ satisfies the functional equation (2), then f is called an additive mapping, a quadratic mapping, and a cubic mapping, respectively.

Theorem 2. *For a given mapping $f : V \to W$, the followings are equivalent:*

(i) *$Ef(x_1,x_2,x_3,x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$.*

(ii) *$\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ for all $x, x_1, x_2, x_3, x_4 \in V$ with $f(0) = 0$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $f : V \to W$ satisfies $Ef(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. Since $\underset{0}{\Delta}\underset{0}{\Delta}\underset{0}{\Delta}\underset{0}{\Delta} f(0) = 0$, we get $f(0) = -\underset{0}{\Delta}\underset{0}{\Delta}\underset{0}{\Delta}\underset{0}{\Delta} f(0) + f(0) = -Ef(0,0,0,0) = 0$ and

$$\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(0) = Ef(x_1, x_2, x_3, x_4) + f(0) = 0 \tag{3}$$

for all $x_1, x_2, x_3, x_4 \in V$. If we put $x_4 = x$, then it follows (3) that

$$\underset{x_1 x_2 x_3}{\Delta\Delta\Delta} f(x) - \underset{x_1 x_2 x_3}{\Delta\Delta\Delta} f(0) = \underset{x\, x_1 x_2 x_3}{\Delta\Delta\Delta\Delta} f(0) = 0 \tag{4}$$

for all $x, x_1, x_2, x_3, x_4 \in V$. From (3) and (4), we obtain the desired result

$$\begin{aligned} \underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) =& \underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) - \underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(0) \\ =& \underset{x_4}{\Delta}\left(\underset{x_1 x_2 x_3}{\Delta\Delta\Delta} f(x) - \underset{x_1 x_2 x_3}{\Delta\Delta\Delta} f(0) \right) \\ =& \underset{x_4}{\Delta} 0 = 0 \end{aligned}$$

for all $x, x_1, x_2, x_3, x_4 \in V$ with $f(0) = 0$.

(ii) $\Rightarrow$ (i) If $f : V \to W$ satisfies $\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ for all $x, x_1, x_2, x_3, x_4 \in V$ with $f(0) = 0$, then $Ef(x_1, x_2, x_3, x_4) = \underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(0) - f(0) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. □

According to Theorem 1 and Theorem 2, we know that a mapping $f : V \to W$ satisfies the functional equation $Ef(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$ if and only if there exist an additive mapping A, a quadratic mapping Q, and a cubic mapping C such that $f(x) = A(x) + Q(x) + C(x)$ for all $x \in V$.

A very useful equality for proving the main theorem is given in the following lemma.

Lemma 1. *For any mapping $f : V \to W$, the equality*

$$\Gamma f(x) = Ef(2x, 2x, 2x, 2x) + 4Ef(2x, 2x, x, x) + 8Ef(2x, x, x, x) + 8Ef(x, x, x, x) \tag{5}$$

holds for all $x \in V$.

Proof. We obtain the equality (5) from the equalities

$$\begin{aligned} Ef(2x, 2x, 2x, 2x) =& f(8x) - 4f(6x) + 6f(4x) - 4f(2x), \\ 4Ef(2x, 2x, x, x) =& 4f(6x) - 8f(5x) - 4f(4x) + 16f(3x) - 4f(2x) - 8f(x) \\ 8Ef(2x, x, x, x) =& 8f(5x) - 24f(4x) + 16f(3x) + 16f(2x) - 24f(x), \\ 8Ef(x, x, x, x) =& 8f(4x) - 32f(3x) + 48f(2x) - 32f(x) \end{aligned}$$

for all $x \in V$. □

From the definitions of f_1, f_2, f_3, and Γf, we obtain the following lemma.

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Lemma 2. *For any mapping $f : V \to W$, the equalities*

$$f_1(x) - \frac{f_1(2x)}{2} = -\frac{1}{24}\Gamma f(x), \tag{6}$$
$$f_2(x) - \frac{f_2(2x)}{4} = \frac{1}{32}\Gamma f(x), \tag{7}$$
$$f_3(x) - \frac{f_3(2x)}{8} = -\frac{1}{192}\Gamma f(x), \tag{8}$$
$$f_1(x) - 2f_1\left(\frac{x}{2}\right) = \frac{1}{12}\Gamma f\left(\frac{x}{2}\right), \tag{9}$$
$$f_2(x) - 4f_2\left(\frac{x}{2}\right) = -\frac{1}{8}\Gamma f\left(\frac{x}{2}\right), \tag{10}$$
$$f_3(x) - 8f_k\left(\frac{x}{2}\right) = \frac{1}{24}\Gamma f\left(\frac{x}{2}\right), \tag{11}$$
$$f(x) = \sum_{k=1}^{3} f_k(x) \tag{12}$$

hold for all $x \in V$ and each $k \in \{1, 2, 3\}$.

Proof. The calculation process is omitted because the equalities (6)–(12) can be shown simply by calculation. □

Lemma 3. *If a mapping $f : V \to W$ satisfies the functional equation $Ef(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$, then the mappings $f_k : V \to W$ satisfies*

$$f_k(2x) = 2^k f_k(x) \tag{13}$$

for all $x \in V$ and each $k \in \{1, 2, 3\}$.

Proof. If a mapping $f : V \to W$ satisfies the functional equation $Ef(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$, then f satisfies the functional equation $\Gamma f(x) = 0$ by (5). Therefore, the equality (13) follows from the equality (6)–(8). □

From now on, let X be a real normed space and Y be a real Banach space.

According to Corollary 6 in [9], we obtain following lemma.

Lemma 4. *For a given mapping $f : V \to Y$, assume that there exist a mapping $F : V \to Y$ and a function $\phi : V \to [0, \infty)$ that satisfy either*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^i}\phi(2^i x) < \infty \quad \textit{or} \tag{14}$$
$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x) + \sum_{i=0}^{\infty} 2^i \phi\left(\frac{1}{2^i}x\right) < \infty \quad \textit{or} \tag{15}$$
$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{8^i}\phi(2^i x) + \sum_{i=0}^{\infty} 4^i \phi\left(\frac{1}{2^i}x\right) < \infty \quad \textit{or} \tag{16}$$
$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} 8^i \phi\left(\frac{1}{2^i}x\right) < \infty \tag{17}$$

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for all $x \in V$. And also if there exist mappings $F_1, F_2, F_3 : V \to Y$ such that $F(x) = \sum_{k=1}^{3} F_k(x)$ and $F_k(2x) = 2^k F_k(x)$ for all $x \in V$ and $k \in \{1,2,3\}$, then the mapping F is uniquely determined.

As a corollary of Lemma 3 and Lemma 4, we obtain the following theorem.

Theorem 3. *For a given mapping $f : V \to Y$, if there exists an additive-quadratic-cubic-mapping $F : V \to Y$ and a function $\phi : V \to [0,\infty)$ that satisfy either (14) or (15) or (16) or (17) for all $x \in V$, then the mapping F is uniquely determined.*

The following inequalities and identities are needed to prove the main theorem.

Lemma 5. *The following inequalities hold:*

$$\left| \frac{1}{12 \cdot 2^i} - \frac{1}{8 \cdot 4^i} + \frac{1}{24 \cdot 8^i} \right| \leq \frac{1}{12 \cdot 2^i}, \tag{18}$$

$$\left| -\frac{1}{8 \cdot 4^i} + \frac{1}{24 \cdot 8^i} \right| \leq \frac{1}{8 \cdot 4^i}, \tag{19}$$

$$\left| \frac{2^i}{12} - \frac{4^i}{8} + \frac{8^i}{24} \right| \leq \frac{8^i}{24}, \tag{20}$$

$$\left| \frac{2^i}{12} - \frac{4^i}{8} \right| \leq \frac{4^i}{8} \tag{21}$$

for all $i \in \mathbb{N} \cup \{0\}$. Moreover, the following identity holds:

$$\frac{2^i}{12} - \frac{4^i}{8} + \frac{8^i}{24} = 0 \tag{22}$$

when $i \in \{0,1\}$.

Proof. The verification of the above identity and inequalities requires only tedious calculations, so the proofs are omitted. □

Now, as the main theorem, we will show generalized stability of the additive-quadratic-cubic functional equation in the sense of Găvruţa.

Theorem 4. *Assume that a function $\varphi : V^4 \to [0,\infty)$ satisfies either*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{2^i} < \infty \quad or \tag{23}$$

$$\sum_{i=0}^{\infty} \left(\frac{\varphi(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{4^i} + 2^i \varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i} \right) \right) < \infty \quad or \tag{24}$$

$$\sum_{i=0}^{\infty} \left(\frac{\varphi(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{8^i} + 4^i \varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i} \right) \right) < \infty \quad or \tag{25}$$

$$\sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i} \right) < \infty \tag{26}$$

for all $x_1, x_2, x_3, x_4 \in V$. If a mapping $f : V \to Y$ satisfies the inequality

$$\|Ef(x_1, x_2, x_3, x_4)\| \leq \varphi(x_1, x_2, x_3, x_4) \tag{27}$$

for all $x_1, x_2, x_3, x_4 \in V$, then there exists a unique additive-quadratic-cubic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{1}{12}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^{i+1}} & (\text{if } \varphi \text{ satisfies } (23)), \\ \frac{1}{12}\sum_{i=0}^{\infty} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{8}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{4^{i+1}} & (\text{if } \varphi \text{ satisfies } (24)), \\ \frac{1}{8}\sum_{i=0}^{\infty} 4^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{24}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{8^{i+1}} & (\text{if } \varphi \text{ satisfies } (25)), \\ \frac{1}{24}\sum_{i=0}^{\infty} 8^i \Phi\left(\frac{x}{2^{i+1}}\right) & (\text{if } \varphi \text{ satisfies } (26)) \end{cases} \tag{28}$$

for all $x \in V$, where $\Phi : V \to [0, \infty)$ is the function defined by

$$\Phi(x) := \varphi(2x, 2x, 2x, 2x) + 4\varphi(2x, 2x, x, x) + 8\varphi(2x, x, x, x) + 8\varphi(x, x, x, x).$$

Proof. From (5), (27), and the definition of Φ, we obtain that

$$\begin{aligned} \|\Gamma f(x)\| =& \|Ef(2x, 2x, 2x, 2x) + 4Ef(2x, 2x, x, x) + 8Ef(2x, x, x, x) + 8Ef(x, x, x, x)\| \\ \leq& \Phi(x) \end{aligned} \tag{29}$$

for all $x \in V$.

(1) If $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (23) and $k \in \{1, 2, 3\}$, then

$$\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^{k(i+1)}} \leq \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^i} < \infty \tag{30}$$

for all $x \in V$. It follows from (6) and (29) that

$$\begin{aligned} \left\| f_1(x) - \frac{f_1(2^m x)}{2^m} \right\| \leq& \sum_{i=0}^{m-1} \left\| \frac{f_1(2^i x)}{2^i} - \frac{f_1(2^{i+1} x)}{2^{i+1}} \right\| \\ =& \frac{1}{12} \sum_{i=0}^{m-1} \left\| \frac{\Gamma f(2^i x)}{2^{i+1}} \right\| \\ \leq& \frac{1}{12} \sum_{i=0}^{m-1} \frac{\Phi(2^i x)}{2^{i+1}} \end{aligned}$$

for all $x \in V$. In the same way, we get the inequalities

$$\begin{aligned} \left\| f_2(x) - \frac{f_2(2^m x)}{4^m} \right\| \leq& \frac{1}{8} \sum_{i=0}^{m-1} \frac{\Phi(2^i x)}{4^{i+1}}, \\ \left\| f_3(x) - \frac{f_3(2^m x)}{8^m} \right\| \leq& \frac{1}{24} \sum_{i=0}^{m-1} \frac{\Phi(2^i x)}{8^{i+1}} \end{aligned}$$

for all $x \in V$ from (7) and (8). Together with the equality

$$\frac{f_k(2^m x)}{2^{km}} - \frac{f_k(2^{m+l} x)}{2^{k(m+l)}} = \sum_{i=m}^{m+l-1} \left(\frac{f_k(2^i x)}{2^{ki}} - \frac{f_k(2^{i+1} x)}{2^{k(i+1)}} \right)$$

for all $x \in V$ and all $k \in \{1,2,3\}$, we obtain the inequalities

$$\left\| \frac{f_1(2^m x)}{2^m} - \frac{f_1(2^{m+l} x)}{2^{m+l}} \right\| \le \frac{1}{12} \sum_{i=m}^{m+l-1} \frac{\Phi(2^i x)}{2^{i+1}}, \tag{31}$$

$$\left\| \frac{f_2(2^m x)}{4^m} - \frac{f_2(2^{m+l} x)}{4^{m+l}} \right\| \le \frac{1}{8} \sum_{i=m}^{m+l-1} \frac{\Phi(2^i x)}{4^{i+1}}, \tag{32}$$

$$\left\| \frac{f_3(2^m x)}{8^m} - \frac{f_3(2^{m+l} x)}{8^{m+l}} \right\| \le \frac{1}{24} \sum_{i=m}^{m+l-1} \frac{\Phi(2^i x)}{8^{i+1}} \tag{33}$$

for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$. It follows from (30), (31), (32), and (33) that the sequence $\left\{\frac{f_k(2^i x)}{2^{ki}}\right\}$ is a Cauchy sequence for all $x \in V$ when $k \in \{1,2,3\}$. Since Y is complete, the sequence $\left\{\frac{f_k(2^i x)}{2^{ki}}\right\}$ converges for all $x \in V$ when $k \in \{1,2,3\}$. Hence we can define a mapping $F_k : V \to Y$ by

$$F_k(x) := \lim_{i\to\infty} \frac{f_k(2^i x)}{2^{ki}}$$

for all $x \in V$ when $k \in \{1,2,3\}$. Since

$$\begin{aligned}
&\|EF_1(x_1,x_2,x_3,x_4)\| \\
&= \lim_{i\to\infty} \left\| \frac{Ef_1(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{2^i} \right\| \\
&= \frac{1}{12} \lim_{i\to\infty} \left\| \frac{Ef(2^{2+i} x_1, 2^{2+i} x_2, 2^{2+i} x_3, 2^{2+i} x_4)}{2^i} \right. \\
&\qquad \left. - \frac{12Ef(2^{1+i} x_1, 2^{1+i} x_2, 2^{1+i} x_3, 2^{1+i} x_4)}{2^i} + \frac{32Ef(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4}{2^i} \right\| \\
&= \frac{1}{12} \lim_{i\to\infty} \left(\frac{\varphi(2^{2+i} x_1, 2^{2+i} x_2, 2^{2+i} x_3, 2^{2+i} x_4)}{2^i} \right. \\
&\qquad \left. + \frac{12\varphi(2^{1+i} x_1, 2^{1+i} x_2, 2^{1+i} x_3, 2^{1+i} x_4)}{2^i} + \frac{32\varphi(2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{2^i} \right) \\
&= 0
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in V$, we obtain the equation $EF_1(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. In the same way, we can obtain the equation $EF_k(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$ when $k \in \{2,3\}$. If we put $F(x) := \sum_{k=1}^{3} F_k(x)$ for all $x \in V$, then $F(x)$ satisfies

$$EF(x_1, x_2, x_3, x_4) = \sum_{k=1}^{3} EF_k(x_1, x_2, x_3, x_4) = 0$$

for all $x_1, x_2, x_3, x_4 \in V$, i.e, F is an additive-quadratic-cubic mapping. From (6), (7), (8), (12), (18), and (29), we get

$$\begin{aligned}
\|f(x) - F(x)\| =& \left\| \sum_{k=1}^{3} (f_k(x) - F_k(x)) \right\| \\
\leq& \sum_{i=0}^{\infty} \left\| \frac{f_1(2^i x)}{2^i} - \frac{f_1(2^{i+1}x)}{2^{i+1}} + \frac{f_2(2^i x)}{4^i} - \frac{f_2(2^{i+1}x)}{4^{i+1}} + \frac{f_3(2^i x)}{8^i} - \frac{f_3(2^{i+1}x)}{8^{i+1}} \right\| \\
=& \sum_{i=0}^{\infty} \left\| \left(\frac{1}{12 \cdot 2^{i+1}} - \frac{1}{8 \cdot 4^{i+1}} + \frac{1}{24 \cdot 8^{i+1}} \right) \Gamma f(2^i x) \right\| \\
\leq& \sum_{i=0}^{\infty} \frac{1}{12} \left\| \frac{\Gamma f(2^i x)}{2^{i+1}} \right\| \\
\leq& \frac{1}{12} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^{i+1}}
\end{aligned}$$

for all $x \in V$. According to Theorem 3, F is a unique additive-quadratic-cubic mapping satisfying the inequality (28) for all $x \in V$.

(2) If $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (24) for all $x \in V$ and $k \in \{2, 3\}$, then

$$\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^{k(i+1)}} \leq \sum_{i=0}^{\infty} \left(\frac{\Phi(2^i x)}{4^i} + 2^i \Phi\left(\frac{x}{2^i}\right) \right) < \infty \tag{34}$$

for all $x \in V$. By using the inequality (34), we can obtain the inequalities (32) and (33) for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$. Hence we can define a mapping $F_k : V \to Y$ by

$$F_k(x) := \lim_{i \to \infty} \frac{f_k(2^i x)}{2^{ki}}$$

for all $x \in V$ when $k \in \{2, 3\}$. Also we can obtain equation

$$EF_2(x_1, x_2, x_3, x_4) = EF_3(x_1, x_2, x_3, x_4) = 0$$

for all $x_1, x_2, x_3, x_4 \in V$.

On the other hand, if $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (24) for all $x \in V$ and $k = 1$, then

$$\sum_{i=0}^{\infty} 2^i \Phi\left(\frac{x}{2^i}\right) \leq \sum_{i=0}^{\infty} \left(\frac{\Phi(2^i x)}{4^i} + 2^i \Phi\left(\frac{x}{2^i}\right) \right) < \infty \tag{35}$$

for all $x \in V$. It follows from (13) and (29) that

$$\begin{aligned}
\left\| f_1(x) - 2^m f_1\left(\frac{x}{2^m}\right) \right\| \leq& \frac{1}{12} \sum_{i=0}^{m-1} \left\| 2^i \Gamma f\left(\frac{x}{2^{i+1}}\right) \right\| \\
\leq& \frac{1}{12} \sum_{i=0}^{m-1} 2^i \Phi\left(\frac{x}{2^{i+1}}\right)
\end{aligned}$$

for all $x \in V$. Together with the equality

$$2^m f_1\left(\frac{x}{2^m}\right) - 2^{m+l} f_1\left(\frac{x}{2^{m+l}}\right) = \sum_{i=m}^{m+l-1} \left(2^i f_1\left(\frac{x}{2^i}\right) - 2^{i+1} f_1\left(\frac{x}{2^{i+1}}\right) \right)$$

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for all $x \in V$, we obtain the inequality

$$\left\|2^m f_1\left(\frac{x}{2^m}\right) - 2^{m+l} f_1\left(\frac{x}{2^{m+l}}\right)\right\| \leq \frac{1}{12} \sum_{i=m}^{m+l-1} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) \tag{36}$$

for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$. From (35) and (36), we can define a mapping $F_1 : V \to Y$ by

$$F_1(x) := \lim_{i\to\infty} 2^i f_1\left(\frac{x}{2^i}\right)$$

for all $x \in V$. We observe that

$$\begin{aligned}
&\|EF_1(x_1, x_2, x_3, x_4)\| \\
&= \lim_{i\to\infty} \left\|2^i E f_1\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right\| \\
&= \frac{1}{12} \lim_{i\to\infty} \left\|2^i E f\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right) - 12 \cdot 2^i E f\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right)\right. \\
&\qquad \left. + 32 \cdot 2^i E f\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right\| \\
&\leq \frac{1}{12} \lim_{i\to\infty} \left(2^i \varphi\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right) + 12 \cdot 2^i \varphi\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right)\right. \\
&\qquad \left. + 32 \cdot 2^i \varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right) \\
&= 0
\end{aligned}$$

by (24), i.e, $EF_1(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. If we put $F(x) := \sum_{k=1}^{3} F_k(x)$ for all $x \in V$, then F is an additive-quadratic-cubic mapping.

From (7), (8), (13), (12), (19), and (29), we get

$$\begin{aligned}
\|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \left\|2^i f_1\left(\frac{x}{2^i}\right) - 2^{i+1} f_1\left(\frac{x}{2^{i+1}}\right)\right. \\
&\qquad \left. + \frac{f_2(2^i x)}{4^i} - \frac{f_2(2^{i+1}x)}{4^{i+1}} + \frac{f_3(2^i x)}{8^i} - \frac{f_3(2^{i+1}x)}{8^{i+1}}\right\| \\
&\leq \sum_{i=0}^{\infty} \left\|\frac{2^i}{12} \Gamma f\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{\infty} \left\|\left(-\frac{1}{8 \cdot 4^{i+1}} + \frac{1}{24 \cdot 8^{i+1}}\right) \Gamma f(2^i x)\right\| \\
&\leq \frac{1}{12} \sum_{i=0}^{\infty} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{8} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{4^{i+1}}
\end{aligned}$$

for all $x \in V$. According to Theorem 3, F is a unique additive-quadratic-cubic mapping satisfying the inequality (28) for all $x \in V$.

(3) If $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (25) for all $x \in V$ and $k = 3$, then

$$\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{8^{i+1}} \leq \sum_{i=0}^{\infty} \left(\frac{\Phi(2^i x)}{8^i} + 4^i \Phi\left(\frac{x}{2^i}\right)\right) < \infty \tag{37}$$

for all $x \in V$. By using the inequality (37), we can obtain the inequality (33) for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$. Hence we can define a mapping $F_3 : V \to Y$ by

$$F_3(x) := \lim_{i\to\infty} \frac{f_3(2^i x)}{8^i}$$

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for all $x \in V$. Also we can obtain the equation

$$EF_3(x_1, x_2, x_3, x_4) = 0$$

for all $x_1, x_2, x_3, x_4 \in V$.

On the other hand, if $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (25) for all $x \in V$ and $k \in \{1, 2\}$, then

$$\sum_{i=0}^{\infty} 2^{ki}\Phi\left(\frac{x}{2^i}\right) \le \sum_{i=0}^{\infty}\left(\frac{\Phi(2^i x)}{8^i} + 4^i\Phi\left(\frac{x}{2^i}\right)\right) < \infty \tag{38}$$

for all $x \in V$. By using the inequality (13), (10), and (38), we can obtain the inequalities (36) and

$$\left\|4^m f_2\left(\frac{x}{2^m}x\right) - 4^{m+l} f_2\left(\frac{x}{2^{m+l}}\right)\right\| \le \frac{1}{8}\sum_{i=m}^{m+l-1} 4^i\Phi\left(\frac{x}{2^{i+1}}\right) \tag{39}$$

for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$.

From (36), (38), and (39), we can define a mapping $F_k : V \to Y$ by

$$F_k(x) := \lim_{i\to\infty} 2^{ki} f_k\left(\frac{x}{2^i}\right)$$

for all $x \in V$ when $k \in \{1, 2\}$. We observe that

$$\begin{aligned}
&\|EF_2(x_1, x_2, x_3, x_4)\| \\
&= \lim_{i\to\infty}\left\|4^i Ef_2\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right\| \\
&= \frac{1}{8}\lim_{i\to\infty}\Bigg\|4^i Ef\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right) \\
&\qquad - 10\cdot 4^i Ef\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right) + 16\cdot 4^i Ef\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\Bigg\| \\
&\le \frac{1}{8}\lim_{i\to\infty}\Bigg(4^i\varphi\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right) + 10\cdot 4^i\varphi\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right) \\
&\qquad + 16\cdot 4^i\varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\Bigg) \\
&= 0
\end{aligned}$$

by (24), i.e, $EF_2(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. In the same way, we can obtain the equation $EF_1(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. If we put $F(x) := \sum_{k=1}^{3} F_k(x)$ for all $x \in V$, then F is an additive-quadratic-cubic mapping.

From (8), (13), (10), (12), (21), and (29), we get

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty}\Bigg\|2^i f_1\left(\frac{x}{2^i}\right) - 2^{i+1} f_1\left(\frac{x}{2^{i+1}}\right) + 4^i f_2\left(\frac{x}{2^i}\right) - 4^{i+1} f_2\left(\frac{x}{2^{i+1}}\right) \\
&\qquad + \frac{f_3(2^i x)}{8^i} - \frac{f_3(2^{i+1}x)}{8^{i+1}}\Bigg\| \\
&\le \sum_{i=0}^{\infty}\left\|\left(\frac{2^i}{12} - \frac{4^i}{8}\right)\Gamma f\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{\infty}\left\|\frac{1}{24\cdot 8^{i+1}}\Gamma f(2^i x)\right\| \\
&\le \frac{1}{8}\sum_{i=0}^{\infty} 4^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{24}\sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{8^{i+1}}
\end{aligned}$$

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for all $x \in V$. According to Theorem 3, F is a unique additive-quadratic-cubic mapping satisfying the inequality (28) for all $x \in V$.

(4) If $\varphi : V^4 \to [0, \infty)$ satisfies the inequality (26) for all $x \in V$ and $k \in \{1,2,3\}$, then

$$\sum_{i=0}^{\infty} 2^{ki}\Phi\left(\frac{x}{2^i}\right) \le \sum_{i=0}^{\infty} 8^i\Phi\left(\frac{x}{2^i}\right) < \infty \tag{40}$$

for all $x \in V$. By using the inequality (13), (10), (11), and (40), we can obtain the inequalities (36), (39), and

$$\left\| 8^m f_3\left(\frac{x}{2^m}x\right) - 8^{m+l} f_3\left(\frac{x}{2^{m+l}}\right)\right\| \le \frac{1}{24}\sum_{i=m}^{m+l-1} 8^i\Phi\left(\frac{x}{2^{i+1}}\right) \tag{41}$$

for all $x \in V$ and $m, l \in \mathbb{N} \cup \{0\}$.

From (36), (39), (40), and (41), we can define a mapping $F_k : V \to Y$ by

$$F_k(x) := \lim_{i\to\infty} 2^{ki} f_k\left(\frac{x}{2^i}\right)$$

for all $x \in V$ when $k \in \{1,2,3\}$. We observe that

$$\begin{aligned}
&\|EF_3(x_1, x_2, x_3, x_4)\| \\
&= \lim_{i\to\infty}\left\| 8^i E f_3\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right\| \\
&= \frac{1}{24}\lim_{i\to\infty}\left\| 8^i E f\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right)\right. \\
&\qquad \left. - 6\cdot 8^i E f\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right) + 8\cdot 8^i E f\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right\| \\
&\le \frac{1}{24}\lim_{i\to\infty}\left(8^i \varphi\left(\frac{4x_1}{2^i}, \frac{4x_2}{2^i}, \frac{4x_3}{2^i}, \frac{4x_4}{2^i}\right)\right. \\
&\qquad \left. + 6\cdot 8^i \varphi\left(\frac{2x_1}{2^i}, \frac{2x_2}{2^i}, \frac{2x_3}{2^i}, \frac{2x_4}{2^i}\right) + 8\cdot 8^i \varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\right)\right) \\
&= 0
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in V$, i.e, $EF_3(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. In the same way, we can obtain the equation $EF_k(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$ when $k \in \{1,2\}$. If we put $F(x) := \sum_{k=1}^{3} F_k(x)$ for all $x \in V$, then F is an additive-quadratic-cubic mapping.

From (8), (13), (10), (12), (20), (22), and (29), we get

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty}\left\| 2^i f_1\left(\frac{x}{2^i}\right) - 2^{i+1} f_1\left(\frac{x}{2^{i+1}}\right) + 4^i f_2\left(\frac{x}{2^i}\right) - 4^{i+1} f_2\left(\frac{x}{2^{i+1}}\right)\right. \\
&\qquad \left. + 8^i f_3\left(\frac{x}{2^i}\right) - 8^{i+1} f_3\left(\frac{x}{2^{i+1}}\right)\right\| \\
&= \left(\sum_{i=0}^{1} + \sum_{i=2}^{\infty}\right)\left\|\left(\frac{2^i}{12} - \frac{4^i}{8} + \frac{8^i}{24}\right)\Gamma f\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\le \frac{1}{24}\sum_{i=2}^{\infty} 8^i\Phi\left(\frac{x}{2^{i+1}}\right)
\end{aligned}$$

for all $x \in V$. According to Theorem 3, F is a unique additive-quadratic-cubic mapping satisfying the inequality (28) for all $x \in V$. □

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From Theorem 4, we obtain the generalized stability of the functional equation $\underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ for all $x, x_1, x_2, x_3, x_4 \in V$ in the sense of Găvruţa.

Corollary 1. *Assume that a function $\bar{\varphi} : V^5 \to [0, \infty)$ satisfies either*

$$\sum_{i=0}^{\infty} \frac{\bar{\varphi}(0, 2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{2^i} < \infty \quad or \tag{42}$$

$$\sum_{i=0}^{\infty} \frac{\bar{\varphi}(0, 2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{4^i} + 2^i \bar{\varphi}\Big(0, \frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\Big) < \infty \quad or \tag{43}$$

$$\sum_{i=0}^{\infty} \frac{\bar{\varphi}(0, 2^i x_1, 2^i x_2, 2^i x_3, 2^i x_4)}{8^i} + 4^i \bar{\varphi}\Big(0, \frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\Big) < \infty \quad or \tag{44}$$

$$\sum_{i=0}^{\infty} 8^i \bar{\varphi}\Big(0, \frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{x_4}{2^i}\Big) < \infty \tag{45}$$

for all $x_1, x_2, x_3, x_4 \in V$. If a mapping $f : V \to Y$ satisfies the inequality

$$\Big\| \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x) \Big\| \le \bar{\varphi}(x, x_1, x_2, x_3, x_4) \tag{46}$$

for all $x, x_1, x_2, x_3, x_4 \in V$, then there exists a unique additive-quadratic-cubic mapping $F : V \to Y$ such that

$$\|\tilde{f}(x) - F(x)\| \le \begin{cases} \frac{1}{12} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^{i+1}} & (\text{if } \bar{\varphi} \text{ satisfies } (42)), \\ \frac{1}{12} \sum_{i=0}^{\infty} 2^i \Phi\Big(\frac{x}{2^{i+1}}\Big) + \frac{1}{8} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{4^{i+1}} & (\text{if } \bar{\varphi} \text{ satisfies } (43)), \\ \frac{1}{8} \sum_{i=0}^{\infty} 4^i \Phi\Big(\frac{x}{2^{i+1}}\Big) + \frac{1}{24} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{8^{i+1}} & (\text{if } \bar{\varphi} \text{ satisfies } (44)), \\ \frac{1}{24} \sum_{i=0}^{\infty} 8^i \Phi\Big(\frac{x}{2^{i+1}}\Big) & (\text{if } \bar{\varphi} \text{ satisfies } (45)) \end{cases} \tag{47}$$

for all $x \in V$, where $\tilde{f}(x)$ is the mapping given by $\tilde{f}(x) = f(x) - f(0)$ for all $x \in V$ and $\Phi : V \to [0, \infty)$ is the function defined by

$$\Phi(x) := \bar{\varphi}(0, 2x, 2x, 2x, 2x) + 4\bar{\varphi}(0, 2x, 2x, x, x) + 8\bar{\varphi}(0, 2x, x, x, x) + 8\bar{\varphi}(0, x, x, x, x).$$

Proof. Let $\varphi : V^4 \to (0, \infty)$ be the function given by $\varphi(x_1, x_2, x_3, x_4) = \bar{\varphi}(0, x_1, x_2, x_3, x_4)$ for all $x_1, x_2, x_3, x_4 \in V$ and let $\tilde{f}(x)$ be the mapping given by $\tilde{f}(x) = f(x) - y_0$ for all $x \in V$, where $y_0 = f(0) \in Y$. Since

$$\underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} \tilde{f}(x) = \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x) - \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} y_0 = \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x)$$

for all $x, x_1, x_2, x_3, x_4 \in V$ with $\tilde{f}(0) = 0$, we know that

$$E\tilde{f}(x_1, x_2, x_3, x_4) = \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} \tilde{f}(0) + \tilde{f}(0) = \underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(0)$$

for all $x_1, x_2, x_3, x_4 \in V$. From inequalities (42), (43), (44), (45), and (46), $\tilde{f}$ and φ satisfy inequalities (23), (24), (25), (26), and (27) in Theorem 4, so there exists a unique additive-quadratic-cubic mapping $F : V \to Y$ that satisfies the inequality (47). □

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From Theorem 4, we obtain the Hyers-Ulam-Rassias stability of the additive-quadratic-cubic functional equation (1) in the sense of Rassias.

Theorem 5. *Let θ and p be positive real constants such that $p \neq 1,2,3$. If $f : X \to Y$ satisfies the inequality*

$$\left\|Ef(x_1, x_2, x_3, x_4)\right\| \leq \theta(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p)$$

for all $x_1, x_2, x_3, x_4 \in X$, then there exists a unique additive-quadratic-cubic mapping F such that

$$\|f(x) - F(x)\| \leq \begin{cases} \dfrac{1}{12(2-2^p)}(20 \cdot 2^p + 64)\theta\|x\|^p & (\text{if } \ p < 1), \\ \left(\dfrac{1}{12(2^p-2)} + \dfrac{1}{8(4-2^p)}\right)(20 \cdot 2^p + 64)\theta\|x\|^p & (\text{if } \ 1 < p < 2), \\ \left(\dfrac{1}{8(2^p-4)} + \dfrac{1}{24(8-2^p)}\right)(20 \cdot 2^p + 64)\theta\|x\|^p & (\text{if } \ 2 < p < 3), \\ \dfrac{8}{3 \cdot 4^p(2^p-8)}(20 \cdot 2^p + 64)\theta\|x\|^p & (\text{if } \ 3 < p) \end{cases}$$

for all $x \in X$.

Proof. If we substitute $\theta(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p)$ for $\varphi(x_1, x_2, x_3, x_4)$ in Theorem 4, then we get $\Phi(x) = (20 \cdot 2^p + 64)\theta\|x\|^p$. From this, we can easily obtain this theorem as a corollary of Theorem 4. □

From Theorem 4, we obtain the stability of the additive-quadratic-cubic functional equation (1) in the sense of Hyers.

Theorem 6. *Let δ be a positive real constant. If $f : V \to Y$ satisfies the inequality*

$$\|Ef(x_1, x_2, x_3, x_4)\| \leq \delta$$

for all $x_1, x_2, x_3, x_4 \in V$, then there exists a unique additive-quadratic-cubic mapping F such that

$$\|f(x) - F(x)\| \leq \frac{7}{4}\delta$$

for all $x \in X$.

Proof. If we substitute δ for $\phi(x_1, x_2, x_3, x_4)$ in Theorem 4, then we get $\Phi(x) = 21\delta$. Applying the case in Theorem 4 where p satisfies (23), we obtain this theorem. □

From Corollary 1, we obtain the Hyers-Ulam-Rassias stability of the functional equation $\underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ in the sense of Rassias.

Corollary 2. *Let θ and p be positive real numbers such that $p \neq 1,2,3$. If $f : X \to Y$ satisfies the inequality*

$$\left\|\underset{x_1x_2x_3x_4}{\Delta\Delta\Delta\Delta} f(x)\right\| \leq \theta(\|x\|^p + \|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_4\|^p)$$

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for all $x, x_1, x_2, x_3, x_4 \in X$, then there exists a unique additive-quadratic-cubic mapping F such that

$$\|\tilde{f}(x) - F(x)\| \leq \begin{cases} \frac{1}{12(2-2^p)}(20\cdot 2^p + 64)\theta\|x\|^p & (\text{if } p < 1), \\ \left(\frac{1}{12(2^p-2)} + \frac{1}{8(4-2^p)}\right)(20\cdot 2^p + 64)\theta\|x\|^p & (\text{if } 1 < p < 2), \\ \left(\frac{1}{8(2^p-4)} + \frac{1}{24(8-2^p)}\right)(20\cdot 2^p + 64)\theta\|x\|^p & (\text{if } 2 < p < 3), \\ \frac{8}{3\cdot 4^p(2^p-8)}(20\cdot 2^p + 64)\theta\|x\|^p & (\text{if } 3 < p) \end{cases}$$

for all $x \in X$.

From Corollary 1, we obtain the stability of the functional equation $\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x) = 0$ in the sense of Hyers.

Corollary 3. *Let δ be a positive real constant. If a mapping $f : V \to Y$ satisfies the inequality*

$$\left\|\underset{x_1 x_2 x_3 x_4}{\Delta\Delta\Delta\Delta} f(x)\right\| \leq \delta$$

for all $x, x_1, x_2, x_3, x_4 \in V$, then there exists a unique additive-quadratic-cubic mapping F such that

$$\|\tilde{f}(x) - F(x)\| \leq \frac{7}{4}\delta$$

for all $x \in X$.

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