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Orbital Spectrum of Generators of an $SU(N)$ Symmetry Group: Interpretation as Spin Angular Momentum Operators

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Abstract

We have developed a quantum structure of an $SU(N)$ symmetry group characterized by orthonormal group basis vectors. An accurate mathematical method is developed for enumerating and determining generators of the $SU(N)$ symmetry group as symmetric and antisymmetric pairs of tensor products of the group basis vectors. It is established that in the familiar Gell-Mann basis and the corresponding spin angular momentum basis, the generators are systematically distributed in an orbital spectrum composed of $N-1$ configuration shells each containing a definite number of generators specified by quantum numbers $l = 1, ..., N-1$; $m = 0, 1, ..., l$. In particular, in a standard orbital spectrum, a configuration shell specified by a quantum number $l = 1, ..., N-1$ contains $2l + 1$ traceless generators specified by the two quantum numbers l , m in a form precisely similar to the orbital spectrum of angular momentum states specified by orbital and magnetic quantum numbers $l = 0, 1, ..., n-1$; $m = 0, \pm 1, ..., \pm l$ in the n^{th}-energy level of an atom. In the spin angular momentum basis, each configuration shell may be interpreted as a Cartan-Weyl subspace containing generators satisfying Cartan subalgebra; the Lie algebra of the $SU(N)$ symmetry group is determined through the algebra of the Cartan-Weyl basis within and across the subspaces (configuration shells). Considering applications to gauge field theories, we easily establish that $SU(N)$ gauge fields have quantum structure corresponding directly to the orbital spectrum of generators of the $SU(N)$ symmetry group.

Keywords: spin angular momentum; SU(N) symmetry group; Cartan-Weyl subspace

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1. Introduction

In two earlier articles [1 , 2], we developed an exact mathematical method for determining $SU(N)$ symmetry group generators and established that the generators occur in a spectrum composed of $N-1$ state transition subspaces each containing a definite number of specified generators, similar to the electronic state configuration shells in the energy level spectrum of an atom. In particular, we developed an important algebraic property that the $SU(N)$ symmetry group characterizes an N-dimensional (integer $N = 2, 3, 4, ...$) quantum state space spanned by N mutually orthonormal state vectors $|n\rangle$, $n = 1, 2, 3, ..., N$, which we call group basis vectors, defined as column matrices, i.e., $N \times 1$ matrices, with entries 0 in all rows except entry 1 in the n-th row according to

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} ; \quad \ldots ; \quad |N-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} ; \quad |N\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{1}$$

satisfying orthonormalization relation

$$\langle n|m\rangle = \delta_{nm} \tag{2}$$

Generators of the $SU(N)$ symmetry group are determined as tensor products of pair-wise coupled group basis vectors defined in hermitian symmetric and antisymmetric forms within the respective state transition subspaces. We established in [2] that a state transition subspace contains a definite number of traceless diagonal and off-diagonal symmetric and antisymmetric generators specified by a quantum number.

In the present article, we follow the orbital shell interpretation suggested in [2] to provide a complete orbital spectrum of the *general* $SU(N)$ symmetry group generators, which includes $N-1$ traceful diagonal symmetric generators defined within each of the $N-1$ state transition subspaces. By taking a weighted sum of the $N-1$ traceful diagonal symmetric generators to determine the $N \times N$ identity matrix I, we reduce the general spectrum to the *standard* $SU(N)$ generator spectrum composed of the $N \times N$ identity matrix and the standard N^2-1 traceless diagonal and off-diagonal generators. The emerging picture suggests a reinterpretation of the state transition subspaces as *generator configuration shells* containing definite numbers of specified generators. This is now a comprehensive quantum structure in which each of the $N-1$ configuration shells in an $SU(N)$ generator spectrum is specified by a quantum number l taking $N-1$ values $l=1,...,N-1$ and contains a definite number of generators determined in symmetric and antisymmetric pairs specified by a quantum number m taking $l+1$ values $m=0,1,...,l$. The quantum structure of an $SU(N)$ generator spectrum is thus similar to the quantum structure of an orbital spectrum of angular momentum states in the n^{th}-energy level of an atom, which is composed of orbital state configuration shells specified by orbital angular momentum quantum number $l=0,...,n-1$, where each orbital shell contains a definite number $2l+1$ of angular momentum states specified by magnetic quantum numbers $m=0,\pm1,\pm2,...,\pm l$. In particular, developing a spin angular momentum interpretation leads to an important physical picture that an $SU(N)$ symmetry group has a quantum structure composed of an *orbital spectrum of generators* corresponding precisely to the standard orbital spectrum of angular momentum states in an atomic energy level, as we describe in detail below.

For clarity, we have chosen to develop the quantum structure and algebraic properties of an $SU(N)$ generator spectrum in three interrelated stages. The purpose here is to open up our minds to the general composition of an $SU(N)$ generator spectrum, which, in addition to the familiar N^2-1 traceless generators denoted in the Gell-Mann basis by $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$, also contains $N-1$ traceful diagonal symmetric generators arising from the basic algebraic definition of $SU(N)$ generators. An appropriately weighted sum of these $N-1$ traceful diagonal generators constitutes the full $N \times N$ identity matrix I, which completes the specification of the generator spectrum of the $SU(N)$ symmetry group.

In section 2, we present a general method for enumerating and determining $SU(N)$ symmetry group generators in the Gell-Mann basis specified by quantum numbers $l=1,...,N-1$, $m=0,1,...,l$. The enumeration defines the distribution of the generators in an orbital spectrum composed of $N-1$ configuration shells specified by an orbital quantum number $l=1,...,N-1$, where each configuration shell contains $2l+1$ traceless generators and 1 traceful diagonal generator in a *general* $SU(N)$ symmetry group. In section 3, we take an appropriately weighted sum of the $N-1$ traceful diagonal generators to obtain the full $N \times N$ identity matrix, thus reducing the general generator spectrum to a standard orbital spectrum. The property that each shell specified by a quantum number $l=0,1,...,N-1$ contains $2l+1$ generators specified by a quantum number $m=0,1,...,l$ brings the quantum structure of the standard $SU(N)$ generator spectrum into direct correspondence with the quantum structure of orbital angular momentum state spectrum in an atomic n^{th}-energy level composed of configuration shells each containing $2l+1$ ($l=0,1,...,n-1$) orbital angular momentum states. The similarity of the orbital spectrum of generators of the standard $SU(N)$ symmetry group and the orbital spectrum of angular momentum states in the n^{th}-energy level of an atom leads to a reinterpretation of $SU(N)$ symmetry group generators as spin angular momentum operators in section 4. The $SU(N)$ generators in the Gell-Mann basis are now enumerated and determined as symmetric and antisymmetric pairs of spin angular momentum operators specified by quantum numbers l , m in each of the $N-1$

configuration shells $l = 1, ..., N-1$, in one-to-one correspondence with orbital angular momentum states similarly specified by a corresponding pair of quantum numbers l , m in an atom. The $2l+1$ traceless generators in the l^{th}-shell are interpreted as components of a $(2l+1)$-component l^{th}-shell spin angular momentum vector, which we square to determine a quadratic spin angular momentum operator. The sum of generators raised to even and odd powers takes simple forms which we use to introduce l^{th}-shell quadratic and spin state superposition operators of general order. We introduce the Cartan-Weyl basis, which we use to generate basic algebraic relations for general $SU(N)$ symmetry groups. We give some remarks on the implications of the distribution of generators in an orbital spectrum in the formulation of $SU(N)$ gauge theories of particle physics in section 5. The Conclusion is given in section 6.

2. General Orbital Spectrum of Generators of $SU(N)$ Symmetry Groups

Following the picture originally introduced in [2], we develop a complete quantum structure of an $SU(N)$ symmetry group. The N orthonormal state vectors which characterize the N-dimensional quantum state space are now interpreted as $SU(N)$ symmetry group basis vectors defined in equation (1). As explained in [1 , 2], the group generators are determined as symmetric and antisymmetric tensor products of coupled pairs of the group basis vectors. In developing a detailed quantum structure in which the generators are distributed in an orbital quantum spectrum, we specify the N group basis vectors by two quantum numbers l , m taking integer values $l = 1, 2, ..., N-1$, $m = 0, 1, ..., l$. Consequently, the generators determined through tensor products of the group basis vectors are enumerated by the quantum numbers l , m as defined below.

The orbital spectrum of generators of the $SU(N)$ symmetry group is composed of $N-1$ *configuration shells* specified by the quantum number $l = 1, 2, ..., N-1$. The l^{th}-shell contains $2l+1$ traceless and 1 traceful generators. In the standard Gell-Mann notation λ, there are l pairs of traceless off-diagonal symmetric and antisymmetric generators enumerated by the quantum numbers l , m as $(\lambda_{l^2+2m} \, , \, \lambda_{l^2+2m+1})$ and 1 corresponding pair of diagonal generators enumerated at $m = l$ as $(\lambda_{l^2+2l} \, , \, I_{l^2+2l})$. Here, we define l as the shell quantum number l and m as the symmetric-antisymmetric pair quantum number, noting that m runs through values $m = 0, 1, ..., l-1$ in enumerating the l pairs of the off-diagonal generators and takes the value $m = l$ in enumerating the diagonal generators.

The l traceless off-diagonal symmetric and antisymmetric generator pairs $(\lambda_{l^2+2m} \, , \, \lambda_{l^2+2m+1})$ are determined as tensor products of the pair-wise coupled group basis vectors $|m+1\rangle$ and $|l+1\rangle$ obtained in hermitian form

$$\lambda_{l^2+2m} = |m+1\rangle\langle l+1| + |l+1\rangle\langle m+1| \quad ; \quad \lambda_{l^2+2m+1} = -i(\, |m+1\rangle\langle l+1| - |l+1\rangle\langle m+1| \,)$$
$$l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l-1 \tag{3}$$

The traceless and traceful diagonal pair $(\lambda_{l^2+2l} \, , \, I_{l^2+2l})$ at $m = l$ are determined as normalized superpositions of tensor products of the pair-wise coupled group basis vectors $|m+1\rangle$ and $|l+1\rangle$ obtained in hermitian form

$$\lambda_{l^2+2l} = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} (\, |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1| \,)$$
$$I_{l^2+2l} = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} (\, |m+1\rangle\langle m+1| + |l+1\rangle\langle l+1| \,) \, ; \qquad l = 1, ..., N-1 \tag{4}$$

Using equations (3)-(4), we run through all the $l+1$ values of the symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$ to enumerate and determine in explicit forms all the $2l+1$ traceless symmetric and antisymmetric generators $\lambda_l^2, \lambda_{l^2+1}, \lambda_{l^2+2}, ..., \lambda_{l^2+2l}$ and 1 traceful symmetric generator I_{l^2+2l}, making a total $2(l+1)$ generators in the l^{th}-shell of a general $SU(N)$ symmetry group. The orbital spectrum of generators in the l^{th}-shell is presented in the general form in equation (5) below.

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Orbital spectrum of generators in the l^{th}-shell of a general $SU(N)$ symmetry group

$$l^{\text{th}}\text{-shell}: \begin{cases} m=0: & \lambda_{l^2} = |1\rangle\langle l+1| + |l+1\rangle\langle 1| \\ & \lambda_{l^2+1} = -i(|1\rangle\langle l+1| - |l+1\rangle\langle 1|) \\ m=1: & \lambda_{l^2+2} = |2\rangle\langle l+1| + |l+1\rangle\langle 2| \\ & \lambda_{l^2+3} = -i(|2\rangle\langle l+1| - |l+1\rangle\langle 2|) \\ & \vdots \\ m=l-1: & \lambda_{l^2+2(l-1)} = |l\rangle\langle l+1| + |l+1\rangle\langle l| \\ & \lambda_{l^2+2l-1} = -i(|l\rangle\langle l+1| - |l+1\rangle\langle l|) \\ m=l: & \lambda_{l^2+2l} = \sqrt{\frac{2}{l(l+1)}}\sum_{m=0}^{l-1}(|m+1\rangle\langle m+1| - |l+1\rangle\langle l+1|) \\ & I_{l^2+2l} = \sqrt{\frac{2}{l(l+1)}}\sum_{m=0}^{l-1}(|m+1\rangle\langle m+1| + |l+1\rangle\langle l+1|) \end{cases} \quad (5)$$

The $2l+1$ traceless generators λ_{l^2+2m} , λ_{l^2+2m+1} , λ_{l^2+2l} in the l^{th}-shell are correlated by an algebra obtained in the general form

$$[\lambda_{l^2+2l}\ ,\ \lambda_{l^2+2m}] = i\sqrt{\frac{2l(l+1)}{l}}\ \lambda_{l^2+2m+1} \quad ; \quad [\lambda_{l^2+2m+1}\ ,\ \lambda_{l^2+2l}] = i\sqrt{\frac{2l(l+1)}{l}}\ \lambda_{l^2+2m}$$

$$\sum_{m=0}^{l-1}[\lambda_{l^2+2m}\ ,\ \lambda_{l^2+2m+1}] = i\sqrt{2l(l+1)}\ \lambda_{l^2+2l} \quad (6)$$

In the Gell-Mann basis, the generators expressed in tensor product forms are evaluated in the standard $N \times N$ matrix forms. As examples, we now use equation (5) to determine the orbital spectrum of the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups, which play important roles in the formulation of models of unified gauge theories of particle physics in quantum field theory [3-13].

2.1. Orbital Spectrum of $SU(2)$ Generators

The orbital spectrum of the $SU(2)$ symmetry group is composed of $2-1=1$ configuration shell specified by

$$N = 2\ :\quad \text{no. of shells} = 1\ :\quad \text{shell quantum numbers } l = 1$$

$$\text{group basis vectors}:\quad |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}\ ;\quad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad (7)$$

The single configuration shell is specified by orbital quantum number $l = 1$ and symmetric-antisymmetric generator pair quantum numbers $m = 0, 1$. This single 1^{st}-shell contains $l = 1$ pair of off-diagonal symmetric and antisymmetric generators specified by pair quantum number $m = 0$, and 1 pair of corresponding diagonal generators specified by $m = 1$. The $m = 0$ off-diagonal pair are enumerated and determined by setting $l = 1, m = 0$ in equation (5) as $\lambda_1 = |1\rangle\langle 2| + |2\rangle\langle 1|$, $\lambda_2 = -i(\ |1\rangle\langle 2| - |2\rangle\langle 1|\ )$, while the $m = 1$ diagonal pair are enumerated and determined by setting $l = 1$ in the last pair in equation (5) as $\lambda_3 = \sqrt{\frac{2}{1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1| - |2\rangle\langle 2|\ ) = (|1\rangle\langle 1| - |2\rangle\langle 2|)$, $I_3 = \sqrt{\frac{2}{1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1| + |2\rangle\langle 2|\ ) = (|1\rangle\langle 1| + |2\rangle\langle 2|)$. We use the definitions of the $SU(2)$ group basis vectors given above in equation (7) to evaluate the tensor products explicitly as 2×2 matrices. The $2(1+1) = 4$ symmetric and antisymmetric generators $\lambda_1, \lambda_2, \lambda_3, I_3$ contained in the single orbital configuration shell of the general $SU(2)$ generator spectrum are presented below.

Orbital spectrum of $SU(2)$ generators : single shell , $l = 1$

$$1^{\text{st}}\text{-shell, } l=1: \quad \begin{cases} m=0: \quad \lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ m=1: \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \quad I_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases} \tag{8}$$

2.2. *Orbital Spectrum of $SU(3)$ Generators*

The orbital spectrum of the $SU(3)$ symmetry group is composed of $3-1=2$ configuration shells specified by $l=1\,,\,2$.

$$N=3 \;:\quad \text{no. of shells} = 2 \;:\quad \text{shell quantum numbers } l=1\,,\,2$$

$$\text{group basis vectors}: \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

The 1^{st}-shell ($l=1$) contains $1+1=2$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1$; the $m=0$ pair are enumerated and determined by setting $l=1$ in equation (5) as $\lambda_1 = |1\rangle\langle 2| + |2\rangle\langle 1|$, $\lambda_2 = -i(\,|1\rangle\langle 2| - |2\rangle\langle 1|\,)$, while the $m=1$ pair are enumerated and determined by setting $l=1$ in the last pair in equation (5) as $\lambda_3 = \sqrt{\frac{2}{1(1+1)}}(\,\sum_{m=0}^{1-1}|m+1\rangle\langle m+1| - |2\rangle\langle 2|\,) = (|1\rangle\langle 1| - |2\rangle\langle 2|)$, $I_3 = \sqrt{\frac{2}{1(1+1)}}(\,\sum_{m=0}^{1-1}|m+1\rangle\langle m+1| + |2\rangle\langle 2|\,) = (|1\rangle\langle 1| + |2\rangle\langle 2|)$.

The 2^{nd}-shell ($l=2$) contains $2+1=3$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1,2,$ where $m=0,1$ specify off-diagonal and $m=2$ specifies the diagonal pairs. The $m=0$ off-diagonal pair are enumerated and determined by setting $l=2$ in equation (5) as $\lambda_4 = |1\rangle\langle 3| + |3\rangle\langle 1|$, $\lambda_5 = -i(\,|1\rangle\langle 3| - |3\rangle\langle 1|\,)$, the $m=1$ off-diagonal pair are enumerated and determined by setting $l=2$ in the second pair in equation (5) as $\lambda_6 = |2\rangle\langle 3| + |3\rangle\langle 2|$, $\lambda_7 = -i(\,|2\rangle\langle 3| - |3\rangle\langle 2|\,)$, while the $m=2$ diagonal pair are enumerated and determined by setting $l=2$ in the last pair in equation (5) as $\lambda_8 = \sqrt{\frac{2}{2(2+1)}}(\,\sum_{m=0}^{2-1}|m+1\rangle\langle m+1| - |3\rangle\langle 3|\,) = \frac{1}{\sqrt{3}}(\,(\,|1\rangle\langle 1| - |3\rangle\langle 3|\,) + (\,|2\rangle\langle 2| - |3\rangle\langle 3|\,)\,)$, $I_8 = \sqrt{\frac{2}{2(2+1)}}(\,\sum_{m=0}^{2-1}|m+1\rangle\langle m+1| + |3\rangle\langle 3|\,) = \frac{1}{\sqrt{3}}(\,(\,|1\rangle\langle 1| + |3\rangle\langle 3|\,) + (\,|2\rangle\langle 2| + |3\rangle\langle 3|\,)\,)$. We use the definitions of the $SU(3)$ group basis vectors given above in equation (9) to evaluate the tensor products explicitly as 3×3 matrices. The $(3-1)(3+2)=10$ symmetric and antisymmetric generators $\lambda_1, \lambda_2, \lambda_3, I_3, \ldots, \lambda_8, I_8$ contained in the 2 orbital configuration shells of the general $SU(3)$ symmetry group are presented below.

Orbital spectrum of SU(3) generators : 2 shells , $l=1,2$

$$1^{\text{st}}\text{-shell, } l=1: \quad \begin{cases} m=0: \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ m=1: \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$$

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$$
2^{\text{nd}}\text{-shell, } l=2: \quad \begin{cases} m=0: \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; ; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\ m=1: \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; ; \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ m=2: \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \; ; \quad I_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{cases} \tag{10}
$$

Without giving details of the straightforward enumeration and determination of the respective tensor products, we have obtained the orbital spectra of generators of the $SU(4)$, $SU(5)$ symmetry groups presented below.

2.3. *SU*(4) *Symmetry Group*

$$
N = 4 \;: \qquad \text{number of shells} = 3 \;: \quad \text{shell quantum numbers } l = 1,2,3
$$

$$
\text{group basis vectors}: \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

$$
\textit{Orbital spectrum of } SU(4) \textit{ generators}: \; 3 \text{ shells}, \; l = 1,2,3
$$

$$
1^{\text{st}}\text{-shell, } l=1: \quad \begin{cases} m=0: \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ m=1: \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \quad I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}
$$

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$$2^{\text{nd}}\text{-shell, } l=2: \begin{cases} m=0: & \lambda_4 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix} \\ m=1: & \lambda_6 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}; \quad \lambda_7 = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix} \\ m=2: & \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0 \end{pmatrix}; \quad I_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0 \end{pmatrix} \end{cases}$$

$$3^{\text{rd}}\text{-shell, } l=3: \begin{cases} m=0: & \lambda_9 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}; \quad \lambda_{10} = \begin{pmatrix} 0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix} \\ m=1: & \lambda_{11} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix}; \quad \lambda_{12} = \begin{pmatrix} 0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0 \end{pmatrix} \\ m=2: & \lambda_{13} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}; \quad \lambda_{14} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix} \\ m=3: & \lambda_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3 \end{pmatrix}; \quad I_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3 \end{pmatrix} \end{cases} \tag{11}$$

2.4. $SU(5)$ Symmetry Group

$$N=5: \quad \text{number of shells} = 4: \quad \text{shell quantum numbers } l = 1,2,3,4$$

$$\text{group basis vectors}: \; |1\rangle = \begin{matrix}1\\0\\0\\0\\0\end{matrix}; \; |2\rangle = \begin{matrix}0\\1\\0\\0\\0\end{matrix}; \; |3\rangle = \begin{matrix}0\\0\\1\\0\\0\end{matrix}; \; |4\rangle = \begin{matrix}0\\0\\0\\1\\0\end{matrix}; \; |5\rangle = \begin{matrix}0\\0\\0\\0\\1\end{matrix}$$

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$$\textit{Orbital spectrum of } SU(5) \textit{ generators}: \text{ 4 shells }, \ l = 1,2,3,4$$

$$1^{\text{st}} - \text{shell} \ : \ l = 1 \ : \ \begin{cases} m = 0 : \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m = 1 : \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$2^{\text{nd}} - \text{shell} \ : \ l = 2 \ : \ \begin{cases} m = 0 : \lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m = 1 : \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m = 2 : \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

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$$
3^{\mathrm{rd}} - \mathrm{shell}\ :\ l = 3\ :\ \begin{cases}
m=0:\lambda_9 = \begin{pmatrix} 0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0&0&0&-i&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\\0&0&0&0&0 \end{pmatrix}\\
m=1:\lambda_{11} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&-i&0\\0&0&0&0&0\\0&i&0&0&0\\0&0&0&0&0 \end{pmatrix}\\
m=2:\lambda_{13} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&-i&0\\0&0&i&0&0\\0&0&0&0&0 \end{pmatrix}\\
m=3:\lambda_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&-3&0\\0&0&0&0&0 \end{pmatrix}, \quad I_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&3&0\\0&0&0&0&0 \end{pmatrix}
\end{cases}
$$

$$4^{\text{th}} - \text{shell} \ : \ l = 4 \ : \ \begin{cases} m=0: \lambda_{16} = \begin{pmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0 \end{pmatrix}, \quad \lambda_{17} = \begin{pmatrix} 0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0 \end{pmatrix} \\ m=1: \lambda_{18} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0 \end{pmatrix}, \quad \lambda_{19} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0 \end{pmatrix} \\ m=2: \lambda_{20} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&1&0&0 \end{pmatrix}, \quad \lambda_{21} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0 \end{pmatrix} \\ m=3: \lambda_{22} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0 \end{pmatrix}, \quad \lambda_{23} = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&i&0 \end{pmatrix} \\ m=4: \lambda_{24} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-4 \end{pmatrix}, \quad I_{24} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&4 \end{pmatrix} \end{cases} \tag{12}$$

We have thus established that the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry group generators are distributed within configuration shells specified by appropriately defined quantum numbers l , m in an orbital spectrum. We observe that the $SU(2)$, $SU(3)$, $SU(4)$ generators in equations (8) , (10) , (11) have been determined in the familiar correct form [3-6 , 14 , 15], while the correct form of the $SU(5)$ generators obtained here in equation (12) rectify the errors arising from the arbitrary methods applied in studies of $SU(5)$ grand unification theory [3 , 6 , 10-13]. Similar enumeration and determination of the respective tensor products yield the orbital spectrum of generators of higher dimension $SU(N)$ $(N \geq 5)$ symmetry groups.

3. The Standard Orbital Spectrum of Generators

We observe that a complete specification of the orbital spectrum is composed of $N^2 - 1$ traceless generators $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ and $N - 1$ traceful generators $I_1, I_2, ..., I_{N-1}$, a total of $(N-1)(N+2)$, as seen in the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ examples in equations (8) , (10) , (11) , (12) above. But, in standard definition, an $SU(N)$ symmetry group is defined by the full $N \times N$ identity I and the $N^2 - 1$ traceless generators [3]. To reduce the general orbital spectrum of generators to the standard form, we apply an important algebraic property that an appropriately weighted sum of the $N - 1$ traceful diagonal symmetric generators I_{l^2+2l} in each of the $N-1$ shells $l = 1, 2, ..., N-1$ constitutes the full $N \times N$ identity generator I of the $SU(N)$ symmetry group according to

$$\frac{1}{N-1}\sum_{l=1}^{N-1}\sqrt{\frac{1}{2}l(l+1)}\; I_{l^2+2l} = I \tag{13}$$

We may then combine the $N-1$ traceful generators I_{l^2+2l} from the respective $N-1$ shells and effectively represent them by the resultant group identity generator I, which we denote in the Gell-Mann basis by $I = \lambda_0$ determined according to equation (13) as

$$I = \lambda_0\;;\quad \lambda_0 = \frac{1}{N-1}\sum_{l=1}^{N-1}\sqrt{\frac{1}{2}l(l+1)}\; I_{l^2+2l}\;;\quad N \geq 2 \tag{14}$$

We then obtain a standard definition of an $SU(N)$ symmetry group composed of one traceful identity generator λ_0 and N^2-1 traceless generators $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$.

Due to its unique algebraic property that it is traceful and commutes with all the other N^2-1 traceless generators, the identity generator λ_0 is placed in a separate shell, which we classify as the 0^{th}-shell, specified by the lowest orbital quantum number $l = 0$. We thus determine the standard orbital spectrum composed of N configuration shells specified by the quantum number $l = 0, 1, ..., N-1$. In the standard orbital spectrum of generators, the l^{th}-shell contains $2l+1$ generators specified by the quantum number $m = 0, 1, ..., l$, where $m = 0, 1, ..., l-1$ enumerates l pairs of traceless off-diagonal symmetric and antisymmetric generators, while $m = l$ enumerates the traceless diagonal generator. We now present the distribution of the generators $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ in the 0^{th} and l^{th} $(l = 1, 2, ..., N-1)$ shells in a standard orbital spectrum of an $SU(N)$ symmetry group in equation (15) below.

Standard orbital spectrum of generators of an $SU(N)$ symmetry group

$$0^{th} - \text{shell}\,,\; l=0\,,\; m=0: \qquad m=0: \qquad \lambda_0 = I$$

$$l^{\text{th}} - \text{shell},\; m = 0,1,\ldots,l: \begin{cases} m=0: & \lambda_{l^2} = |1\rangle\langle l+1| + |l+1\rangle\langle 1|, \\ & \lambda_{l^2+1} = -i\left(|1\rangle\langle l+1| - |l+1\rangle\langle 1|\right) \\ m=1: & \lambda_{l^2+2} = |2\rangle\langle l+1| + |l+1\rangle\langle 2|, \\ & \lambda_{l^2+3} = -i\left(|2\rangle\langle l+1| - |l+1\rangle\langle 2|\right) \\ \vdots & \quad\vdots \\ m=l-1: & \lambda_{l^2+2(l-1)} = |l\rangle\langle l+1| + |l+1\rangle\langle l|, \\ & \lambda_{l^2+2l-1} = -i\left(|l\rangle\langle l+1| - |l+1\rangle\langle l|\right) \\ m=l: & \lambda_{l^2+2l} = \sqrt{\frac{2}{l(l+1)}}\sum_{m=0}^{l-1}\Big(|m+1\rangle\langle m+1| - |l+1\rangle\langle l+1|\Big) \end{cases} \tag{15}$$

With each of the N shells containing $2l+1$ generators, the total number of generators in the standard orbital spectrum of an $SU(N)$ symmetry group is obtained as $\sum_{l=0}^{N-1}(2l+1) = N^2$, consisting of the single (1) identity generator λ_0 and the N^2-1 traceless generators $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$.

Using the explicit forms of the $N-1$ traceful generators I_{l^2+2l}, $l = 1, ..., N-1$ already determined in the general $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ orbital spectra in equations (8) , (10) , (11) , (12) to determine the respective identity generators λ_0 according to the definition in equation (13), we take

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account of the explicit forms of the respective $N^2 - 1$ traceless generators λ_1 , λ_2 , ... , λ_{N^2-1} already determined in the examples to present the standard orbital spectra of the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups below.

3.1. Standard Orbital Spectra of $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ Generators

Absorbing all the traceful generators I_{l^2+2l} from the $N-1$ shells into the $N \times N$ identity $\lambda_0 = I$ according to equation (14), we obtain the explicit form of the standard orbital spectrum of generators in equation (15) for $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups as examples below.

$$SU(2)$$

Using equation (13) with the traceful generator I_3 already determined in equation (8), we determine the $SU(2)$ identity generator λ_0 in the form

$$\lambda_0 = I_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

which, taken together with the $2^2 - 1 = 3$ generators $\lambda_1, \lambda_2, \lambda_3$ also determined in equation (8), constitute the standard orbital spectrum of generators of the $SU(2)$ symmetry group below.

$$\begin{gathered} 0^{th} - \text{shell} \; , \; l = 0 \; : \qquad m = 0 \qquad\qquad \lambda_0 \\ 1^{\text{st}} - \text{shell}, \; l = 1 : \quad \begin{cases} m = 0 : & \lambda_1 \; ; \; \lambda_2 \\ m = 1 : & \lambda_3 \end{cases} \end{gathered} \tag{17}$$

$$SU(3)$$

Using equation (13) with the traceful generators I_3 , I_8 already determined in equation (10), we determine the $SU(3)$ identity generator λ_0 in the form

$$\begin{gathered} 0^{\text{th}} - \text{shell}, \; l = 0 : \quad m = 0 \qquad \lambda_0 \\ 1^{\text{st}} - \text{shell}, \; l = 1 : \quad \begin{cases} m = 0 : & \lambda_1 \; ; \; \lambda_2 \\ m = 1 : & \lambda_3 \end{cases} \\ 2^{\text{nd}} - \text{shell}, \; l = 2 : \quad \begin{cases} m = 0 : & \lambda_4 \; ; \; \lambda_5 \\ m = 1 : & \lambda_6 \; ; \; \lambda_7 \\ m = 2 : & \lambda_8 \end{cases} \end{gathered} \tag{19}$$

$$SU(4)$$

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Using equation (13) with the traceful generators I_3 , I_8 , I_{15} already determined in equation (11), we determine the $SU(4)$ identity generator λ_0 in the form

$$\lambda_0 = \frac{1}{3}\left(I_3 + \sqrt{3}\, I_8 + \sqrt{6}\, I_{15}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

which, taken together with the $4^2 - 1 = 15$ generators $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, ..., \lambda_{13}, \lambda_{14}, \lambda_{15}$ also determined in equation (11), constitute the standard orbital spectrum of the $SU(4)$ symmetry group below.

$$0^{th} - \text{shell}\,,\ l = 0\ :\quad m = 0 \qquad\qquad \lambda_0$$

$$1^{\text{st}}\text{-shell},\ l = 1 : \quad \begin{cases} m = 0 : \lambda_1\ ;\ \lambda_2 \\ m = 1 : \lambda_3 \end{cases}$$

$$2^{\text{nd}}\text{-shell},\ l = 2 : \quad \begin{cases} m = 0 : \lambda_4\ ;\ \lambda_5 \\ m = 1 : \lambda_6\ ;\ \lambda_7 \\ m = 2 : \lambda_8 \end{cases}$$

$$3^{\text{rd}}\text{-shell},\ l = 3 : \quad \begin{cases} m = 0 : \lambda_9\ ;\ \lambda_{10} \\ m = 1 : \lambda_{11}\ ;\ \lambda_{12} \\ m = 2 : \lambda_{13}\ ;\ \lambda_{14} \\ m = 3 : \lambda_{15} \end{cases} \tag{21}$$

$SU(5)$

Using equation (13) with the traceful generators I_3 , I_8 , I_{15} , I_{24} already determined in equation (12), we determine the $SU(5)$ identity generator λ_0 in the form

$$\lambda_0 = \frac{1}{4}\left(I_3 + \sqrt{3}\, I_8 + \sqrt{6}\, I_{15} + \sqrt{10}\, I_{24}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

which, taken together with the $5^2 - 1 = 24$ generators $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, ..., \lambda_{22}, \lambda_{23}, \lambda_{24}$ also determined in equation (12), constitute the standard orbital spectrum of the $SU(5)$ symmetry group below.

$$0^{th} - \text{shell}\,,\ l = 0\ :\quad m = 0 \qquad\qquad \lambda_0$$

$$1^{\text{st}}\text{-shell}, \ l=1: \quad \begin{cases} m=0: & \lambda_1, \lambda_2 \\ m=1: & \lambda_3 \end{cases}$$

$$2^{\text{nd}}\text{-shell}, \ l=2: \quad \begin{cases} m=0: & \lambda_4, \lambda_5 \\ m=1: & \lambda_6, \lambda_7 \\ m=2: & \lambda_8 \end{cases}$$

$$3^{\text{rd}}\text{-shell}, \ l=3: \quad \begin{cases} m=0: & \lambda_9, \lambda_{10} \\ m=1: & \lambda_{11}, \lambda_{12} \\ m=2: & \lambda_{13}, \lambda_{14} \\ m=3: & \lambda_{15} \end{cases}$$

$$4^{\text{th}}\text{-shell}, \ l=4: \quad \begin{cases} m=0: & \lambda_{16}, \lambda_{17} \\ m=1: & \lambda_{18}, \lambda_{19} \\ m=2: & \lambda_{20}, \lambda_{21} \\ m=3: & \lambda_{22}, \lambda_{23} \\ m=4: & \lambda_{24} \end{cases}$$

The orbital spectrum of generators determined explicitly in the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups in equations (17) , (19) , (21) , (23) clearly display a complete shell structure of a standard orbital spectrum of $SU(N)$ symmetry group generators. Each shell specified by a quantum number $l = 0, 1, ..., N-1$ contains a definite number $2l+1$ of generators specified by the symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$. The generators in each shell satisfy the algebra obtained in the general form in equation (6).

We observe that the standard orbital spectrum of generators of an $SU(N)$ symmetry group is precisely similar to the spectrum of orbital angular momentum states specified by the standard orbital and magnetic quantum numbers $l = 0, 1, ..., n-1$, $m = 0, \pm1, \pm2, ..., \pm l$ in the n^{th}-energy level of an atom. Within the atomic energy level, each orbital state configuration shell specified by the orbital angular momentum quantum number l contains a definite number $2l+1$ of orbital angular momentum states specified by the magnetic quantum number m. The orbital spectrum of generators characterizes an important algebraic property that an $SU(N)$ symmetry group space has a quantum structure similar to the quantum structure of the n^{th}-energy level of an atom. This quantum structure gives motivation to reinterpret the generators in the Gell-Mann basis as *spin angular momentum operators*, composed of hermitian conjugate spin state raising and lowering operators specified by quantum numbers l , m. This reinterpretation of the generators in the Gell-Mann basis as spin angular momentum operators explicitly specified by the quantum numbers l , m ensures that the orbital spectrum of the $SU(N)$ symmetry group generators corresponds precisely to the orbital angular momentum state spectrum, also specified by a corresponding pair of quantum numbers l , m taking similar values in the n^{th}-energy level of an atom, as we now establish in the next section.

4. Orbital Spectrum of $SU(N)$ Symmetry Group Generators in Spin Angular Momentum Interpretation

The algebraic property that generators in an $SU(N)$ are distributed in a standard orbital spectrum composed of configuration shells specified by quantum numbers $l = 0, 1, ..., N-1$ and symmetric-antisymmetric generator pair quantum numbers $m = 0, 1, ..., l$, similar to the specification of orbital angular momentum states by orbital shell quantum numbers $l = 0, 1, ..., n-1$ and magnetic quantum numbers $m = -l, -(l-1), ..., 0, 1, ..., l-1, l$ in the n^{th}-energy level in an atom, provides the physical basis for seeking a precise correspondence between an $SU(N)$ symmetry group orbital spectrum and the orbital angular momentum state spectrum in the n^{th}-energy level in an atom.

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An atomic energy level specified by a principal quantum number n, normally referred to as the n^{th}-energy level, is composed of n orbital state configuration shells specified by an orbital angular momentum quantum number l taking n values $l = 0, 1, ..., n-1$. Each orbital shell contains $2l+1$ orbital angular momentum states, each described by a spherical harmonic function Y_l^m specified by orbital shell quantum number l and a magnetic quantum number m taking $2l+1$ integer values $m = -l, -(l-1), ..., 0, 1, ..., (l-1), l$.

Noting that the specifications of the configuration shell quantum numbers l in both $SU(N)$ generator and atomic n^{th}-energy level orbital state spectra are precisely consistent, we harmonize the specifications of the $SU(N)$ symmetric-antisymmetric generator pair and the atomic magnetic quantum numbers, m, by considering that in the atomic orbital state spectrum, the single $l = 0$, $m = 0$ state Y_0^0 takes a symmetrically neutral unit value obtained as

$$l = 0 \quad ; \quad m = 0 \; : \qquad Y_0^0 = 1 \tag{24}$$

while the remaining $2l$ orbital states Y_l^m , $l \neq 0$, specified by $m = \mp 1, ..., \mp l$ can be reinterpreted as l conjugate pairs (Y_l^{-m} , Y_l^m), now specified by l values of the magnetic quantum number $m = 1, ..., l$ according to the standard relation [16 , 17]

$$Y_l^{-m} = (-1)^m Y_l^{m*} \qquad ; \qquad m = 1, ..., l \tag{25}$$

Taking the single symmetrically neutral orbital state Y_0^0 in equation (24) and the l conjugate pairs $Y_l^{\pm m}$ related according to equation (25) together, we now redefine the atomic magnetic quantum number m as a conjugate orbital state pair quantum number taking $l+1$ values $m = 0, 1, ..., l$ including the unit state Y_0^0, which is now precisely consistent with the specification of the $SU(N)$ symmetric-antisymmetric generator pair quantum number m also taking $l+1$ values $m = 0, 1, ..., l$ including the identity generator λ_0.

To achieve complete harmony in the comparison of the shell structures of the orbital spectra of $SU(N)$ generators and atomic n^{th}-energy level orbital angular momentum states, we reorganize the notation for the atomic orbital angular momentum states Y_0^0 , $Y_l^{\pm m}$ in the equivalent form Y_{00} , $Y_{lm}^{\pm}$ according to the redefinitions

$$Y_{00} = Y_0^0 \; , \quad Y_{lm}^+ = Y_l^m \; , \quad Y_{lm}^- = Y_l^{-m} \quad \Rightarrow \quad Y_{lm}^- = (-1)^m \, Y_{lm}^{+*} , \quad m = 1, \ldots, l \tag{26}$$

We now redefine the $SU(N)$ generators and introduce an appropriate notation specified by the quantum numbers l , m corresponding directly to the specification and notation of the atomic orbital angular momentum states Y_{00} , $Y_{lm}^{\pm}$. Such a redefinition of $SU(N)$ generators is easily achieved in the spin angular momentum basis, where we follow the formulae for enumerating and determining $SU(N)$ generators in symmetric-antisymmetric pairs in equations (3) , (4) to introduce hermitian conjugate spin angular momentum state raising and lowering operators $S_{lm}^{\pm}$ defined by

$$S_{lm}^+ = |m+1\rangle\langle l+1| \; ; \quad S_{lm}^- = |l+1\rangle\langle m+1| \; ; \quad S_{lm}^- = (S_{lm}^+)^\dagger \; ; \quad l = 1, ..., N-1 \; ; \; m = 0, 1, ..., l-1 \tag{27}$$

Using the $SU(N)$ symmetry group basis state vector orthonormality relation given in equation (2), noting

$$m = 0, ..., l-1 \quad \Rightarrow \quad l+1 > m+1 \quad ; \quad \langle m+1|l+1\rangle = 0 \quad ; \quad \langle l+1|m+1\rangle = 0 \tag{28}$$

we obtain the algebraic relations

$$S_{lm}^{+\,2} = 0 \quad ; \quad S_{lm}^{-\,2} = 0 \qquad ; \qquad S_{lm}^+ S_{lm}^- = |m+1\rangle\langle m+1| \quad ; \quad S_{lm}^- S_{lm}^+ = |l+1\rangle\langle l+1| \tag{29}$$

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The $SU(N)$ generators in the Gell-Mann basis $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ are now interpreted as hermitian spin angular momentum operators specified by quantum numbers l , m. In particular, the off-diagonal symmetric-antisymmetric generator pair $(\lambda_{l^2+2m}\ ,\ \lambda_{l^2+2m+1})$ enumerated and determined according to the formula in equation (3) is now interpreted as the off-diagonal symmetric-antisymmetric hermitian spin operator pair $(\sigma^x_{lm}\ ,\ \sigma^y_{lm})$ determined in the form

$$\lambda_{l^2+2m} = \sigma^x_{lm} \quad ; \quad \lambda_{l^2+2m+1} = \sigma^y_{lm}$$

$$\sigma^x_{lm} = |m+1\rangle\langle l+1| + |l+1\rangle\langle m+1| \quad ; \quad \sigma^y_{lm} = -i(\ |m+1\rangle\langle l+1| - |l+1\rangle\langle m+1|\ ) \tag{30}$$

while the diagonal symmetric-antisymmetric generator pair $(\lambda_{l^2+2l}\ ,\ I_{l^2+2l})$ enumerated and determined according to the formula in equation (4) is now interpreted as the diagonal symmetric-antisymmetric generator hermitian spin operator pair $(\sigma^z_l\ ,\ \sigma^0_l)$ determined according to equation (4) in the form

$$\lambda_{l^2+2l} = \sigma^z_l \quad ; \quad I_{l^2+2l} = \sigma^0_l$$

$$\sigma^z_l = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} \sigma^z_{lm} \quad ; \quad \sigma^z_{lm} = |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1|$$

$$\sigma^0_l = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} I_{lm} \quad ; \quad I_{lm} = |m+1\rangle\langle m+1| + |l+1\rangle\langle l+1| \tag{31}$$

It follows from the definitions in equations (28) , (29) , (30) , (31) that the off-diagonal generators σ^x_{lm} , σ^y_{lm} and the diagonal generators are expressed in terms of the spin state raising and lowering operators $S^{\pm}_{lm}$ in the form

$$\sigma^x_{lm} = S^+_{lm} + S^-_{lm}\ ; \quad \sigma^y_{lm} = -i(\ S^+_{lm} - S^-_{lm}\ )\ ; \quad \sigma^z_l = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} (S^+_{lm}S^-_{lm} - S^-_{lm}S^+_{lm})$$

$$\sigma^0_l = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} (S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}) \tag{32}$$

We interpret the $2l$ traceless off-diagonal generators σ^x_{lm} , σ^y_{lm} (enumerated by $m = 0, 1, ..., l-1$) and the single traceless diagonal generator σ^z_l (enumerated by $m = l$) as components of a $(2l+1)$-component l^{th}-shell spin angular momentum vector $\vec{\sigma}_l$ defined by

$$\vec{\sigma}_l = (\ \sigma^x_{l0}\ ,\ \sigma^y_{l0}\ ,\ \sigma^x_{l1}\ ,\ \sigma^y_{l1}\ ,\ ...\ ,\ \sigma^x_{l\,l-1}\ ,\ \sigma^y_{l\,l-1}\ ,\ \sigma^z_l\ ) \quad ; \quad l = 1, ..., N-1 \tag{33}$$

We then introduce an l^{th}-shell quadratic spin angular momentum operator σ^2_l obtained as

$$\sigma^2_l = \vec{\sigma}_l \cdot \vec{\sigma}_l = \sum_{m=0}^{l-1} ((\sigma^x_{lm})^2 + (\sigma^y_{lm})^2)\ +\ (\sigma^z_l)^2 \tag{34}$$

Using σ^x_{lm} , σ^y_{lm} from equation (32) gives

$$(\sigma^x_{lm})^2 + (\sigma^y_{lm})^2 = 2(\ S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}\ ) \tag{35}$$

which we substitute into equation (34) to obtain the form

$$\sigma^2_l = \sum_{m=0}^{l-1} 2(\ S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}\ )\ +\ (\sigma^z_l)^2 \tag{36}$$

To introduce some higher order spin operators, we use the algebraic relations in equations (28) , (29) , (30) , (31) to obtain the following algebraic relations

$$I_{lm} = S^{+}_{lm}S^{-}_{lm} + S^{-}_{lm}S^{+}_{lm} \; ; \quad [S^{\pm}_{lm} \, , \, I_{lm}] = 0 \; ; \quad I^{k}_{lm} = I_{lm} \; ; \quad k = 1, 2, 3, ... \tag{37}$$

$$(\sigma^{x}_{lm})^2 = I_{lm} \; ; \quad [\, \sigma^{x}_{lm} \, , \, I_{lm} \,] = 0 \quad \Rightarrow \quad (\sigma^{x}_{lm})^{2k} = I_{lm} \; ; \quad (\sigma^{x}_{lm})^{2k+1} = \sigma^{x}_{lm}$$

$$(\sigma^{y}_{lm})^2 = I_{lm}; \quad [\, \sigma^{y}_{lm} \, , \, I_{lm} \,] = 0 \quad \Rightarrow \quad (\sigma^{y}_{lm})^{2k} = I_{lm}; \quad (\sigma^{y}_{lm})^{2k+1} = \sigma^{y}_{lm} \; ; \quad k = 1, 2, 3, ... \tag{38}$$

We can now use these general algebraic properties of the $SU(N)$ generators in the spin angular momentum basis to introduce generalizations of the l^{th}-shell quadratic spin angular momentum σ^2_l to higher order spin operators. Noting that σ^2_l as defined in equation (34) is an even-power spin operator, we introduce generalizations to l^{th}-shell *even-power spin operator* $Q_{l:2n}$ and *odd-power spin operator* $F_{l:2n+1}$ of n^{th}-order, $n = 0, 1, 2, 3, ...$ defined by

$$Q_{l:2n} = \sum_{m=0}^{l-1} ((\sigma^{x}_{lm})^{2n} + (\sigma^{y}_{lm})^{2n}) \; + \; (\sigma^{z}_{l})^{2n}$$

$$F_{l:2n+1} = \sum_{m=0}^{l-1} ((\sigma^{x}_{lm})^{2n+1} + (\sigma^{y}_{lm})^{2n+1}) \; + \; (\sigma^{z}_{l})^{2n+1} \; ; \quad n = 0, 1, 2, 3, ... \tag{39}$$

Using equations (37) , (38), we obtain

$$(\sigma^{x}_{lm})^{2n} + (\sigma^{y}_{lm})^{2n} = 2I_{lm} = 2(S^{+}_{lm}S^{-}_{lm} + S^{-}_{lm}S^{+}_{lm})$$

$$(\sigma^{x}_{lm})^{2n+1} + (\sigma^{y}_{lm})^{2n+1} = \sigma^{x}_{lm} + \sigma^{y}_{lm} = \alpha S^{+}_{lm} + \alpha^{*} S^{-}_{lm} \; ; \quad \alpha = 1 - i = \sqrt{2}\, e^{-i\frac{\pi}{4}} \tag{40}$$

which we substitute into equation (39) as appropriate to express the even-power and odd-power spin operators in the form

$$Q_{l:2n} = \sum_{m=0}^{l-1} 2(S^{+}_{lm}S^{-}_{lm} + S^{-}_{lm}S^{+}_{lm}) \; + \; (\sigma^{z}_{l})^{2n}$$

$$F_{l:2n+1} = \sum_{m=0}^{l-1} \sqrt{2}(e^{-i\frac{\pi}{4}} S^{+}_{lm} + e^{i\frac{\pi}{4}} S^{-}_{lm}) \; + \; (\sigma^{z}_{l})^{2n+1} \; ; \quad n = 0, 1, 2, 3, ... \tag{41}$$

Setting $n = 0$ in equation (41) gives 0^{th}-order even and odd power spin operators $Q_{l:0}$, $F_{l;1}$ in the l^{th}-shell in the form

$$n = 0 \; : \quad Q_{l:0} = \sum_{m=0}^{l-1} 2(S^{+}_{lm}S^{-}_{lm} + S^{-}_{lm}S^{+}_{lm}) \; + \; I \; ; \quad F_{l:1} = \sum_{m=0}^{l-1} \sqrt{2}(\, e^{-i\frac{\pi}{4}} S^{+}_{lm} + e^{i\frac{\pi}{4}} S^{-}_{lm}) \; + \; \sigma^{z}_{l} \tag{42}$$

where I is the full $N \times N$ identity generator. We observe that the l^{th}-shell odd-power spin operator of 0^{th}-order, $\mathcal{F}_{l:1}$, obtained here in equation (42) takes a form precisely similar to the form of a semiclassical spin Hamiltonian operator, which we define here as a spin state superposition operator. In general, we may now identify the l^{th}-shell odd-power spin operator $F_{l:2n+1}$ in equation (41) as the l^{th}-shell spin state superposition operator of n^{th}-order. On the other hand, the property that the full $N \times N$ identity I commutes with all the $N^2 - 1$ generators, the sub-identity $I_{lm} = S^{+}_{lm}S^{-}_{lm} + S^{-}_{lm}S^{+}_{lm}$ commutes with all the corresponding generators within the l^{th}-shell means that the 0^{th}-order even-power spin operator $Q_{l;0}$ commutes with all the $2l + 1$ generators within the l^{th}-shell, meaning that $Q_{l;0}$ may be interpreted

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as a Casimir operator within the l^{th}-shell. In addition, it is important to note that setting $n = 1$ in $Q_{l:2n}$ in equation (41) gives the 1^{st}-order even-power spin operator $Q_{l:2}$ in the form

$$n = 1 \;:\quad Q_{l:2} = \sum_{m=0}^{l-1} (S^{+}_{lm} S^{-}_{lm} + S^{-}_{lm} S^{+}_{lm}) \;+\; (\sigma^{z}_{l})^2 \quad \Rightarrow \quad Q_{l:2} = \sigma^2_l \tag{42}$$

which we identify as the l^{th}-shell quadratic spin angular momentum operator σ^2_l obtained in equation (36). We may then interpret the general l^{th}-shell even-power spin operator $Q_{l:2n}$ in equation (41) as the l^{th}-shell quadratic spin angular momentum operator of n^{th}-order.

We have now defined the full content of the l^{th}-shell of the orbital spectrum of an $SU(N)$ symmetry group in spin angular momentum interpretation. Having $2l + 1$ traceless generators expressed as spin angular momentum operators σ^{x}_{lm} , σ^{y}_{lm} , σ^{z}_{l}, together with the corresponding n^{th}-order quadratic spin angular momentum operator $Q_{l;2n}$ and spin state superposition operator $F_{l;2n+1}$, the l^{th}-shell of an $SU(N)$ symmetry group orbital spectrum of generators now takes the form presented in equation (43) below.

Standard orbital spectrum of generators in an $SU(N)$ symmetry group in spin angular momentum basis

$$0^{th} - \text{shell} \;:\quad m=0 \;: \qquad\qquad I$$

$$l^{\text{th}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{l0} = S^+_{l0} + S^-_{l0}, \quad \sigma^y_{l0} = -i(S^+_{l0} - S^-_{l0}) \\ m=1: & \sigma^x_{l1} = S^+_{l1} + S^-_{l1}, \quad \sigma^y_{l1} = -i(S^+_{l1} - S^-_{l1}) \\ & \vdots \\ m=l-1: & \sigma^x_{l,l-1} = S^+_{l,l-1} + S^-_{l,l-1}, \quad \sigma^y_{l,l-1} = -i(S^+_{l,l-1} - S^-_{l,l-1}) \\ m=l: & \sigma^z_l = \sqrt{\frac{2}{l(l+1)}} \sum_{m=0}^{l-1} (S^+_{lm}S^-_{lm} - S^-_{lm}S^+_{lm}) \\ & Q_{l:2n} = \sum_{m=0}^{l-1} 2(S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}) + (\sigma^z_l)^{2n} \\ & F_{l:2n+1} = \sum_{m=0}^{l-1} \sqrt{2}(e^{-i\pi/4}S^+_{lm} + e^{i\pi/4}S^-_{lm}) + (\sigma^z_l)^{2n+1} \end{cases} \tag{43}$$

In the spin angular momentum basis, the $2l+1$ traceless generators within the l^{th}-shell are correlated by an algebra obtained in the general form

$$[\sigma^z_l \,,\, \sigma^x_{lm}] = i\sqrt{\frac{2l(l+1)}{l}}\,\sigma^y_{lm} \quad ; \quad [\sigma^y_{lm} \,,\, \sigma^z_l] = i\sqrt{\frac{2l(l+1)}{l}}\,\sigma^x_{lm}$$

$$\sum_{m=0}^{l-1} [\sigma^x_{lm} \,,\, \sigma^y_{lm}] = i\sqrt{2l(l+1)}\,\sigma^z_l \tag{44}$$

corresponding to the l^{th}-shell algebra in the Gell-Mann basis in equation (6).

As examples, we present the standard orbital spectra of generators of $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups in the spin angular momentum basis, including the respective l^{th}-shell quadratic spin angular momentum and spin state superposition operators of n^{th}-order in equations (45)-(48) below.

$$SU(2)$$

$$0^{th} - \text{shell} \;:\quad m=0 \;: \qquad\qquad I$$

$$1^{\text{st}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{10} = S^+_{10} + S^-_{10}, \quad \sigma^y_{10} = -i(S^+_{10} - S^-_{10}) \\ m=1: & \sigma^z_1 = S^+_{10}S^-_{10} - S^-_{10}S^+_{10} \\ & Q_{1:2n},\ F_{1:2n+1}, \quad n=0,1,2,\dots \end{cases} \tag{45}$$

$$SU(3)$$

$$0^{th} - \text{shell} \;:\quad m=0 \;: \qquad\qquad I$$

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$$1^{\text{st}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{10} = S^+_{10} + S^-_{10}, \quad \sigma^y_{10} = -i(S^+_{10} - S^-_{10}) \\ m=1: & \sigma^z_1 = S^+_{10}S^-_{10} - S^-_{10}S^+_{10} \\ & Q_{1:2n},\ F_{1:2n+1}, \quad n = 0,1,2,\ldots \end{cases}$$

$$2^{\text{nd}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{20} = S^+_{20} + S^-_{20}, \quad \sigma^y_{20} = -i(S^+_{20} - S^-_{20}) \\ m=1: & \sigma^x_{21} = S^+_{21} + S^-_{21}, \quad \sigma^y_{21} = -i(S^+_{21} - S^-_{21}) \\ m=2: & \sigma^z_2 = \frac{1}{\sqrt{3}}\sum_{m=0}^{1}(S^+_{2m}S^-_{2m} - S^-_{2m}S^+_{2m}) \\ & Q_{2:2n},\ F_{2:2n+1}, \quad n = 0,1,2,\ldots \end{cases} \tag{46}$$

$$SU(4)$$

$$0^{th} - \text{shell} \ : \quad m = 0 \ : \qquad I$$

$$1^{\text{st}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{10} = S^+_{10} + S^-_{10}, \quad \sigma^y_{10} = -(S^+_{10} - S^-_{10}) \\ m=1: & \sigma^z_1 = S^+_{10}S^-_{10} - S^-_{10}S^+_{10} \\ & Q_{1:2n},\ F_{1:2n+1}, \quad n = 0,1,2,\ldots \end{cases}$$

$$2^{\text{nd}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{20} = S^+_{20} + S^-_{20}, \quad \sigma^y_{20} = -i(S^+_{20} - S^-_{20}) \\ m=1: & \sigma^x_{21} = S^+_{21} + S^-_{21}, \quad \sigma^y_{21} = -i(S^+_{21} - S^-_{21}) \\ m=2: & \sigma^z_2 = \frac{1}{\sqrt{3}}\sum_{m=0}^{1}(S^+_{2m}S^-_{2m} - S^-_{2m}S^+_{2m}) \\ & Q_{2:2n},\ F_{2:2n+1}, \quad n = 0,1,2,\ldots \end{cases}$$

$$3^{\text{rd}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{30} = S^+_{30} + S^-_{30}, \quad \sigma^y_{30} = -i(S^+_{30} - S^-_{30}) \\ m=1: & \sigma^x_{31} = S^+_{31} + S^-_{31}, \quad \sigma^y_{31} = -i(S^+_{31} - S^-_{31}) \\ m=2: & \sigma^x_{32} = S^+_{32} + S^-_{32}, \quad \sigma^y_{32} = -i(S^+_{32} - S^-_{32}) \\ m=3: & \sigma^z_3 = \frac{1}{\sqrt{6}}\sum_{m=0}^{2}(S^+_{3m}S^-_{3m} - S^-_{3m}S^+_{3m}) \\ & Q_{3:2n},\ F_{3:2n+1}, \quad n = 0,1,2,\ldots \end{cases} \tag{47}$$

$$SU(5)$$

$$0^{th} - \text{shell} \ : \quad m = 0 \ : \qquad I$$

$$1^{\text{st}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{10} = S^+_{10} + S^-_{10}, \quad \sigma^y_{10} = -i(S^+_{10} - S^-_{10}) \\ m=1: & \sigma^z_1 = S^+_{10}S^-_{10} - S^-_{10}S^+_{10} \\ & Q_{1:2n},\ F_{1:2n+1}, \quad n = 0,1,2,\ldots \end{cases}$$

$$
2^{\text{nd}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{20} = S^+_{20} + S^-_{20}, \quad \sigma^y_{20} = -i(S^+_{20} - S^-_{20}) \\ m=1: & \sigma^x_{21} = S^+_{21} + S^-_{21}, \quad \sigma^y_{21} = -i(S^+_{21} - S^-_{21}) \\ m=2: & \sigma^z_2 = \frac{1}{\sqrt{3}} \sum_{m=0}^{1} (S^+_{2m} S^-_{2m} - S^-_{2m} S^+_{2m}) \\ & Q_{2:2n}, \; F_{2:2n+1}, \quad n = 0,1,2,\ldots \end{cases}
$$

$$
3^{\text{rd}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{30} = S^+_{30} + S^-_{30}, \quad \sigma^y_{30} = -i(S^+_{30} - S^-_{30}) \\ m=1: & \sigma^x_{31} = S^+_{31} + S^-_{31}, \quad \sigma^y_{31} = -i(S^+_{31} - S^-_{31}) \\ m=2: & \sigma^x_{32} = S^+_{32} + S^-_{32}, \quad \sigma^y_{32} = -i(S^+_{32} - S^-_{32}) \\ m=3: & \sigma^z_3 = \frac{1}{\sqrt{6}} \sum_{m=0}^{2} (S^+_{3m} S^-_{3m} - S^-_{3m} S^+_{3m}) \\ & Q_{3:2n}, \; F_{3:2n+1}, \quad n = 0,1,2,\ldots \end{cases}
$$

$$
4^{\text{th}}\text{-shell}: \begin{cases} m=0: & \sigma^x_{40} = S^+_{40} + S^-_{40}, \quad \sigma^y_{40} = -i(S^+_{40} - S^-_{40}) \\ m=1: & \sigma^x_{41} = S^+_{41} + S^-_{41}, \quad \sigma^y_{41} = -i(S^+_{41} - S^-_{41}) \\ m=2: & \sigma^x_{42} = S^+_{42} + S^-_{42}, \quad \sigma^y_{42} = -i(S^+_{42} - S^-_{42}) \\ m=3: & \sigma^x_{43} = S^+_{43} + S^-_{43}, \quad \sigma^y_{43} = -i(S^+_{43} - S^-_{43}) \\ m=4: & \sigma^z_4 = \frac{1}{\sqrt{10}} \sum_{m=0}^{3} (S^+_{4m} S^-_{4m} - S^-_{4m} S^+_{4m}) \\ & Q_{4:2n}, \; F_{4:2n+1}, \quad n = 0,1,2,\ldots \end{cases} \tag{48}
$$

The generators are determined in explicit forms using the definitions of the spin state raising and lowering operators $S^{\pm}_{lm}$ in equation (27) for $l = 1, ..., N-1$, $m = 0, 1, ..., l$, with the respective group basis vectors defined in equations (7) , (9) , (11) , (12). The resulting matrix forms are exactly the same as the corresponding Gell-Mann matrices obtained in the respective orbital spectra of $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ in equations (8) , (10) , (11) , (12), enumerated according to the relations in equations (30) , (31).

4.1. $SU(N)$ Symmetry Group Algebra in the Cartan-Weyl Basis

Let us now determine the basic algebraic properties of the $N^2 - 1$ traceless generators distributed among the $N - 1$ configuration shells in the standard orbital spectrum of an $SU(N)$ symmetry group. We use the property that in the spin angular momentum basis, the l^{th}-shell contains 1 traceless diagonal generator σ^z_l, which we identify as a Cartain generator $H_l = \sigma^z_l$, and $2l$ traceless off-diagonal generators σ^x_{lm} , σ^y_{lm}, composed of two hermitian conjugate spin angular momentum operators S^+_{lm} , S^-_{lm} according to equation (30). We identify the set σ^z_l , S^+_{lm} , S^-_{lm} as the Cartan-Weyl basis in the l^{th}-shell [18 , 19]. The algebra generated by the Cartan-Weyl basis constitutes the Cartan subalgebra of the $2l + 1$ traceless generators within the l^{th}-shell. The l^{th}-shell and therefore, each of the $N - 1$ ($l = 1, ..., N - 1$) configuration shells in the orbital spectrum of an $SU(N)$ symmetry group, may be interpreted as a Cartan-Weyl subspace. The complete algebraic properties of the $SU(N)$ symmetry group is then determined by the algebra of the Cartan-Weyl basis in each of the $N - 1$ shells (subspaces) and the algebra of correlations across the shells, which we have determined in the general forms as presented below.

The $N - 1$ Cartan generators $\sigma^z_l = H_l$ in the orbital spectrum of generators mutually commute, satisfying

$$
[\, \sigma^z_l \, , \, \sigma^z_{l'} \,] = 0 \, ; \quad l \neq l' \; = 1, ..., N-1 \tag{49}
$$

where the two different quantum numbers $l \neq l'$ specify two different shells. , The Cartan-Weyl subalgebra of the basis σ_l^z , S_{lm}^+, S_{lm}^- within the l^{th}-shell is obtained in the general form governing correlations among all the $2l+1$ traceless generators as

$$[\,\sigma_l^z\,,\,S_{lm}^{\pm}\,] = \pm\sqrt{\frac{2(1+l)}{l}}\,S_{lm}^{\pm}\,;\quad \sum_{m=0}^{l-1}[\,S_{lm}^{+}\,,\,S_{lm}^{-}\,] = \sqrt{\frac{1}{2}l(l+1)}\,\sigma_l^z$$

$$[\,S_{lm'}^{\pm}\,,\,S_{lm}^{\mp}\,] = \pm\,|m+1\rangle\langle m'+1|\,;\quad [\,S_{lm'}^{\pm}\,,\,S_{lm}^{\pm}\,] = 0$$
$$l = 1,...,N-1 \quad ; \quad m' < m \ : \quad m' = 0,...,l-2 \quad ; \quad m = 1,...,l-1 \tag{50}$$

where two different pair quantum numbers $m \neq m'$ specify two different generators within the l^{th}-shell.

Across the configuration shells, the Cartan generators σ_l^z in *higher* shells specified by $l = 2,...,N-1$ commute with all operators $S_{l'm'}^{\pm}$ in the *lower* shells specified by $l' = 1,...,l-1$ according to the algebraic relations

$$[\,\sigma_l^z\,,\,S_{l'm'}^{\pm}\,] = 0$$
$$l > l' \ : \qquad l = 2,...,N-1 \quad ; \quad l' = 1,...,l-1\,; \quad m' = 0,...,l'-1 \tag{51}$$

while Cartan generators σ_l^z in lower shells specified by $l = 1,...,l'-1$ have mixed algebraic relations with the operators $S_{l'm'}^{\pm}$ in the upper shells specified by $l' = 2,...,N-1\,;\, m' = 0,...,l'-1$ according to

$$[\sigma_l^z, S_{l'm'}^{\pm}] = \begin{cases} \pm\sqrt{\dfrac{2}{l(l+1)}}\,S_{l'm'}^{\pm}, & m' < l \\ \mp\dfrac{2l}{\sqrt{2l(l+1)}}\,S_{l'm'}^{\pm}, & m' = l \\ 0, & m' > l \end{cases}$$

$$l < l' : \quad l = 1,\ldots,l'-1, \quad l' = 2,\ldots,N-1, \quad m' = 0,\ldots,l'-1 \tag{52}$$

Finally, the commutation brackets across shells between operators $S_{lm}^{\pm}$ in lower shells ($l = 1,\ldots,l'-1$; $m = 0,\ldots,l-1$) and operators $S_{l'm'}$ in higher shells ($l' = 2,\ldots,N-1$; $m' = 0,\ldots,l'-1$) *vanish*, except for the case $m' = m, l$, giving the general forms:

$$m' = m,\, l : \quad [S_{lm}^{\pm}, S_{l'm}^{\mp}] = \mp S_{l'l}^{\mp},$$
$$[S_{lm}^{\pm}, S_{l'l}^{\pm}] = \pm S_{l'm}^{\pm}$$

$$l < l' : \quad l = 1,\ldots,l'-1, \quad l' = 2,\ldots,N-1, \quad m = 0,\ldots,l'-1 \tag{53}$$

The full set of algebraic properties in equations (49)-(53) captures the complete quantum structure of an $SU(N)$ symmetry group with generators systematically distributed among $N-1$ configuration shells (Cartan-Weyl subspaces) in an orbital spectrum specified by quantum numbers $l = 0,1,...,N-1$, $m = 0,1,...,l$. Note that in the familiar Gell-Mann basis, the generators in the Cartan-Weyl basis within the l^{th}-shell (or l^{th}-subspace) are defined by $H_l = \lambda_{l^2+2l}$, $S_{lm}^{\pm} = \lambda_{l^2+2m} \pm i\lambda_{l^2+2m+1}$. It is straightforward to check that in the respective Cartan-Weyl basis, the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generators determined explicitly in equations (8) , (10) , (11) , (12) satisfy the algebraic properties in equations (49)-(53).

5. Some Remarks on Implications of the Orbital Spectrum of Generators in Models of $SU(N)$ Gauge Field Theories

We begin by observing that theoretical models of elementary particle interactions driven by electromagnetic, weak nuclear and strong nuclear forces, acting separately or as unified forces, in quantum field theory have generally been formulated as gauge field theories based on the algebraic properties of $U(1)$ and $SU(N)$ symmetry groups. For $SU(N)$ gauge field theories such as the unified electroweak interaction, strong nuclear interaction and grand unified interaction models, the driving gauge field forces are characterized by the respective vector bosons (quanta of the gauge field), which are identified directly with the generators of the Lie algebra of the chosen $SU(N)$ symmetry group. In particular, an $SU(N)$ gauge field is specified by N^2-1 vector boson components $\mathcal{A}_j^\mu$, $j=1,2,...,N^2-1$, corresponding to the N^2-1 traceless generators of the $SU(N)$ symmetry group enumerated in the Gell-Mann basis $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$, such that the $SU(N)$ gauge field potential four-vector $\mathcal{A}^\mu$ is obtained as $\mathcal{A}^\mu = \sum_{j=1}^{N^2-1} \lambda_j \mathcal{A}_j^\mu$ [6 , 8 , 10-20]. The general dynamics in an $SU(N)$ gauge model is generated by interaction energy arising from the coupling of particle currents and the gauge field. The formulation of an $SU(N)$ gauge field theory is completed by including appropriately specified scalar fields, generally identified as the Higgs fields, which also couple to the particle currents and the gauge field [6 , 8 , 10-20].

In the present study, we have established an important property that an $SU(N)$ symmetry group has a well-defined quantum structure in which the identity and all the N^2-1 generators are systematically distributed among $N-1$ configuration shells in an orbital spectrum specified by quantum numbers l , m, taking integer values $l = 0,1,...,N-1$, $m = 0,1,...,l$, precisely similar to the orbital spectrum of angular momentum states in an atomic energy level. In the familiar Gell-Mann basis and the corresponding spin angular momentum basis, the enumeration and determination of the $SU(N)$ symmetry group generators is explicitly specified by the quantum numbers l , m. The important physical implication is that the formulation of a gauge field theory must be based on the complete quantum structure of the $SU(N)$ symmetry group. As the group generators are distributed among configuration shells in an orbital spectrum, the corresponding gauge field vector boson components are also distributed among the same configuration shells specified by the same quantum numbers l , m. The quantum structure means that the $SU(N)$ gauge field four-vector $\mathcal{A}^\mu$ is composed of a spectrum of N^2-1 vector bosons $\mathcal{A}_j^\mu$, $j = 1,...,N^2-1$, distributed in definite numbers among the $N-1$ configuration shells of the orbital spectrum, as we demonstrate in the more transparent spin angular momentum basis below.

In particular, in the l^{th}-shell where the $2l+1$ generators σ_{lm}^x , σ_{lm}^y , σ_l^z are specified explicitly by quantum numbers $l = 1,...,N-1$, $m = 0,1,...,l$, the corresponding gauge field vector bosons are also specified by the same quantum numbers l , m as $\mathcal{A}_{lm}^{\mu x}$, $\mathcal{A}_{lm}^{\mu y}$, $\mathcal{A}_l^{\mu z}$, such that the gauge field potential four-vector $\mathcal{A}_l^\mu$ in the l^{th}-shell is obtained as $\mathcal{A}_l^\mu = \sum_{m=0}^{l-1} \sigma_{lm}^j \mathcal{A}_{lm}^{\mu j} + \sigma_l^z \mathcal{A}_l^{\mu z}$, $j = x,y$, and the total $SU(N)$ gauge field potential four-vector $\mathcal{A}^\mu$ is then obtained as the sum of the l^{th}-shell potentials $\mathcal{A}_l^\mu$ from all the $N-1$ shells in the form $\mathcal{A}^\mu = \sum_{l=1}^{N-1} \mathcal{A}_l^\mu$.

We present an illustrative picture of the expected quantum structure of an $SU(N)$ gauge field model, using the QCD $SU(3)$ and grand unified $SU(5)$ theories as examples, which capture all features of the quantum structure of gauge field models in equations (54) , (55) below. The generators are in the spin angular momentum basis, enumerated and determined according to the general $SU(N)$ generator spectrum in equation (43) and the $SU(3)$, $SU(5)$ examples in equations (46) , (48). We present the standard orbital spectrum of the corresponding generators and gauge field components side by side. Noting that the grand unified $SU(5)$ gauge field model is composed of strong, leptoquark and weak interaction sectors [3 , 6 , 10-13], the gauge field vector bosons corresponding to the traceless generators σ_{lm}^x , σ_{lm}^y , σ_l^z in the l^{th}-shell within these sectors are identified as gluons G^μ , leptoquark bosons X^μ and weak bosons $\mathcal{W}^\mu$. In each model, we have provided the l^{th}-shell component and the full gauge field is obtained as the sum of all the $N-1$ $(N = 3,5)$ components. For convenience, we have dropped the Greek index μ on the four-vector boson notations.

Shell structure of $SU(3)$ gauge field model

$$
\begin{aligned}
&1^{\text{st}}\text{-shell},\ l=1: &&\begin{cases} m=0: & \sigma^x_{10},\ \sigma^y_{10} \quad \mapsto \quad G^x_{10},\ G^y_{10} \\ m=1: & \sigma^z_1 \quad \mapsto \quad G^z_1 \\ & \mathcal{A}_1 = \sigma^z_1 G^z_1 + \sigma^x_{10} G^x_{10} + \sigma^y_{10} G^y_{10} \end{cases} \\
&2^{\text{nd}}\text{-shell},\ l=2: &&\begin{cases} m=0: & \sigma^x_{20},\ \sigma^y_{20} \quad \mapsto \quad G^x_{20},\ G^y_{20} \\ m=1: & \sigma^x_{21},\ \sigma^y_{21} \quad \mapsto \quad G^x_{21},\ G^y_{21} \\ m=2: & \sigma^z_2 \quad \mapsto \quad G^z_2 \\ & \mathcal{A}_2 = \sigma^z_2 G^z_2 + \sum_{m=0}^{1} (\sigma^x_{2m} G^x_{2m} + \sigma^y_{2m} G^y_{2m}) \end{cases}
\end{aligned} \tag{54}
$$

The QCD $SU(3)$ gauge field drives strong coupling interactions of elementary particles. Interestingly, the property that the strong interaction dynamics is composed of two orbital components $\mathcal{A}_1 = \sigma^z_1 G^z_1 + \sigma^x_{10} G^x_{10} + \sigma^y_{10} G^y_{10}$ in the 1^{st}-shell and $\mathcal{A}_2 = \sigma^z_2 G^z_2 + \sum_{m=0}^{1} (\sigma^x_{2m} G^x_{2m} + \sigma^y_{2m} G^y_{2m})$ in the 2^{nd}-shell seems consistent with Gell-Mann's proposal that the strong interaction may be divided into two parts, namely, medium strong interaction and very strong interaction, noting that the medium strong interaction breaks $SU(3)$ symmetry, but conserves isospin generated by $\lambda_1 = \sigma^x_{10}$, $\lambda_2 = \sigma^y_{10}$, $\lambda_3 = \sigma^z_1$ and hypercharge generated by $\lambda_8 = \sigma^z_2$ [20]. The full QCD $SU(3)$ gauge field is obtained as the sum of the two components, $\mathcal{A} = \sum_{l=1}^{2} \mathcal{A}_l$.

Shell structure of $SU(5)$ gauge field model

$$
\begin{aligned}
&1^{\text{st}}\text{-shell},\ l=1: &&\begin{cases} m=0: & \sigma^x_{10},\ \sigma^y_{10} \mapsto G^x_{10},\ G^y_{10} \\ m=1: & \sigma^z_1 \mapsto G^z_1 \\ & \mathcal{A}_1 = \sigma^z_1 G^z_1 + \sigma^x_{10} G^x_{10} + \sigma^y_{10} G^y_{10} \end{cases} \\
&2^{\text{nd}}\text{-shell},\ l=2: &&\begin{cases} m=0: & \sigma^x_{20},\ \sigma^y_{20} \mapsto G^x_{20},\ G^y_{20} \\ m=1: & \sigma^x_{21},\ \sigma^y_{21} \mapsto G^x_{21},\ G^y_{21} \\ m=2: & \sigma^z_2 \mapsto G^z_2 \\ & \mathcal{A}_2 = \sigma^z_2 G^z_2 + \sum_{m=0}^{1} (\sigma^x_{2m} G^x_{2m} + \sigma^y_{2m} G^y_{2m}) \end{cases} \\
&3^{\text{rd}}\text{-shell},\ l=3: &&\begin{cases} m=0: & \sigma^x_{30},\ \sigma^y_{30} \mapsto X^x_{30},\ X^y_{30} \\ m=1: & \sigma^x_{31},\ \sigma^y_{31} \mapsto X^x_{31},\ X^y_{31} \\ m=2: & \sigma^x_{32},\ \sigma^y_{32} \mapsto X^x_{32},\ X^y_{32} \\ m=3: & \sigma^z_3 \mapsto X^z_3 \\ & \mathcal{A}_3 = \sigma^z_3 X^z_3 + \sum_{m=0}^{2} (\sigma^x_{3m} X^x_{3m} + \sigma^y_{3m} X^y_{3m}) \end{cases} \\
&4^{\text{th}}\text{-shell},\ l=4: &&\begin{cases} m=0: & \sigma^x_{40},\ \sigma^y_{40} \mapsto \mathcal{W}^x_{40},\ \mathcal{W}^y_{40} \\ m=1: & \sigma^x_{41},\ \sigma^y_{41} \mapsto \mathcal{W}^x_{41},\ \mathcal{W}^y_{41} \\ m=2: & \sigma^x_{42},\ \sigma^y_{42} \mapsto \mathcal{W}^x_{42},\ \mathcal{W}^y_{42} \\ m=3: & \sigma^x_{43},\ \sigma^y_{43} \mapsto \mathcal{W}^x_{43},\ \mathcal{W}^y_{43} \\ m=4: & \sigma^z_4 \mapsto \mathcal{W}^z_4 \\ & \mathcal{A}_4 = \sigma^z_4 \mathcal{W}^z_4 + \sum_{m=0}^{3} (\sigma^x_{4m} \mathcal{W}^x_{4m} + \sigma^y_{4m} \mathcal{W}^y_{4m}) \end{cases}
\end{aligned} \tag{55}
$$

An important physical feature which emerges in the expected quantum structure of the $SU(5)$ gauge field model ($SU(5)$ GUT) in equation (54) is that the 24-component $SU(5)$ gauge field potential four-vector $\mathcal{A}^{\mu} = (G^{\mu}\ , X^{\mu}\ , \mathcal{W}^{\mu})$ is composed of $2\times 1+1 = 3$ gluons $G^x_{10}\ , G^y_{10}\ , G^z_1$ corresponding to the 3 traceless generators $\sigma^x_{10}\ , \sigma^y_{10}\ , \sigma^z_1$, constituting the component $\mathcal{A}_1 = \sigma^z_1 G^z_1 + \sigma^x_{10}G^x_{10} + \sigma^y_{10}G^y_{10}$ in the 1^{st}-shell; $2\times 2+1 = 5$ gluons $G^x_{20}\ , G^y_{20}\ , G^x_{21}\ , G21^y\ , G^z_2$ corresponding to the 5 traceless generators $\sigma^x_{20}\ , \sigma^y_{20}\ , \sigma^x_{21}\ , \sigma^y_{21}\ , \sigma^z_2$, constituting the component $\mathcal{A}_2 = \sigma^z_2 G^z_2 + \sum_{m=0}^{1}(\sigma^x_{2m}G^x_{2m} + \sigma^y_{2m}G^y_{2m})$ in the 2^{nd}-shell; $2\times 3+1 = 7$ leptoquark (or *intermediate interaction*) bosons $X^x_{30}\ , X^y_{30}\ , X^x_{31}\ , X^y_{31}\ , X^x_{32}\ , X^y_{32}\ , X^z_3$ corresponding to the 7 traceless generators $\sigma^x_{30}\ , \sigma^y_{30}\ , \sigma^x_{31}\ , \sigma^y_{31}\ , \sigma^x_{32}\ , \sigma^y_{32}\ , \sigma^z_3$, constituting the component $\mathcal{A}_3 = \sigma^z_3 X^z_3 + \sum_{m=0}^{2}(\sigma^x_{3m}X^x_{3m} + \sigma^y_{3m}X^y_{3m})$ in the 3^{rd}-shell and $2\times 4+1 = 9$ weak interaction gauge field bosons $\mathcal{W}^x_{40}\ , \mathcal{W}^y_{40}\ , \mathcal{W}^x_{41}\ , \mathcal{W}^y_{41}\ , \mathcal{W}^x_{42}\ , \mathcal{W}^y_{42}\ , \mathcal{W}^x_{43}\ , \mathcal{W}^y_{43}\ , \mathcal{W}^z_4$ corresponding to the 9 traceless generators $\sigma^x_{40}\ , \sigma^y_{40}\ , \sigma^x_{41}\ , \sigma^y_{41}\ , \sigma^x_{42}\ , \sigma^y_{42}\ , \sigma^x_{43}\ , \sigma^y_{43}\ , \sigma^z_4$, constituting the component $\mathcal{A}_4 = \sigma^z_4\mathcal{W}^z_4 + \sum_{m=0}^{3}(\sigma^x_{4m}\mathcal{W}^x_{4m} + \sigma^y_{4m}\mathcal{W}^y_{4m})$ in the 4^{th}-shell, making a total of $5^2-1 = 24$ gauge field vector bosons in the full $SU(5)$ gauge field $\mathcal{A} = \sum_{l=1}^{4}\mathcal{A}_l$. The $SU(5)$ gauge field model is thus composed of a strong interaction sector consisting of two sub-sectors driven by 3 gluon vector bosons $\mathcal{A}^{\mu}_1 \equiv (G^x_{10}, G^y_{10}, G^z_1)$ in the 1^{st}-shell and 5 gluon vector bosons $\mathcal{A}^{\mu}_2 \equiv (G^x_{20}, G^y_{20}, G^x_{21}, G^y_{21}, G^z_2)$ in the 2^{nd}-shell; an intermediate interaction sector driven by 7 leptoquark vector bosons $\mathcal{A}^{\mu}_3 \equiv (X^x_{30}, X^y_{30}, X^x_{31}, X^y_{31}, X^x_{32}, X^y_{32}, X^z_3)$ in the 3^{rd}-shell and a weak interaction sector driven by 9 weak vector bosons $\mathcal{A}^{\mu}_4 \equiv (\mathcal{W}^x_{40}, \mathcal{W}^y_{40}, \mathcal{W}^x_{41}, \mathcal{W}^y_{41}, \mathcal{W}^x_{42}, \mathcal{W}^y_{42}, \mathcal{W}^x_{43}, \mathcal{W}^y_{43}, \mathcal{W}^z_4)$ in the 4^{th}-shell. We note that the weak interaction sector in the 4^{th}-shell is generalized such that, of the 9 weak vector bosons, the first 6, namely, $\mathcal{W}^x_{40}\ , \mathcal{W}^y_{40}\ , \mathcal{W}^x_{41}\ , \mathcal{W}^y_{41}\ , \mathcal{W}^x_{42}\ , \mathcal{W}^y_{42}$, are identified with the (6) Y vector bosons introduced in the original $SU(5)$ grand unified theory [3 , 6 , 10-13], while the last three, namely, $\mathcal{W}^x_{43}\ , \mathcal{W}^y_{43}\ , \mathcal{W}^z_4$ are identified with the (3) well known $W^{\pm}\ , W_3$ vector bosons in the $SU(2)$ component of the standard $SU(2)\times U(1)$ electroweak gauge theory [6 , 8 , 10 , 11 , 14].

6. Conclusions

We have established that an $SU(N)$ symmetry group has a quantum structure characterized by a set of N orthonormal group basis vectors. The mathematical formula for enumerating and determining $SU(N)$ symmetry group generators provides an important property that the generators, obtained in symmetric and antisymmetric pairs specified by quantum numbers $l = 1, ..., N-1\ ; m = 0, 1, ..., l$ in both Gell-Mann and spin angular momentum bases, are distributed in an orbital spectrum composed of $N-1$ configuration shells. In a general orbital spectrum where the generators are determined in symmetric-antisymmetric pairs, the quantum number l taking integer values $l = 1, ..., N-1$ specifies a configuration shell, while the quantum number m taking integer values $m = 0, 1, ..., l$ specifies a symmetric-antisymmetric generator pair. Generators in the l^{th}-shell are specified by both l and m. Combining the $N-1$ traceful symmetric diagonal generators in a weighted sum constituting the full $N\times N$ identity generator reduces the generator spectrum to a standard orbital spectrum now composed of N configuration shells specified by $l = 0, 1, ..., N-1$. The l^{th}-shell in a standard orbital spectrum contains $2l+1$ generators, where the $l = 0$ shell contains only the traceful identity generator, i.e., $2\times 0+1 = 1$, while an $l = 1, ..., N-1$ shell contains $2l+1$ traceless generators. The $2l+1$ traceless generators within a shell satisfy a subalgebra. In the spin angular momentum basis, we have introduced generalized n^{th}-order even-power and odd-power spin angular momentum operators in each shell, which can be useful in determining the spectrum of states within the shell. Redefining the traceless generators in the Cartan-Weyl basis where the $N-1$ traceless diagonal generators are identified as Cartan generators, the configuration shells may be interpreted as Cartan-Weyl subspaces, meaning that the standard orbital spectrum of an $SU(N)$ symmetry group consists of $N-1$ Cartan-Weyl subspaces; the $2l+1$ traceless generators in the l^{th}-subspace satisfy Cartan subalgebra. The Lie algebra of the full $SU(N)$ symmetry group is easily generated through the Cartan-Weyl basis within and across the $N-1$ configuration shells (subspaces). We note that, with each of the N shells in the standard orbital spectrum containing $2l+1$ generators, the total number of generators of an $SU(N)$ symmetry group is obtained as $\sum_{l=0}^{N-1}(2l+1) = N^2$, consisting of the single (1) traceful identity

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generator and the $\sum_{l=0}^{N-1}(2l+1) = N^2 - 1$ traceless generators. The property that the $N^2 - 1$ traceless generators of an $SU(N)$ symmetry group are systematically distributed in $N-1$ configuration shells has important physical implications in the formulation of $SU(N)$ gauge theories of elementary particle physics. As we have demonstrated using the $SU(3)$ and $SU(5)$ examples, the $N^2 - 1$ components of an $SU(N)$ gauge field are distributed in $N-1$ configuration shells specified by two quantum numbers $l = 1, ..., N-1$, $m = 0, 1, ..., l$ in a standard orbital spectrum. Details of the quantum structure and further insight into the dynamical properties of existing or new models of gauge field theories of particle interactions based on $SU(N)$ symmetry groups will be presented in later work.

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