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Not peer-reviewed version

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Posted Date: 3 July 2025

doi: 10.20944/preprints202507.0315.v1

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Cut Elimination Versus Logic of Paradox

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Abstract

A Cut-/Reflexivity-free version $LK^{-\mathrm{C}/\mathrm{R}}$ of the propositional fragment of Gentzen calculus LK for the classical propositional logic PC endowed with propositional rules inverse to its logical ones as well as rules of constant elimination is proved to be equivalent to the bounded version of the *"logic of paradox"/"Kleene three-valued logic"* $(LP/K3)_{01}$ under the standard interpretation of propositional sequents by propositional clauses and inverse interpretation of propositional formulas by premise-less single-conclusion sequents, "with same theorems as PC, implying that LK has same derivable sequents as $LK^{-\mathrm{R}}$, and so yielding a new semantic insight into Cut Elimination in LK"/. As a by-product of the discovered equivalence and absence of proper consistent extensions of $(LP/K3)_{01}$ other than PC "and that relatively axiomatized by the *Ex Contradictione Quodlibet* rule"/, proved here upon the basis of the universal algebraic technique elaborated in an earlier work of ours, we prove that $LK^{-\mathrm{C}/\mathrm{R}}$ has no proper consistent extension other than LK "and the one relatively axiomatized by the context-free restriction of Cut"/.

Keywords: equent calculus; logical matrix; logic of paradox; disjunctive logic; Kleene algebra; extension; cut; structural rule; logical rule

MSC: 03B50; 03B53; 03F05; 06D30; 8C15

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1. Introduction

According to [14], the constant-free propositional empty-sequent-less fragment of LK [4] endowed with rules inverse to logical ones is equivalent (in the sense of [11]) to the *logic of paradox* LP [9], having the same theorems as the classical propositional logic PC, in view of [13, Lemma 4.14], that has yielded both a novel semantic insight into Cut Elimination in LK and the fact that the only proper consistent extension of the sequent calculus involved distinct from LK is the one relatively axiomatized by Cut with minimal non-empty context (viz., having just a single formula either on the right or on the left but not on both sides). The primary objective of this work is to expand [14] to the full propositional fragment of LK upon proper expanding underlying works [10,13].

The rest of the work is as follows. Section 2 is a brief summary of basis issues underlying the work. Section 3 is devoted to key universal issues, then used in the main part presented in Section 4.

2. General background

2.1. Set-theoretical background

Non-negative integers are identified with sets/ ordinals of lesser ones, their set/ordinal being denoted by ω. Unless any confusion is possible, one-element/-component sets/sequences are identified with their elements/components. As usual, functions are treated as binary relations.

Given any sets A, B and an infix $\diamond : A^2 \to A$, let $\wp_{(\omega)}([B,]A)$ be the set of all (finite) subsets of A [including B], $\epsilon_A \triangleq \{\langle a,a\rangle \mid a \in A\}$ the equality relation on A, $A^{*|+} \triangleq (\bigcup_{m\in(\omega\setminus(0|1))} A^m)$ and $\diamond_+ : A^+ \to A, \langle\{\langle\bar{a},b\rangle,\}c\rangle \mapsto (\{\diamond_+(\langle\bar{a},b\rangle)\diamond\}c)$, while A-tuples ⟨viz., functions with domain A⟩ are written in the sequence form $\bar{t}$ with t_a, where $a \in A$, stands for $\pi_a(\bar{t})$, as well as, in case $A = ([n+]1)$

 doi:10.20944/preprints202507.0315.v1

[where $n \in \omega$] written also in the standard finite tuple/sequence form $[(\langle)]t_0[\{,\}\ldots\{,\}t_n(\rangle)]$ [and identified with $\langle \bar{t}\upharpoonright n, t_n\rangle$] under identification of $B^{[n+]1}$ with $[B^n\times]B$ whereas $*:(A^*)^2\to A^*, \langle\bar{a},\bar{b}\rangle\mapsto(\bar{a}\cup((((+(\mathrm{dom}\,\bar{a})))\upharpoonright(\mathrm{dom}\,\bar{b}))^{-1}\circ\bar{b}))$ the *concatenation* binary operation.

An $X\in Y\subseteq\wp(A)$ is said to be *meet-irreducible in* Y, if $\forall Z\in\wp(Y):((A\cap(\bigcap Z))=X)\Rightarrow(X\in Z)$, their set being denoted by $\mathrm{MI}(Y)$. A $\mathcal{U}\subseteq\wp(A)$ is said to be *upward-directed*, if $\forall\mathcal{S}\in\wp_\omega(\mathcal{U}):\exists T\in(\mathcal{U}\cap\wp(\bigcup\mathcal{S},A))$, subsets of $\wp(A)$ closed under unions of upward directed subsets being called *inductive*. A *[finitary] closure operator over* A is any unary operation on $\wp(A)$ such that $\forall X\in\wp(A),\forall Y\in\wp(X):(X\cup \mathrm{C}(\mathrm{C}(X))\cup\mathrm{C}(Y))\subseteq\mathrm{C}(X)[=(\bigcup\mathrm{C}[\wp_\omega(X)])]$. A *closure system over* A is any $\{\langle$inductive$\rangle\}$ $\mathcal{C}\subseteq\wp(A)$ containing A and closed under intersections of subsets containing A, any $\mathcal{B}\subseteq\wp(A)$ {such that $\mathcal{C}=\{A\cap(\bigcap\mathcal{S})\mid\mathcal{S}\subseteq\mathcal{B}\}$ being called a *(closure) basis of* $\mathcal{C}$ and} determining the $\{\langle$finitary$\rangle\}$ closure operator $\mathrm{C}_{\mathcal{B}}\triangleq\{\langle Z,A\cap(\bigcap(\mathcal{B}\cap\wp(Z,A)))\rangle\mid Z\in\wp(A)\}\{=\mathrm{C}_{\mathcal{C}}\}$ over A such that $\mathcal{B}\subseteq(\mathrm{img}\,\mathrm{C}_{\mathcal{B}})\{=\mathcal{C}\}$. Conversely, $\mathrm{img}\,\mathrm{C}$ is a[n inductive] closure system over A such that $\mathrm{C}_{\mathrm{img}\,\mathrm{C}}=\mathrm{C}$, C and $\mathrm{img}\,\mathrm{C}$ being called *dual to* one another.

Remark 1. *Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element,* $\mathrm{MI}(\mathcal{C})$ *is a basis of any inductive closure system* $\mathcal{C}$. $\square$

A *[dual] Galois connection/retraction between/of* a poset $\mathcal{Q}=\langle Q,\precsim\rangle$ *and/onto* a poset $\mathcal{P}=\langle P,\lesssim\rangle$ is any $\langle f,g\rangle\in(Q^P\times P^Q)$ such that:

$$(a\lesssim c)\quad\Rightarrow\quad(f(c)\precsim^{[-1]}f(a)), \tag{1}$$

$$(b\precsim^{[-1]}d)\quad\Rightarrow\quad(g(d)\lesssim g(b)), \tag{2}$$

$$(a\lesssim g(b))\quad\Leftrightarrow\quad(b/a)\preceq^{[-1]}/\lesssim(f(a)/g(b)), \tag{3}$$

$$(f(a)/b)\quad=/\preceq^{[-1]}\quad f(g(f(a)/b)), \tag{4}$$

$$g(f(g(b)/a))\quad=\quad(g(b)/a), \tag{5}$$

for all $a,c\in P$ and $b,d\in Q$, [dual] Galois retractions of $\mathcal{Q}$ onto $\mathcal{P}$ being exactly [dual] Galois connections between $\mathcal{Q}$ and $\mathcal{P}$ with either injective left or surjective right component.

2.2. Algebraic background

Unless otherwise specified, we deal with a fixed but arbitrary finitary algebraic (viz., functional) signature L, viewed as a *propositional language* consisting of *(propositional) connectives*, L-algebras/"their carriers|class" being denoted by "/respective capital Fraktur/Italic letters [with /same indices], unless otherwise specified"|A_L. Then, $\mathrm{Tm}_L^{\{\alpha\}}$ {where $\alpha\in((\omega(\setminus 1))|\{\omega\})$ (unless L has a constant)|} is the set of L-terms, viewed as $\langle$*propositional*$\rangle$ *L-formulas*, with $\lceil$propositional$\rceil$ variables in $\mathrm{Var}_{\{\alpha\}}\triangleq(\mathrm{img}\,\bar{x}_{\{\alpha\}})$, where $\bar{x}_{\{\alpha\}}\triangleq\{\langle i,x_i\rangle\mid i\in(\omega\{\cap\alpha\})\}$, viz., the carrier of the absolutely-free L-algebra $\mathfrak{Tm}_L^{\{\alpha\}}$, freely-generated by $\mathrm{Var}_{\{\alpha\}}$ |{whose endomorphisms are viewed as $\lfloor$*propositional*$\rfloor$ *L-substitutions*, their set being denoted by $\mathrm{Sb}_L\ni\sigma_{+n}\triangleq[x_j/x_{j+n}]_{j\in\omega}$, where $n\in\omega$}. Any m-ary connective $\mathsf{c}\in L$, where $m\in\omega$, is identified with $\mathsf{c}(\bar{x}_m)\in\mathrm{Tm}_L^m$. As usual, the class of all "isomorphic copies"/subalgebras/"[ultra-]products of tuples" of members of a $\mathsf{K}\subseteq\mathsf{A}_L$ is denoted by $(\mathbf{I}/\mathbf{S}/\mathbf{P}^{[\mathrm{U}]})\mathsf{K}$.

2.3. Logical background

Here, we mainly follow [11] but allow infinitary logics and calculi as well as adopt more conventional terminology and notations.

Let $F=\langle L,P\rangle$ be a *[first-order] language* (viz., finitary signature), where $P\neq\varnothing$ is a relational one, any $\sigma\in\mathrm{Sb}_L$ being extended to the equally-denoted unary operation on the set $\mathrm{Fm}_{F|L}^{|P}$ of *F-formulas/-axioms* (viz., first-order atomic formulas of the signature F with variables in Var) via setting $\sigma(\Phi)\triangleq\mathsf{p}(\bar{\varphi}\circ\sigma)$, for all $\Phi=\mathsf{p}(\bar{\varphi})\in\mathrm{Fm}_F$. Then, any $\mathcal{R}=\langle\Gamma,\Psi\rangle\in\mathrm{Ru}_F^{[\omega]}\triangleq(\wp_{[\omega]}(\mathrm{Fm}_F)\times\mathrm{Fm}_F)$ is called a *(non-axiomatic||proper) [finitary] F-rule* with "elements of Γ"/Ψ called its *premises/conclusion*, written as (either $\frac{\Gamma}{\Phi}$ or $\Gamma\leadsto$)Φ and identified with $\langle$the universal closure of$\rangle$ $((\bigwedge\Gamma)\to)\Phi$ (iff $\Gamma\neq\varnothing$), those of

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the form $\Psi \rightsquigarrow \Upsilon$, where $\Upsilon \in \Gamma$, being said to be *inverse to* $\mathcal{R}$, while any $f : \mathrm{Fm}_F \to \lceil \wp_{[\omega]}(\rceil \mathrm{Fm}_{F\lfloor'\rfloor} \lceil)\rceil$ $\lfloor$where $F' = \langle L', P' \rangle$ is a language$\rfloor$ is extended to the equally-denoted $f : \mathrm{Ru}_F^{[\omega]} \to \lceil \wp_{[\omega]}(\rceil \mathrm{Ru}_{F\lfloor'\rfloor}^{[\omega]} \lceil)\rceil$ via setting $f(\mathcal{R}) \triangleq (\{\lceil \bigcup \rceil f[\Gamma]\} \times f(\Phi))$ under proper identifying singletons with their elements in the non-$\lceil\rceil$-optional case covering L-substitutions, whereas sets of {non-proper} [finitary] F-rules are called *{axiomatic} "[finitary] F-calculi"[/"deductive bases over F"]* [/[11]].

A closure operator C [with non-one-element range-image] over Fm_F is said to be *structural*, if, for all $\sigma \in \mathrm{Sb}_L$ and $X \subseteq \mathrm{Fm}_F$, $\sigma[C(X)] \subseteq C(\sigma[X])$, i.e., $\mathrm{img}\, C$ is *closed under inverse substitutions* in the sense that, for all $\sigma \in \mathrm{Sb}_L$ and $T \in (\mathrm{img}\, C)$, $\sigma^{-1}[T] \in (\mathrm{img}\, C)$, in which case it is called a *[consistent] F-logic(al system) {satisfying* an F-rule $\langle \Gamma, \Phi \rangle$, if $\Phi \in C(\Gamma)$}, elements of $\mathrm{Thm}(C) \triangleq C(\varnothing)$ being called its *theorems*, while any F-logic C' such that $(\mathrm{img}\, C')[= ((\mathrm{img}\, C) \cap \wp(\mathrm{Thm}(C'), \mathrm{Fm}_F))] \subseteq (\mathrm{img}\, C)\{\neq (\mathrm{img}\, C')\}$ is said to be an *[axiomatic] {proper} extension of* C, F-logics forming a complete lattice poset under extension partial ordering $\leqq$, intersection of dual closure systems as join and point-wise intersection of F-logics as meet, whereas C is an extension of the theorem-less F-logic C^{+0} dual to the closure system $(\mathrm{img}\, C) \cup 1$ over Fm_F, called the *theorem-less version of* C. Then, the least F-logic $\mathrm{Cn}_{\mathcal{C}}^{[C]}$ [being an extension of C and] satisfying all rules in a (finitary) F-calculus $\mathcal{C}$ is said to be *axiomatized by* $\mathcal{C}$ *[relatively to* C] ($\mathrm{Cn}_{\mathcal{C}}$ being finitary), in which case C is axiomatized by the set of all {finitary} F-rules satisfied in it {if it is finitary}, and so C is finitary iff it is axiomatized by a finitary F-calculus, while axiomatic extensions of C are exactly its extensions relatively axiomatized by axiomatic F-calculi, whereas any F-rule $\langle \Gamma, \Phi \rangle$ is satisfied in $\mathrm{Cn}_{\mathcal{C}}$ iff it is *derivable in* $\mathcal{C}$ in the sense that there is a *$\mathcal{C}$-derivation of Φ from Γ*, i.e., a mapping $\overline{\Psi}$ from a (finite) ordinal α, called its *length*, to Fm_F such that $\Phi \in (\mathrm{img}\, \overline{\Psi})$ and, for each $\beta \in \alpha$, either $\Psi_\beta \in \Gamma$ or there is some $\Xi \subseteq \overline{\Psi}[\beta]$ such that $\langle \Xi, \Psi_\beta \rangle \in \mathrm{Sb}_L[\mathcal{C}] \triangleq (\bigcup\{\sigma[\mathcal{C}] \mid \sigma \in \mathrm{Sb}_L\})$, as well as:

$$\mathrm{Cn}_{\mathcal{C}}^{[C]+0} = \mathrm{Cn}_{\mathcal{C}^{+0}}^{[C^{+0}]}, \tag{6}$$

where $\mathcal{C}^{+0} \triangleq \{\langle \Gamma \cup \{\Psi\}, \Phi \rangle \mid \Psi = \mathsf{p}(\bar{x}_n) \in \mathrm{Fm}_{\varnothing}^{P}, n \in \omega, \langle \Gamma, \Phi \rangle \in \sigma_{+n}[\mathcal{C}]\}$. Given a *sublanguage* $F' = \langle L', P' \rangle$ of F, where $L' \subseteq L$ and $\varnothing \neq P' \subseteq P$, the *$L'$-fragment of* C is the L'-logic $(C \restriction F') \triangleq \{\langle X, C(X) \cap \mathrm{Fm}_{L'} \rangle \mid X \subseteq \mathrm{Fm}_{L'}\}$.

2.3.1. Basic kinds of languages

Sentential languages

Let D be a unary *truth predicate* relation symbol, $L^D \triangleq \langle L, D \rangle$ the *sentential L-language* and $(\mathrm{Fm} \,|\, \mathrm{Ru})_L \triangleq (\mathrm{Fm} \,|\, \mathrm{Ru})_{L^D}$ "identified with Tm_L under identification of any $\varphi \in \mathrm{Tm}_L$ with $D(\varphi)$"|, L^D-formulas/-rules/-axioms/-calculi/-logics being called *[sentential] L-formulas/-rules/-axioms/-calculi/-logics*.

First-order L^D-structures (viz., *algebraic systems* of the signature L^D; cf. [7]) [with truth predicate distinct from its carrier] are called *[consistent] (logical) L-matrices* (cf. [6]), identified with {the left components of} the couples constituted by their *underlying algebras* (viz., L-reducts) and truth predicates {whenever these are are empty, L-algebras being thus viewed as L-matrices with empty truth predicates} as well as denoted by capital Calligraphic letters ⟨with indices⟩, their underlying algebras being denoted by respective capital Gothic letters ⟨with same indices⟩. Any class M of L-matrices *defines* its L-logic Cn_{M} dual to the closure system over Fm_L with closure basis $\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Tm}_L, \mathfrak{A})\}$, satisfying any L-rule iff this is satisfied in M in the usual-model-theoretic sense, as well as being finitary, whenever both M and all its members are finite (cf. [6]), but, otherwise, not necessarily being so,[1] such that

$$\mathrm{Cn}_{\mathsf{M}}^{+0} = \mathrm{Cn}_{\mathsf{M} \cup \{\langle \mathfrak{O}, \varnothing \rangle\}}, \tag{7}$$

where $\mathfrak{O}$ is any one-element L-algebra. Then, L-matrices defining extensions of an L-logic C are called its *models*, their class being denoted by $\mathrm{Mod}(C)$.

1 This is mainly why we have extended here the finitary framework of [11].

 doi:10.20944/preprints202507.0315.v1

Given L-matrices $\mathcal{A}$ and $\mathcal{B}$, a[n] *[injective] /surjective |strict homomorphism from $\mathcal{A}$ to/onto $\mathcal{B}$* is any [injective] $h \in \hom(\mathfrak{A},\mathfrak{B})$ such that $D^{\mathcal{A}} \subseteq|= h^{-1}[D^{\mathcal{B}}]$ and $h[A] \subseteq/= B$ (their set being denoted by $\hom^{[\mathrm{I}]/[\mathrm{I},]\mathrm{S}}_{|\mathrm{S}}(\mathcal{A},\mathcal{B})$) |[also called an *embedding/isomorphism of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, $\mathcal{A}$ being said to be *embedable/isomorphic into/to $\mathcal{B}$ under h* as well as, in case $h = \epsilon_A$, called a‖the *submatrix*‖*restriction of $\mathcal{B}$* ‖"*on A* with setting $(\mathcal{B}{\restriction}A) \triangleq \mathcal{A}$"], in which case

$$\hom(\mathfrak{Tm}_L,\mathfrak{B}) \supseteq/= \{g \circ h \mid g \in \hom(\mathfrak{Tm}_L,\mathfrak{A})\}, \tag{8}$$

and so:

$$\mathrm{Cn}_{(\mathcal{B}/\mathcal{A})|\mathcal{A}}(\varnothing|\Gamma) \subseteq/(\subseteq|=)\ \mathrm{Cn}_{\mathcal{B}}(\varnothing|\Gamma), \tag{9}$$

for all $\Gamma \subseteq \mathrm{Fm}_L$.

Equational languages

Let $\approx$ be an infix binary *equality* relation symbol, $L^{\approx} \triangleq \langle L,\approx\rangle$ the *equational L-language* and $\mathrm{Eq}_L \triangleq \mathrm{Fm}_{L^{\approx}}$ the set of *L-equations/-identities* identified with Fm^2_L under identification of any $\phi \approx \psi$ with $\langle\phi,\psi\rangle$, $L^{\approx}$-rules/-axioms/-calculi/-logics being called *equational L-rules/-axioms/-calculi/-logics*. Then, given a class K of L-algebras, we have the equational L-logic $\mathrm{Cn}^{\approx}_{\mathsf{K}}$ dual to the closure system over Eq_L with closure basis $\{\ker h \mid \mathfrak{A} \in \mathsf{K}, h \in \hom(\mathfrak{Tm}_L,\mathfrak{A})\}$, called the *equational logic of* K, equal to that of

$$\mathbf{I}[\mathbf{SP}]\mathsf{K}[= \{\mathfrak{A} \in \mathsf{A}_L \mid (A^2 \cap (\bigcap\{\ker h \mid h \in \hom(\mathfrak{A},\mathfrak{B}), \mathfrak{B} \in \mathsf{K}\})) = \epsilon_A\}], \tag{10}$$

satisfying any equational L-rule iff this is satisfied in K in the usual model-theoretic sense, as well as being finitary, whenever $\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{I}\mathsf{K}$ (in view of the Compactness Theorem; cf. [7]) {in particular, both K and all its members are finite; cf. [3, Corollary 2.3]}, but, otherwise, not necessarily being so,[2] in which case L-algebras defining extensions of an equational logic C are called its *models*, their class being denoted by $\mathrm{Mod}(C)$, and so the mappings $C' \mapsto \mathrm{Mod}(C')$ and $\mathsf{P} \mapsto \mathrm{Cn}^{\approx}_{\mathsf{P}}$ form a Galois retraction of the poset of *pre-varieties* (in the sense of [15]; viz., classes closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$, $\mathbf{ISP}\mathsf{K}$ being the least one including K and said to be *generated by* K) of L-algebras onto the one of the equational logics of classes of L-algebras. Clearly, the latter poset is closed under axiomatic extensions, the reservation "axiomatic" appearing redundant, in view of the following observation:

Remark 2. *Given any class K of L-algebras and any $\theta \in (\mathrm{img}\,\mathrm{Cn}^{\approx}_{\mathsf{K}})$, there are some set I, some $\bar{\mathfrak{A}} \in \mathsf{K}^I$ and some $\bar{h} \in (\prod_{i\in I}\hom(\mathfrak{Tm}_L,\mathfrak{A}_i))$ such that $\theta = (\bigcap_{i\in I}(\ker h_i))$, in which case $h : \mathrm{Fm}_L \to (\prod_{i\in I} A_i), \varphi \mapsto \langle h_i(\varphi)\rangle_{i\in I}$ is a [surjective] homomorphism from $\mathfrak{Tm}_L$ [on]to $[\mathfrak{A}_\theta \triangleq]((\prod_{i\in I}\mathfrak{A}_i)[{\restriction}(\mathrm{img}\,h)])[\in \mathbf{SP}\mathsf{K}]$ such that $(\ker h) = \theta$, and so, by the right alternative of (8), every equational L-rule $\mathcal{R}$, such that each one in $\mathrm{Sb}_L[\{\mathcal{R}\}]$ is true in $\mathfrak{A}_\theta$ under h, is true in $\mathfrak{A}_\theta$, any extension C of $\mathrm{Cn}^{\approx}_{\mathsf{K}}$ being then the equational logic of $\{\mathfrak{A}_\theta \mid \theta \in (\mathrm{img}\,C)\} \subseteq \mathbf{SP}\mathsf{K}$.* □

Sequential languages

Let $\vdash^m_n$, where $\langle m,n\rangle \in [\ell \subseteq]\omega^2$, be an $(m+n)$-ary *sequent* relation symbol, $P^{\vdash}_{[\ell]} \triangleq \{\vdash^m_n \mid \langle m,n\rangle \in (\omega^2[\cap\ell])\}$ the *[ℓ-]sequent(ial) relation signature*, $L^{\vdash}_{[\ell]} \triangleq \langle L, P^{\vdash}_{[\ell]}\rangle$ the *[ℓ-]sequent(ial) L-language* and $\mathrm{Seq}^{[\ell]}_L \triangleq \mathrm{Fm}_{L^{\vdash}_{[\ell]}}$ the set of *L-sequents [of rank ℓ]*, any one $\vdash^m_n(\bar{\varphi})$ being written in the standard form $(\bar{\varphi}{\restriction}m) \vdash (((+m){\restriction}n) \circ \bar{\varphi})$. Then, $L^{\vdash}_{[\ell]}$-rules/-axioms/-calculi/-logics are called *[ℓ-]sequent(ial) L-rules/-axioms/-calculi/-logics*.

[2] This is one more reason of our going beyond the finitary framework of [11].

 doi:10.20944/preprints202507.0315.v1

3. Preliminaries

3.1. Disjunctive sentential logics

Fix any $\Delta \in \wp_{[\omega]}(\mathrm{Tm}_L^2)$. Given any $X, Y \subseteq \mathrm{Fm}_L$, set $\Delta(X,Y) \triangleq (\bigcup\{\Delta(\phi,\psi) \mid \phi \in X, \psi \in Y\}$. Then, an L-logic C is said to be *weakly/⟨strongly⟩ [finitely] Δ-disjunctive*, if, for all $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}_L$, $C(X \cup \Delta(\phi,\psi)) \subseteq/= (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$ /"in which case:

$$C(\Delta(\phi,\psi)) = C(\Delta(\psi,\phi)), \tag{11}$$
$$C(\Delta(\phi,\phi)) = C(\{\phi\})". \tag{12}$$

Likewise, it is said to be Δ*-multiplicative*, if, for all $(X \cup \{\varphi\}) \subseteq \mathrm{Fm}_L$, $\Delta(C(X),\varphi) \subseteq C(\Delta(X,\varphi))$. Finally, an L-matrix $\mathcal{A}$ is said to be Δ*-disjunctive*, if, for all $a, b \in A$, $((\{a,b\} \cap D^{\mathcal{A}}) \neq \varnothing) \Leftrightarrow (\{\delta^{\mathfrak{A}}(a,b) \mid \delta \in \Delta\} \subseteq D^{\mathcal{A}})$.

Theorem 1. *A (finitary) L-logic C is Δ-disjunctive if(f) it is both weakly Δ-disjunctive and Δ-multiplicative, while both* (11) *and* (12) *hold {whereas:*

$$C(\Delta(\varphi,\Delta(\phi,\psi))) = C(\Delta(\Delta(\varphi,\phi),\psi)), \tag{13}$$

for all $\varphi,\phi,\psi \in \mathrm{Fm}_L$*} if(f) C is defined by a class of [consistent] Δ-disjunctive L-matrices.*

Proof. {The second "if" part is immediate.} Now, assume C is both weakly Δ-disjunctive and Δ-multiplicative, while both (11) and (12) hold. Consider any $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}_L$ and any $\varphi \in (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, in which case $\varphi \in C(\Delta(\varphi,\varphi)) \subseteq C(\Delta(C(X \cup \{\psi\}),\varphi) \subseteq C(\Delta(X \cup \{\psi\},\varphi)) \subseteq C(X \cup \Delta(\psi,\varphi)) = C(X \cup \Delta(\varphi,\psi)) \subseteq C(X \cup \Delta(C(\{\varphi\}),\psi)) \subseteq C(X \cup C(\Delta(X,\psi) \cup \Delta(\phi,\psi))) = C(X \cup \Delta(\phi,\psi))$, and so C, being weakly Δ-disjunctive, is Δ-disjunctive. (Finally, assume C is Δ-disjunctive. Then, by Remark 1, it, being finitary and structural, is defined by $\mathsf{M} \triangleq (\{\mathfrak{Tm}_L\} \times \mathrm{MI}(\mathrm{img}\, C))$. Consider any $T \in \mathrm{MI}(\mathrm{img}\, C) \not\ni \mathrm{Fm}_L$ and any $\phi,\psi \in \mathrm{Fm}_L$ such that $\Delta(\phi,\psi) \subseteq T$, in which case $T = C(T) = C(T \cup \Delta(\phi,\psi)) = (C(T \cup \{\phi\}) \cap C(T \cup \{\psi\}))$, and so either $T = C(T \cup \{\phi\}) \ni \phi$ or $T = C(T \cup \{\psi\}) \ni \psi$, members of M being thus both consistent and Δ-disjunctive, by the weak Δ-disjunctivity of C.) □

3.1.1. Multiplicative sentential calculi

Given any $\mathcal{R} = \langle \Gamma, \phi \rangle \in \mathrm{Ru}_L$ and $\psi \in \mathrm{Fm}_L$, put $\Delta(\mathcal{R},\psi) \triangleq \{\langle \Delta(\Gamma,\psi), \varphi \rangle \mid \varphi \in \Delta(\phi,\psi)\}$, elements of $\Delta(\sigma_{+1}(\mathcal{R}), x_0)$ being called Δ*-multiplications of* $\mathcal{R}$. Then, an L-calculus $\mathcal{C}$ is said to be Δ*-multiplicative*, if each multiplication of every rule of it is derivable in it, in which case, by induction on the length of $\mathcal{C}$-derivations, $\mathrm{Cn}_{\mathcal{C}}$ is Δ-multiplicative, and so, by the structurality of L-logics and Theorem 1, we get:

Corollary 1. *A [finitary] L-logic C is Δ-disjunctive if[f] it is both weakly Δ-disjunctive and axiomatized by a Δ-multiplicative L-calculus, while both* (11) *and* (12) *(as well as* (13)*) hold.*

3.2. Extensions versus interpretations

Here, we entirely follow the conventions adopted in Chapter 2 but allowing not necessarily finitary logics as well as *finitary* translations (viz., those with finite values). Fix any propositional language L, first-order ones $F['] = \langle L, P['] \rangle$, translations $\tau|\rho$ from $P|P'$ to $P'|P$ over L and an $F[']$-logic $C[']$. Then, τ is said to be *compatible with* C', if the condition (ii) of Definition 2.1 of [11] holds, that is (in view of the structurality of C'), $C'(\sigma[\tau[\Gamma]]) = C'(\tau[\sigma[\Gamma]])$, for all $\sigma \in \mathrm{Sb}_L$ and all [one-element] $\Gamma \subseteq \mathrm{Fm}_F$ (cf. Proposition 2.2 therein), i.e., for all $\sigma \in \mathrm{Sb}_L$ and all $\Theta \in (\mathrm{img}\, C')$, $\sigma^{-1}[\tau^{-1}[\Theta]] = \tau^{-1}[\sigma^{-1}[\Theta]]$, in which case τ is compatible with any extension of C'. In that case, τ is called an *interpretation of C in* C', if the condition (i) of Definition 2.1 of [11] holds too, that is, $(\mathrm{img}\, C) = \tau^{-1}[\mathrm{img}\, C'] \triangleq \{\tau^{-1}[\Theta] \mid \Theta \in (\mathrm{img}\, C')\}$ (cf. Proposition 2.4 therein). We start from presenting the following almost immediate observation:

 doi:10.20944/preprints202507.0315.v1

Lemma 1. *Let $C''(\gtreqq C')$ be an F'-logic (and $C'''[\gtreqq C]$ an F-logic as well as $\mathcal{C}$ an F-calculus). Suppose τ is compatible with C'' (resp., with C' [in particular, τ is an interpretation of C in C' {more specifically, C and C' are equivalent with $\langle$respect to$\rangle$ τ and ρ}]). Then, $\tau^{-1}[\mathrm{img}\,C'']$ is a closure system over Fm_F closed under inverse substitutions, in which case τ is an interpretation of the F-logic $\tau^{-1}(C'')$ dual to $\tau^{-1}[\mathrm{img}\,C'']$ in C'' ([while $C \leqq \tau^{-1}(C'')$ {whereas $C'' = \rho^{-1}(\tau^{-1}(C''))$, and so $\tau^{-1}(C'')$ and C'' are equivalent with $\langle$respect to$\rangle$ τ and ρ}]). (Conversely, $\tau_{C'}[\mathrm{img}\,C'''] \triangleq \{\Theta \in (\mathrm{img}\,C') \mid \tau^{-1}[\Theta] \in (\mathrm{img}\,C''')\}$ is a closure system over $\mathrm{Fm}_{F'}$ closed under inverse substitutions, in which case the F'-logic $\tau_{C'}(C''')$ dual to $\tau_{C'}[\mathrm{img}\,C''']$ is an extension of C', while $\tau_{C'}[\mathrm{Cn}^{C'''}_{\mathcal{C}}] = \mathrm{Cn}^{\tau_{C'}(C''')}_{\tau[\mathcal{C}]}$ [and $\tau^{-1}(\tau_{C'}(C''')) = C'''$, τ being an interpretation of C''' in $\tau_{C'}(C''')$ {whereas $\tau_{C'}(C''') = \rho^{-1}(C''')$, C''' and $\tau_{C'}(C''')$ being equivalent with $\langle$respect to$\rangle$ τ and ρ}]).*

This, first, immediately yields the following infinitary extension of [11, Theorem 2.20]:

Theorem 2. *Suppose C and C' are equivalent with (respect to) τ and ρ. Then, $\rho^{-1}|\tau_{C'}$ and $\tau^{-1}\|\rho_C$ form inverse to one another isomorphisms between the complete lattices of [axiomatic] extensions of C and C', corresponding ones being equivalent with (respect to) τ and ρ.*

And what is more, as an equally immediate consequence of Lemma 1, we have the following important result, being formally beyond the scopes of [11] but implicitly contained, though not explicitly presented, therein:

Theorem 3. *Suppose τ is an interpretation of C in C'. Then, $\tau_{C'}$ and τ^{-1} form a dual Galois retraction of the poset of extensions of C' onto that of C, τ being an interpretation of any extension C'' of C (relatively axiomatized by an F-calculus $\mathcal{C}$) in the extension $\tau_{C'}(C'')$ of C' (relatively axiomatized by $\tau[\mathcal{C}]$).*

This, in its turn, by Remark 2, yields a more canonical insight into the main universal result of [13] in the spirit of the outstanding work [11] plagiarized (like [13]) more and more by such crooks as Font, Pigozzi, et al.:

Corollary 2 (cf. [13, Theorem 3.3])**.** *Let ∇ be a translation from $\{D\}$ to $\{\approx\}$ over L, C an L-logic, K a class of L-algebras, $\mathsf{P} \triangleq$ **ISP**K and $C' \triangleq \mathrm{Cn}^{\approx}_{\mathsf{K}} = \mathrm{Cn}^{\approx}_{\mathsf{P}}$. Suppose ∇ is an interpretation of C in C', i.e., C is defined by $\mathsf{K}^{\nabla} \triangleq \{\langle\mathfrak{A}, \{a \in A \mid \mathfrak{A} \models (\bigwedge\nabla_D)[x_0/a]\}\rangle \mid \mathfrak{A} \in \mathsf{K}\}$, viz., by P^{∇}. Then, the mappings $C'' \mapsto (\mathsf{P} \cap \mathrm{Mod}(\nabla_{C'}(C'')))$ and $\mathsf{S} \mapsto \mathrm{Cn}_{\mathsf{S}^{\nabla}}$ form a Galois retraction of the poset of sub-pre-varieties of P onto the one of extensions of C such that, for any L-calculus $\mathcal{C}$, $(\mathsf{P} \cap \mathrm{Mod}(\nabla_{C'}(\mathrm{Cn}^{C}_{\mathcal{C}}))) = (\mathsf{P} \cap \mathrm{Mod}(\nabla[\mathcal{C}]))$, while, for any $\mathsf{C} \subseteq \mathsf{P}$, $\mathrm{Cn}_{(\mathbf{ISP}\mathsf{C})^{\nabla}} = \mathrm{Cn}_{\mathsf{C}^{\nabla}}$.*

4. Main issues

Here, we deal with the propositional languages $L^{(-)}_{+[,01]} \triangleq \{\wedge, \vee[, \bot, \top](, \neg)\}$, where $\wedge$ and $\vee$ are binary [while $\bot$ and $\top$ are nullary] (whereas $\neg$ is unary) with [bounded] lattices {cf. [1]} viewed as $L_{+[,01]}$-algebras, $\phi \lessapprox \psi$ standing for $\phi \approx (\phi \wedge \psi)$. Then, a *[bounded] (De)/ Morgan/Kleene lattice* [traditionally called a *(De)/ Morgan/Kleene algebra*; cf., e.g., [1]] is any $L^{-}_{+[,01]}$-algebra with [bounded] distributive lattice $L_{+[,01]}$-reduct, satisfying:

$$\neg\neg x_0 \approx x_0, \tag{14}$$
$$\neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_1), \tag{15}$$
$$(x_0 \wedge \neg x_0) \lessapprox (x_{0/1} \vee \neg x_{0/1}), \tag{16}$$

their variety being denoted by $[\mathsf{B}](\mathsf{M}/\mathsf{K})\mathsf{L}$. Let $\mathfrak{L}_{n[,01]}$ be the chain [bounded] lattice over $n \in (\omega \setminus 2)$, while $\mathfrak{K}_{n[,01]}$ the [bounded] Kleene lattice with [bounded] lattice reduct $\mathfrak{L}_{n[,01]}$ and $\neg^{\mathfrak{K}_{n[,01]}} \triangleq \{\langle i, n-1-i\rangle \mid i \in n\}$, whereas $\mathfrak{M}_{4[,01]}$ the [bounded] Morgan lattice with [bounded] lattice reduct $\mathfrak{L}^2_{2[,01]}$ and $\neg^{\mathfrak{M}_{n[,01]}} : 2^2 \to 2^2, \langle j,k\rangle \mapsto \langle 1-k, 1-j\rangle$, the following standard notations of

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elements of 2^2 being used in this connection $(\mathsf{f}|\mathsf{t}) \triangleq \langle 0|1, 0|1\rangle$ and $(\mathsf{n}\|\underline{)} \triangleq \langle 0\|1, 1\|0\rangle$, as well as $(FDE/(LP|K3)/PC)_{[01]}$ the logic of $(\mathcal{M}/\mathcal{K}/\mathcal{K})^{/|\top/}_{(4/3/2)[,01]} \triangleq \langle(\mathfrak{M}/\mathfrak{K}/\mathfrak{K})_{(4/3/2)[,01]}, \{\underline{/}(1|2)/1, \mathsf{t}/2/1\}\rangle$, being [the bounded (version of the)] "{relevance} first-degree entailment"/("logic of paradox"|"Kleene three-valued logic")/ "classical logic" [2]/ ([9]|[5])/[8], in which case the truth predicate of $\mathcal{K}^{[|\top]/}_{3/2[,01]}$ is equationally definable by the translation $\nabla^{[|\top]} \triangleq \{\langle D, \{(\neg x_0[|\top]) \lessapprox x_0\}\rangle\}$ from $\{D\}$ to $\{\approx\}$ over $L^-_{+[,01]}$ in the sense that:

$$\mathcal{K}^{[|\top]/}_{3/2[,01]} \models \forall x_0(D(x_0) \leftrightarrow \nabla^{[|\top]}_D), \tag{17}$$

and so the universal elaboration of [13] is equally applicable to the bounded versions of both the logic of paradox and Kleene three-valued logic. Then, for any $L^-_{+[,01]}$-algebras $\mathfrak{A}, \mathfrak{B}$ and any $L^-_{+[,01]}$-rule $\mathcal{R}$:

$$\begin{aligned} \hom^{(\mathrm{I}\{,)\mathrm{S}\}}(\mathfrak{A}, \mathfrak{B}) &\subseteq \hom^{(\mathrm{I}\{,)\mathrm{S}\}}_{(\mathrm{S})}(\mathfrak{A}^{\nabla^{[|\top]}}, \mathfrak{B}^{\nabla^{[|\top]}}), & (18)\\ (\mathfrak{A} \times \mathfrak{B})^{\nabla^{[|\top]}} &= (\mathfrak{A}^{\nabla^{[|\top]}} \times \mathfrak{B}^{\nabla^{[|\top]}}), & (19)\\ (\mathfrak{A} \in \mathrm{Mod}(\nabla^{[\top]}(\mathcal{R}))) &\Leftrightarrow (\mathfrak{A}^{\nabla^{[|\top]}} \in \mathrm{Mod}(\mathcal{R})), & (20) \end{aligned}$$

so, for any $\mathsf{K} \subseteq \mathsf{A}_{L^-_{+[,01]}}$:

$$\mathrm{Cn}_{\mathsf{K}^{\nabla^{[|\top]}}} = \mathrm{Cn}_{(\mathbf{ISPK})^{\nabla^{[|\top]}}}. \tag{21}$$

4.1. *An axiomatization of the bounded version of FDE*

Let $\mathcal{C}_{\mathrm{H}[,01]}$ be the $L^-_{+[,01]}$-calculus, constituted the L^-_{+}-rules given by [10, Definition 5.1]:

$$(R_1)\ \frac{x_0 \wedge x_1}{x_0} \qquad (R_2)\ \frac{x_0 \wedge x_1}{x_1 \wedge x_0} \qquad (R_3)\ \frac{x_0 \quad x_1}{x_0 \wedge x_1}$$

$$(R_4)\ \frac{x_0}{x_0 \vee x_1} \qquad (R_5)\ \frac{x_0 \vee x_1}{x_1 \vee x_0} \qquad (R_6)\ \frac{x_0 \vee (x_1 \vee x_2)}{(x_0 \vee x_1) \vee x_2}$$

$$(R_7)\ \frac{x_0 \vee (x_1 \wedge x_2)}{(x_0 \vee x_1) \wedge (x_0 \vee x_2)} \qquad (R_8)\ \frac{(x_0 \vee x_1) \wedge (x_0 \vee x_2)}{x_0 \vee (x_1 \wedge x_2)} \qquad (R_9)\ \frac{x_0 \vee x_0}{x_0}$$

$$(R_{10})\ \frac{x_0 \vee x_2}{\neg\neg x_0 \vee x_2} \qquad (R_{11})\ \frac{\neg\neg x_0 \vee x_2}{x_0 \vee x_2} \qquad (R_{12})\ \frac{(\neg x_0 \wedge \neg x_1) \vee x_2}{\neg(x_0 \vee x_1) \vee x_2}$$

$$(R_{13})\ \frac{\neg(x_0 \vee x_1) \vee x_2}{(\neg x_0 \wedge \neg x_1) \vee x_2} \qquad (R_{14})\ \frac{(\neg x_0 \vee \neg x_1) \vee x_2}{\neg(x_0 \wedge x_1) \vee x_2} \qquad (R_{15})\ \frac{\neg(x_0 \wedge x_1) \vee x_2}{(\neg x_0 \vee \neg x_1) \vee x_2}$$

[and the following additional $L^-_{+,01}$-rules and -axioms:

$$(R_{16})\ \frac{\bot \vee x_0}{x_0} \qquad (R_{17})\ \frac{\neg\top \vee x_0}{x_0} \qquad (A_1)\ \ \top \qquad (A_2)\ \ \neg\bot].$$

Let $\mathcal{EM} \triangleq (x_0 \vee \neg x_0)$ be the *Excluded Middle* axiom, $\mathcal{RS} \triangleq (\{x_0 \vee x_1, \neg x_0 \vee x_1\} \rightsquigarrow x_1)$ the *Resolution* rule, $\mathcal{C}^{\overline{\mathcal{R}}}_{\mathrm{H}[,01]} \triangleq (\mathcal{C}_{\mathrm{H}[,01]} \cup (\mathrm{img}\,\overline{\mathcal{R}}))$, where $\overline{\mathcal{R}} \in \{\mathcal{EM}, \mathcal{RS}\}^*$, and $\mathcal{M}_{4-\bar{c}[,01]} \triangleq (\mathcal{M}_{4[,01]} \restriction (2^2 \setminus (\mathrm{img}\,\bar{c})))$, where $\bar{c} \in (2^2 \setminus \epsilon_2)^*$.

Lemma 2. *$\mathcal{C}^{(\mathcal{EM})\{\mathcal{RS}\}}_{\mathrm{H}[,01]}$ is $\vee$-multiplicative.*

Proof. According to [10, Theorem 5.2], *FDE* is axiomatized by $\mathcal{C}_{\mathrm{H}}$. Then, since its defining matrix $\mathcal{M}_4$ is $\vee$-disjunctive, by Theorem 1, it is $\vee$-multiplicative, while (13) holds for it, in which case both the rule $\overline{R}_6$ inverse to R_6 and the $\vee$-multiplication of any rule of $\mathcal{C}_{\mathrm{H}}$, being satisfied in *FDE*, are derivable in $\mathcal{C}_{\mathrm{H}}$, and so in $\mathcal{C}^{(\mathcal{EM})\{\mathcal{RS}\}}_{\mathrm{H}[,01]}$. Moreover, by R_4, the $\vee$-multiplication of any axiom in $\varnothing[\cup\{A_1, A_2\}](\cup\{\mathcal{EM}\})$ is derivable in $\mathcal{C}^{(\mathcal{EM})\{\mathcal{RS}\}}_{\mathrm{H}[,01]}$. [Finally, due to the following demonstration:

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1. $((\bot|\neg\top) \vee x_1) \vee x_0$ — Hypothesis;
2. $(\bot|\neg\top) \vee (x_1 \vee x_0)$ — $\overline{R}_6[x_0/(\bot|\neg\top), x_2/x_0] : (1)$;
3. $x_1 \vee x_0$ — $R_{16|17}[x_0/(x_1 \vee x_0)] : (2)$;

the $\vee$-multiplication of $R_{16|17}$, being derivable in $\{R_{16|17}, \overline{R}_6\}$, is so in $\mathcal{C}_{\mathrm{H},01}^{(\mathcal{EM})\{\mathcal{RS}\}}$.] {Likewise, due to the following one:

1. $(x_1 \vee x_2) \vee x_0$ — Hypothesis;
2. $(\neg x_1 \vee x_2) \vee x_0$ — Hypothesis;
3. $x_1 \vee (x_2 \vee x_0)$ — $\overline{R}_6[x_0/x_1, x_1/x_2, x_2/x_0] : (1)$;
4. $\neg x_1 \vee (x_2 \vee x_0)$ — $\overline{R}_6[x_0/\neg x_1, x_1/x_2, x_2/x_0] : (2)$;
5. $x_2 \vee x_0$ — $\mathcal{RS}[x_0/x_1, x_1/(x_2 \vee x_0)] : (3), (4)$;

the $\vee$-multiplication of $\mathcal{RS}$, being derivable in $\{\mathcal{RS}, \overline{R}_6\}$, is so in $\mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\mathcal{RS}}$.} □

An $L^-_{+[,01]}$-matrix $\mathcal{A}$ is said to be *($\wedge$-)conjunctive*, if $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$ is $\wedge$-disjunctive.

Theorem 4. $\mathrm{Cn}_{\mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\{\mathcal{RS}\}}} = \mathrm{Cn}_{\mathcal{M}_{4-(\mathsf{n})\{\underline{\}}[,01]}}$.

Proof. Clearly, since $\mathfrak{M}_{4[,01]} \in [\mathsf{B}]\mathsf{ML}$, $\mathcal{M}_{4-(\mathsf{n})\{\underline{\}}[,01]}$, being both conjunctive and $\vee$-disjunctive, is a model of $\mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\{\mathcal{RS}\}}$, i.e., $C \triangleq \mathrm{Cn}_{\mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\{\mathcal{RS}\}}} \leqq \mathrm{Cn}_{\mathcal{M}_{4-(\mathsf{n})\{\underline{\}}[,01]}}$. Conversely, by Theorem 1, Corollary 1, Lemma 2 and the inclusion $\{R_i \mid i \in \{4,5,9\}\} \subseteq \mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\{\mathcal{RS}\}}$, C, being both finitary and $\vee$-disjunctive, is defined by a class M of consistent $\vee$-disjunctive $L^-_{+[,01]}$-matrices. Consider any $\mathcal{A} \in \mathsf{M} \subseteq \mathrm{Mod}(\mathcal{C}_{\mathrm{H}[,01]}^{(\mathcal{EM})\{\mathcal{RS}\}})$ and take any $a \in (A \setminus D^{\mathcal{A}}) \neq \varnothing$, in which case, by the truth of R_1, R_2 and R_3 in $\mathcal{A}$, this is conjunctive [while, by that of A_1, $\top^{\mathfrak{A}} \in D^{\mathcal{A}}$, whereas by that of R_{16} under $[x_0/a]$, $\bot^{\mathfrak{A}} \notin D^{\mathcal{A}}$]. Then, [by the truth of A_2 in $\mathcal{A}$, $\bot^{\mathfrak{A}} \notin E \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[D^{\mathcal{A}}])$, while, by that of R_{17} under $[x_0/a]$, $\top^{\mathfrak{A}} \in E$, whereas] by that of R_{12}, R_{13}, R_{14} and R_{15} under assignments containing $\langle x_2, a \rangle$, $\langle \mathfrak{A}, E \rangle$ is both conjunctive and $\vee$-disjunctive, for $\mathcal{A}$ is so. Finally, consider any $b \in A$, in which case, by the truth of R_{10} and R_{11} in $\mathcal{A}$ under $[x_0/b, x_2/a]$, $(b \in D^{\mathcal{A}}) \Leftrightarrow (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \in D^{\mathcal{A}})$ (while, by that of $\mathcal{EM}$ in $\mathcal{A}$ under $[x_0/b]$, $(\{b, \neg^{\mathfrak{A}}b\} \cap D^{\mathcal{A}}) \neq \varnothing$) {whereas, by that of $\mathcal{RS}$ in $\mathcal{A}$ under $[x_0/b, x_1/a]$, $\{b, \neg^{\mathfrak{A}}b\} \nsubseteq D^{\mathcal{A}}$}, and so $e \triangleq \chi_A^{D^{\mathcal{A}}} \in \mathrm{hom}(\mathfrak{A} \upharpoonright L_{+[,01]}, \mathfrak{L}_{2[,01]}) \ni f \triangleq \chi_A^E$, $\{\langle c, \langle e(c), f(c) \rangle \rangle \mid c \in A\}$ being in $\mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{M}_{4-(\mathsf{n})\{\underline{\}}[,01]})$, as required, in view of (9). □

This, by (9) and the fact that $((+|\pi_0) \upharpoonright M_{4-((\mathsf{n}|\underline{)}/(\mathsf{n}\underline{)})})$ is an isomorphism from $\mathcal{M}_{4-((\mathsf{n}|\underline{)}/(\mathsf{n}\underline{)})[,01]}$ onto $\mathcal{K}^{|\top/}_{(3/2)[,01]}$), immediately yields:

Corollary 3. *$(FDE/(LP|K3)/PC)_{[01]}$ is axiomatized by $\mathcal{C}_{\mathrm{H}[,01]}^{/(\mathcal{EM}|\mathcal{RS})/(\mathcal{EM},\mathcal{RS})}$. In particular, $K3_{[01]}$ is the extension of $FDE_{[01]}$, relatively axiomatized by $\mathcal{RS}$, while $(LP\|PC)_{[01]}$ is the axiomatic extension of $(FDE\|K3)_{[01]}$ relatively axiomatized by $\mathcal{EM}$.*

This subsumes [10, Corollary 5.3].

4.2. Extensions of the bounded logic of paradox and Kleene's three-valued logic versus pre-varieties of Kleene algebras

Key observations enabling one to expand [13] onto the bounded case *almost* immediately are as follows:

Lemma 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be bounded lattices and $h \in \mathrm{hom}(\mathfrak{A} \upharpoonright L_+, \mathfrak{B} \upharpoonright L_+)$. Suppose $(\bot|\top)^{\mathfrak{B}} \in h[A]$ (in particular, $h[A] = B$). Then, $h((\bot|\top)^{\mathfrak{A}}) = (\bot|\top)^{\mathfrak{B}}$.*

Proof. Take any $(a|b) \in A$ such that $h(a|b) = (\bot|\top)^{\mathfrak{B}}$, in which case $((a|b)(\wedge|\vee)^{\mathfrak{A}} (\bot|\top)^{\mathfrak{A}}) = (\bot|\top)^{\mathfrak{A}}$, so $h((\bot|\top)^{\mathfrak{A}}) = ((\bot|\top)^{\mathfrak{B}}(\wedge|\vee)^{\mathfrak{B}} h((\bot|\top)^{\mathfrak{A}})) = (\bot|\top)^{\mathfrak{B}}$. □

Lemma 4. *For any {2-element} [bounded] Morgan lattice $\mathfrak{A}$ and any (distinct) $a, b \in A$, $\{\langle 0, ((a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \wedge^{\mathfrak{A}} (b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b))[\wedge^{\mathfrak{A}} \bot$ $\neg^{\mathfrak{A}} a) \vee^{\mathfrak{A}} (b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b))[\vee^{\mathfrak{A}} \top^{\mathfrak{A}}]\rangle \in \mathrm{hom}^{(\mathrm{I}\{,\mathrm{S}\})}(\mathfrak{K}_{2[,01]}, \mathfrak{A})$.*

Since $\mathfrak{K}_3$ has no non-one-element subalgebra not retaining bounds, by (10), Lemma 3 and [12, Proposition 3.4], we, first, have the following well-known fact (cf., e.g., [1]):

Corollary 4. $\mathsf{BKL} = \mathbf{ISP}(\mathbf{P}^{\mathrm{U}})\mathfrak{K}_{3,01}$.

Let $N(L)P_{01}$ be the extension of LP_{01} relatively axiomatized by the *Ex Contradictione Quodlibet* rule $\mathcal{NP} \triangleq (\{x_0, \neg x_0\} \rightsquigarrow x_1)$, viz., the least non-paraconsistent extension of $LP_{[01]}$. Then, a [bounded] Kleene lattice is said to be *non-paraconsis-tent*, if it satisfies $\nabla(\mathcal{NP})$, i.e., satisfies [13, (13)]:

$$\{\neg x_0 \approx x_0\} \rightsquigarrow (\nabla[x_0/x_1]), \tag{22}$$

in which it satisfies (22)$[x_1/\neg x_1]$, and so, by the right alternative of (16), satisfies [12, (8)]:

$$\{\neg x_0 \approx x_0\} \rightsquigarrow (x_0 \approx x_1), \tag{23}$$

i.e., it is *non-idempotent* in the sense of [12, Definition 4.1]. Conversely, any [bounded] Kleene lattice, satisfying (23), satisfies (23)$[x_1/\neg x_1]$, in which case it is non-paraconsistent, and so non-paraconsistent [bounded] Kleene lattices are exactly non-idempotent ones, their quasi-variety being denoted by N[B]KL.

Let $\mathfrak{NK}_{6[(+2),01]} \triangleq (((\mathfrak{K}_{3[,01]} \times \mathfrak{K}_{2[,01]})[(\times \mathfrak{K}_{3,01})])[(\upharpoonright(((3 \times 2) \times \{1\}) \cup \{\langle 2 \cdot i, i, 2 \cdot i\rangle \mid i \in 2\}))])$ and $\mathcal{NK}_{6,01} \triangleq (\mathcal{K}_{3,01} \times \mathcal{K}_{2,01})$. Then, since the only non-one-element subalgebra of $\mathfrak{NK}_6$ not retaining bounds is that with two-element carrier $\{1\} \times 2$, by (10), Lemmas 3, 4 and [12, Proposition 4.5]/[13, Theorem 4.10], we immediately have:

Corollary 5. $\mathsf{NBKL} = \mathbf{ISP}(\mathbf{P}^{\mathrm{U}})\mathfrak{NK}_{6,01}$.

Let $MP_{[01]}$ be the extension of $(N)LP_{[01]}$ relatively axiomatized by the *Modus Ponens* rule for material implication $\mathcal{MP} \triangleq (\{x_0, \neg x_0 \vee x_1\} \rightsquigarrow x_1)$ (in view of the $\vee$-disjunctivity of $\mathcal{K}_{3[,01]}$ and Theorem 1). Then, a [bounded] Kleene lattice is said to be *regular/classical* (cf. Definition 4.6/4.11 of [12]/[13]), if it satisfies $\nabla(\mathcal{MP})$, i.e., satisfies (10/14) of [12]/[13]:

$$(\nabla \cup \{(x_0 \wedge \neg x_1) \lessapprox (\neg x_0 \vee x_1)\}) \rightsquigarrow (\nabla[x_0/x_1]), \tag{24}$$

their quasi-variety being denoted by $\mathsf{R[B]KL} \subseteq \mathsf{N[B]KL}$. Since the only non-one-element subalgebra of $\mathfrak{K}_4$ not retaining bounds is that with two-element carrier $\{1, 2\}$, by (10), Lemmas 3, 4 and [12, Proposition 4.7], we immediately get:

Corollary 6. $\mathsf{R[B]KL} = \mathbf{ISP}(\mathbf{P}^{\mathrm{U}})\mathfrak{K}_{4[,01]}$.

From now on, we use (17), (18), (19), (20), (21) and Corollary 2 tacitly. Then, by Corollaries 4 and 5, we, first, have:

Theorem 5. *(An arbitrary exrension C of) $[N]LP_{01} = \mathrm{Cn}_{[\mathcal{N}]\mathcal{K}_{3[+3],01}}$ is defined by* $\mathsf{[N]BKL}^{\nabla}(\cap \mathrm{Mod}(C))$.

Theorem 6. $(\mathrm{Thm})(MP_{[01]}) = (\mathrm{Thm})(PC_{[01]})(= \mathrm{Thm}(LP_{[01]}))$.

 doi:10.20944/preprints202507.0315.v1

Proof. Then, by Theorem 5 and Corollary 6, $MP_{[01]}$, being defined by R[B]KL, is defined by $\mathcal{K}_{4[,01]} \triangleq \mathfrak{K}^{\nabla}_{4[,01]} = \langle \mathfrak{K}_{4[,01]}, \{2,3\} \rangle$, while $\chi_4^{\{2,3\}} \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}_{4[,01]}, \mathcal{K}_{2[,01]})$, whereas $\{\langle i, \chi_4^{\{3\}}(i) + \chi_4^{4\setminus 1}(i) \rangle\} \in \mathrm{hom}^{\mathrm{S}}(\mathcal{K}_{4[,01]}, \mathcal{K}_{3[,01]})$, (9) ending the proof.[3] □

Theorem 7. *Proper consistent extensions of LP_{01} form the two-element chain $NP_{01} \lneqq PC_{01}$.*

Proof. First, $\mathcal{K}_{3,01} \not\models \mathcal{NP}[x_i/(1-i)]_{i \in 2}$, while $\mathcal{NK}_{6,01} \not\models \mathcal{MP}[x_j/\langle 1-j, 1 \rangle]_{j \in 2}$, whereas $\mathcal{K}_{2,01} \not\models x_0[x_0/0]$. Then, by Theorems 5 and 6, NP_{01} and PC_{01} are proper consistent extensions of LP_{01} forming the chain involved. Finally, consider any consistent extension C of LP_{01}, in which case, by Theorem 5, there is a non-one-element $\mathfrak{A} \in$ BKL such that $\mathfrak{A}^{\nabla} \in \mathrm{Mod}(C)$, and so, by (9) and Lemma 4, $C \leqq PC_{01}$. In particular, $C = PC_{01}$, whenever $PC_{01} \leqq C$. Otherwise, consider the following complementary cases:

- $NP_{01} \leqq C$,
  in which case, by Theorem 5, there is some $\mathfrak{A} \in (\mathrm{NBKL} \setminus \mathrm{RBKL})$ such that $\mathfrak{A}^{\nabla} \in \mathrm{Mod}(C)$, and so $\mathfrak{B} \triangleq (\mathfrak{A} \upharpoonright L^{-}_{+}) \in (\mathrm{NKL} \setminus \mathrm{RKL})$. Then, by the case 4/3 of the proof of Theorem 4.8/4.11 of [12]/[13], there is an $e \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{NK}_6, \mathfrak{B})$. Consider the following complementary subcases:
  - $e(\langle 0,0 \rangle) = \bot^{\mathfrak{A}}$,
    in which case $e(\langle 2,1 \rangle) = \top^{\mathfrak{A}}$, and so $e \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{NK}_{6,01}, \mathfrak{A})$. Then, by (9), $\mathcal{NK}_{6,01} = \mathfrak{NK}^{\nabla}_{6,01} \in \mathrm{Mod}(C)$.
  - $e(\langle 0,0 \rangle) \neq \bot^{\mathfrak{A}}$,[4]
    in which case $e(\langle 2,1 \rangle) \neq \top^{\mathfrak{A}}$, so $(((\pi_0 \upharpoonright (NK_6 \times \{1\})) \circ e) \cup \{\langle 0,0,0,\bot^{\mathfrak{A}} \rangle, \langle 2,1,2,\top^{\mathfrak{A}} \rangle\}) \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{NK}_{8,01}, \mathfrak{A})$. Also, $(\pi_0 \upharpoonright NK_{8,01}) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathfrak{NK}^{\nabla}_{8,01}, \mathcal{NK}_{6,01})$. Then, by (9), $\mathcal{NK}_{6,01} \in \mathrm{Mod}(C)$.

  Thus, anyway, $\mathcal{NK}_{6,01} \in \mathrm{Mod}(C)$, in which case, by Theorem 5, $C \leqq NP_{01}$, and so $C = NP_{01}$.
- $NP_{01} \nleqq C$,
  in which case, by Theorem 5, there is some $\mathfrak{A} \in (\mathrm{BKL} \setminus \mathrm{NBKL})$ such that $\mathfrak{A}^{\nabla} \in \mathrm{Mod}(C)$, and so there is some $a \in A \neq \{a\}$ such that $\neg^{\mathfrak{A}} a = a$. Then, $\{\langle 0, \bot^{\mathfrak{A}} \rangle, \langle 1, a \rangle, \langle 2, \top^{\mathfrak{A}} \rangle\} \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{K}_{3,01}, \mathfrak{A})$, in which case, by (9), $\mathcal{K}_{3,01} = \mathfrak{K}^{\nabla}_{3,01} \in \mathrm{Mod}(C)$, and so $C = LP_{01}$. □

□

Theorem 8. *PC_{01} is the only proper consistent extension of $K3_{01}$.*

Proof. Consider a consistent extension C of $K3_{01}$ distinct from PC_{01}, in which case, by Theorem 7, $PC_{01} \nleqq C$, and so, by Corollaries 3 and 4, there is some $\mathfrak{A} \in$ BKL such that $\mathfrak{A}^{\nabla^{\top}} \in (\mathrm{Mod}(C) \setminus \mathrm{Mod}(\mathcal{EM}))$. Then, there is some $a \in A$ such that $b \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \neq \top^{\mathfrak{A}}$. Consider the following complementary cases:

- $\neg^{\mathfrak{A}} b = b$,
  in which case $\{\langle 0, \bot^{\mathfrak{A}} \rangle, \langle 1, b \rangle, \langle 2, \top^{\mathfrak{A}} \rangle\} \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{K}_{3,01}, \mathfrak{A})$, and so, by (9), $\mathcal{K}^{\top}_{3,01} \in \mathrm{Mod}(C)$.
- $\neg^{\mathfrak{A}} b \neq b$,
  in which case $\{\langle 0, \bot^{\mathfrak{A}} \rangle, \langle 1, \neg^{\mathfrak{A}} b \rangle, \langle 2, b \rangle, \langle 3, \top^{\mathfrak{A}} \rangle\} \in \mathrm{hom}^{\mathrm{I}}(\mathfrak{K}_{4,01}, \mathfrak{A})$, and so, by (9), $\mathcal{K}^{\top}_{3,01} \in \mathrm{Mod}(C)$, for $\{\langle i, \chi_4^{4\setminus 1}(i) + \chi_4^{4\setminus 3}(i) \rangle \mid i \in 4\} \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathfrak{K}^{\nabla^{\top}}_{4,01}, \mathcal{K}^{\top}_{3,01})$.

Thus, in any case, $\mathcal{K}^{\top}_{3,01} \in \mathrm{Mod}(C)$, i.e., $C \leqq K3_{01}$, C being equal to $K3_{01}$, as required, in view of Corollary 3, for $\mathcal{K}^{\top}_{3,01} \not\models \mathcal{EM}[x_0/1]$. □

If, for any $\nabla' \subseteq (\mathrm{Tm}^1_L)^2$, $\forall x_0 (D(x_0) \leftrightarrow (\bigwedge \nabla'))$ was true in $\mathcal{K}^{\top}_3$, then it would be true in $\mathcal{K}_1 \triangleq (\mathcal{K}_3 \upharpoonright \{1\})$, in which case, since $\mathcal{K}_1 \models \forall x_0 (\bigwedge \nabla')$, $\forall x_0 D(x_0)$ would be true in $\mathcal{K}_1$, and so $1 \in K_1$ would

[3] Though [13, Lemma 4.14] is expandable onto the bounded case, we have presented here more immediate and transparent model-theoretic proofs of both it and the axiomatizability of the [bounded] classical logic relatively to the [bounded] logic of paradox by Modus Ponens for material implication.

[4] It is this subcase that justifies the reservation "almost" in the first sentence of this subsection.

be in $D^{\mathcal{K}_1} = \varnothing$. Nevertheless, though the universal algebraic approach developed in [13] is thus not applicable to K_3, Theorem 8 is still so as follows.

Lemma 5. *Any extension C of K3 with(out) theorems is (the theorem-less version of) the $L_+^{-,D}$-fragment of an extension of $K3_{01}$.*

Proof. Consider any $T \in ((\mathrm{img}\,C) \setminus 1)$, in which case there exists some $H_T \subseteq \hom(\mathfrak{Tm}_{L_+^-}, \mathfrak{K}_3)$ such that $T = (\mathrm{Fm}_{L_+^-} \cap (\bigcap_{h \in H_T} h^{-1}[\{2\}]))$, and so $g_T : \mathrm{Fm}_{L_+^-} \to 3^{H_T}, \varphi \mapsto \langle f(\varphi)\rangle_{f \in H_T}$ is a [surjective] strict homomorphism from $\langle \mathfrak{Tm}_{L_+^-}, T\rangle$ [on]to $[\mathcal{A}_T \triangleq]((\mathcal{K}_3^{\top})^{H_T}[\upharpoonright(\mathrm{img}\,g_T)])$. Take any $\phi \in T \neq \varnothing$, in which case $A_T \ni g_T(\phi) = (H_T \times \{2\})$, and so $A_T \ni g_T(\neg\phi) = (H_T \times \{0\})$. Then, A_T forms a subalgebra of $\mathfrak{D}_T \triangleq \mathfrak{K}_{3,01}^{H_T}$, in which case, by (9), $\mathcal{B}_T \triangleq \langle \mathfrak{D}_T \upharpoonright A_T, D^{\mathcal{A}_T}\rangle$, being a submatrix of $(\mathcal{K}_{3,01}^{\top})^{H_T}$, is a model of $K3_{01}$, and so $K3_{01} \leqq C' \triangleq \mathrm{Cn}_{\{\mathcal{B}_T \mid T \in ((\mathrm{img}\,C)\setminus 1)\}}$. Thus, by (9), $C = \mathrm{Cn}^{(+0)}_{\{\mathfrak{Tm}_{L_+^-}\} \times ((\mathrm{img}\,C)\setminus 1)} = \mathrm{Cn}^{(+0)}_{\{\mathcal{A}_T \mid T \in ((\mathrm{img}\,C)\setminus 1)\}} = (C' \upharpoonright L_+^{-,D})^{(+0)}$. □

Let IC be the inconsistent L_+^--logic.

Corollary 7. *Proper extensions of K3 form the diamond lattice, isomorphic to $\mathfrak{L}_2^2$ under ι^{-1}, where $\iota : 2^2 \to \wp(\mathrm{Fm}_{L_+^-})^{\wp(\mathrm{Fm}_{L_+^-})}, \langle 0|1, 1[-1]\rangle) \mapsto (P|I)C^{[+0]} = \mathrm{Cn}^{LP|(LP)}_{\{[x_1 \rightsquigarrow](\mathcal{EM}|x_0)\}} = \mathrm{Cn}_{(\{\mathcal{K}_2\}|\varnothing)[\cup\{\mathcal{K}_1\}]}$ is injective.*

Proof. Clearly, no axiom is true in the {sub}matrix $\mathcal{K}_1$ {of $\mathcal{K}_3^{\top}$} with empty truth predicate, while $\mathcal{K}_{3\|2}^{\top\|} \not\models (\langle x_1 \rightsquigarrow\rangle(\mathcal{EM}\|x_0)[x_0/(1\|0)\langle, x_1/(2\|1)\rangle]$, (6), (7), (9), Corollary 3, Theorem 8 and Lemma 5 completing the argument. □

4.3. Cut-free versions of Gentzen calculus

Let $LK^{(-\mathrm{R}/\langle\mathrm{R}\rangle\mathrm{C}])}_{(/\{+\mathrm{CFC}\})[01]}$ be the $(\omega^2 \setminus (1[-1])^2)$-sequent $L^-_{+[,01]}$-calculus constituted by the following *strucural* rules (except for Reflexivity/"⟨both Reflexivity and⟩ Cut {with non-(1[−1])-ary sequent predicate in conclusion}"):

Reflexivity	$x_0 \vdash x_0$	
(/{Context-Free}) Cut	$\dfrac{\{\Lambda, x_0 \vdash \Xi; \Lambda \vdash \Xi, x_0\}}{\Lambda \vdash (\Xi * (\bar{x}_{\Bbbk} \circ \sigma_{+1}))}$	
	Left	Right
Enlargement	$\dfrac{\Gamma \vdash \Delta}{\Gamma, x_0 \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, x_0}$
Contraction	$\dfrac{\Gamma, x_0, x_0 \vdash \Delta}{\Gamma, x_0 \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta, x_0, x_0}{\Gamma \vdash \Delta, x_0}$
Permutation	$\dfrac{\Gamma, x_0, x_1, \Theta \vdash \Delta}{\Gamma, x_1, x_0, \Theta \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta, x_0, x_1, \Theta}{\Gamma \vdash \Delta, x_1, x_0, \Theta}$

where $\Gamma, \Delta, \Theta \in \mathrm{Var}^* \ni \Lambda, \Xi(\notin (\varnothing/(\varnothing\{\cup\,\mathrm{Var}^+\})))$ and $\Bbbk \triangleq ((1 - \min(1, (\mathrm{dom}\,\Lambda) + (\mathrm{dom}\,\Xi)))[\cdot 0])$, together with the following *logical* rules [and axioms]:

	Left	Right
	$[\bot \vdash$	$\vdash \top]$
$(\neg)$	$\dfrac{\Gamma \vdash \Delta, x_0}{\Gamma, \neg x_0 \vdash \Delta}$	$\dfrac{\Gamma, x_0 \vdash \Delta}{\Gamma \vdash \Delta, \neg x_0}$
$(\wedge)$	$\dfrac{\Gamma, x_0, x_1 \vdash \Delta}{\Gamma, x_0 \wedge x_1 \vdash \Delta}$	$\dfrac{\{\Gamma \vdash \Delta, x_0; \Gamma \vdash \Delta, x_1\}}{\Gamma \vdash \Delta, x_0 \wedge x_1}$
$(\vee)$	$\dfrac{\{\Gamma, x_0 \vdash \Delta; \Gamma, x_1 \vdash \Delta\}}{\Gamma, x_0 \vee x_1 \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta, x_0, x_1}{\Gamma \vdash \Delta, x_0 \vee x_1}$

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where $\Gamma, \Delta \in \mathrm{Var}^*$ (as well as [both] the rules inverse to logical ones [and the following *constant elimination* rules:

$$\begin{array}{cc} \text{Left} & \text{Right} \\ \dfrac{\Gamma, \top \vdash \Delta}{\Gamma \vdash \Delta} & \dfrac{\Gamma \vdash \Delta, \bot}{\Gamma \vdash \Delta} \end{array}$$

where $\Gamma, \Delta \in \mathrm{Var}^*$]) its rules being derivable in $LK_{[01]}$.

Let

$$\tau_{[01]} \triangleq \{\langle \vdash^m_n, \{\wedge_+([\langle]\bar{x}_m * (\bar{x}_n \circ \sigma_{+m})[,\bot\rangle])\}\rangle \mid \langle m,n\rangle \in (\omega^2 \setminus (1[-1])^2)\}$$

and $\rho \triangleq \{\langle D, \{\vdash x_0\}\rangle\}$ be parameter-less translations from $P^{\vdash}_{\omega^2\setminus(1[-1])^2}$ to $\{D\}$ and *vice versa* over $L^-_{+[,01]}$ in the sense of the fundamental work [11] we follow here tacitly.

Lemma 6. $\mathrm{Cn}_{LK^{-\mathrm{RC}}_{[01]}}$ *and* $FDE_{[01]}$ *are equivalent with (respect to)* $\tau_{[01]}$ *and* ρ.

Proof. First, for any $(\Gamma \rightsquigarrow \Psi) \in LK^{-\mathrm{RC}}_{[01]}$ and any $\overline{\Phi} \in \Gamma^{|\Gamma|}$ with range-image Γ, $(\wedge_+([\langle]\overline{\Phi} \circ \tau_{[01]}[,\top\rangle])) \lessapprox \tau_{[01]}(\Psi)$ is true in $[\mathrm{B}]\mathrm{ML} \ni \mathfrak{M}_{4[,01]}$, so $\tau_{[01]}[\Gamma] \rightsquigarrow \tau_{[01]}(\Psi)$ is true in $\mathcal{M}_{4[,01]}$, for $D^{\mathcal{M}_{4[,01]}}[\ni \top^{\mathfrak{M}_{4,01}}]$ is a filter of $\mathfrak{M}_{4[,01]}\restriction L_+$. Likewise, $x_0 \approx \rho(\tau_{[01]}(D))$ is true in $[\mathrm{B}]\mathrm{ML} \ni \mathfrak{M}_{4[,01]}$, so both $x_0 \rightsquigarrow \tau_{[01]}(\rho(D))$ and $\tau_{[01]}(\rho(D)) \rightsquigarrow x_0$ are true in $\mathcal{M}_{4[,01]}$. Conversely, for any $(\Gamma \rightsquigarrow \varphi) \in \mathcal{C}_{\mathrm{H}[,01]}$, $\rho[\Gamma] \rightsquigarrow \rho(\varphi)$ is derivable in $LK^{-\mathrm{RC}}_{[01]}$. Finally, for all $\langle m,n\rangle \in (\omega^2 \setminus (1[-1])^2)$, both $\Upsilon \rightsquigarrow \rho(\tau_{[01]}(\vdash^m_n))$ and $\rho(\tau_{[01]}(\vdash^m_n)) \rightsquigarrow \Upsilon$, where $\Upsilon \triangleq (\bar{x}_m \vdash (\bar{x}_n \circ \sigma_{+m}))$, are derivable in $LK^{-\mathrm{RC}}_{[01]}$. Then, Theorems 2.24 of [11] and 4 complete the argument. □

Theorem 9. $\mathrm{Cn}_{LK^{-(\mathrm{C/R})}_{[01]}}$ *and* $(LP/K3)_{[01]}$ *are equivalent with (respect to)* $\tau_{[01]}$ *and* ρ.

Proof. Clearly, $\tau_{[01]}(\mathrm{Reflexivity/Cut})$ is true in $\mathcal{K}^{/\top}_{3[,01]}$, while $\rho(\mathcal{EM}/\mathcal{RS})$ is derivable in $LK^{-(\mathrm{C/R})}_{[01]}$, Corollaries 2.27 of [11], 3 and Lemma 6 ending the proof. □

Corollary 8 (cf. [13, Theorem 4.13] for the non-[]-optional case)**.** *Proper consistent extensions of* $\mathrm{Cn}_{LK^{-\mathrm{C}}_{[01]}}$ *form the two-element chain* $\mathrm{Cn}_{LK^{-\mathrm{C}}_{+\mathrm{CFC}[,01]}} \lneqq \mathrm{Cn}_{LK_{[01]}}$*, the lesser/greater being equivalent to* $(NP/PC)_{[01]}$ *with (respect to)* $\tau_{[01]}$ *and* ρ.

Proof. Clearly, $\tau_{[01]}[\{x_0 \vdash; \vdash x_0\}] \rightsquigarrow \tau_{[01]}(\vdash (\bar{x}_{1[\cdot 0]} \circ \sigma_{+1}))$ is true in $\mathcal{NK}_{6[,01]}$. Conversely, $\rho[\{x_0, \neg x_0\}] \rightsquigarrow \rho(x_1)$ is derivable in $LK^{-\mathrm{C}}_{+\mathrm{CFC}[,01]}$. Likewise, for any instance $\Gamma \rightsquigarrow \Phi$ of Cut $\tau_{[01]}[\Gamma] \rightsquigarrow \tau_{[01]}(\Phi)$ is true in $\mathcal{K}_{2[,01]}$. Finally, $\rho[\{x_0, \neg x_0 \vee x_1\}] \rightsquigarrow \rho(x_1)$ is derivable in $LK_{[01]}$. Then, Theorems 2, 5, 6, 7, [13, 4.10], 9 and [11, Corollary 2.27] complete the argument. □

Corollary 9. $\mathrm{Cn}_{LK_{01}}$ *is the only proper consistent extension of* $\mathrm{Cn}_{LK^{-\mathrm{C}}_{01}}$.

Proof. Clearly, $\tau_{01}(\mathrm{Reflexivity})$ is true in $\mathcal{K}_{2,01}$. Conversely, $\rho(\mathcal{EM})$ is derivable in LK_{01}. Then, Theorems 2, 8, 9 and [11, Corollary 2.27] complete the argument. □

Likewise, by Theorems 2 and 9 as well as Corollaries [11, 2.28] and 7, we eventually get:

Corollary 10. *Proper extensions of* $\mathrm{Cn}_{LK^{-\mathrm{R}}}$ *form the diamond lattice, isomorphic to the one of those of* $K3$ *under* $\kappa \triangleq \{\langle \mathrm{Cn}_{LK^{-\mathrm{R}} \cup \{(\vdash x_1) \rightsquigarrow (\vdash x_0 \vee \neg x_0)\}}, PC^{+0}\rangle, \langle \mathrm{Cn}_{LK^{-\mathrm{R}} \cup \{(\vdash x_1) \rightsquigarrow (\vdash x_0)\}}, IC^{+0}\rangle, \langle \mathrm{Cn}_{LK^{-\mathrm{R}} \cup \{\vdash x_0\}}, IC\rangle, \langle \mathrm{Cn}_{LK}, PC\rangle\}$*, any* $C \in (\mathrm{dom}\ \kappa)$ *and* $\kappa(C)$ *being equivalent with [respect to]* τ *and* ρ.

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