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Gamaliel Morales *

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On k-Lichtenberg Matrix Sequence: Another Way of Demonstrating Their Properties

Gamaliel Morales

Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile; gamaliel.cerda.m@mail.pucv.cl

Abstract

In a recent study, Morales introduced matrices derived from Gaussian Lichtenberg and modified Lichtenberg numbers. In the present work, through the identification of certain special matrices with generalized Lichtenberg coefficients, we can identify other forms of demonstration of its properties and also the description of commutative matrix properties for negative indices. A new generalization of this sequence is used for our purpose.

Keywords: Lichtenberg number; matrix sequence; recurrence relation

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1. Introduction

The Lichtenberg numbers was studied by Hinz in his article that was published in 2017 [6]. Subsequently, Stockmeyer [8] presented several characterizations of this sequence and numerous examples of its nature. Recently, Gaussian Lichtenberg and modified Lichtenberg numbers have recently been studied by Morales [9]. Also Morales [10] introduced Lichtenberg hybrid numbers and Lichtenberg hybrid quaternions and studies several of their properties.

Lichtenberg numbers are named after Georg Christoph Lichtenberg, who studied these numbers in the 17th century. Lichtenberg numbers are denoted by ℓ_n, defined mathematically by the recurrence $\ell_n + \ell_{n-1} = 2^n - 1$ and have the form

$$\ell_n = \frac{1}{6}\left[(-1)^{n+1} + 2^{n+2} - 3\right]. \tag{1}$$

The first few terms of the Lichtenberg sequence are:

$$0, 1, 2, 5, 10, 21, 42, 85, 170, 341, 682, \ldots (A000975).$$

The Lichtenberg sequence $\{\ell_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$\ell_{n+3} = 2\ell_{n+2} + \ell_{n+1} - 2\ell_n, \tag{2}$$

with $\ell_0 = 0$, $\ell_1 = 1$ and $\ell_2 = 2$ (see for example [6,8]). Also, the Binet formula for Lichtenberg numbers is defined in two different ways, including well-known sequences of order 2,

$$\ell_n = \frac{1}{2}\left[\frac{2^{n+2} - (-1)^{n+2}}{3} - 1\right] = \frac{1}{2}[J_{n+2} - 1]$$

and

$$\ell_n = \frac{1}{3}\left[2^{n+1} - 1 - \frac{(-1)^n + 1}{2}\right] = \frac{1}{3}\left[M_{n+1} - \frac{(-1)^n + 1}{2}\right],$$

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where J_n is the n-th Jacobsthal number (see [7]) and M_n is the n-th Mersenne number (see [1]). The relation (1) can be rewritten as

$$\ell_n = \frac{1}{3}\left[2^{n+1} - (2\varepsilon(n+1) + \varepsilon(n))\right], \tag{3}$$

where $\varepsilon(n)$ is the sequence $\varepsilon(n) = \frac{1-(-1)^n}{2}$.

On the other hand, in Civciv's Ph.D. thesis titled 'Fibonacci and Lucas matrix sequences and their properties" new Fibonacci matrix sequence have been introduced and studied. Also, many of the properties of these sequences are proved (see [2,3]). Many other authors have used this technique to study properties of integer sequences. For example, Coskun and Taskara [4], Gulec and Taskara [5], Morales [11], Yilmaz [12] and several others.

In the next section, a generalization for the Lichtenberg sequence using a fixed integer parameter $k \geq 2$ is defined. Using this definition, the k-Lichtenberg matrix sequences are introduced and several of their properties are studied.

2. k-Lichtenberg Numbers and Their Matrix Sequences

Definition 1. *Let $k \geq 2$ be a fixed integer, the k-Lichtenberg sequence $(\ell_n(k))_{n\geq 0}$ is defined by*

$$\ell_{n+3}(k) = k\ell_{n+2}(k) + \ell_{n+1}(k) - k\ell_n(k),$$

where $\ell_0(k) = 0$, $\ell_1(k) = 1$ and $\ell_2(k) = k$. Also, the modified k-Lichtenberg sequence $(\ell^*_n(k))_{n\geq 0}$ is defined by

$$\ell^*_{n+3}(k) = k\ell^*_{n+2}(k) + \ell^*_{n+1}(k) - k\ell^*_n(k),$$

where $\ell^*_0(k) = 3$, $\ell^*_1(k) = k$ and $\ell^*_2(k) = k^2 + 2$.

Especially, when $k = 2$, then $\ell_n(2) = \ell_n$ (the n-th Lichtenberg number) and $\ell^*_n(2) = \ell^*_n$ (the n-th modified Lichtenberg number). Next, let us look at two mathematical definitions that will be used in this paper.

Definition 2. *Let $k \geq 2$ be a fixed integer, the k-Lichtenberg matrix sequence $(M_{k,n})_{n\geq 0}$ is defined by*

$$M_{k,n+3} = kM_{k,n+2} + M_{k,n+1} - kM_{k,n},$$

where $M_{k,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_{k,1} = \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *and*
$M_{k,2} = \begin{bmatrix} k^2+1 & 0 & -k^2 \\ k & 1 & -k \\ 1 & 0 & 0 \end{bmatrix}$.

Definition 3. *Let $k \geq 2$ be a fixed integer, the modified k-Lichtenberg matrix sequence $(N_{k,n})$ is defined by*

$$N_{k,n+3} = kN_{k,n+2} + N_{k,n+1} - kN_{k,n},$$

$N_{k,0} = \begin{bmatrix} k & 2 & -3k \\ 3 & -2k & -1 \\ \frac{1}{k} & 2 & -\frac{1}{k} - 2k \end{bmatrix}$, $N_{k,1} = \begin{bmatrix} k^3 & 2 & -k^3 - 2k \\ k^2+2 & -2k & -k^2 \\ k & 2 & -3k \end{bmatrix}$ *and*
$N_{k,2} = \begin{bmatrix} k^4+2 & -2k & -k^4 \\ k^3 & 2 & -k^3 - 2k \\ k^2+2 & -2k & -k^2 \end{bmatrix}$.

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In this way, we will state the following result.

Proposition 1. *For any integer $n \geq 1$, we obtain*

$$(\mathrm{M}_{k,1})^n = \begin{bmatrix} \ell_{n+1}(k) & \Delta\ell_{n-1}(k) & -k\ell_n(k) \\ \ell_n(k) & \Delta\ell_{n-2}(k) & -k\ell_{n-1}(k) \\ \ell_{n-1}(k) & \Delta\ell_{n-3}(k) & -k\ell_{n-2}(k) \end{bmatrix},$$

where $\Delta\ell_n(k) = \ell_{n+1}(k) - k\ell_n(k)$, $\ell_{-1}(k) = 0$ and $\ell_{-2}(k) = -\frac{1}{k}$.

Proof. We will use mathematical induction on n. The result holds for $n = 1$:

$$(\mathrm{M}_{k,1})^1 = \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \ell_2(k) & \Delta\ell_0(k) & -k\ell_1(k) \\ \ell_1(k) & \Delta\ell_{-1}(k) & -k\ell_0(k) \\ \ell_0(k) & \Delta\ell_{-2}(k) & -k\ell_{-1}(k) \end{bmatrix}.$$

By mathematical induction, we assume that

$$(\mathrm{M}_{k,1})^n = \begin{bmatrix} \ell_{n+1}(k) & \Delta\ell_{n-1}(k) & -k\ell_n(k) \\ \ell_n(k) & \Delta\ell_{n-2}(k) & -k\ell_{n-1}(k) \\ \ell_{n-1}(k) & \Delta\ell_{n-3}(k) & -k\ell_{n-2}(k) \end{bmatrix},$$

with $\Delta\ell_n(k) = \ell_{n+1}(k) - k\ell_n(k)$. Next, consider the following matrix power

$$\begin{aligned}
(\mathrm{M}_{k,1})^{n+1} &= \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1} \\
&= \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \ell_{n+1}(k) & \Delta\ell_{n-1}(k) & -k\ell_n(k) \\ \ell_n(k) & \Delta\ell_{n-2}(k) & -k\ell_{n-1}(k) \\ \ell_{n-1}(k) & \Delta\ell_{n-3}(k) & -k\ell_{n-2}(k) \end{bmatrix} \begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k\ell_{n+1}(k) + \Delta\ell_{n-1}(k) & \ell_{n+1}(k) - k\ell_n(k) & -k\ell_{n+1}(k) \\ k\ell_n(k) + \Delta\ell_{n-2}(k) & \ell_n(k) - k\ell_{n-1}(k) & -k\ell_n(k) \\ k\ell_{n-1}(k) + \Delta\ell_{n-3}(k) & \ell_{n-1}(k) - k\ell_{n-2}(k) & -k\ell_{n-1}(k) \end{bmatrix} \\
&= \begin{bmatrix} \ell_{n+2}(k) & \Delta\ell_n(k) & -k\ell_{n+1}(k) \\ \ell_{n+1}(k) & \Delta\ell_{n-1}(k) & -k\ell_n(k) \\ \ell_n(k) & \Delta\ell_{n-2}(k) & -k\ell_{n-1}(k) \end{bmatrix},
\end{aligned}$$

where $\Delta\ell_n(k) = \ell_{n+1}(k) - k\ell_n(k)$, which proves what is requested. □

In addition, we will take the following matrix products indicated in the expression $\mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n$. Then, we consider the following result.

Proposition 2. *For any integer $n \geq 1$,we obtain*

$$\mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n = \begin{bmatrix} \ell^*_{n+1}(k) & \Delta\ell^*_{n-1}(k) & -k\ell^*_n(k) \\ \ell^*_n(k) & \Delta\ell^*_{n-2}(k) & -k\ell^*_{n-1}(k) \\ \ell^*_{n-1}(k) & \Delta\ell^*_{n-3}(k) & -k\ell^*_{n-2}(k) \end{bmatrix},$$

where $\Delta\ell^*_n(k) = \ell^*_{n+1}(k) - k\ell^*_n(k)$, $\ell^*_{-1}(k) = \frac{1}{k}$ and $\ell^*_{-2}(k) = \frac{1}{k^2} + 2$.

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Proof. Similar to the previous proposition, we have

$$\begin{aligned}
\mathrm{N}_{k,0}\mathrm{M}_{k,1} &= \begin{bmatrix} k & 2 & -3k \\ 3 & -2k & -1 \\ \frac{1}{k} & 2 & -\frac{1}{k}-2k \end{bmatrix}\begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k^2+2 & -2k & -k^2 \\ k & 2 & -3k \\ 3 & -2k & -1 \end{bmatrix} = \begin{bmatrix} \ell_2^*(k) & \Delta\ell_0^*(k) & -k\ell_1^*(k) \\ \ell_1^*(k) & \Delta\ell_{-1}^*(k) & -k\ell_0^*(k) \\ \ell_0^*(k) & \Delta\ell_{-2}^*(k) & -k\ell_{-1}^*(k) \end{bmatrix}.
\end{aligned}$$

and the result holds for $n=1$. Now, by mathematical induction on n, it is enough to verify

$$\begin{aligned}
\mathrm{N}_{k,0}(\mathrm{M}_{k,1})^{n+1} &= \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^{n}\mathrm{M}_{k,1} \\
&= \begin{bmatrix} \ell_{n+1}^*(k) & \Delta\ell_{n-1}^*(k) & -k\ell_n^*(k) \\ \ell_n^*(k) & \Delta\ell_{n-2}^*(k) & -k\ell_{n-1}^*(k) \\ \ell_{n-1}^*(k) & \Delta\ell_{n-3}^*(k) & -k\ell_{n-2}^*(k) \end{bmatrix}\begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k\ell_{n+1}^*(k)+\Delta\ell_{n-1}^*(k) & \ell_{n+1}^*(k)-k\ell_n^*(k) & -k\ell_{n+1}^*(k) \\ k\ell_n^*(k)+\Delta\ell_{n-2}^*(k) & \ell_n^*(k)-k\ell_{n-1}^*(k) & -k\ell_n^*(k) \\ k\ell_{n-1}^*(k)+\Delta\ell_{n-3}^*(k) & \ell_{n-1}^*(k)-k\ell_{n-2}^*(k) & -k\ell_{n-1}^*(k) \end{bmatrix} \\
&= \begin{bmatrix} \ell_{n+2}^*(k) & \Delta\ell_n^*(k) & -k\ell_{n+1}^*(k) \\ \ell_{n+1}^*(k) & \Delta\ell_{n-1}^*(k) & -k\ell_n^*(k) \\ \ell_n^*(k) & \Delta\ell_{n-2}^*(k) & -k\ell_{n-1}^*(k) \end{bmatrix},
\end{aligned}$$

where $\Delta\ell_n^*(k) = \ell_{n+1}^*(k) - k\ell_n^*(k)$, which proves what is requested. □

In addition, we can also verify the behavior of the following determinants indicated by

$$\begin{aligned}
\det\left[(\mathrm{M}_{k,1})^n\right] &= \det\begin{bmatrix} \ell_{n+1}(k) & \Delta\ell_{n-1}(k) & -k\ell_n(k) \\ \ell_n(k) & \Delta\ell_{n-2}(k) & -k\ell_{n-1}(k) \\ \ell_{n-1}(k) & \Delta\ell_{n-3}(k) & -k\ell_{n-2}(k) \end{bmatrix}, \\
\det\left[\mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n\right] &= \det\begin{bmatrix} \ell_{n+1}^*(k) & \Delta\ell_{n-1}^*(k) & -k\ell_n^*(k) \\ \ell_n^*(k) & \Delta\ell_{n-2}^*(k) & -k\ell_{n-1}^*(k) \\ \ell_{n-1}^*(k) & \Delta\ell_{n-3}^*(k) & -k\ell_{n-2}^*(k) \end{bmatrix},
\end{aligned}$$

where $\Delta\ell_n(k)$ and $\Delta\ell_n^*(k)$ as in Propositions (1) and (2).

Corollary 1. *For any integer $n \geq 1$, we obtain*

$$\det\left[(\mathrm{M}_{k,1})^n\right] = (-k)^n, \tag{4}$$

$$\det\left[\mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n\right] = -4(k^2-1)^2(-k)^{n-1}. \tag{5}$$

Proof. We note that $\det[\mathrm{M}_{k,1}] = -k$. By mathematical induction, let us write

$$\begin{aligned}
\det\left[(\mathrm{M}_{k,1})^{n+1}\right] &= \det\left[(\mathrm{M}_{k,1})^n \cdot \mathrm{M}_{k,1}\right] \\
&= \det\left[(\mathrm{M}_{k,1})^n\right] \cdot \det[\mathrm{M}_{k,1}] \\
&= (-k)^n \cdot (-k) = (-k)^{n+1}.
\end{aligned}$$

Similarly, let us admit that $\det\left[N_{k,0}(M_{k,1})^n\right] = (k+1)^2(k^2+k+2)k^{n-1}$. Thus, we can see that

$$\begin{aligned}\det\left[N_{k,0}(M_{k,1})^{n+1}\right] &= \det\left[N_{k,0}(M_{k,1})^n \cdot M_{k,1}\right] \\ &= \det\left[N_{k,0}(M_{k,1})^n\right] \cdot \det[M_{k,1}] \\ &= -4(k^2-1)^2(-k)^{n-1}. \cdot (-k) \\ &= -4(k^2-1)^2(-k)^n.\end{aligned}$$

□

3. Other Demonstrations for the Commutative Properties

For any integer $n \geq 1$, for simplicity of notation let

$$J_n(k) = (M_{k,1})^n$$

and

$$j_n(k) = N_{k,0}(M_{k,1})^n,$$

are two matrices of order 3, whose coefficients are real numbers.

In this section, we will discuss other simpler and more immediate ways to demonstrate the matrix properties of the matrices previously defined.

Theorem 1. *For $m,\ n \geq 1$, the following results hold:*

$$J_{m+n}(k) = J_m(k) \cdot J_n(k) = J_n(k) \cdot J_m(k), \tag{6}$$

$$j_m(k) \cdot j_n(k) = j_n(k) \cdot j_m(k), \tag{7}$$

$$J_m(k) \cdot j_n(k) = j_n(k) \cdot J_m(k), \tag{8}$$

Proof. For the first item (6), we see that the following commutative properties occur

$$\begin{aligned}J_{m+n}(k) = (M_{k,1})^{m+n} &= (M_{k,1})^m \cdot (M_{k,1})^n \\ &= (M_{k,1})^n \cdot (M_{k,1})^m \\ &= J_{n+m}(k).\end{aligned}$$

For the equation (7) of this theorem, let us also see that

$$\begin{aligned}j_m(k) \cdot j_n(k) &= N_{k,0}(M_{k,1})^m \cdot N_{k,0}(M_{k,1})^n \\ &= N_{k,0}(M_{k,1})^n \cdot N_{k,0}(M_{k,1})^m \\ &= j_n(k) \cdot j_m(k)\end{aligned}$$

and we record the commutative property of $N_{k,0} \cdot M_{k,1} = M_{k,1} \cdot N_{k,0}$. To conclude, let us easily see that

$$\begin{aligned}J_m(k) \cdot j_n(k) &= (M_{k,1})^m \cdot N_{k,0}(M_{k,1})^n \\ &= N_{k,0}(M_{k,1})^n \cdot (M_{k,1})^m = j_n(k) \cdot J_m(k),\end{aligned}$$

as desired. □

The following result gives a relation between matrices $J_m(k)$ and $j_m(k)$.

Theorem 2. *For $n \geq 1$, the following result holds*

$$\mathrm{j}_n(k) = 3\mathrm{J}_{n+1}(k) - 2k\mathrm{J}_n(k) - \mathrm{J}_{n-1}(k). \tag{9}$$

Proof. For any integer $n \geq 1$ and $\mathrm{J}_n(k) = (\mathrm{M}_{k,1})^n$, we get

$$\begin{aligned} 3\mathrm{J}_{n+1}(k) - 2k\mathrm{J}_n(k) - \mathrm{J}_{n-1}(k) &= 3(\mathrm{M}_{k,1})^{n+1} - 2k(\mathrm{M}_{k,1})^n - (\mathrm{M}_{k,1})^{n-1} \\ &= (\mathrm{M}_{k,1})^n \left[3\mathrm{M}_{k,1} - 2k \cdot \mathrm{I}_3 - (\mathrm{M}_{k,1})^{-1}\right], \end{aligned}$$

where I_3 is the third-order identity matrix. Also, we can determine that

$$\begin{aligned} &3\mathrm{M}_{k,1} - 2k \cdot \mathrm{I}_3 - (\mathrm{M}_{k,1})^{-1} \\ &= 3\begin{bmatrix} k & 1 & -k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - 2k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{k} & 1 & \frac{1}{k} \end{bmatrix} \\ &= \begin{bmatrix} k & 2 & -3k \\ 3 & -2k & -1 \\ \frac{1}{k} & 2 & -\frac{1}{k} - 2k \end{bmatrix} \\ &= \mathrm{N}_{k,0}. \end{aligned}$$

Finally, we determine equality

$$3\mathrm{J}_{n+1}(k) - 2k\mathrm{J}_n(k) - \mathrm{J}_{n-1}(k) = (\mathrm{M}_{k,1})^n \cdot \mathrm{N}_{k,0} = \mathrm{N}_{k,0} \cdot (\mathrm{M}_{k,1})^n = \mathrm{j}_n(k),$$

as desired. □

Other properties of the matrix $\mathrm{j}_n(k)$ are proved in the following result.

Theorem 3. *For $n \geq 1$, the following results holds*

$$(\mathrm{j}_{n+1}(k))^2 = (\mathrm{j}_1(k))^2 \cdot \mathrm{J}_{2n}(k), \tag{10}$$

$$\mathrm{j}_{2n+1}(k) = \mathrm{J}_n(k) \cdot \mathrm{j}_{n+1}(k). \tag{11}$$

Proof. (10) In this case, we can determine directly from the Definition 3 that

$$\begin{aligned} (\mathrm{j}_{n+1}(k))^2 &= \mathrm{j}_{n+1}(k) \cdot \mathrm{j}_{n+1}(k) \\ &= \mathrm{N}_{k,0} \cdot (\mathrm{M}_{k,1})^{n+1} \cdot \mathrm{N}_{k,0} \cdot (\mathrm{M}_{k,1})^{n+1} \\ &= \mathrm{N}_{k,0}\mathrm{M}_{k,1} \cdot (\mathrm{M}_{k,1})^n \cdot \mathrm{N}_{k,0}\mathrm{M}_{k,1} \cdot (\mathrm{M}_{k,1})^n \\ &= \mathrm{j}_1(k) \cdot (\mathrm{M}_{k,1})^n \cdot \mathrm{j}_1(k) \cdot (\mathrm{M}_{k,1})^n \\ &= (\mathrm{j}_1(k))^2 \cdot (\mathrm{M}_{k,1})^{2n} \\ &= (\mathrm{j}_1(k))^2 \cdot \mathrm{J}_{2n}(k). \end{aligned}$$

(11) Similarly, we see that

$$\begin{aligned} \mathrm{j}_{2n+1}(k) &= \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^{2n+1} \\ &= \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n \cdot (\mathrm{M}_{k,1})^{n+1} \\ &= (\mathrm{M}_{k,1})^n \cdot \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^{n+1} \\ &= \mathrm{J}_n(k) \cdot \mathrm{j}_{n+1}(k). \end{aligned}$$

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□

Theorem 4. *For any integers $m, n \geq 1$, we obtain*

$$\mathrm{j}_{m+n}(k) = \mathrm{j}_m(k) \cdot \mathrm{J}_n(k) = \mathrm{J}_m(k) \cdot \mathrm{j}_n(k).$$

Proof. Let us consider that

$$\mathrm{j}_{m+n}(k) = \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^{m+n} = \mathrm{N}_{k,0}(\mathrm{M}_{k,1})^n \cdot (\mathrm{M}_{k,1})^m = \mathrm{j}_n(k)\mathrm{J}_m(k).$$

In addition, we can write immediately that

$$\mathrm{j}_{m+n}(k) = \mathrm{j}_0(k)\mathrm{J}_{m+n}(k) = \mathrm{j}_0(k)\mathrm{J}_n(k) \cdot \mathrm{J}_m(k) = \mathrm{J}_n(k) \cdot \mathrm{j}_0(k)\mathrm{J}_m(k) = \mathrm{J}_n(k) \cdot \mathrm{j}_m(k)$$

since, we know the commutativity of the matrix product $\mathrm{J}_1(k) \cdot \mathrm{j}_0(k) = \mathrm{j}_0(k) \cdot \mathrm{J}_1(k)$. The proof is completed. □

4. Matrix Sequence Properties for Negative Indices

In this section, we will develop the study of certain properties determined by the following inverse k-Lichtenberg matrix indicated by $(\mathrm{M}_{k,1})^{-n}$. We can immediately determine some particular cases

$$(\mathrm{M}_{k,1})^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{k} & 1 & \frac{1}{k} \end{bmatrix},$$

and

$$(\mathrm{M}_{k,1})^{-2} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{k} & 1 & \frac{1}{k} \\ -\frac{1}{k^2} & 0 & \frac{k^2+1}{k^2} \end{bmatrix}.$$

We have observed that the elements of the type $\ell_{-n}(k)$, for a positive integer $n \geq 0$ can be determined directly from the recurrence relation indicated by

$$\ell_{-n}(k) = \frac{1}{k}\Big[\ell_{-(n-1)}(k) + k\ell_{-(n-2)}(k) - \ell_{-(n-3)}(k)\Big],$$

where $\ell_{-1}(k) = 0$ and $\ell_{-2}(k) = -\frac{1}{k}$.

From these preliminary examples, we will state the following theorem.

Theorem 5. *For any integer $n \geq 1$, we obtain*

$$\mathrm{J}_{-n}(k) = \begin{bmatrix} \ell_{-(n-1)}(k) & \Delta\ell_{-(n+1)}(k) & -k\ell_{-n}(k) \\ \ell_{-n}(k) & \Delta\ell_{-(n+2)}(k) & -k\ell_{-(n+1)}(k) \\ \ell_{-(n+1)}(k) & \Delta\ell_{-(n+3)}(k) & -k\ell_{-(n+2)}(k) \end{bmatrix},$$
$$\mathrm{J}_{-n}(k) = (\mathrm{M}_{k,1})^{-n} = \Big[(\mathrm{M}_{k,1})^{-1}\Big]^n = \big[(\mathrm{M}_{k,1})^n\big]^{-1},$$

where $\Delta\ell_{-n}(k) = \ell_{-(n-1)}(k) - k\ell_{-n}(k)$.

Proof. The result holds for $n = 1$:

$$\mathrm{J}_{-1}(k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{k} & 1 & \frac{1}{k} \end{bmatrix} = \begin{bmatrix} \ell_0(k) & \Delta\ell_{-2}(k) & -k\ell_{-1}(k) \\ \ell_{-1}(k) & \Delta\ell_{-3}(k) & -k\ell_{-2}(k) \\ \ell_{-2}(k) & \Delta\ell_{-4}(k) & -k\ell_{-3}(k) \end{bmatrix}.$$

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By mathematical induction on n, we assume that

$$\mathrm{J}_{-n}(k)=\begin{bmatrix}\ell_{-(n-1)}(k) & \Delta\ell_{-(n+1)}(k) & -k\ell_{-n}(k)\\ \ell_{-n}(k) & \Delta\ell_{-(n+2)}(k) & -k\ell_{-(n+1)}(k)\\ \ell_{-(n+1)}(k) & \Delta\ell_{-(n+3)}(k) & -k\ell_{-(n+2)}(k)\end{bmatrix},$$

with $\Delta\ell_{-n}(k)=\ell_{-(n-1)}(k)-k\ell_{-n}(k)$. Furthermore, consider the following matrix power

$$\begin{aligned}\mathrm{J}_{-(n+1)}(k)&=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{k} & 1 & \frac{1}{k}\end{bmatrix}^{n+1}\\ &=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{k} & 1 & -\frac{1}{k}\end{bmatrix}^{n}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{k} & 1 & \frac{1}{k}\end{bmatrix}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\mathrm{J}_{-(n+1)}(k)&=\begin{bmatrix}\ell_{-(n-1)}(k) & \Delta\ell_{-(n+1)}(k) & -k\ell_{-n}(k)\\ \ell_{-n}(k) & \Delta\ell_{-(n+2)}(k) & -k\ell_{-(n+1)}(k)\\ \ell_{-(n+1)}(k) & \Delta\ell_{-(n+3)}(k) & -k\ell_{-(n+2)}(k)\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{k} & 1 & \frac{1}{k}\end{bmatrix}\\ &=\begin{bmatrix}\ell_{-n}(k) & \ell_{-(n-1)}(k)-k\ell_{-n}(k) & \Delta\ell_{-(n+1)}(k)-\ell_{-n}(k)\\ \ell_{-(n+1)}(k) & \ell_{-n}(k)-k\ell_{-(n+1)}(k) & \Delta\ell_{-(n+2)}(k)-\ell_{-(n+1)}(k)\\ \ell_{-(n+2)}(k) & \ell_{-(n+1)}(k)-k\ell_{-(n+2)}(k) & \Delta\ell_{-(n+3)}(k)-\ell_{-(n+2)}(k)\end{bmatrix}\\ &=\begin{bmatrix}\ell_{-n}(k) & \Delta\ell_{-(n+2)}(k) & -k\ell_{-(n+1)}(k)\\ \ell_{-(n+1)}(k) & \Delta\ell_{-(n+3)}(k) & -k\ell_{-(n+2)}(k)\\ \ell_{-(n+2)}(k) & \Delta\ell_{-(n+4)}(k) & -k\ell_{-(n+3)}(k)\end{bmatrix},\end{aligned}$$

and using the following relation

$$\ell_{-n}(k)=\frac{1}{k}\Big[\ell_{-(n-1)}(k)+k\ell_{-(n-2)}(k)-\ell_{-(n-3)}(k)\Big],$$

note that $\Delta\ell_{-n}(k)=\ell_{-(n-1)}(k)-k\ell_{-n}(k)$. In this way, we will write

$$\mathrm{J}_{-n}(k)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{k} & 1 & \frac{1}{k}\end{bmatrix}^{n}=\Big[(\mathrm{M}_{k,1})^{-1}\Big]^{n}.$$

□

In the following theorem, we will reduce the Binet formula corresponding to the terms of negative indices for k-Lichtenberg numbers.

Lemma 1. *For any integer $n\geq 1$, we obtain*

$$\ell_{-n}(k)=\frac{1}{k^2-1}\left[k\left(\frac{1}{k}\right)^{n}-k\varepsilon(n+1)-\varepsilon(n)\right], \qquad (12)$$

where $\varepsilon(n)=n-2\lfloor\frac{n}{2}\rfloor$.

Proof. The Binet formula for k-Lichtenberg numbers is given by

$$\ell_n(k)=\frac{1}{k^2-1}\Big[k^{n+1}-(k\varepsilon(n+1)+\varepsilon(n))\Big],$$

where $\varepsilon(n) = \frac{1-(-1)^n}{2}$. Using the relation $\varepsilon(-n) = \varepsilon(n)$ and $\varepsilon(n+2) = \varepsilon(n)$ for any integer $n \geq 1$, we will make the following substitutions

$$\begin{aligned}\ell_{-n}(k) &= \frac{1}{k^2-1}\left[k^{-n+1} - (k\varepsilon(-n+1) + \varepsilon(-n))\right] \\ &= \frac{1}{k^2-1}\left[\frac{k}{k^n} - (k\varepsilon(-(n-1)) + \varepsilon(-n))\right] \\ &= \frac{1}{k^2-1}\left[\frac{k}{k^n} - (k\varepsilon(n-1) + \varepsilon(n))\right] \\ &= \frac{1}{k^2-1}\left[k\left(\frac{1}{k}\right)^n - (k\varepsilon(n+1) + \varepsilon(n))\right],\end{aligned}$$

as desired. □

Let us consider the following equation $\mathrm{j}_n(k) = 3\mathrm{J}_{n+1}(k) - 2k\mathrm{J}_n(k) - \mathrm{J}_{n-1}(k)$ from Theorem 2. We will reduce the corresponding identity to the terms of negative indices.

Theorem 6. *For any integer $n \geq 1$, we obtain that*

$$\ell^*_{-n}(k) = 3\ell_{-(n-1)}(k) - 2k\ell_{-n}(k) - \ell_{-(n+1)}(k),$$

where $\ell^*_n(k)$ is the n-th modified Lichtenberg number.

Proof. From Lemma 1, we can observe that

$$\begin{aligned}&(k^2-1)\cdot\left[3\ell_{-(n-1)}(k) - 2k\ell_{-n}(k) - \ell_{-(n+1)}(k)\right] \\ &= 3k\left(\frac{1}{k}\right)^{n-1} - 3k\varepsilon(n) - 3\varepsilon(n-1) \\ &\quad - 2k^2\left(\frac{1}{k}\right)^n + 2k^2\varepsilon(n+1) + 2k\varepsilon(n) \\ &\quad - k\left(\frac{1}{k}\right)^{n+1} + k\varepsilon(n+2) + \varepsilon(n+1) \\ &= (k^2-1)\left(\frac{1}{k}\right)^n + 2\left(k^2-1\right)\varepsilon(n+1) \\ &= (k^2-1)\left[\left(\frac{1}{k}\right)^n + 2\varepsilon(n+1)\right].\end{aligned}$$

Further, we can directly verify that $\varepsilon(n+1) = \frac{1-(-1)^{n+1}}{2}$. Finally, we deduce $2\varepsilon(n+1) = 1 + (-1)^n$ and $\left(\frac{1}{k}\right)^n + (-1)^n + 1 = \ell^*_{-n}(k)$. The result is completed. □

Now, we see the following theorem that allows determining the generating matrices for the family of matrices $\{\mathrm{j}_{-n}(k)\}_{n\geq 0}$, which we have preliminarily defined by

$$\mathrm{j}_{-n}(k) = \begin{bmatrix} \ell^*_{-(n-1)}(k) & \Delta\ell^*_{-(n+1)}(k) & -k\ell^*_{-n}(k) \\ \ell^*_{-n}(k) & \Delta\ell^*_{-(n+2)}(k) & -k\ell^*_{-(n+1)}(k) \\ \ell^*_{-(n+1)}(k) & \Delta\ell^*_{-(n+3)}(k) & -k\ell^*_{-(n+2)}(k) \end{bmatrix},$$

where $\Delta\ell^*_{-n}(k) = \ell^*_{-(n-1)}(k) - k\ell^*_{-n}(k)$.

Theorem 7. *For any integer $n \geq 1$, we obtain that*

$$\mathrm{j}_{-n}(k) = (\mathrm{J}_1(k))^{-n} \cdot \mathrm{j}_0(k) = \mathrm{j}_0(k) \cdot (\mathrm{J}_1(k))^{-n}, \tag{13}$$

$$(\mathrm{j}_n(k))^{-1} = (\mathrm{j}_0(k))^{-1} \cdot \mathrm{j}_{-n}(k) \cdot (\mathrm{j}_0(k))^{-1}. \tag{14}$$

Proof. (13): We know that $\mathrm{J}_{-n}(k) = (\mathrm{M}_{k,1})^{-n}$. In this way, we will make the corresponding substitutions to determine that

$$\mathrm{j}_{-n}(k) = \left(\mathrm{M}_{k,1}^{(3)}\right)^{-n} \cdot \mathrm{N}_{k,0}^{(3)} = \mathrm{N}_{k,0}^{(3)} \cdot \left(\mathrm{M}_{k,1}^{(3)}\right)^{-n} = \mathrm{j}_0(k) \cdot \mathrm{J}_{-n}(k).$$

(14): From equation (13) and $\mathrm{j}_n(k) = \mathrm{N}_{k,0}^{(3)} \cdot \left(\mathrm{M}_{k,1}^{(3)}\right)^{n}$, we know that

$$\begin{aligned}(\mathrm{j}_n(k))^{-1} &= \left(\mathrm{M}_{k,1}^{(3)}\right)^{-n} \cdot \left(\mathrm{N}_{k,0}^{(3)}\right)^{-1} \\ &= \left(\mathrm{N}_{k,0}^{(3)}\right)^{-1} \cdot \mathrm{N}_{k,0}^{(3)} \left(\mathrm{M}_{k,1}^{(3)}\right)^{-n} \cdot \left(\mathrm{N}_{k,0}^{(3)}\right)^{-1} \\ &= (\mathrm{j}_0(k))^{-1} \cdot \mathrm{j}_{-n}(k) \cdot (\mathrm{j}_0(k))^{-1}.\end{aligned}$$

The proof is completed. □

5. Conclusion

In this work, we defined the k-Lichtenberg and modified k-Lichtenberg matrix sequences using the definition of a new sequence that generalizes the well-known Lichtenberg sequence. Moreover, we also presented the relation between this matrices and the k-Lichtenberg sequences. This matrix representation allows us to visualize new identities for this new sequence. In the future, we could investigate Lichtenberg numbers with arbitrary initial values, and even their applications to coding theory and cryptography.

References

1. P. Catarino, H. Campos, P. Vasco, *On the Mersenne sequence*, Annales Mathematicae et Informaticae, **46**, (2016), 37-53.
2. H. Civciv, R. Türkmen, *On the (s,t)-Fibonacci matrix sequences*, Ars Combinatoria, **87**, (2008), 161-173.
3. H. Civciv, R. Türkmen, *Notes on the (s,t)-Lucas and Lucas matrix sequences*, Ars Combinatoria, **89**, (2008), 271-285.
4. A. Coskun, N. Taskara, *A note on the bi-periodic Fibonacci and Lucas matrix sequences*, Applied Mathematics and Computation, **320**, (2018), 400-406.
5. H. H. Gulec, N. Taskara, *On the (s,t)-Pell and (s,t)-Pell–Lucas sequences and their matrix representations*, Applied Mathematics Letters, **25**(10), (2012), 1554-1559.
6. A. M. Hinz, *The Lichtenberg sequence*, Fibonacci Quart., **55**(2), (2017), 2-12.
7. A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart., **43**(1), (1996), 40-54.
8. P. K. Stockmeyer, *An exploration of sequence A000975*, Fibonacci Quart., **55**(5), (2017), 174-185.
9. G. Morales, *Properties between Lichtenberg, Jacobsthal and Mersenne Gaussian numbers*, Konuralp Journal of Mathematics, **13**(1), (2025), 21-27.
10. G. Morales, *On the Lichtenberg hybrid quaternions*, Mathematica Moravica, **29**(1), (2025), 31-41.
11. G. Morales, *On the third-order Jacobsthal and third-order Jacobsthal-Lucas sequences and their matrix representations*, Mediterr. J. Math., **16**(32), (2019), 1-12.
12. N. Yilmaz, *The generalized bivariate Fibonacci and Lucas matrix polynomials*, Mathematica Montisnigri, **53** (2022), 33-44.