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Article

Wavefront Fitting Over Arbitrary Freeform Apertures via CSF-Guided Progressive Quasi-Conformal Mapping

Tong Yang *, Chengxiang Guo, Lei Yang and Hongbo Xie *

School of Precision Instruments and Opto-electronics Engineering, Tianjin University, Key Laboratory of Optoelectronics Information Technology, Ministry of Education, Tianjin, 300072, China

* Correspondence: tongy@tju.edu.cn (T.Y.); hbxie@tju.edu.cn (H.X.)

Abstract

In freeform optical metrology, wavefront fitting over non-circular apertures is hindered by the loss of Zernike polynomial orthogonality and severe sampling grid distortion inherent in standard conformal mappings. To address the resulting numerical instability and fitting bias, we propose a unified framework curve shortening flow (CSF)-guided progressive quasi-conformal mapping (CSF-QCM), which integrates geometric boundary evolution with topology-aware parameterization. CSF-QCM first smooths complex boundaries via curve-shortening flow, then solves a sparse Laplacian system for harmonic interior coordinates, thereby establishing a stable diffeomorphism between physical and canonical domains. For doubly connected apertures, it preserves topology by computing the conformal modulus via Dirichlet energy minimization and simultaneously mapping both boundaries. Benchmarked against state-of-the-art methods (e.g., Fornberg, Schwarz-Christoffel and Ricci flow) on representative irregular apertures, CSF-QCM suppresses area distortion and restores discrete orthogonality of the Zernike basis, reducing the Gram matrix condition number from >900 to <8 . This enables high-precision reconstruction with RMS residuals as low as $3 \times 10^{-3} \lambda$ and up to 92% lower fitting errors than baselines. The framework provides a unified, computationally efficient, and numerically stable solution for wavefront reconstruction in complex off-axis and freeform optical systems.

Keywords: freeform surface metrology; wavefront fitting; quasi-conformal mapping; curve shortening flow; non-circular aperture

1. Introduction

Advanced optical manufacturing increasingly demands nanometer surface accuracy for freeform optical elements [1,2]. Wavefront testing is therefore a critical component of the metrology pipeline, and reconstruction accuracy directly affects the reliability of closed-loop fabrication. Conventional wavefront analysis commonly relies on Zernike polynomials defined on the unit disk, where orthogonality and physical interpretability are advantageous [3–5]. However, for arbitrary freeform apertures (e.g., non-convex regions, high-aspect-ratio pupils, or apertures with obscurations), a mismatch between the physical support and the unit disk induces significant modal crosstalk and coefficient coupling [6–8].

Two broad strategies have been used to mitigate these limitations.

First, customized orthogonal bases can be constructed by Gram–Schmidt orthogonalization or singular value decomposition (SVD) on sampled apertures [9,10]. Recent developments in this direction have further optimized orthogonal fitting algorithms for aberration removal on arbitrary shaped apertures [11] and data-driven approaches utilizing deep neural networks [12] have emerged. While effective, numerical bases are aperture-specific and complicate standardization, whereas deep learning models often lack physical interpretability and require extensive training datasets.

Second, conformal maps, such as the Fornberg algorithm [13], variants for slender regions [14], or the Schwarz–Christoffel transformation [15], can map a non-circular region to a canonical domain

before Zernike fitting. Notably, the Schwarz–Christoffel mapping has been recently adapted for modal wavefront reconstruction on noncircular pupils[16], demonstrating the enduring relevance of mapping-based approaches. However, standard conformal maps may be numerically fragile on non-smooth boundaries due to derivative singularities (the crowding phenomenon)[17], producing severe grid distortion.

Although advances in computational geometry, such as Ricci flow [18,19] and optimal mass transport [20], have improved mesh quality for complex topologies, for doubly connected apertures (e.g., annular pupils), mapping must also respect topological invariants; particularly, preserving the conformal modulus is necessary to avoid excessive shear.

Quasi-conformal mapping (QCM) provides additional flexibility by allowing for bounded angular distortion, which is quantitatively characterized by the Beltrami coefficient [21]. When the distortion magnitude is bounded ($\|\mu\|_\infty < 1$), the mapping remains a homeomorphism; the conformal case is recovered when $\mu \equiv 0$. However, leveraging this flexibility to automatically construct a low-distortion, topology-consistent, and boundary-robust quasi-conformal parameterization for arbitrary freeform apertures remains a significant challenge.

This paper introduces a CSF-guided progressive quasi-conformal mapping framework (CSF-QCM). By coupling boundary smoothing through geometric evolution with topology-aware canonicalization and interior harmonic relaxation, the method yields a unified low-distortion parameterization for simply and doubly connected apertures. The resulting diffeomorphic map improves sampling uniformity on the canonical domain, reduces design-matrix ill-conditioning, suppresses boundary ringing artifacts, and improves numerical stability in wavefront fitting.

2. Methodology

2.1. Notation and Problem Setup

Let $\Omega_{\text{phy}} \subset \mathbb{R}^2$ denote the physical aperture domain, sampled on a triangular mesh with vertices $\mathcal{V}$ and faces $\mathcal{F}$ using standard discrete geometric processing techniques. The boundary may be simply connected (only an outer boundary $\partial\Omega_{\text{out}}$) or doubly connected (an outer boundary $\partial\Omega_{\text{out}}$ and an inner boundary $\partial\Omega_{\text{in}}$ representing an obscuration).

We seek a diffeomorphic parameterization $\Phi : \Omega_{\text{phy}} \rightarrow \Omega_{\text{can}}$, where Ω_{can} is a canonical domain:

- Simply connected: $\Omega_{\text{can}} = \mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$ (unit disk).
- Doubly connected: $\Omega_{\text{can}} = \mathbb{A} = \{w \in \mathbb{C} : R_{\text{in}} \leq |w| \leq 1\}$ (concentric annulus).

Wavefront measurements are given as samples $W(x_i, y_i)$ on Ω_{phy} . After mapping to Ω_{can} , we fit W using an orthogonal basis on Ω_{can} .

2.2. Framework overview

The CSF-QCM pipeline (Figure 1) consists of four distinct steps:

1. **Boundary geometric regularization (CSF).** We evolve the physical boundary $\partial\mathcal{M}$ under discrete Curve Shortening Flow. This iterative process smooths high-curvature features (e.g., sharp corners of a star-shaped aperture) and generates a sequence of regularized boundary frames that converge to a canonical circle, minimizing grid crowding effects.
2. **Topology-aware canonical domain specification.** The target domain Ω_{can} is determined by the aperture's topology. For simply connected domains, Ω_{can} is set to the unit disk $\mathbb{D}$. For doubly connected domains (e.g., apertures with central obscurations like a triangular hole), we solve the harmonic equation to compute the Dirichlet energy and derive the conformal modulus. This uniquely defines the inner radius R_{in} of the target annulus $\mathbb{A}$, ensuring a conformal bijection.
3. **Interior mesh optimization via Laplacian construction.** We construct the discrete Laplacian matrix $\mathbf{L}$ using cotangent weights to approximate the harmonic energy. The interior parameterization is obtained by solving the sparse linear system $\mathbf{L}\mathbf{u} = -\mathbf{L}_{\text{bd}}\mathbf{u}_{\text{bd}}$, where the boundary conditions $\mathbf{u}_{\text{bd}}$ are updated progressively using the CSF frames. This step relaxes the internal mesh vertices to minimize angular distortion.

4. **Wavefront resampling and orthogonal fitting.** Using the computed bijection Φ , we map the wavefront data into the canonical domain. Depending on the topology determined in step 2, the wavefront is fitted using standard Zernike polynomials (for disk $\mathbb{D}$) or Annular Zernike polynomials (for annulus $\mathbb{A}$), allowing for high-precision reconstruction over arbitrary free-form apertures.

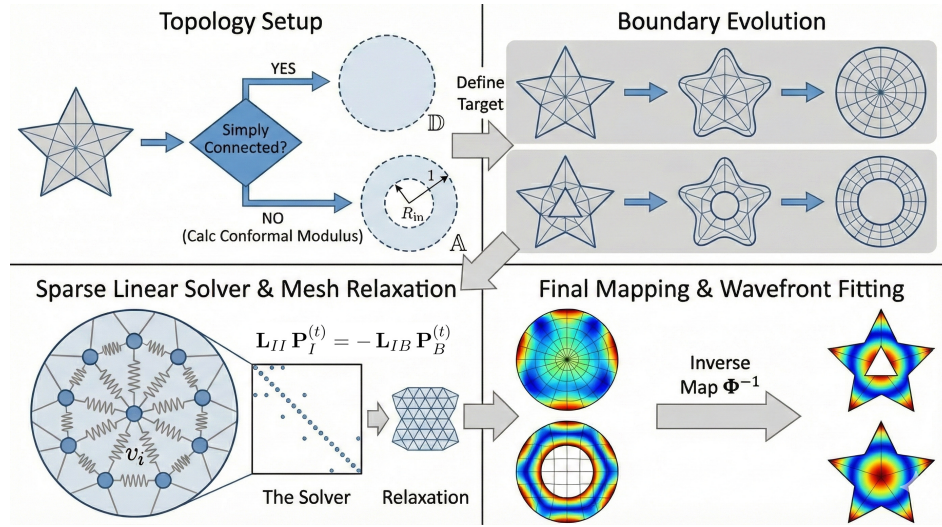


Figure 1. Pipeline of the CSF-QCM framework

2.3. Boundary Regularization via Curve Shortening Flow

Conformal maps can be numerically fragile on non-smooth boundaries because corners and sharp curvature variations induce large derivatives and crowding. We therefore smooth the boundary geometry prior to parameterization.

Let $\mathcal{C}(u, t) : S^1 \times [0, T] \rightarrow \mathbb{R}^2$ denote a parameterized boundary curve, where t is evolution time and $\mathbf{N}(u, t)$ is the inward unit normal. Under curve shortening flow (CSF), the boundary evolves by[22]

$$\frac{\partial \mathcal{C}(u, t)}{\partial t} = \kappa(u, t) \mathbf{N}(u, t), \quad (1)$$

where $\kappa(u, t)$ is curvature. Intuitively, high-curvature regions move faster, smoothing the curve.

Stopping criterion.

Because CSF ultimately shrinks embedded plane curves to a round point, the evolution must be stopped early. We monitor a circularity deviation metric $E_{\text{circ}}^{(k)}$ (defined on the discrete boundary at iteration k) and terminate when the relative change is sufficiently small:

$$\left| \frac{E_{\text{circ}}^{(k+1)} - E_{\text{circ}}^{(k)}}{E_{\text{circ}}^{(k)} + \varepsilon} \right| < \epsilon_{\text{tol}}, \quad (2)$$

where $\epsilon_{\text{tol}} = 10^{-6}$ and ε is a small constant to avoid division by zero. This criterion indicates that high-frequency boundary irregularities have been sufficiently attenuated while preserving topology.

For doubly connected apertures, the same evolution is applied to both $\partial\Omega_{\text{out}}$ and $\partial\Omega_{\text{in}}$ (with appropriate inward normals defined with respect to the domain).

2.4. Progressive Quasi-Conformal Mapping and Distortion

Let $z = x + iy$ denote the complex coordinate in the physical plane and $w = \Phi(z)$ denote the canonical coordinate. The Beltrami coefficient of Φ is defined by[23]

$$\mu_{\Phi}(z) = \frac{\partial\Phi/\partial\bar{z}}{\partial\Phi/\partial z}, \quad \|\mu_{\Phi}\|_{\infty} < 1, \quad (3)$$

where $\mu_{\Phi} \equiv 0$ corresponds to conformality.

Progressive strategy.

To avoid instability under large deformations, we decompose the mapping into a sequence of small transitions. The intermediate boundary states are provided by the CSF frames. At each frame t , we impose Dirichlet boundary positions on $\partial\Omega_{\text{can}}$ and compute interior vertex positions by harmonic relaxation. To ensure mesh robustness and prevent triangle flips, we employ the *graph Laplacian*[24], denoted by $\mathbf{L}$, which is used here purely as a numerical smoothing operator rather than a geometric Laplace–Beltrami discretization.

The interior coordinates are solved via the linear system:

$$\mathbf{L}_{II} \mathbf{P}_I^{(t)} = -\mathbf{L}_{IB} \mathbf{P}_B^{(t)}, \quad (4)$$

where $\mathbf{P}_B^{(t)}$ stores the fixed boundary vertex coordinates on $\partial\Omega_{\text{can}}$, and $\mathbf{P}_I^{(t)}$ stores the unknown interior coordinates. The matrices $\mathbf{L}_{II}$ and $\mathbf{L}_{IB}$ are the sub-blocks of $\mathbf{L}$ corresponding to the interior (I) and boundary (B) vertex indices, respectively.

The rationale for employing the Laplacian operator lies in its connection to Dirichlet energy minimization. The solution to the harmonic equation $\Delta u = 0$ corresponds to the configuration that minimizes the stretching energy of the mapping, analogous to the equilibrium state of a stretched rubber sheet or a spring network.

In our framework, once the boundary geometric tension is released by the CSF evolution, the Laplacian solver naturally propagates this relaxation into the domain interior. By minimizing the local Dirichlet energy, the Laplacian operator promotes smoothness and uniformity in the coordinate field. This intrinsic smoothing property prevents the formation of singular gradients, thereby ensuring that the resulting Beltrami coefficient μ_{Φ} —which depends on the coordinate derivatives—remains low and spatially continuous.

Boundary correspondence.

Rather than prescribing point-to-point matching manually, boundary correspondence is induced by CSF vertex trajectories with arc-length redistribution. This allocates parameter-space resolution to regions that are geometrically difficult (high curvature or near concavities), mitigating the crowding problem typical of static conformal maps.

Area uniformity and monitoring.

We monitor distortion using both $|\mu_{\Phi}|$ (angular distortion) and an area-scaling statistic derived from the Jacobian determinant $J_{\Phi}(z) = |\det D\Phi(z)|$. In discrete form, J_{Φ} is evaluated per triangle as the area ratio between mapped and physical triangles. Ideally, the mapping should preserve the area element to maintain the orthogonality of the polynomial basis defined on the physical domain. Significant variations in the Jacobian determinant J_{Φ} would effectively introduce a non-uniform weight function in the inner product integral, destroying the orthogonality and necessitating computationally expensive re-orthogonalization processes like the Gram-Schmidt procedure[9]. Therefore, maintaining a quasi-uniform J_{Φ} allows us to use standard bases with improved discrete orthogonality under uniform sampling.

2.5. Topology-Aware Canonical Annulus for Doubly Connected Domains

For an aperture with an internal obscuration, mapping to the unit disk is topologically invalid[25]. We therefore map to a concentric annulus

$$\mathbb{A} = \{w \in \mathbb{C} : R_{\text{in}} \leq |w| \leq 1\}.$$

The inner radius R_{in} is determined by the *conformal modulus* of the physical domain. Assigning an arbitrary radius would introduce significant shear distortion[26].

We compute this conformal invariant by solving a harmonic Dirichlet problem on the physical domain Ω_{phy} [27,28]:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega_{\text{phy}}, \\ u = 0, & \text{on } \partial\Omega_{\text{out}}, \\ u = 1, & \text{on } \partial\Omega_{\text{in}}. \end{cases} \quad (5)$$

On a triangular mesh, we discretize the Laplace-Beltrami operator Δ using the cotangent weights[29], leading to the stiffness matrix $\mathbf{W}$. We then solve the resulting sparse linear system associated with $\mathbf{W}$ under Dirichlet boundary constraints. The corresponding solution minimizes the Dirichlet energy, defined as follows[30]:

$$E = \int_{\Omega_{\text{phy}}} \|\nabla u\|^2 d\sigma \approx \mathbf{u}^T \mathbf{W} \mathbf{u}. \quad (6)$$

For the canonical annulus, the analytic energy is given by $E = 2\pi / \ln(1/R_{\text{in}})$. By equating the discrete and analytic energies, we obtain the unique radius:

$$R_{\text{in}} = \exp\left(-\frac{2\pi}{E}\right). \quad (7)$$

This ensures the canonical annulus preserves the conformal modulus of the original domain.

2.6. Wavefront Reconstruction and Numerical Stability

After obtaining $\Phi : \Omega_{\text{phy}} \rightarrow \Omega_{\text{can}}$, wavefront samples are mapped as[31]

$$w_i = \Phi(x_i, y_i), \quad W_i = W(x_i, y_i).$$

We then fit W on the canonical domain using an orthogonal basis.

Simply connected (disk).

On $\mathbb{D}$, we use the standard Zernike basis $\{Z_j\}_{j=1}^M$ (Noll indexing)[4]. The expansion is

$$W \approx \sum_{j=1}^M c_j Z_j(w), \quad w \in \mathbb{D}. \quad (8)$$

Doubly connected (annulus).

On $\mathbb{A}$, we use annular Zernike polynomials $\{Z_j^{\text{ann}}(\cdot; R_{\text{in}})\}_{j=1}^M$, which reduce to standard Zernike polynomials as $R_{\text{in}} \rightarrow 0$ [32]. The expansion is

$$W \approx \sum_{j=1}^M c_j Z_j^{\text{ann}}(w; R_{\text{in}}), \quad w \in \mathbb{A}. \quad (9)$$

Discrete least squares.

With N samples, define the design matrix $\mathbf{H} \in \mathbb{R}^{N \times M}$ by

$$H_{ij} = \Psi_j(w_i),$$

where Ψ_j denotes either Z_j (disk) or $Z_j^{\text{ann}}(\cdot; R_{\text{in}})$ (annulus). The coefficients are obtained by a numerically stable least-squares solve (QR or SVD):

$$\mathbf{c} = \arg \min_{\mathbf{c} \in \mathbb{R}^M} \|\mathbf{H}\mathbf{c} - \mathbf{W}\|_2^2. \quad (10)$$

Conditioning and discrete orthogonality.

In exact arithmetic with continuous uniform sampling, these bases are orthogonal on their canonical domains. In discrete computations, stability is governed by the conditioning of $\mathbf{H}$ (or the Gram matrix $\mathbf{G} = \mathbf{H}^T \mathbf{H}$). Severe area distortion (high variance of J_Φ) effectively introduces non-uniform sampling weights on Ω_{can} , degrading discrete orthogonality and increasing $\kappa(\mathbf{H})$ [6]. CSF-QCM leverages bounded quasi-conformal relaxation to mitigate extreme area distortion, thereby improving conditioning and preventing error amplification.

The overall procedure is summarized in Algorithm 1.

3. Results

We evaluate three representative freeform aperture types exhibiting pronounced geometric challenges, corresponding to specific optical metrology scenarios:

1. **Type I:** a non-convex butterfly-shaped aperture, typical of the irregular interference regions encountered in speckle metrology, as shown in Figure 2(a);
2. **Type II:** a high-aspect-ratio rounded rectangle, representing the geometry of primary or secondary mirrors in wide-field-of-view off-axis three-mirror anastigmat (TMA) systems, as shown in Figure 2(b);
3. **Type III:** a highly eccentric doubly connected annulus, modeling the pupil in fundus aberration interferometry where the central macular region creates an off-center obscuration, as shown in Figure 2(c).

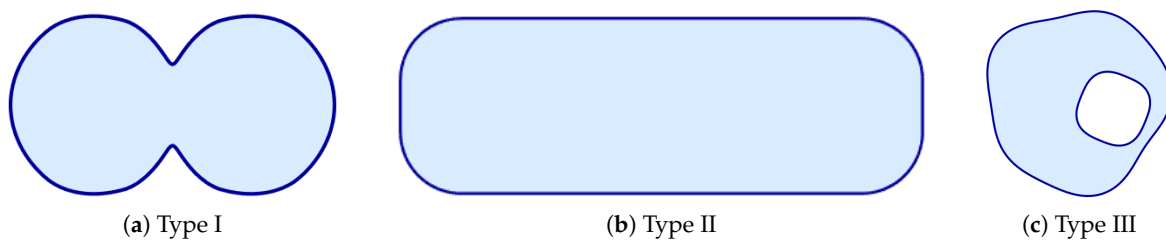


Figure 2. Benchmark aperture geometries for validation. (a) Type I (Butterfly-shaped), (b) Type II (Rounded Rectangle), (c) Type III (Annulus doubly Connected)

3.1. Mesh Distribution and Sampling Uniformity

In mapping-based wavefront fitting, local cell areas determine discrete quadrature weights and influence numerical stability. Excessive area compression or stretching leads to oversampling or undersampling and degrades least-squares conditioning.

3.1.1. Visual Assessment of Parameterization Meshes

Figure 3 illustrates that strict conformality can impose geometric rigidity, producing highly non-uniform meshes on irregular boundaries:

- **Type I (Butterfly):** Fornberg-type conformal mapping exhibits strong crowding near concave regions, yielding redundant sampling.
- **Type II (Rounded rectangle):** Schwarz–Christoffel (SC) mapping compresses the grid near the ends of the long axis due to crowding, producing disproportionate sampling.
- **Type III (Annulus):** Discrete Ricci flow preserves angles but can introduce severe area distortion in narrow eccentric gaps.

By contrast, CSF-QCM smooths high-frequency boundary features and redistributes arc length during CSF evolution, resulting in quasi-uniform meshes across all cases while preserving topology.

Algorithm 1 CSF-Guided Progressive Quasi-Conformal Mapping (CSF-QCM)

Require: Physical domain mesh Ω_{phy} with vertices $\mathcal{V}$ and faces $\mathcal{F}$; Wavefront samples vector $\mathbf{W}$; Boundary definition $\partial\Omega_{\text{out}}$ (and $\partial\Omega_{\text{in}}$ if doubly connected); Tolerance ϵ_{tol} .

Ensure: Canonical mapping Φ , Fitted coefficients $\mathbf{c}$.

Step 1: Topology-Aware Target Specification ▷ See Sec. 2.5

- 1: **if** domain is simply connected **then**
- 2: Set canonical domain $\Omega_{\text{can}} \leftarrow \mathbb{D}$ (Unit Disk)
- 3: **else** ▷ Doubly connected case
- 4: Solve harmonic Eq. (5): $\Delta u = 0$ s.t. $u|_{\partial\Omega_{\text{out}}} = 0, u|_{\partial\Omega_{\text{in}}} = 1$
- 5: Compute Dirichlet energy $E \approx \mathbf{u}^T \mathbf{W} \mathbf{u}$ ▷ Eq. (6)
- 6: Compute conformal modulus: $R_{\text{in}} \leftarrow \exp(-2\pi/E)$ ▷ Eq. (7)
- 7: Set canonical domain $\Omega_{\text{can}} \leftarrow \mathbb{A}$ (Annulus with radii $R_{\text{in}}, 1$)
- 8: **end if**

Step 2: Boundary Geometric Regularization (CSF) ▷ See Sec. 2.3

- 9: Initialize $k \leftarrow 0$, boundary configuration $\mathcal{C}^{(0)}$
- 10: **repeat**
- 11: Compute curvature κ and normals $\mathbf{N}$
- 12: Evolve boundary: $\frac{\partial \mathcal{C}}{\partial t} \leftarrow \kappa \mathbf{N}$ ▷ Eq. (1)
- 13: Update circularity metric $E_{\text{circ}}^{(k+1)}$
- 14: $k \leftarrow k + 1$
- 15: **until** Stopping criterion Eq. (2) is met: $|\Delta E_{\text{circ}}| < \epsilon_{\text{tol}}$
- 16: Store sequence of boundary frames $\{\mathcal{C}^{(t)}\}_{t=0}^k$

Step 3: Progressive Interior Optimization ▷ See Sec. 2.4

- 17: Construct Graph Laplacian $\mathbf{L}$
- 18: **for** each time step $t = 0 \dots k$ **do**
- 19: Update boundary positions $\mathbf{P}_B^{(t)}$ mapping $\mathcal{C}^{(t)}$ to $\partial\Omega_{\text{can}}$
- 20: Solve linear system for interior positions $\mathbf{P}_I^{(t)}$:
- 21: $\mathbf{L}_{II} \mathbf{P}_I^{(t)} = -\mathbf{L}_{IB} \mathbf{P}_B^{(t)}$ ▷ Eq. (4)
- 22: Compute Beltrami coefficient μ_Φ for validation ▷ Eq. (3)
- 23: **end for**
- 24: Construct final map Φ from $\mathbf{P}^{(k)}$

Step 4: Wavefront Fitting ▷ See Sec. 2.6

- 25: Map sample points: $w_i \leftarrow \Phi(x_i, y_i)$
 - 26: **if** $\Omega_{\text{can}} = \mathbb{D}$ **then**
 - 27: Construct $\mathbf{H}$ using Standard Zernike $\{Z_j\}$ ▷ Eq. (8)
 - 28: **else**
 - 29: Construct $\mathbf{H}$ using Annular Zernike $\{Z_j^{\text{ann}}(\cdot; R_{\text{in}})\}$ ▷ Eq. (9)
 - 30: **end if**
 - 31: Solve $\mathbf{c} = \arg \min \|\mathbf{H}\mathbf{c} - \mathbf{W}\|_2^2$ via QR/SVD
 - 32: **return** $\Phi, \mathbf{c}$
-

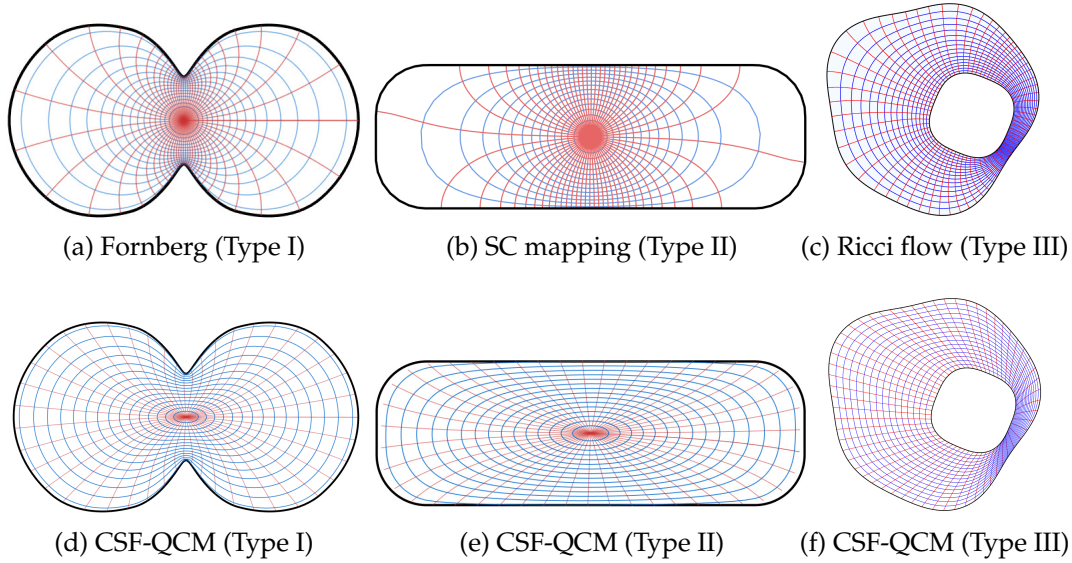


Figure 3. Comparison of parameterization meshes on physical domains. Top row (a–c): baseline conformal approaches with visible crowding or stretching. Bottom row (d–f): CSF-QCM achieves improved global area uniformity.

3.1.2. Quantitative Statistics of Area Distortion

For each triangle τ , define an area scaling factor

$$s_\tau = \frac{\text{Area}(\Phi(\tau))}{\text{Area}(\tau)} \approx J_\Phi.$$

Uniform sampling on Ω_{can} is promoted when the distribution of s_τ avoids extreme tails. We summarize the distribution using the empirical cumulative distribution function (ECDF). Curves closer to a steep transition around a constant scale indicate more uniform area scaling.

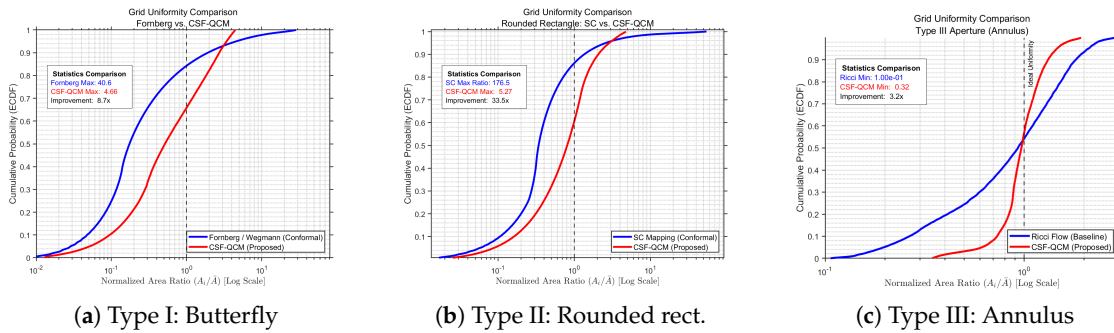


Figure 4. ECDF comparison of the local area scaling factor s_τ . Curves with reduced tail spread indicate improved area uniformity.

3.2. Characterization of Quasi-Conformal Distortion and Discrete Orthogonality

The key mechanism of CSF-QCM is to implicitly accommodate local angular distortion to improve global area uniformity. This section examines the Beltrami coefficient distribution as a post-mapping indicator and the resulting recovery of discrete orthogonality.

3.2.1. Beltrami Coefficient Distribution

Strict conformality enforces $\mu_\Phi \equiv 0$, which can be overly restrictive on irregular boundaries. CSF-QCM relaxes this constraint via progressive quasi-conformal deformation. In geometrically constrained regions (e.g., concave turns in Type I or long-edge endpoints in Type II), moderate $|\mu_\Phi|$ values arise to relieve boundary-induced tension and reduce area distortion, while maintaining $|\mu_\Phi| < 1$ to preserve

diffeomorphism. Figure 5 illustrates the spatial distributions (a–c) and statistical histograms (d–f) of the Beltrami coefficients for all three aperture types.

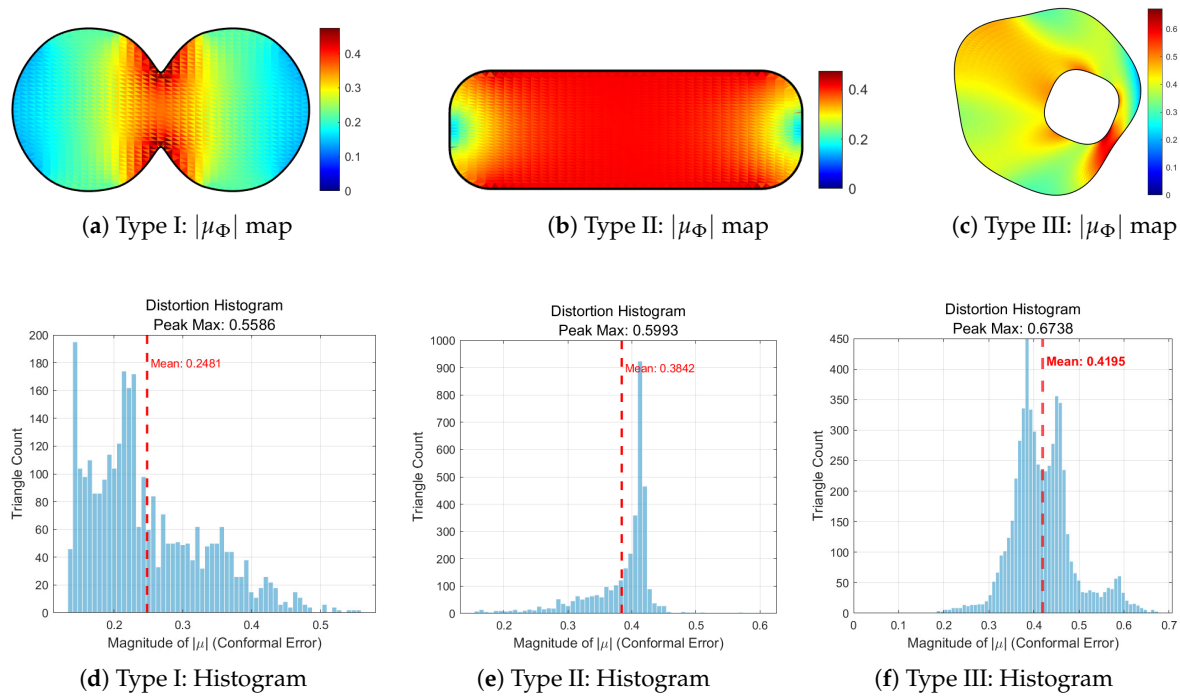


Figure 5. Analysis of Beltrami coefficients. Top row (a)–(c): Spatial distributions of the Beltrami magnitude $|\mu_\Phi|$ for the three aperture types. Bottom row (d)–(f): Corresponding statistical histograms. The distribution of $|\mu_\Phi|$ characterizes the degree of quasi-conformal relaxation: non-zero values emerge in particular regions to accommodate the geometry, facilitating global area uniformity while preserving the diffeomorphic property ($|\mu_\Phi| < 1$).

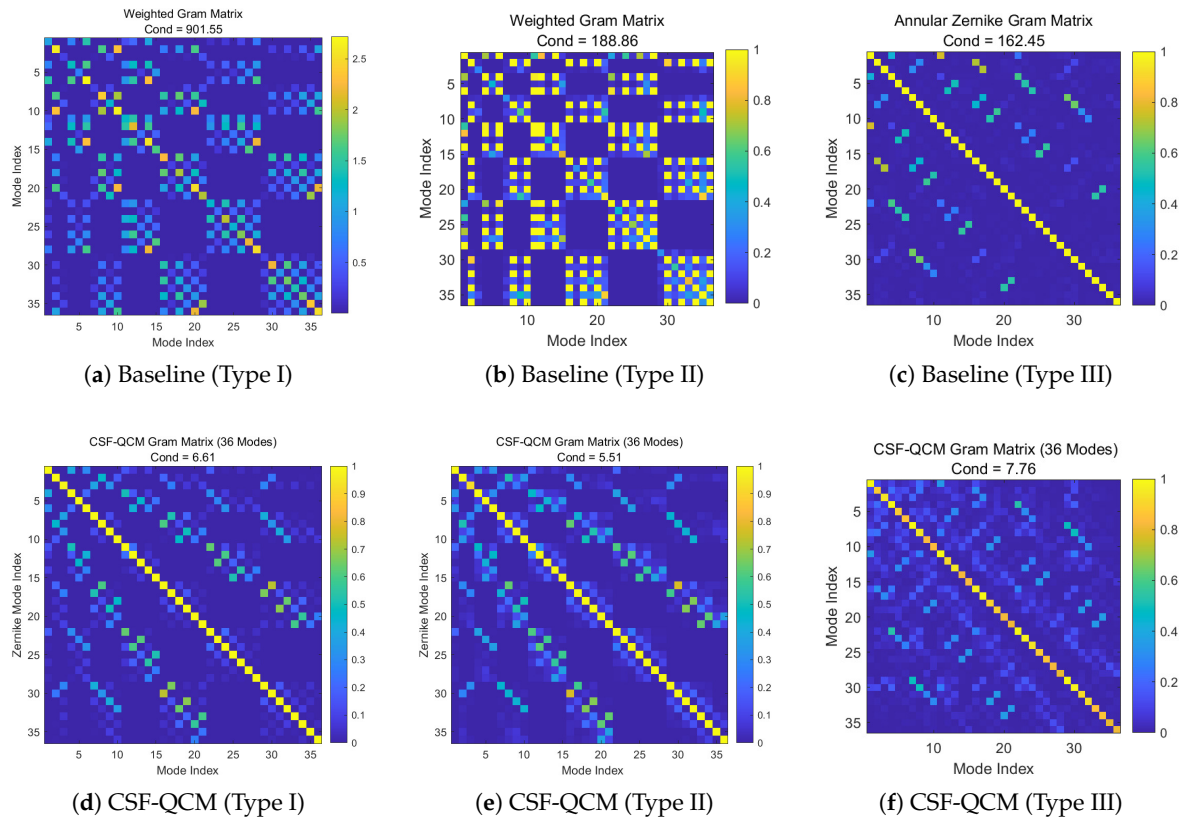
3.2.2. Recovery of Discrete Orthogonality

When the mapping Φ induces highly non-uniform area scaling, the discrete orthogonality of the canonical basis is compromised. This degradation is visualized by the structure of the Gram matrix $\mathbf{G} = \mathbf{H}^T \mathbf{H}$. In an ideal scenario with uniform sampling, $\mathbf{G}$ should be an identity matrix (up to normalization). However, as shown in the top row of Figure 6, baseline conformal mappings produce Gram matrices with significant off-diagonal components (modal crosstalk), particularly for the high-aspect-ratio Type II and eccentric Type III apertures. This structure indicates that the basis functions have become numerically linearly dependent on the sampled grid, leading to an ill-conditioned inverse problem.

By contrast, CSF-QCM explicitly optimizes for sampling uniformity. As illustrated in the bottom row of Figure 6, the resulting Gram matrices are strictly diagonally dominant with suppressed off-diagonal terms, confirming the recovery of discrete orthogonality. We quantify this improvement using the condition number $\kappa(\mathbf{G})$ for the first 36 Zernike modes (Table 1). While baseline methods yield condition numbers ranging from ~ 162 to over 900, implying amplification of measurement noise, CSF-QCM consistently reduces $\kappa(\mathbf{G})$ to single digits (< 8) across all aperture types. This near-optimal conditioning ensures that the wavefront fitting remains numerically stable and minimizes the propagation of potential measurement errors.

Table 1. Condition number $\kappa(\mathbf{G})$ of the Gram matrix (first 36 modes) under different mappings.

Method	Type I (Butterfly)	Type II (Rounded rect.)	Type III (Annulus)
Baseline (Fornberg/SC/Ricci)	901.55	188.86	162.45
CSF-QCM (proposed)	6.61	5.51	7.76

**Figure 6. Comparison of Gram matrices $\mathbf{G}$ (first 36 Zernike modes).** Top row (a)–(c): Baseline approaches (Fornberg, SC, and Ricci flow) exhibit significant off-diagonal energy (crosstalk), indicating loss of orthogonality due to non-uniform sampling. Bottom row (d)–(f): CSF-QCM effectively recovers discrete orthogonality, yielding diagonally dominant matrices with significantly reduced condition numbers.

3.3. Wavefront Fitting Accuracy and Computational Efficiency

To validate the reconstruction fidelity, we generated a synthetic wavefront composed of the first 36 Zernike modes (using Noll indexing for simply connected domains and the corresponding annular basis for Type III), with a peak-to-valley (PV) amplitude of $1/20\lambda$. This setup mimics typical high-precision testing scenarios where minimizing residual fitting error is critical.

Figure 7 presents a comprehensive visual comparison of the reconstructed surfaces and their associated error maps. The baseline methods (Figure 7(b, c)) exhibit distinct boundary ringing and structural artifacts, particularly near high-curvature regions and narrow gaps. These errors are consistent with the ill-conditioning analysis in Section 3.2: the local sampling sparsity caused by conformal crowding prevents the orthogonal basis from resolving edge behaviors accurately. In contrast, CSF-QCM (Figure 7(d, e)) yields a highly accurate reconstruction with spatially homogeneous residuals. The absence of localized error spikes confirms that the quasi-uniform sampling strategy effectively stabilizes the numerical fit.

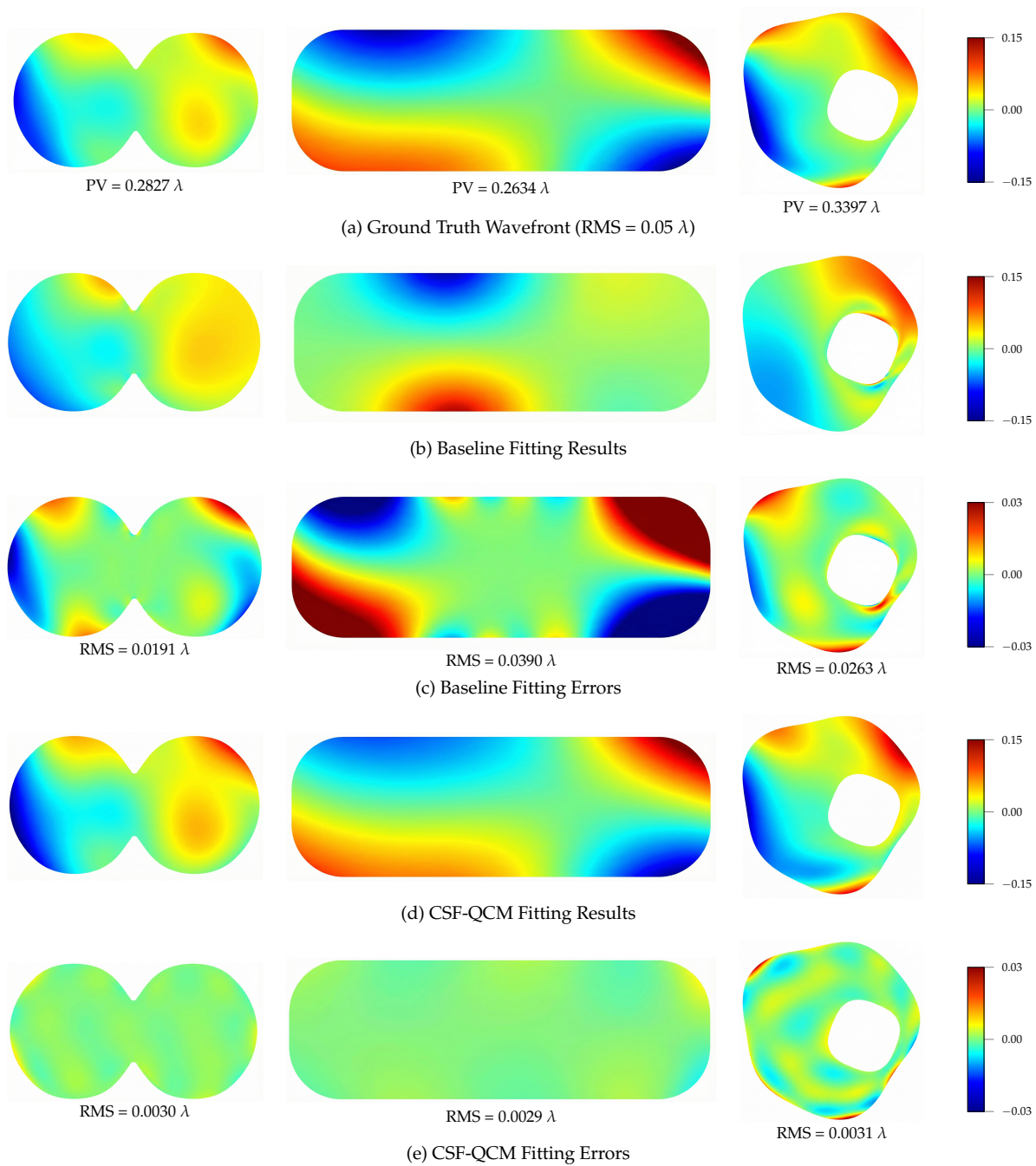


Figure 7. Comparison of wavefront fitting performance. (a) Ground truth wavefronts. (b, c) Reconstructed wavefronts and error maps for baseline methods. (d, e) Reconstructed wavefronts and error maps for the proposed CSF-QCM. The aspect ratios are preserved for accurate visualization.

Table 2 quantitatively summarizes the Peak-to-Valley (PV) and Root-Mean-Square (RMS) residuals. Across all three aperture types, CSF-QCM significantly outperforms the baseline approaches. For visible wavelengths (e.g., $\lambda \approx 633$ nm), the method achieves nanometer RMS accuracy. Notably, for the most challenging Type II (rounded rectangle) and Type III (eccentric annulus) apertures, the RMS fitting error is reduced by over 90%, demonstrating the method's robustness against complex boundary topologies.

Table 2. Wavefront reconstruction residuals (PV and RMS in λ) comparing baselines to CSF-QCM.

Aperture type	Method	PV residual		RMS residual	
		Value	Reduction	Value	Reduction
Type I (Butterfly)	Fornberg-type	0.1576	–	0.0191	–
	CSF-QCM	0.0255	83.82%	0.0030	84.3%
Type II (Rounded rect.)	SC mapping	0.2687	–	0.0390	–
	CSF-QCM	0.0392	85.41%	0.0029	92.56%
Type III (Annulus)	Discrete Ricci flow	0.3317	–	0.0263	–
	CSF-QCM	0.0448	86.49%	0.0031	88.21%

Beyond accuracy, practical metrology demands computational efficiency. We evaluated the runtime on a standard PC (Intel i5 CPU, 16 GB RAM) using meshes with approximately 15,000 vertices. Table 3 details the computational cost broken down by processing stage. Although CSF-QCM introduces an iterative boundary evolution step, the total runtime remains competitive (~ 3 seconds). This is achieved because the subsequent interior mapping relies on solving sparse linear systems, which is computationally inexpensive compared to the nonlinear optimization required by Ricci flow. The proposed framework thus offers a favorable trade-off, providing high-precision reconstruction with a computational cost suitable for routine laboratory testing.

Table 3. Runtime comparison on typical meshes ($\sim 15k$ vertices). Time unit: seconds.

Aperture type	Method	Pre-processing	Mapping	Total
Type I (Butterfly)	Fornberg	0.25	4.82	5.07
	CSF-QCM	0.85	1.95	2.80
Type II (Rounded rect.)	SC mapping	0.10	7.45	7.55
	CSF-QCM	0.90	2.10	3.00
Type III (Annulus)	Ricci flow	0.00	8.30	8.30
	CSF-QCM	1.20	2.45	3.65

4. Discussion

CSF-QCM improves accuracy and numerical stability for wavefront fitting on simply and doubly connected apertures by effectively managing the trade-off between conformality (angle preservation) and area uniformity (sampling regularity). Several aspects merit further investigation.

4.1. Extension to Multiply Connected Apertures

The current framework uses a capacity-based invariant to determine the canonical annulus for doubly connected domains. For more complex topologies (e.g., triply connected apertures, segmented mirrors, or pupils with multiple obscurations), the canonical domain is no longer a simple annulus. Possible extensions include circle-domain parameterizations leveraging fast boundary integral equation methods[33] or analytical approaches based on the Schottky–Klein prime function[34]. These modern numerical tools offer superior convergence rates compared to classical Koebe-type iterative constructions for general n -connected planar domains. CSF-based boundary smoothing remains applicable and can be combined with these multi-boundary canonicalization techniques to handle complex segmented pupil geometries.

4.2. Interaction Between Quasi-Conformal Distortion and Aberration Estimation

The present optimization emphasizes geometric uniformity, while mapping distortion may interact with finite sampling and specific aberration modes. A useful next step is to quantify the sensitivity of estimated coefficients to local quasi-conformal distortion, e.g., through a distortion–aberration sensitivity matrix. Such a model could enable distortion allocation strategies that prioritize regions most influential to the targeted aberration terms, conceptually analogous to the weighted Zernike decomposition strategies employed in high-contrast imaging.[35]

4.3. Acceleration Toward Real-Time Metrology

Although CSF-QCM achieves second-level runtime on typical meshes, online metrology may require remeshing and remapping when the valid aperture changes dynamically. The dominant cost arises from large sparse linear solves (Eq. (4)). While GPU-accelerated sparse factorizations can reduce latency, a more transformative direction is the adoption of physics-informed machine learning. Specifically, Physics-Informed Neural Networks (PINNs)[36] or Deep Operator Networks (DeepONet)[37] could serve as real-time surrogate models to predict the mapping functions directly by learning the underlying Laplace operator, potentially bypassing the iterative mesh generation process entirely.

5. Conclusion

We presented a CSF-guided progressive quasi-conformal mapping framework (CSF-QCM) to address the loss of orthogonality and numerical instability in wavefront fitting on non-circular freeform apertures. By introducing boundary evolution as a geometric preprocessing step, the framework constructs low-distortion parameterizations from irregular physical apertures—including non-convex, high-aspect-ratio, and doubly connected domains—to canonical computational domains.

The quantitative validations presented in this study highlight three key advancements. Firstly, the numerical stability is drastically improved; by managing the conformality–uniformity trade-off, CSF-QCM reduces the condition number of the Gram matrix from severe levels (e.g., > 900 for butterfly apertures) to single digits (< 8) across all tested geometries, effectively resolving the ill-conditioning caused by crowding. Secondly, this stability translates into superior reconstruction accuracy. The method eliminates boundary ringing artifacts and achieves nanometer-level precision ($\text{RMS} \approx 0.003\lambda$), reducing fitting errors by over 80% (up to 92.5% for high-aspect-ratio shapes) compared to baseline Fornberg and Ricci flow algorithms. Thirdly, the framework maintains computational efficiency, processing typical meshes ($\sim 15\text{k}$ vertices) in approximately 3 seconds, which is faster than iterative Ricci flow and suitable for routine laboratory testing.

In summary, CSF-QCM provides a unified, robust, and fast preprocessing route for wavefront fitting over arbitrary apertures, addressing critical challenges in high-precision freeform metrology and the alignment of complex off-axis optical systems.

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References

1. Malacara, D. *Optical shop testing*; John Wiley & Sons, 2007.
2. Ye, J.; Chen, L.; Li, X.; Yuan, Q.; Gao, Z. Review of optical freeform surface representation technique and its application. *Optical Engineering* **2017**, *56*, 110901–110901.
3. Niu, K.; Tian, C. Zernike polynomials and their applications. *Journal of Optics* **2022**, *24*, 123001.

4. Noll, R.J. Zernike polynomials and atmospheric turbulence. *Journal of the Optical Society of America* **1976**, 66, 207–211.
5. Zernike, F. Diffraction theory of the knife-edge test and its improved form, the phase-contrast method. *Monthly Notices of the Royal Astronomical Society*, Vol. 94, p. 377–384 **1934**, 94, 377–384.
6. Navarro, R.; López, J.L.; Díaz, J.A.; Sinusía, E.P. Generalization of Zernike polynomials for regular portions of circles and ellipses. *Optics Express* **2014**, 22, 21263–21279.
7. Ferreira, C.; López, J.L.; Navarro, R.; Sinusia, E.P. Orthogonal systems of Zernike type in polygons and polygonal facets. *arXiv preprint arXiv:1506.07396* **2015**.
8. Ye, J.; Li, X.; Gao, Z.; Wang, S.; Sun, W.; Wang, W.; Yuan, Q. Modal wavefront reconstruction over general shaped aperture by numerical orthogonal polynomials. *Optical Engineering* **2015**, 54, 034105–034105.
9. Swantner, W.; Chow, W.W. Gram–Schmidt orthonormalization of Zernike polynomials for general aperture shapes. *Applied optics* **1994**, 33, 1832–1837.
10. Chang, L.; Wei, Z.; Shen, W.; Lin, Z. Wavefront fitting of Interferogram with Zernike polynomials based on SVD. In Proceedings of the 2nd International Symposium on Advanced Optical Manufacturing and Testing Technologies: Optical Test and Measurement Technology and Equipment. SPIE, 2006, Vol. 6150, pp. 90–95.
11. Chai, X.; Zhang, H.; Lin, X.; Zhou, Y.; Yu, Y. Method for orthogonal fitting of arbitrary shaped aperture wavefront and aberration removal. *Optical Engineering* **2024**, 63, 054112–054112.
12. Zhang, Y.; An, Q.; Yang, M.; Ma, L.; Wang, L. A Review of Wavefront Sensing and Control Based on Data-Driven Methods. *Aerospace* **2025**, 12, 399.
13. Fornberg, B. A numerical method for conformal mappings. *SIAM Journal on Scientific and Statistical Computing* **1980**, 1, 386–400.
14. DeLillo, T.K.; Elcrat, A.R. A Fornberg-like conformal mapping method for slender regions. *Journal of computational and applied mathematics* **1993**, 46, 49–64.
15. Driscoll, T.A.; Trefethen, L.N. *Schwarz-christoffel mapping*; Vol. 8, Cambridge university press, 2002.
16. Yang, D.; Yang, Z.; Zhang, Y. Modal wavefront reconstruction by Schwarz-Christoffel mapping and Zernike circle polynomials for noncircular pupils. *Optics and Lasers in Engineering* **2025**, 184, 108643.
17. Trefethen, L.N. Numerical computation of the Schwarz–Christoffel transformation. *SIAM Journal on Scientific and Statistical Computing* **1980**, 1, 82–102.
18. Jin, M.; Kim, J.; Luo, F.; Gu, X. Discrete surface Ricci flow. *IEEE Transactions on Visualization and Computer Graphics* **2008**, 14, 1030–1043.
19. Zeng, W.; Samaras, D.; Gu, D. Ricci flow for 3D shape analysis. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **2010**, 32, 662–677.
20. Su, Z.; Wang, Y.; Shi, R.; Zeng, W.; Sun, J.; Luo, F.; Gu, X. Optimal mass transport for shape matching and comparison. *IEEE transactions on pattern analysis and machine intelligence* **2015**, 37, 2246–2259.
21. Zeng, W.; Luo, F.; Yau, S.T.; Gu, X.D. Surface quasi-conformal mapping by solving Beltrami equations. In Proceedings of the IMA International Conference on Mathematics of Surfaces. Springer, 2009, pp. 391–408.
22. Gage, M.; Hamilton, R.S. The heat equation shrinking convex plane curves. *Journal of Differential Geometry* **1986**, 23, 69–96.
23. Astala, K.; Iwaniec, T.; Martin, G. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane (PMS-48)*; Princeton University Press, 2008.
24. Grone, R.; Merris, R.; Sunder, V.S. The Laplacian spectrum of a graph. *SIAM Journal on matrix analysis and applications* **1990**, 11, 218–238.
25. Conway, J.B. *Functions of one complex variable II*; Vol. 159, Springer Science & Business Media, 2012.
26. Forster, O. *Lectures on Riemann surfaces*; Vol. 81, Springer Science & Business Media, 2012.
27. Nehari, Z. *Conformal mapping*; Courier Corporation, 2012.
28. Hakula, H.; Rasila, A.; Vuorinen, M. Conformal modulus on domains with strong singularities and cusps. *arXiv preprint arXiv:1501.06765* **2015**.
29. Pinkall, U.; Polthier, K. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics* **1993**, 2, 15–36.
30. Iwaniec, T.; Koh, N.T.; Kovalev, L.V.; Onninen, J. Existence of energy-minimal diffeomorphisms between doubly connected domains. *Inventiones mathematicae* **2011**, 186, 667–707.
31. Tyson, R.K.; Frazier, B.W. *Principles of adaptive optics*; CRC press, 2022.
32. Mahajan, B.V.N. Zernike annular polynomials and optical aberrations of systems with annular pupils. *Applied optics* **1994**, 33, 8125–8127.

33. Nasser, M.M. Fast computation of the circular map. *Computational Methods and Function Theory* **2015**, *15*, 187–223.
34. Crowdy, D. *Solving problems in multiply connected domains*; SIAM, 2020.
35. Allan, G.; Kang, I.; Douglas, E.S.; Barbastathis, G.; Cahoy, K. Deep residual learning for low-order wavefront sensing in high-contrast imaging systems. *Optics Express* **2020**, *28*, 26267–26283.
36. Romanenko, T.; Razgulin, A.; Iroshnikov, N.; Larichev, A. Wavefront Reconstruction by its Slopes via Physics-Informed Neural Networks. *The International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences* **2025**, *48*, 233–240.
37. Zhang, H.; Chen, C.; Li, F.; Cai, J.; Yao, L.; Dong, F.; Wei, Y.; Liu, Y.; Zhang, X.; Zhou, Y.; et al. Single-Pass Wavefront Reconstruction via Depth Heterogeneity Self-Supervised Neural Operator for Turbulence Correction. *Laser & Photonics Reviews* **2025**, *19*, e00909.

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