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## Article

# Underlying Geometric Flow in Hamiltonian Evolution

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**Abstract:** In this paper an underlying perturbed Ricci flow construction is made, within the metric operator space, originated from the Heisenberg dynamical equations, to formulate a method which appears to provide a new geometric approach for the geometric formulation of the quantum mechanical dynamics. A quantum mechanical notion of stability and local instability is introduced within the quantum mechanical theory, based on the quantum mechanical dynamical equations governing the evolution of the tensor metric operator. The stability analysis is carried in the topology of little Hölder spaces of metrics which the tensor metric operator acts on. Finally, a conjecture is introduced in attempt to characterize the stability properties of the quantum mechanical system such that it brings in the quantum mechanical dynamics into the analysis.

**Keywords:**

## 1. Introduction

We briefly introduce the work of Horwitz et al. [1], attempting to characterize chaotic Hamiltonian systems in terms of the curvature associated with a Riemannian metric tensor through definition of a conformal metric.

Given the classical Hamiltonian of form (1) in a curved space [1]

$$H_G := \frac{1}{2m} g_{ij}(x) p^i p^j \quad (1)$$

Following the Hamilton equations results in the geodesic equation

$$\ddot{x}_l = -\Gamma_l^{mn} \dot{x}_m \dot{x}_n \quad (2)$$

where  $\Gamma_l^{mn}$  is the connection form.

Studying the separation of two nearby trajectories  $\xi_l = x'_l - x_l$  at  $t$ , which are the components of the geodesic deviation vector. Let  $\gamma$  be the parameter for a family of geodesics in the neighborhood of the coordinates  $x_l(t)$  of a point on a geodesic defined by eq.(2) and  $\xi_l(t) = \frac{\partial x_l(\gamma, t)}{\partial \gamma} |_{\gamma=0}$ .

Then, the second covariant derivative,  $D/Dt$ , of the geodesic deviation results in the following geodesic deviation equation [1]

$$\frac{D^2 \xi_i}{Dt^2} = R_i^{jlk} \dot{x}_j \dot{x}_k \xi_l \quad (3)$$

where  $R_i^{jlk}$  are the components of the Riemann curvature tensor. The stability of the geodesic flow is locally determined by the geodesic deviation equation (3) and therefore is locally characterized by the curvature of the manifold [1].

Next, to study the stability of the Hamiltonian evolution generated by

$$H = \delta_{ij} \frac{p^i p^j}{2m} + V(y), \quad (4)$$

a geometric embedding of the Hamiltonian dynamics is constructed by the Hamiltonian  $H_G$  of form of eq.(1) with a conformal transformation of eq.(4), where the conformal factor results in [1]

$$g_{ij}(x) := \Phi(x)\delta_{ij}, \quad \Phi(x) := \frac{E}{E - V(y)} \equiv F(y) \quad (5)$$

A relation is made with the set of coordinates  $\{y^j\}$  of eq.(4) with the set of coordinates  $\{x_i\}$  through the conformal factor. Two assumptions are made; 1.  $E$  is assumed to be the conserved value of  $H$  and  $H_G$  2. the momentum is the same for both of the dynamics governed by the respective Hamiltonians  $H_G$  and  $H$  (curved and flat spaces, respectively).

Therefore, they are related by

$$E - V(y) = \delta_{ij} \frac{p^i p^j}{2m} \quad (6)$$

Next, following Hamilton equations of both curved and flat spaces

$$\begin{aligned} \dot{x}_i &= \frac{\partial H_G}{\partial p^i} = \frac{1}{m} g_{ij} p^j \\ \dot{y}^i &:= \frac{1}{m} p^i = \frac{\partial H}{\partial p^i} \end{aligned} \quad (7)$$

and equating the momenta results in

$$\dot{y}^j = g^{ji} \dot{x}_i \quad (8)$$

as a local tangent space transformation which reflects a geometric embedding of the original Hamiltonian motion. Clearly, this map is not integrable so the global relation between  $\{x\}$  and  $\{y\}$  is not established by this map [1].

It follows from eq.(8) that

$$\ddot{x}_l = g_{lj} \ddot{y}^j + \frac{\partial g_{lj}}{\partial x_n} \dot{x}_n \dot{y}^j \quad (9)$$

Then, substituting eq.(9) in eq.(2) results in

$$\ddot{y}^l = -M_{mn}^l \dot{y}^m \dot{y}^n, \quad \text{where} \quad M_{mn}^l := \frac{1}{2} g^{lk} \frac{\partial g_{nm}}{\partial y^k} \quad (10)$$

which has the form of a geodesic equation, with a reduced connection form that is completely covariant. Following Horwitz, Yahalom et al. [3] prove, by power series expansions, one may argue [1] that the two coordinate systems are involved with two coordinatizations. As a coordinate space, the  $\{y^l\}$  are called the *Hamilton manifold* and  $\{x_i\}$  are called the *Gutzwiller manifold* [1], with two different connection forms and related locally by  $\delta y^j := g^{ji} \delta x_i$ .

A curvature corresponding to the Hamilton manifold may be derived by the covariant derivative for a covariant tensor on the Gutzwiller manifold [1] with the connection form, to derive a covariant derivative in the Hamilton manifold (for the formula for curvature)

$$A^{m;q} = \frac{\partial A^m}{\partial x_q} - \Gamma_k^{mq} A^k \quad (11)$$

and induced connection form (lowering the index  $q$  with  $g_{lq}$ ),

$$\Gamma_{lk}^m \equiv g_{lq} \Gamma_k^{mq} = \frac{1}{2} g^{mq} \left( \frac{\partial g_{lq}}{\partial y^k} - \frac{\partial g_{kq}}{\partial y^l} - \frac{\partial g_{kl}}{\partial y^q} \right) \quad (12)$$

Since it is antisymmetric in its lower indices ( $l, k$ ) (implying the existence of torsion), along geodesic motion, it results in the symmetric connection form (10) [1].

These derived expressions for the geodesic equation and connection form corresponding to the

Hamilton manifold (not metric compatible connections) may be explained by a parallel transport on the local flat tangent space of the Gutzwiller manifold (whose tensor metric is  $g_{ij}$ ) which results in the "truncated" connection (10) [1] if a transformation is made to reach the Hamilton manifold (by raising the tensor index).

Next, to compute the geodesic deviation in the Hamilton manifold, one can construct a covariant derivative such that the second order geodesic deviation equations results in [1]

$$\frac{D^2 \xi^l}{Dt^2} = R_{qmn}^l \dot{y}^q \dot{y}^n \xi^m \quad (13)$$

where

$$R_{qmn}^l = \frac{\partial M_{qm}^l}{\partial y^n} - \frac{\partial M_{qn}^l}{\partial y^m} + M_{qm}^k M_{nk}^l - M_{qn}^k M_{mk}^l \quad (14)$$

to be called the *dynamical curvature* [1].

Horwitz et al. conjectured that this structure of eq.(13) can be used as a local criterion to characterize the stability of the original Hamiltonian motion [1][2], picturing local regions of instability which are conjectured to drive the motion into a chaotic behavior. Instability in the classical case occurs if at least one of the eigenvalues of the dynamical curvature is negative [1][2].

In a previous work [4], we suggested to extend these ideas to a quantum mechanical framework. We studied the quantum theory associated with a general operator valued Hermitian Riemannian Hamiltonian

$$\hat{H}_G := \frac{1}{2m} p^i g_{ij}(x) p^j \quad (15)$$

with canonical commutation relations

$$[x_i, p^j] = i\hbar \delta_i^j \quad (16)$$

Following the Heisenberg picture results in

$$\dot{x}_k = \frac{1}{2m} \{p^i, g_{ik}\} \quad (17)$$

and

$$p^l = \frac{m}{2} \{\dot{x}_k, g^{kl}\} \quad (18)$$

We derived from the Heisenberg equations the quantum mechanical form of the "geodesic" equation for  $\ddot{x}_l$  generated by the Hamiltonian  $\hat{H}_G$ ,

$$\ddot{x}_l = \frac{1}{16} (\{ \{ \{ g^{nm}, \dot{x}_m \}, \frac{\partial g_{ln}}{\partial x_i} \}, g_{ij} \{ g^{jp}, \dot{x}_p \} \} - 2 \{ g^{ip}, \dot{x}_p \} g_{ln} \frac{\partial g_{ij}}{\partial x_n} \{ g^{jp}, \dot{x}_q \} ) \quad (19)$$

where the classical limit of eq.(19) results in the same conventional structure of a geodesic flow generated by a classical Hamiltonian of the form (1) [1]

$$\ddot{x}_l = -\Gamma_l^{pq} \dot{x}_p \dot{x}_q \quad (20)$$

with

$$\Gamma_l^{pq} = \frac{1}{2} g_{ln} \left( \frac{\partial g^{nq}}{\partial x_p} + \frac{\partial g^{np}}{\partial x_q} - \frac{\partial g^{pq}}{\partial x_n} \right) \quad (21)$$

In analogy to the classical case, quantization of eq.(8), yields a new set of operators

$$\dot{y}^l := \frac{1}{2} \{\dot{x}_k, g^{kl}\} \quad (22)$$

Substituting e.q.(18) in eq.(22) results in the structure analogous to the classical form

$$p^l = m\dot{y}^l \quad (23)$$

Following the Heisenberg picture, the second order equation for the dynamical variable  $\{y\}$

$$\ddot{y}^l = -\frac{1}{2}\dot{y}^i \frac{\partial g_{ij}}{\partial x_l} \dot{y}^j \quad (24)$$

closely related to the form obtained in the classical case for the "geodesic" equation (eq.(10)) with reduced connection [1].

Moreover, the quantum "geodesic" formula eq.(19) reflects a property of the quantum mechanical nature. This can be seen by expressing explicitly in terms of the canonical momenta using (22) and (23), to get a form of a bilinear in momentum ordered to bring momenta to the left and right, to obtain

$$\ddot{x}_l = \frac{1}{2m^2} p^i \left( \frac{\partial g_{li}}{\partial x_n} g_{nj} + \frac{\partial g_{lj}}{\partial x_n} g_{ni} - \frac{\partial g_{ij}}{\partial x_n} g_{ln} \right) p^j + \frac{1}{4m^2} \frac{\partial}{\partial x_j} \left( \frac{\partial^2 g_{ln}}{\partial x_i \partial x_n} g_{ij} \right) \quad (25)$$

where the first term is closely related to the classical connection form, and the second term is an essentially quantum effect. This property is consistently manifested, originated by the natural property of quantum mechanics, as it was observed in our latest work, as well [11].

We have introduced a criterion for unstable behavior, where for "geodesic deviation" [4], we suggested to induce a deviation in the Ehrenfest approximation to the trajectory, for a smooth function  $\psi_t(x)$ ,

$$\psi_t(x) \rightarrow \psi_t(x + \xi) \quad (26)$$

by the generator of translation,

$$\psi_t(x + \xi) = e^{i p^l \xi_l} \psi_t(x) \quad (27)$$

In the approximation of an "adiabatic" (infinitesimal) translation,  $\delta(\psi_t, \dot{y}^l \psi_t)(t)$  results in

$$-\frac{1}{2} \langle \psi_t | \dot{y}^i \left( \frac{\partial}{\partial x_a} \left( \frac{\partial g_{ij}}{\partial x_l} \right) \right) \dot{y}^j | \psi_t \rangle \xi_a := \ddot{\xi}_l(t) \quad (28)$$

where  $\ddot{\xi}_l(t)$  is defined to be the second derivative of  $\xi_l$ , the distance between the two trajectories as a function of time. We then define

$$\hat{\xi}_{al} := -\frac{1}{2} \dot{y}^i \left( \frac{\partial}{\partial x_a} \left( \frac{\partial g_{ij}}{\partial x_l} \right) \right) \dot{y}^j \quad (29)$$

as the *operator for geodesic deviation*.

In the  $\{y^j\}$  set of coordinates, it follows from [4] that the commutation relation between the momenta and the coordinate operators  $\{y^j\}$  is

$$[p^n, y^l] = -i\hbar g^{nl}(x) \quad (30)$$

Then, formally, a transformation between the two coordinate bases,  $\{x_i\}$  and  $\{y^j\}$ , is made to express  $\delta(\psi_t, \dot{y}^l \psi_t)(t)$  in the  $\{y\}$  coordinate system.

We simulated several quantum dynamical systems and investigated the behavior of the orbits of expectation values of  $\{y\}$ . We demonstrated the correlation between the complexity and properties of the orbits pictured by the expectation values and the predictions of local instability corresponding to deviation under small perturbation, in corresponding with the classical problem [4].

In a following recent work, we studied the quantum stability in the geometrization of quantum mechanical evolution in a Finsler geometry [11].

We defined the generalized geometric Hamiltonian operator to be

$$\hat{H}_G := \frac{1}{2m} p^i g_{ij}(x, p) p^j \quad (31)$$

where  $g_{ij}(x, p)$  is a Hermitian function of the operators  $x$  and  $p$  acting on  $\mathcal{H}_G$ .

We followed the Heisenberg picture to express the dynamical equations of the variables corresponding to  $\{x\}$  and  $\{p\}$ . Then, in analogy to DeWitt's work [12], we defined a new set of Hermitian operators  $y^l(x, p)$  and constructed the representation theory in the  $\{y\}$  coordinates, to designate general point transformations between the momentum operator  $p$  expressed in  $\{x\}$  space representation and the  $p'$  expressed in the  $\{y\}$  space representation. In the special case  $\tilde{g}_{ij}(x)$  and  $y(x)$ , it becomes DeWitt's point transformation formula for the quantum transformation law for the momentum operators [12][11]. Therefore, defining two space representations; the Gutzwiller representation and the Hamilton representation.

Next, following our point of view for the representation theory in the  $\{y\}$  coordinates, we studied the Heisenberg algebra with the geometric Hamiltonian of the form (31). Applying the Heisenberg picture for the variables corresponding to  $\{y\}$  as Heisenberg dynamical variables, results in

$$\dot{y}^l = \frac{-1}{m^2} p^n M_{nm}^l p^m + \frac{i}{2m\hbar} \frac{d}{dt} (p^i [g_{ij}, y^l] p^j), \quad M_{nm}^l := \frac{1}{2} \frac{\partial g_{nm}(x, p)}{\partial x_l} \quad (32)$$

closely related to the form of a geodesic equation, with a truncated connection form, obtained in the classical case (Equation (10)) and in the quantum Riemannian Hamiltonian evolution (equation (24)). The second term in Equation (32) is an essentially quantum effect.

A relation between the momentum  $p^l$  and velocity  $\dot{y}^l$  can be established by  $\dot{y}^l = \frac{1}{m} p^l + \frac{i}{2m\hbar} p^i [g_{ij}, y^l] p^j$  to obtain  $-\dot{y}^n M_{nm}^l \dot{y}^m$  with terms that are an essentially quantum effect [11].

Next, we define the adiabatic evolution of a slowly changing momentum operator where  $\dot{p}$  is considered small in a manner which ensures that

$$\frac{i}{2m\hbar} \frac{d}{dt} (p^i [g_{ij}, y^l] p^j) = \frac{i}{2m\hbar} (\dot{p}^i [g_{ij}, y^l] p^j + p^i \frac{d[g_{ij}, y^l]}{dt} p^j + p^i [g_{ij}, y^l] \dot{p}^j) \approx \frac{i}{2m\hbar} p^i \frac{d[g_{ij}, y^l]}{dt} p^j \quad (33)$$

We study the Heisenberg picture in the adiabatic limit which results in the following dynamical equation

$$\begin{aligned} \ddot{y}^l &= \frac{-1}{m^2} p^i M_{ij}^l p^j + \frac{-1}{m^2} p^i \Xi_{ij}^l p^j, \quad M_{ij}^l := \frac{1}{2} \frac{\partial g_{ij}(x, p)}{\partial x_l} \\ \text{where } \frac{d}{dt} \left[ \frac{i}{\hbar} g_{ij}, y^l \right] &= -\frac{2}{m} \Xi_{ij}^l, \quad \Xi_{ij}^l := \frac{-i}{2\hbar} [M_{ij}^n, y^l] g_{nm} p^m + \frac{-i}{2\hbar} p^n g_{nm} [M_{ij}^m, y^l] + \frac{1}{2\hbar^2} p^n [g_{nm}, [g_{ij}, y^l]] p^m \end{aligned} \quad (34)$$

such that two of the indices of the operator  $\Xi_{ij}^l$  are contracted with momentum.

Then, we attempt to find results for suggesting a definition for the unstable behavior of the dynamical evolution.

We followed the same procedure where we define  $\xi$  as a common number such that  $\psi(x, t) \rightarrow \psi(x + \xi, t)$  and computed  $\delta(\psi, \dot{y}^l \psi)(t)$ , assuming now the physical state is subjected to an infinitesimal translation as before with the results

$$\langle \psi | \frac{-1}{m^2} p^i \left( \left[ \frac{i}{\hbar} p^q, (M_{ij}^l + \Xi_{ij}^l) \right] \right) p^j | \psi \rangle \xi_q := \ddot{\xi}^l(t) \quad (35)$$



where the left-hand side of expression (35) is defined as the second derivative with respect to the common number  $\zeta^l$ , the distance between the two trajectories as a function of time. We then define the *geodesic deviation operator*

$$\hat{\zeta}^{ql} := \frac{-i}{m^2 \hbar} p^i ([p^q, (M_{ij}^l + \Xi_{ij}^l)]) p^j \quad (36)$$

and study the expectation values. The first term corresponds to the operator for geodesic deviation Equation (29) [4] and refers to the underlying geometry of the Hamilton manifold, while the second term accounts for the underlying "geometric flow" altering the Hamilton manifold.

Classically, it provides an underlying geometric interpretation associated with the Heisenberg picture for the quantum mechanical "Finslerian evolution" with a quantum mechanical form of the "geodesic flow"  $\dot{y}^l$ .

The classical initial value problem ( $\{\cdot, \cdot\}$  is the Poisson bracket)

$$\begin{aligned} \frac{\partial}{\partial t} \{g_{ij}, y^l\} &= \frac{2}{m} \Xi_{ij}^l \\ \text{where } \Xi_{ij}^l &:= \{M_{ij}^n, y^l\} g_{nm} p^m + \frac{-1}{2} p^n \{g_{nm}, \{g_{ij}, y^l\}\} p^m \\ \text{such that } \{g_{ij}, y^l\}(0, x, p) &= \{g_{ij}, y^l\}(x, p) \end{aligned} \quad (37)$$

has a unique smooth solution  $\{g_{ij}, y^l\}(t, x, p)$  for some short time interval  $[0, \epsilon)$ .

Equation (37) reflects a geometric flow, referring to the geometry of the Finsler manifold, altered by changing  $\{g_{ij}, y^l\}$  (a one-parameter family of  $\{g_{ij}, y^l\}(t)$ ) via a PDE with the initial condition  $\{g_{ij}, y^l\}(0, x, p) = \{g_{ij}, y^l\}(x, p)$ . A unique smooth short-time solution  $\{g_{ij}, y^l\}(t, x, p)$  is governing the local geometric flow. In this sense, Equation (34) accounts for an underlying "geometric evolution" equation and may be thought of as emerging from the underlying alteration of the connection form. Finally, we suggest a conjecture for "local instability" in quantum theory. This new geometric approach is applicable to a Finsler geometry where a "deviation operator" is introduced to define "local instability" in quantum theory. The expectation values contain important diagnostic behavior and could well be incorporated into a new definition of "quantum chaos", corresponding to deviation under small perturbation [11].

In this work, we attempt to address the basic problem of quantum chaos. We provide a geometric structure within the operator space in the Heisenberg picture with detailed analysis based on the geometric flow properties within the operator space. Conjectures are made that relate the stability properties of the geometric flow within the metric operator space with the stability properties of the quantum mechanical dynamics. A corresponding classical counterpart system is introduced in attempt to relate local instability in the classical dynamics with a formal definition we suggest for "local instability" in the quantum theory.

## 2. Emergence of an Underlying Geometric Flow in the Quantum Theory

We seek to develop a method that incorporates quantum mechanical dynamics into the analysis with the dependency on initial conditions, rather than the dynamical equations that govern the evolution of the trajectories defined by the expectation values.

### 2.1. Representation Theory in the $\{x\}$ Coordinates

In the present work we introduce a new approach which concerns the problem of geometrizing the quantum mechanical dynamics involved the evolution of the tensor metric operator  $g_{ij}(x)$  in the Heisenberg picture evolved in time by a geometric flow which essentially by its nature is originated from the evolution generated by the geometric Hamiltonian operator in a form defined by

**Definition 1.** (Geometric Hamiltonian operator)

Let the position and momentum operators  $x$  and  $p$  satisfy the canonical commutation relation

$$xp - px = i\hbar I \quad (38)$$

and

$$[x_i, x_j] = [p^i, p^j] = 0 \quad (39)$$

Let  $\mathcal{H}_G(\mathbb{R}^n) := L^2(\mathbb{R}^n)$  be a Hilbert space corresponding to a given quantum mechanical system and let the tensor metric operator  $g_{ij}(x)$  be a Hermitian function of the position operator  $x$  acting on  $\mathcal{H}_G$  where  $x$  and  $p$  are as above.

We define the geometric Hamiltonian operator valued Hermitian Riemannian Hamiltonian to be

$$\hat{H}_G := \frac{1}{2m} p^i g_{ij}(x) p^j \quad (40)$$

where  $\hat{H}_G$  is a self-adjoint geometric Hamiltonian operator generating the evolution of the system.

In this work we achieve as well a quantum mechanical embedding of the geometrical structure where we follow some basic ideas introduced by us in previous works [4][11].

We start first by distinguish the Schrodinger picture (notated by  $S$ ) from Heisenberg operators (notated by  $H$ ):

$$g_{ij}(t) = \langle \psi(t) | g_{ij} | \psi(t) \rangle_S = \langle \psi(t_0) | U^\dagger g_{ij} U | \psi(t_0) \rangle_S = \langle \psi | g_{ij}(t) | \psi \rangle_H \quad (41)$$

where the operator is defined as

$$\begin{aligned} g_{ijH}(t) &= U^\dagger(t, t_0) g_{ijS} U(t, t_0) \\ g_{ijH}(t_0) &= g_{ijS} \end{aligned} \quad (42)$$

The time-evolution of the operators following the Heisenberg algebra with the geometric Hamiltonian operator of form eq.(40) is:

$$\begin{aligned} \frac{\partial g_{ijH}}{\partial t} &= \frac{\partial}{\partial t} (U^\dagger g_{ijS} U) = \frac{\partial U^\dagger}{\partial t} g_{ijS} U + U^\dagger g_{ijS} \frac{\partial U}{\partial t} + U^\dagger \frac{\partial g_{ijS}}{\partial t} U \\ &= \frac{i}{\hbar} U^\dagger \hat{H}_G g_{ijS} U - \frac{i}{\hbar} U^\dagger g_{ijS} \hat{H}_G U + \left( \frac{\partial g_{ij}}{\partial t} \right)_H \\ &= \frac{i}{\hbar} H_H g_{ijH} - \frac{i}{\hbar} g_{ijH} H_H = -\frac{i}{\hbar} [g_{ij}, \hat{H}_G]_H \end{aligned} \quad (43)$$

where  $H_H = U^\dagger \hat{H}_G U$ , that results in

$$\frac{\partial g_{ijH}}{\partial t} = -\frac{i}{\hbar} [g_{ij}, \hat{H}_G]_H \quad (44)$$

and is known as the Heisenberg equation of motion.

Since  $\hat{H}_G$  is a time-independent Hamiltonian,  $H_H = \hat{H}_G$ , and the Heisenberg equation results in

$$\frac{\partial g_{ijH}}{\partial t} = -\frac{i}{\hbar} [g_{ijH}, \hat{H}_G] \quad (45)$$



For simplicity we drop the notation  $g_{ij_H}$  and write  $g_{ij}$  in the rest of our present work. Next, we calculate explicitly the time evolution of the tensor metric operator  $g_{ij}(x)$  in the Heisenberg picture following eq.(45) which results in

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \frac{i}{\hbar} [\hat{H}_G, g_{ij}] = \frac{i}{2m\hbar} [p^n g_{nm} p^m, g_{ij}] = \\ \frac{i}{2m\hbar} ([p^n, g_{ij}] g_{nm} p^m + p^n g_{nm} [p^m, g_{ij}]) &= \frac{1}{2m} \left( \frac{\partial g_{ij}}{\partial x_n} g_{nm} p^m + p^n g_{nm} \frac{\partial g_{ij}}{\partial x_m} \right)\end{aligned}\quad (46)$$

such that

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \frac{1}{m} (M_{ij}^n g_{nm} p^m + p^n g_{nm} M_{ij}^m) = \frac{1}{m} \{p^n, g_{nm} M_{ij}^m\} \\ \text{where } M_{ij}^n &:= \frac{1}{2} \frac{\partial g_{ij}}{\partial x_n}\end{aligned}\quad (47)$$

Next, expressing eq.(47) explicitly, substituting anticommutator identities, satisfies the following expression

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \frac{1}{m} \{p^n, g_{nm} M_{ij}^m\} = \frac{1}{m} [p^n, g_{nm} M_{ij}^m] + \frac{2}{m} g_{nm} M_{ij}^m p^n \\ &= \frac{-i\hbar}{m} \frac{\partial (g_{nm} M_{ij}^m)}{\partial x_n} + \frac{2}{m} g_{nm} M_{ij}^m p^n \\ &= \frac{-2i\hbar}{m} M_{nm}^n M_{ij}^m + \frac{-i\hbar}{m} g_{nm} \frac{\partial M_{ij}^m}{\partial x_n} + \frac{2}{m} g_{nm} M_{ij}^m p^n\end{aligned}\quad (48)$$

## 2.2. Representation Theory in the $\{y\}$ Coordinates

Following our previous works [4][11], we now define a new set of Hermitian operators  $\{y_H^l(x)\}$  such that the commutation relations between the momenta and the operators  $\{y^j\}$  are given by a form as follows, in correspondence with the formula eq.(30), given in our previous work [4],

$$[p^n, y^l]_H := -2i\hbar g_H^{nl}(x) \quad (49)$$

with a tensor metric operator in a general operator-valued Hermitian form.

With the canonical commutation relations eq.(38), applied to the left side of eq.(49) results in

$$[p^n, y^l]_H = -i\hbar \frac{\partial y^l}{\partial x_n} = -2i\hbar g^{nl}(x, t) \quad (50)$$

Therefore, one may define the following relation (for a full discussion of this point see [4])

$$\frac{\partial g_{ij}}{\partial x_l} := 2g^{lm} \frac{\partial g_{ij}}{\partial y^m} \quad (51)$$

as a local relation between the two sets of coordinates, suggesting that as a result of such a definition one may think formally of a transformation between the two coordinate bases,  $\{x_i\}$  and  $\{y^j\}$  [4][11]. Applying the Heisenberg picture for the variables corresponding to  $\{y^l\}$ , and substituting equation (49), results in

$$\begin{aligned}\dot{y}^l &= \frac{i}{\hbar} [\hat{H}_G, y^l] = \frac{i}{2m\hbar} [p^i g_{ij} p^j, y^l] = \\ \frac{i}{2m\hbar} ([p^i, y^l] g_{ij} p^j + p^i [g_{ij}, y^l] p^j + p^i g_{ij} [p^j, y^l])\end{aligned}\quad (52)$$

such that it satisfies the following relation with the momentum operator

$$\dot{y}^l = \frac{2}{m} p^l = \{\dot{x}_k, g^{kl}\} \quad (53)$$

substituting equation (18), closely related to the form given in definition Equation (22) [4].

In analogy to our previous work [11] and DeWitt's work [12], introducing DeWitt's point transformation formula for the quantum transformation law for the momentum operators [12]; to designate general point transformations between the momentum operator  $p$  expressed in  $\{x\}$  space representation and the  $p'$  expressed in the  $\{y\}$  space representation, covariant under point transformations;

$$p'_l = \frac{\partial x_i}{\partial y^l} p^i + \frac{1}{2} [p^i, \frac{\partial x_i}{\partial y^l}] = \frac{1}{2} \{p^i, \frac{\partial x_i}{\partial y^l}\} \quad (54)$$

Substituting eq.(50-51), a local relation between the two sets of coordinates  $\{x_j\}$  and  $\{y^i\}$ , the momentum operator  $p'_l$  becomes

$$p'_l = -i\hbar \frac{\partial}{\partial y^l} - \frac{1}{2} i\hbar M_{li}^i, \quad \text{where} \quad M_{li}^i := g^{in} \frac{\partial g_{li}}{\partial y^n} \quad (55)$$

so that  $M_{li}^i$  has the same form (up to factor of 2) as the reduced connection form in the classical case (Equation (10)) and consistent with eq.(47), correctly symmetrized so as to make it Hermitian.

Note that  $p'_l(t)$  is time dependent via the dependency of  $M_{li}^i$  on  $g(x, t)$  which gives the origin for the underlying geometry  $(\mathcal{M}, g(x, t))$  involved with the  $\{y\}$  space representation.

Furthermore, substituting eq.(17), so that Equation (54) for the momentum operator  $p'_l$  implies the following relation

$$p'_l = \frac{1}{4} \{g_{li}, p^i\} = \frac{m}{2} \dot{x}_l \quad (56)$$

The relations between the two momentum operators, substituting  $p^i \rightarrow -i\hbar \frac{\partial}{\partial x_i}$  in Equation (54), are

$$p'_l = \frac{1}{2} g_{li} p^i - \frac{1}{2} i\hbar M_{li}^i \quad (57)$$

Then, the commutation relations result in

$$\begin{aligned} [p^n, x_l] &= -i\hbar \delta_l^n \\ [p^n, y^l] &= -2i\hbar g^{nl} \\ [p'_n, x_l] &= -\frac{1}{2} i\hbar g_{nl} \\ [p'_n, y^l] &= -i\hbar \delta_n^l \end{aligned} \quad (58)$$

The relations between the momenta and the velocities  $\{\dot{y}^n\}$  are (from Equation (56) and Equation (53)), so that

$$\begin{aligned} p^l &= \frac{m}{2} \dot{y}^l = \left\{ \frac{m}{2} \dot{x}_k, g^{kl} \right\} \\ p'_l &= \frac{1}{8} m \{\dot{y}^n, g_{ln}\} = \frac{1}{4} \{p^n, g_{ln}\} = \frac{m}{2} \dot{x}_l \end{aligned} \quad (59)$$

consistent with Equations (17) and (18) and closely related with the definition in Equations (22) and (23) [4] along with the definition for  $p'_l$  (Equation (54)).

Therefore, in analogy to our previous work [11] where we worked with two space representations, motivated by the work of Horwitz et al. [1] on the stability of classical Hamiltonian systems by geometrical methods, where as a coordinate space,  $\{y^l\}$ , which is called the *Hamilton manifold*, is endowed with a connection form  $M_{mn}^l$  (Equation (10)), and what is called the dynamical curvature. It is not uniquely defined in terms of the original manifold  $\{x_l\}$ , which is called the *Gutzwiller manifold*. To rigorously complete our mathematical formulation, we define two space representations, in analogy to our previous work [11], corresponding to the  $\{x\}$  space representation and the  $\{y\}$  space representation:

**Definition 2.** (Gutzwiller representation)

Let  $\mathcal{H}_G$  be a Hilbert space corresponding to a given quantum mechanical system, and let  $\hat{H}_G$  be the self-adjoint geometric Hamiltonian generating the evolution of the system as before. Let the position and momentum operators,  $x$  and  $p$ , be as before and satisfy the canonical commutation relations (CCR).

The position space wavefunctions, expressed in the  $\{x\}$  space representation

$$\psi(x, t) = \int_{\mathbb{R}} \psi(x', t) \langle x | x' \rangle dx' \quad (60)$$

where  $\psi(x, t) \in L^2(\mathbb{R}, dx)$ , are then said to be the Gutzwiller representation.

**Definition 3.** (Hamilton representation)

Let  $\mathcal{H}_G$  be a Hilbert space corresponding to a given quantum mechanical system, and let  $\hat{H}_G$  be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.

Let the position and momentum operators,  $x$  and  $p$ , be as before, and let the position and momentum operators,  $y$  and  $p'$ , be as before

$$p'_l(t) := -i\hbar \frac{\partial}{\partial y^l} - \frac{1}{2} i\hbar M_{li}^i, \quad \text{where} \quad M_{li}^i(t) := g^{in} \frac{\partial g_{li}}{\partial y^n} \quad (61)$$

and satisfy the commutation relations

$$[p'_n, y^l]_H = -i\hbar \delta_n^l \quad (62)$$

The position space wavefunctions are said to be the Hamilton representation, when expressed in the  $\{y\}$  space representation

$$\tilde{\varphi}(y, t) := \langle y | \varphi \rangle(t) = \int_{\mathcal{M}} \varphi(y', t) \langle y | y' \rangle d\omega' \quad (63)$$

where  $d\omega'$  denotes the volume element, and the integration is to be carried out over the entire range of coordinate values,  $\tilde{\varphi}(y, t) \in L^2$ .

Note that the Hamilton representation carries an integration over the underlying manifold  $(\mathcal{M}, g(x, t))$  which is induced by the tensor metric operator  $g_H(x, t)$ .

**2.3. Local Existence of an Underlying Ricci Flow in the Quantum Theory**

In this section, encouraged to find a method which brings in the quantum mechanical dynamics into the analysis of quantum chaos, we then construct a generalized form of the dynamical equations in the Heisenberg picture governing the evolution of the tensor metric operator  $g_{ij}(x)$  eq.(48) in a form of a "geometric flow".

Our so obtained construction carries a structure closely related in form to the classical Ricci flow equations which are time-evolution equations, i.e., equations of motion equal to the first-order time derivatives of  $g_{ij}$ .

We start by following our previous analysis of the Heisenberg picture for the tensor metric operator  $g_{ij}(x)$  as Heisenberg dynamical variable (eq.(48)), satisfying the Heisenberg's form for the equations of motion expressed in the  $\{x\}$  representation. Following our previous formal definition for a transformation between the two space representations  $\{x_i\}$  and  $\{y^j\}$ , equation (48) expressed in the  $\{y\}$  space representation results in

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= \frac{-2i\hbar}{m} M_{nm}^n M_{ij}^m + \frac{-i\hbar}{m} g_{nm} \frac{\partial M_{ij}^m}{\partial x_n} + \frac{2}{m} g_{nm} M_{ij}^m p^n \\ &= \frac{-2i\hbar}{m} M_{nm}^n M_{ij}^m + \frac{-2i\hbar}{m} \frac{\partial M_{ij}^m}{\partial y^m} + \frac{2}{m} g_{nm} M_{ij}^m p'_n \end{aligned} \quad (64)$$

Note that the momentum is expressed in the  $\{y\}$  space representation, covariant under point transformations between the two space representations [11].

Then, we express the momentum operator  $p'$  by the substitution of eq.(59) to relate the momentum  $p'$

with the velocity field expressed in the  $\{y\}$  space representation. One results with a relation between the two dynamical evolutions of the operators  $g_H$  and  $y_H$

$$\frac{\partial g_{ij}}{\partial t} = \frac{-2i\hbar}{m} M_{nm}^n M_{ij}^m + \frac{-2i\hbar}{m} \frac{\partial M_{ij}^m}{\partial y^m} + \frac{1}{4} g_{nm} M_{ij}^m \{y^k, g_{nk}\} \quad (65)$$

Next, we define the "Riemann curvature" operator in the formal form

$$R_{qmn}^l := \frac{\partial M_{qm}^l}{\partial y^n} - \frac{\partial M_{qn}^l}{\partial y^m} + M_{qm}^k M_{nk}^l - M_{qn}^k M_{mk}^l \quad (66)$$

expressed in the  $\{y\}$  space representation, to be a Hermitian function of the position operator  $x$  (since  $g_{ij}(x)$  is a c-number valued function of the  $x$  coordinate base? / is a function of the position operator  $x$ ?) acting on  $\mathcal{H}_G$ , equivalent to the classical form of the Riemann curvature tensor in the  $\{y^l\}$  coordinate base (associated with the geodesic deviation equation in the Hamilton manifold, accounting for the dynamical curvature in equation (14)[1]).

We define, as well, the "Ricci curvature" operator in a formal form to be

$$R_{ij} := R_{ilj}^l \quad (67)$$

classically equivalent in form to the Ricci curvature tensor obtained by taking the contraction of the second and fourth indices of the Riemann curvature tensor.

We suggest, as an analogy to the classical case, a formal construction for a quantum mechanical "Ricci flow" in a generalized form, associated with the general operator-valued Hermitian geometric Hamiltonian of the form (40) following the Heisenberg algebra with the geometric Hamiltonian. The quantum mechanical "Ricci flow" equation is expressed by the representation theory in the  $\{y\}$  coordinates on the Hilbert space  $\mathcal{H}_G$  corresponding to a given quantum mechanical system. Substituting Equations (66-67) in Equation (64) results in

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= \frac{2i\hbar}{m} R_{ij} + \frac{-2i\hbar}{m} \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right) + \frac{2}{m} g_{nm} M_{ij}^m p'_n \\ &= \frac{2i\hbar}{m} R_{ij} + \frac{-2i\hbar}{m} \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b - \frac{-i}{\hbar} g_{nm} M_{ij}^m p'_n \right) \end{aligned} \quad (68)$$

where  $R_{ij}$  is the Ricci operator defined in Equation (67).

Next, given the structure of Equation (68), we define a new operator in a generalized form to be an additional term to the Ricci operator in the quantum mechanical dynamical equation (68), formally defined as follows

$$\Xi_{ij} := \frac{1}{2\lambda} \frac{\partial g_{ij}}{\partial t} + R_{ij}, \quad \text{where } \lambda := \frac{i\hbar}{m} \quad (69)$$

and  $\Xi_{ij}$  is a general symmetric operator acting on  $\mathcal{H}_G$ .

In what follows, motivated by the structure implied by Equation (68), we focus on the topic of geometrization of the quantum mechanical dynamics in the study of geometric flows. We discuss only Ricci flow here but analogues of these results may apply more broadly.

First, we define the *adiabatic flow*, then, the tensor metric operator  $g_{ij}$  is used to define a "distance", to be described below, within the operator space, so it becomes a metric space.

The general form of  $\Xi_{ij}$  associated with the evolution generated by the geometric Hamiltonian of form of eq.(40), following Equation (68), is

$$\Xi_{ij} = \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b - \frac{-i}{\hbar} g_{nm} M_{ij}^m p'_n \quad (70)$$

so that, expressed in the  $\{y\}$  space representation (substitution of eq.(59)) results in

$$\Xi_{ij} = \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b - \frac{-im}{8\hbar} g_{nm} M_{ij}^m \{\dot{y}^k, g_{nk}\} \quad (71)$$

Next, we define an adiabatic quantum mechanical evolution. We start by following operator-valued analysis of the  $\Xi_{ij}$  operator eq.(70) which results in

$$\begin{aligned} \Xi_{ij} g^{nl} + g^{nl} \Xi_{ij} &= \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right) g^{nl} + g^{nl} \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right) - \frac{-i}{\hbar} (g_{nm} M_{ij}^m p'_n g^{nl} + g^{nl} g_{nm} M_{ij}^m p'_n) \\ \frac{\hbar}{-i} \{\Xi_{ij}, g^{nl}\} &= \frac{\hbar}{-i} \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} - (g_{nm} M_{ij}^m) (p'_n g^{nl} + g^{nl} p'_n) \\ \frac{\hbar}{-i} (g^{nm} M_{ij}^m) \{\Xi_{ij}, g^{nl}\} &= \frac{\hbar}{-i} (g^{nm} M_{ij}^m) \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} - (p'_n g^{nl} + g^{nl} p'_n) \end{aligned} \quad (72)$$

where we assume that  $M_{ij}^n$  is invertible so that  $M_{ij}^{ij} M_{ij}^m = M_{ij}^m M_{ij}^{ij} = I_n$ , while  $g^{ij} g_{ij} = g_{ij} g^{ij} = I_n$  and  $I_n$  denotes the n-by-n identity matrix.

Substituting Equation (59), i.e.  $\dot{y}^k = \frac{2}{m} \{p'_i, g^{ik}\}$ , in Equation (72) satisfies the following relation

$$\frac{2\hbar}{-mi} (g^{nm} M_{ij}^m) \{\Xi_{ij}, g^{nl}\} = \frac{2i\hbar}{m} (g^{nm} M_{ij}^m) \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} - \dot{y}^l \quad (73)$$

Define  $A^l$  to be a general operator acting on  $\mathcal{H}_G$ , giving eq.(69), such that

$$A^l = \frac{2\hbar}{-mi} (g^{nm} M_{ij}^m) \{\Xi_{ij}, g^{nl}\} \quad (74)$$

In the special case of Equation (70), the  $A^l$  operator, giving eq.(73), results in

$$A^l = \frac{2i\hbar}{m} (g^{nm} M_{ij}^m) \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} - \dot{y}^l \quad (75)$$

We now consider the case of a bounded  $A^l$  operator, considered to be small, in a manner which ensures that

$$\dot{y}^l \approx \frac{2i\hbar}{m} (g^{nm} M_{ij}^m) \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} \quad (76)$$

Equation (76) is in a structure that bounds the momentum  $p$  in case that the right hand side of eq.(76) is bounded (i.e. manifolds with bounded geometry, to be described below).

Substituting Equation (76) in Equation (71) results in

$$\begin{aligned} \Xi_{ij} &= \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b - \frac{-im}{8\hbar} g_{nm} M_{ij}^m \{\dot{y}^k, g_{nk}\} \\ &\approx \frac{m}{4i\hbar} g_{ml} M_{ij}^m \dot{y}^l - \frac{-im}{8\hbar} g_{nm} M_{ij}^m \{\dot{y}^k, g_{nk}\} \\ &= \frac{-im}{4\hbar} (g_{ml} M_{ij}^m \dot{y}^l - \frac{1}{2} g_{nm} M_{ij}^m \{\dot{y}^k, g_{nk}\}) \\ &= \frac{-im}{4\hbar} (g_{ml} M_{ij}^m \dot{y}^l - \frac{1}{2} g_{nm} M_{ij}^m [\dot{y}^k, g_{nk}] - g_{nm} M_{ij}^m g_{nk} \dot{y}^k) \\ &= \frac{im}{8\hbar} g_{nm} M_{ij}^m [\dot{y}^k, g_{nk}] \end{aligned} \quad (77)$$

which ensures that  $\Xi_{ij}$  is small as well.

Further operator-valued analysis results in

$$\begin{aligned}\Xi_{ij} &\approx \frac{im}{8\hbar} g_{nm} M_{ij}^m [\dot{y}^k, g_{nk}] \\ \Xi_{ij} &\approx \frac{im}{8\hbar} g_{nm} M_{ij}^m \left[ \frac{i}{\hbar} [\hat{H}_G, y^k], g_{nk} \right] = \frac{im}{8\hbar} g_{nm} M_{ij}^m \left[ \frac{\partial g_{nk}}{\partial t}, y^k \right]\end{aligned}\quad (78)$$

implying that the tensor metric operator is *adiabatically* changing in a sense that the commutation relation of the rate of change of the tensor metric operator with the position operator is small.

In our recent work on the subject of quantum mechanical Hamiltonian Evolution on a Finsler Manifold, generated by the self-adjoint geometric Hamiltonian  $\hat{H}_G = \frac{1}{2m} p^i g_{ij}(x, p) p^j$  [11] we found results with dynamical equations governing the evolution of the trajectories defined by the expectation values of the position expressed in the  $\{y(x, p)\}$  space representation in a structure of two terms, where the first term is the quantum mechanical form of a "geodesic flow", with a connection form as in eq.(55) (up to factor of two), and with a second term which is an essentially quantum effect. Note that this term, which was derived from the adiabatic case of a slowly changing momentum operator in a Finsler geometry, is in a form of  $\frac{-m}{2} \frac{d}{dt} [i\hbar g_{ij}, y^l]$ , and carries a structure closely related to eq.(78).

The evolving dynamical equation of  $\dot{y}^l$  derived in [11] results in

$$\dot{y}^l = \frac{-1}{m^2} p^i M_{ij}^l p^j + \frac{1}{2m} p^i \frac{d}{dt} \left[ \frac{i}{\hbar} g_{ij}, y^l \right] p^j, \quad M_{ij}^l := \frac{1}{2} \frac{\partial g_{ij}(x, p)}{\partial x_l} \quad (79)$$

where two of the indices  $i, j$  are contracted with momentum, follows a behavior such that the underlying geodesic flow, as exhibited by the expectation values, is subjected to the presence of an additional kind of "force" term, which is an essentially quantum effect, and has the consequence of contributing to the forces driving the system [11].

This term accounts for the underlying "geometric flow" altering the Hamilton manifold of the underlying geometry [11]. Classically, the sign of the eigenvalues of this term may contribute to the local stability properties of the geodesic flow on the Hamilton manifold, reflecting different behaviors of its geometric evolution [11].

Therefore, our study here on the quantum mechanical Ricci flow is essentially by its nature a geometric flow on a Finsler geometry (a perturbed Ricci flow on a Finsler geometry). The term of eq.(78), in this sense, may affect the stability properties of the perturbed Ricci flow [13][14].

We now turn to the following definitions

**Definition 4.** (*Adiabatic Flow*)

Let  $\mathcal{H}_G$  be a Hilbert space corresponding to a given quantum mechanical system and let  $\hat{H}_G$  be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.

Let the Riemann curvature operator  $R_{qmn}^l$ , the Ricci curvature operator  $R_{ij}$  and the  $\Xi_{ij}$  operator be as before.

Define  $A_l$  to be a bounded operator acting on  $\mathcal{H}_G$ , considered to be small, as before.

An adiabatic evolution is defined to be

$$\dot{y}^l \approx \frac{2i\hbar}{m} (g^{nm} M_m^{ij}) \left\{ \left( \frac{\partial M_{ai}^a}{\partial y^j} + M_{ib}^a M_{aj}^b \right), g^{nl} \right\} \quad (80)$$

A generalized quantum mechanical adiabatic flow equation reads

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= -2\lambda (R_{ij} - \Xi_{ij}), \quad \lambda := \frac{i\hbar}{m} \\ \text{where } \Xi_{ij} &:= \frac{m^2}{(4\hbar)^2} g_{nm} M_{ij}^m \left[ \frac{\partial g_{nk}}{\partial t}, y^k \right]\end{aligned}\quad (81)$$



so that  $R_{ij}$  is a Hermitian function of the tensor metric operator  $g_{ij}$  acting on  $\mathcal{H}_G$  and  $\Xi_{ij}$  is a symmetric operator acting on  $\mathcal{H}_G$ .

The adiabatic flow imposes a restriction on the momentum operator (it bounds the momentum operator) along the evolution of the tensor metric operator  $g_H$  in case that the geometry implied by eq.(80) is bounded (to be specified below).

Classically, following the canonical quantization (Dirac's correspondence principle),  $2\lambda\Xi_{ij}^{classic}$  becomes  $(\{\cdot, \cdot\}_{PB}$  is the Poisson bracket)

$$2\lambda\Xi_{ij}^{classic} = \frac{m}{8}g_{nm}M_{ij}^m\left\{\frac{\partial g_{nk}}{\partial t}, y^k\right\}_{PB} \quad (82)$$

Then, having defined the adiabatic flow, we define "distance" within the operator space and establish the perturbed Ricci flow as the underlying geometry induced by the operator metric space to be

**Definition 5.** (Quantum Mechanical Ricci Flow)

Let  $\mathcal{H}_G$  be a Hilbert space corresponding to a given quantum mechanical system and let  $\hat{H}_G$  be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.

Let the Riemann curvature operator  $R_{qmn}^l$ , the Ricci curvature operator  $R_{ij}$  and the  $\Xi_{ij}$  operator be as before.

Let the adiabatic evolution and the adiabatic flow be as before.

Define the operator space  $\chi$  to be a metric space with a distance function  $||\hat{\delta}_{ij}||_X(t)$  for each  $t \in \mathbb{R}^+$  such that  $||\hat{\delta}_{ij}||_X$  denotes curves from  $[t_s, t_s + \epsilon_s)$  into  $\chi$  by

$$||\hat{\delta}_{ij}||_X(t) := \sup_{t', t'' \in [t_s, t_s + \epsilon_s)} ||g_{ij}(t') - g_{ij}(t'')||_\chi$$

$$\text{where } h_{ij} := g_{ij}(t') - g_{ij}(t''), \quad ||h_{ij}||_\chi := \sup_{\phi, \psi \in \mathcal{H}_G} \sqrt{\langle \phi | h_{ij}^\dagger h_{ij} | \psi \rangle} \quad (83)$$

for all states  $\phi, \psi \in \mathcal{H}_G$  in  $n$ -dimensional  $\mathcal{H}_G(\mathbb{M}^n) := L^2(\mathbb{M}^n)$ , expressed in the Hamilton representation as before. Denote the space of continuous curves for which this norm is finite by  $C([t_0, T]; \chi)$ , for a time interval  $[t_0, T)$ , where  $||\hat{\delta}_{ij}||_X(t)$  lies in the space  $C([t_0, T]; \chi)$ .

Define the trajectory  $\Phi$  corresponding to an initial curve  $||\hat{\delta}||_X(t_0)$  to be [16,17]

$$\Phi_{ij}(t) := \{||\hat{\delta}_{ij}||_X(t) \mid ||\hat{\delta}_{ij}||_X(t), ||\hat{\delta}_{ij}||_X(t_0) \in C([t_0, T]; \chi)\}_{s=0, t \in [t_0, T)} \quad (84)$$

i.e.,  $\Phi(t)$  is the set of  $[t_0, T) \times \{||\hat{\delta}_{ij}||_X(t) : t \in [t_0, T)\}$  reached in the course of the evolution of the system from an initial  $||\hat{\delta}_{ij}||_X(t_0)$ . The trajectory  $\Phi(t)$  is then said to be the quantum mechanical geometric flow within the  $C([t_0, T]; \chi)$  space.

Given a complete metric of bounded geometry  $g_0 = g_{ij}(x)$  on  $n$ -manifold  $(\mathcal{M}^n, g_{ij}(x))$ , a generalized quantum mechanical Ricci flow equation within the metric operator space  $C([t_0, T]; \chi)$  reads

$$\frac{\partial g_{ij}}{\partial t} = -2\lambda(R_{ij} - \Xi_{ij}), \quad \lambda := \frac{i\hbar}{m}$$

$$\text{where } \Xi_{ij} := \frac{m^2}{(4\hbar)^2}g_{nm}M_{ij}^m\left[\frac{\partial g_{nk}}{\partial t}, y^k\right], \quad R_{ij}, \Xi_{ij} \in C([t_0, T]; \chi) \quad (85)$$

$$\text{and } g_{ij}(0) = g_{ij}(x), \quad \Xi_{ij}(0) = \Xi_{ij}(g_{ij}(x)), \quad R_{ij}(0) = R_{ij}(g_{ij}(x))$$

so that  $R_{ij}$  and  $\Xi_{ij}$  are symmetric (0,2)-tensors and has at least a perturbed unique smooth solution  $g_{ij}(t)$  on  $\mathcal{M} \times [0, T)$ , so that  $g(t) \approx (1 - \epsilon)g_{Ric}(t)$  is a small perturbation of the Ricci flow solution  $g_{Ric}(t)$  ( $\epsilon$  is positive or negative), and  $g(t)$  is a smooth one-parameter family of metrics on the  $n$ -manifold  $(\mathcal{M}^n, g_{ij}(t))$ .

Note that in the Hamilton representation the tensor metric operator expressed in the  $\{y\}$  space representation becomes a unitary invariant  $\tilde{g}_H(y)$  operator with self-adjoint geometric Hamiltonian

$\hat{H}_G := \frac{1}{2m} p'_i \tilde{g}_{ij}(y) p'_j$  generating the evolution of the system, however  $g_H(x)$  is not unitary invariant in the Heisenberg sense in the Hamilton representation.

We now turn to the final step, which suggests a definition for "local instability" in the quantum theory. This notion relies on the work of Bahuaud et al. [13][14][15], proving that the Ricci flow for complete metrics with bounded geometry depends continuously on initial conditions for finite time with no loss of regularity [14]. Their work leads to a general theorem (Theorem 3.6. in [14]) about convergence stability around any given converging flow trajectory in any geometric flow i.e. it applies not only to Ricci flow but to more general geometric flows as well.

We suggest to relate the stability of the quantum mechanical system with the principle of convergence stability for geometric flows in the operator metric space, which is the combination of the continuous dependence of the flow on initial conditions, with the stability of fixed points [13][14].

We first express  $\Xi_{ij}$  in the  $\{y^l\}$  coordinate base, substituting e.q.(59) in e.q.(78), expressed in the Hamilton representation

$$\Xi_{ij} \approx \frac{im}{8\hbar} g_{nm} M_{ij}^m [y^k, g_{nk}] = \frac{1}{2} g_{nm} M_{ij}^m g^{ka} \frac{\partial g_{nk}}{\partial y^a} = \frac{1}{2} g_{nm} M_{ij}^m M_{nk}^k \quad (86)$$

such that the generalized *quantum mechanical Ricci flow* equation within the metric operator space  $C([t_0, T]; \chi)$  reads

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2\lambda(R_{ij} - \Xi_{ij}), \quad \lambda := \frac{i\hbar}{m} \\ \text{where } \Xi_{ij} &:= \frac{1}{2} g_{nm} M_{ij}^m M_{nk}^k, \quad R_{ij}, \Xi_{ij} \in C([t_0, T]; \chi) \\ \text{and } g_{ij}(0) &= g_{ij}(x), \quad \Xi_{ij}(0) = \Xi_{ij}(g_{ij}(x)) \end{aligned} \quad (87)$$

We identify this geometric flow, under the adiabatic flow constraint, as the perturbed Ricci flow, where the perturbation term acts as an additional geometric correction.

We now invoke the the work of Bahuaud et al. [13][14][15], studying the convergence stability for Ricci flow on manifolds with bounded geometry and we apply here below their theorem (3.6.) [14].

The Banach spaces on which we let these operators act are little *Hölder* spaces. We work in the topology of little *Hölder* spaces of metrics as in [13][14][15].

Suppose that the geometric flow equation (87) is any ungauged geometric flow which admits a gauging, i.e. the addition of a term  $G(g)$  tangent to the diffeomorphism orbit and such that  $D(F + G)$  is an elliptic geometric operator which is admissible in the sense of [15] (see discussion in [14]).

### Conjecture 1. (Convergence Stability)

Let  $\mathcal{H}_G$  be a Hilbert space corresponding to a given quantum mechanical system and let  $\hat{H}_G$  be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.

Let the quantum mechanical Ricci flow (which is said to be the perturbed Ricci flow) be as before and the geometric flow equation reads

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2\lambda(R_{ij} - \Xi_{ij}), \quad \lambda := \frac{i\hbar}{m} \\ \text{where } \Xi_{ij} &:= \frac{1}{2} g_{nm} M_{ij}^m M_{nk}^k, \quad R_{ij}, \Xi_{ij} \in C([t_0, T]; \chi) \\ \text{and } g_{ij}(0) &= g_{ij}(x), \quad \Xi_{ij}(0) = \Xi_{ij}(g_{ij}(x)), \quad R_{ij}(0) = R_{ij}(g_{ij}(x)) \end{aligned} \quad (88)$$

given that  $g_{ij}(x)$  is a complete metric of bounded geometry on  $n$ -manifold  $(\mathcal{M}^n, g_{ij}(x))$ .

Let  $g_R(0)$  be any metric which is the initial point of a solution  $g_R(t)$  to the ungauged Ricci flow equation defined to be

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2\lambda R_{ij}, \quad \lambda := \frac{i\hbar}{m} \\ \text{where } g_{ij}(0) &= g_{ij}(x), \quad R_{ij}(0) = R_{ij}(g_{ij}(x)) \end{aligned} \quad (89)$$

given that  $g_{ij}(x)$  is as before and suppose that  $g_R(t)$  converges to  $g_h$ , where  $g_h$  is any stationary point of the Ricci flow equation, both ungauged and gauged flows [13][14][15], which is strictly stable, i.e., let  $\mathcal{M} = B^n$  and  $g_h$  be the Poincaré hyperbolic metric.

Then for any metric  $g$  sufficiently near to  $g_R$  in an appropriate weighted Hölder norm  $g - g_R \in C_\mu^{k,\alpha}(\mathcal{M})$  for some  $k \geq 2$  and  $\mu \in (0, n-1)$ , the ungauged perturbed Ricci flow  $g(t)$  is said to be a convergent stability flow if the entire trajectory  $g(t)$  remains close to  $g_R(t)$  for all  $t$ .

The quantum mechanical system is then said to be stable in  $\mathcal{H}_G$  along the evolution.

However, if at some time  $t' : g(t') - g_R(t') \notin C_\mu^{k,\alpha}(\mathcal{M})$ , the quantum mechanical system is then said to be locally unstable in  $\mathcal{H}_G$  at  $t'$ . The entire trajectory  $g(t)$  may not remain close to  $g_R(t)$  for all  $t$  and the unique continuation of  $g(t)$  as a solution may not exist for all  $t > 0$ .

In case that the Ricci flow is not a convergent stability flow in the sense of theorem 3.5. [14], then the quantum mechanical system is said to be unstable in  $\mathcal{H}_G$  along the evolution.

Note that Conjecture 1 is not in the if and only if sense, i.e., in case that the convergence stability holds for the Ricci flow, it may not hold for the perturbed Ricci flow equation.

### 3. Conclusions

This work attempts to address the basic notion of quantum mechanical dynamical stability and attempts to provide definitions for stability, instability and local instability within the quantum mechanical evolution.

A new geometric approach is introduced based on the construction of an underlying perturbed Ricci flow within the metric operator space which provides the basis for the notion of convergence stability in the quantum mechanical theory introduced in this work.

The detailed analysis results in a conjecture which addresses the quantum mechanical evolution defining instability properties of the quantum mechanical dynamical equations in a structure of geometric flows, valid beyond the stability properties of the counterpart classical systems regardless the Ehrenfest theorem.

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