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Central Conics in H^2 are Fibers over the Group of Steiner Conics

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Article

Central Conics in $\mathbb{H}^2$ are Fibers over the Group of Steiner Conics

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Abstract

We provide an intrinsic construction of the central conics in the real hyperbolic plane $\mathbb{H}^2$ whereby each conic C is the composition of a unique pair of Steiner conics (those generated by collineations). The composition is achieved by elliptic curve addition on intersection points of the two components with their orthogonal trajectories, which have a natural representation as genus 1 curves in any inversive model of $\mathbb{H}^2$. The central Steiner conics that have a focal axis L are identified with the subgroup $G(L)$ of collineations generated by reflections in the lines perpendicular to L. We define the *fiber* over $g \in G(L)$ to be the set of compositions C such that $\pi(C) = g$. Here, $\pi(C)$ is the unique Steiner conic tangent to C at the points on L, and we show that $\pi(C)$ is the product of the two elements in $G(L)$ that represent the components of C. The central conics are partitioned into these fibers, which are acted upon transitively by $G(L)$. The geometry and algebra of the fiber bundle are emphasized, without topological considerations.

Keywords: hyperbolic plane; incidence geometry; elliptic curve; fiber bundle

MSC: primary 51M10; secondary 51M15; 51A45

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1. Introduction

An intrinsic construction of conics in the real hyperbolic plane $\mathbb{H}^2$ would require only the incidence properties of the plane, the action of the collineation group, and a conformal metric. In this article, we provide such a construction by viewing $\mathbb{H}^2$ as a linear incidence geometry - *two distinct points are on exactly one line and two distinct lines meet in at most one point*. A first step toward such a construction was taken in [1] where we classified the Steiner conics in $\mathbb{H}^2$, defining them in the manner of [2] (pp. 134-140).

Definition 1. Let T be a collineation of a planar incidence geometry. The *Steiner conic* $E(T;P)$ afforded by T at point P is the locus of intersections $L \cap T(L)$ for the lines L concurrent at P. The conic $E(T_1;P_1)$ is *congruent* to $E(T;P)$ if, for some collineation T_0, $T_1 = T_0TT_0^{-1}$ and $P_1 = T_0(P)$. If the geometry is oriented, $E(T;P)$ is *direct* if T preserves orientation, *opposite* if T reverses orientation. The conic is *degenerate* if $T(P) = P$ or if $T(L) = L$ for some L.

It is possible for all conics in a planar geometry, defined algebraically or by other means, to be Steiner conics. This is the case in a projective plane over a field, as discussed in [3] (p. 80). From this viewpoint, the "conic sections" - ellipses, parabolas, hyperbolas - are classified by the invariants of affine collineations that afford them. By contrast, the collineation group of $\mathbb{H}^2$ consists entirely of isometries if a metric is imposed. This constrains the Steiner conics to the $E(T;P)$ in [1] where we showed them to be proper subsets of seven of the eleven categories of curves described in the classical sources [4,5]. Those categories extended work by Enrico D'Ovidio and others in the 1890s. They were described, non-intrinsically, by intersecting cones with a hyperbolic domain (analogous to the conic

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sections of Apollonius). The numbering and descriptors for these categories in [5] (p. 119), which correspond to the figures in [4] (p. 229), are reprised in Section 2 for the *central* conics, characterized by having two perpendicular lines of symmetry.

Since these are our subject, $E(T;P)$ will always denote a central Steiner conic, and C will always designate a central conic in $\mathbb{H}^2$. We write $\mathcal{D}$ for the set of direct $E(T;P)$ and $\mathcal{O}$ for the set of opposite $E(T;P)$. The set $\mathcal{D}_0$ consists of the members of $\mathcal{D}$ with no absolute points (on the boundary of $\mathbb{H}^2$). Upon choosing a set of congruence representatives $\mathcal{D}_0$ we use elliptic curve addition on their orthogonal trajectories to define the composition C of any two members of $\mathcal{D}_0$. This is our main result, which we state in Section 2. Apart from clarifying references to [1], our aim is to keep this presentation as self-contained as possible and of interest to the pedagogical community. The following outline provides context for the proof of our Main Theorem and its consequences in later sections.

In Section 3 we use the hyperboloid model of $\mathbb{H}^2$ to construct a family of elliptic space curves and describe the addition on their points. In Section 4 we represent points as complex numbers by projecting them into the conformal disk $\mathbb{D}$, where it is easy to visualize these curves and those in $\mathcal{D}_0$ as an orthogonal system. That the composition of a member of $\mathcal{D}_0$ with itself is a circle is proved in Section 5, followed by a formula in $\mathbb{D}$ for the elliptic curve addition. In Section 6 we derive some algebraic identities required for the proof, in Section 7, of the Main Theorem. In Section 8 the remaining categories of central conics listed in Section 2 will be produced from the compositions C. In Section 9 we use the Main Theorem and an intrinsic involution on the half-planes of $\mathbb{H}^2$ that interchanges $\mathcal{D}_0$ and $\mathcal{O}$ to describe a group structure on $\mathcal{D}_0 \cup \mathcal{O}$. We define the fiber over each member of this group in Section 10 and partition the central conics into fibers.

For brevity throughout, we omit the details of routine calculations unless they are essential to the argument at hand. Several of these calculations are characteristically intricate. They have been performed manually and then thoroughly checked with the algebra program in Scientific Workplace 5.5.

2. Background and Statement of Main Theorem

Before stating our main result we establish some basic terminology by way of a brief summary of the classification of the $E(T;P)$. It is convenient to describe the translation component of T in terms of distance, whereby the *eccentricity* ϵ of $E(T;P)$ is a function of the distance s between P and $T(P)$. If $E(T;P) \in \mathcal{D}$ then $\epsilon = \sinh s$. We write E_ϵ for the Steiner conic in this case and will work with a set of congruence representatives parameterized by ϵ. These representatives have common axes of symmetry, hence common center O. We reserve the term *ellipse* for an E_ϵ with $0 \leq \epsilon < 1$, where E_0 is the degenerate ellipse O. The ellipses comprise $\mathcal{D}_0$. Each ellipse is a special case of a locus $C(\kappa, \eta)$ consisting of points the sum of whose distances from two foci is κ, where η is the distance between the foci; for an ellipse, $\epsilon = \tanh \kappa$ and $\tanh \eta = \tanh^2 \kappa$. The *center* of $C(\kappa, \eta)$ is the midpoint of the segment between the foci, its *focal axis* is the line through the foci, and its *transverse axis* is the line perpendicular to the focal axis at the center. If $\epsilon > 1$ we call E_ϵ a *hyper-ellipse*. The conic E_1 is the pair of equidistant curves with angle-of-parallelism $\frac{\pi}{4}$ relative to the focal axis of the ellipses. The members of $\mathcal{O}$ are all central; we refer to them as *hyperbolas* and provide parameters for them in Section 9.

It will suffice to state the Main Theorem in terms of the ellipses because all of the $E(T;P)$ are related to them by the process of *split-inversion*. Split-inversion is a conformal, incidence-preserving involution on the open half-spaces of a hyperbolic space that interchanges complementary angles-of-parallelism. (The term is not standard but varies with context. See [6], where it is used without an explicit name.) In $\mathbb{H}^2$, we have the following definition.

Definition 2. Let $\mathcal{P}$ be an open half-plane for a given line L in $\mathbb{H}^2$. For any $P \in \mathcal{P}$, let L_P be the perpendicular to L through P. The *split-inversion of P in L* is the point $P^\ddagger \in L_P \cap \mathcal{P}$ such that the angle-of-parallelism to L at $P^\ddagger$ is the complement of the angle-of-parallelism to L at P.

For example, if $\epsilon \neq 0$ then split-inversion in the focal axis of the ellipse E_ϵ interchanges it with the hyper-ellipse $E_{\frac{1}{\epsilon}}$. In any inversive model of $\mathbb{H}^2$ (such as the Poincaré disk or upper half-plane) this

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is implemented by inversion in E_1, which sends each point on the axis to two absolute points. In particular, the split-inversion of E_0 is not in the plane, but it will be used in Section 9 to represent reflection in the transverse axis.

The central conics depicted in [4,5] comprise five of the categories in these sources, where they are designated as follows:

C1 Convex hyperbolas
C2 Concave hyperbolas
C4 Ellipses
C9 Equidistant curves
C10 Proper circles.

Since they fit our definition, we will include two other categories, afforded by degenerate cones:

C9A Pairs of intersecting lines
C9B Pairs of ultra-parallel lines.

Except for C9, C9A and C9B, these conics are represented in any inversive model of $\mathbb{H}^2$ by irreducible algebraic curves. We will see that the $C(\kappa,\eta)$ account for C4 and C10. Split-inversion of the $C(\kappa,\eta)$ in their focal axes will account for C2, and split-inversion of the $C(\kappa,\eta)$ in their transverse axes for C1. Our construction will also produce the reducible curves C9, C9A and C9B.

The essential fact (see Section 4) is that the E_ϵ are orthogonal in $\mathbb{H}^2$ to curves F_α defined by the following locus condition:

> Let O be a designated point on a given line L' in $\mathbb{H}^2$. For $\alpha \in \left(-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right)$, let F_α be the locus of points P such that $\measuredangle OPA = \alpha$, where A is the foot of the perpendicular to L' from P.

Each F_α is symmetric about O. Reflection in L' reverses the orientation of $\measuredangle OPA$, interchanging F_α and $F_{-\alpha}$. Our representatives for the E_ϵ are the $E(T;P)$ for which O is the midpoint on L' of P and $T(P)$. For any $\epsilon > 0$, L' is considered to be the transverse axis of E_ϵ, and the line L through O perpendicular to L' is its focal axis. Each F_α meets E_ϵ at two points in $\mathbb{H}^2$, one in each half-plane of the focal axis. The F_α are invariant under split-inversion in L. We will see in Section 3 that the F_α have a natural representation on the Minkowski hyperboloid. Using this model (and later its projection into the disk $\mathbb{D}$) we will derive the elliptic curve addition $P_1 \oplus_\alpha P_2$ on the points of the algebraic curve containing F_α. Our main result, the construction of conics in categories C4 and C10, is as follows.

Main Theorem. *Given ellipses E_{ϵ_1} and E_{ϵ_2} with $0 \le \epsilon_1 \le \epsilon_2$, let P_1 and P_2 be respective intersections with a given F_α. Let $\iota = 1$ if P_1 and P_2 are chosen in the same half-plane of the focal axis L and let $\iota = -1$ if they are chosen in opposite half-planes of L. Then, with $\epsilon_1 = \tanh\kappa_1$ and $\epsilon_2 = \tanh\kappa_2$, the set of points $\left\{P_1 \oplus_\alpha P_2 \mid \alpha \in \left(-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right)\right\}$ for a consistent choice of ι is the locus $C_\iota(\kappa,\eta)$ where $\kappa = \kappa_2 + \iota\kappa_1$ and $\eta = \eta_2 - \eta_1$.*

We refer to $C_\iota(\kappa,\eta)$ as the *composition* of E_{ϵ_1} and E_{ϵ_2} for the value ι, denoting it as convenient by $E_{\epsilon_1} \circ_\iota E_{\epsilon_2}$. The intersections of $C_\iota(\kappa,\eta)$ with L and L' are obtained as limiting cases for α. Note in particular that if $\epsilon_1 = 0$ then $C_\iota(\kappa,\eta) = E_{\epsilon_2}$. Also, if $\epsilon_1 = \epsilon_2$ then $\eta = 0$, that is, the composition of an ellipse with itself is a circle (E_0 if $\iota = -1$); we will prove this special case in advance of the general proof of the theorem. Note that $\tanh\eta = \tanh(\kappa_2+\kappa_1)\tanh(\kappa_2-\kappa_1)$, from which it will follow as a corollary to the theorem that $C_\iota(\kappa,\eta)$ is the composition of a unique pair of ellipses.

3. Constant-Angle Curves on the Minkowski Hyperboloid

The intrinsic view of non-Euclidean geometry promoted by Felix Klein was accelerated in the 'pre-modern' era by the introduction of the hyperboloid model of $\mathbb{H}^2$ in $\mathbb{R}^3$. Recall that this model is obtained from the quadric $\mathcal{H}$,

$$Z^2 - X^2 - Y^2 = 1$$

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whose two sheets are $\mathcal{H}^-$, with $Z \le -1$, and the *Minkowski hyperboloid* $\mathcal{H}^+$, with $Z \ge 1$. Let $N = (0,0,1)$ and $S = (0,0,-1)$. The *points* in this model are those on $\mathcal{H}^+$, and the *lines* are the intersections with $\mathcal{H}^+$ of planes through $(0,0,0)$. Each line, then, is identified with a normal vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, where $v_1^2 + v_2^2 - v_3^2 > 0$, that determines its plane. The *Minkowski inner product* of two lines $\mathbf{v}$ and $\mathbf{w}$ is

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 - v_3 w_3.$$

If $\mathbf{v}$ and $\mathbf{w}$ intersect, the angle θ between them in $\mathcal{H}^+$ is defined by

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}}.$$

Given $\alpha \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, let F_α be the intersection of $\mathcal{H}$ with the quadric $X \cot\alpha = YZ$. We will show that this space curve, which includes points on $\mathcal{H}^-$, is the *complete* form of the locus F_α defined in Section 2 (with $O = N$). First, for any point P on F_α, let P^Z be the reflection of P in the Z-axis. Then P^Z is also on F_α. Reflection in any of the coordinate planes interchanges F_α and $F_{-\alpha}$, so we work with $0 < \alpha < \frac{\pi}{2}$ in this section. For $\phi \in [\alpha - \frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2} - \alpha]$, let

$$Q_\alpha(\phi) = \frac{1}{\sqrt{2}\cos\alpha}\sqrt{\cos 2\alpha + \cos 2\phi}.$$

A one-to-one parameterization of F_α is given by

$$\begin{aligned} X &= \pm Q_\alpha(\phi)\cot\alpha\cot\phi \\ Y &= \pm Q_\alpha(\phi)\cot\alpha \\ Z &= \cot\alpha\cot\phi \end{aligned} \tag{1}$$

whereby $\phi = \frac{\pi}{2} - \alpha$ corresponds to N, and $\phi = \alpha - \frac{\pi}{2}$ to S. Now let P be any point on $F_\alpha \cap \mathcal{H}^+$, so $\phi \in (0, \frac{\pi}{2} - \alpha]$, and let L' be the line $\mathbf{v} = \mathbf{j}$. The line through N and P is $\mathbf{v}_1 = (\sin\phi)\mathbf{i} - (\cos\phi)\,\mathbf{j}$ and the line through P orthogonal to L' is $\mathbf{v}_2 = \mathbf{i} - Q_\alpha(\phi)\mathbf{k}$, so the cosine of the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\frac{\sin\phi}{\sqrt{1 - Q_\alpha^2(\phi)}} = \cos\alpha.$$

Thus, the angle between the line through N and P and the line through P orthogonal to L' is α.

A complete F_α is an example of a real elliptic curve obtained from the intersection of two quadrics. As such, it has an addition structure, with identity N, and P^Z the inverse of P. We are particularly interested in the sum $P_1 \oplus_\alpha P_2$ when P_1 and P_2 are on $F_\alpha \cap \mathcal{H}^+$. This sum is constructed from the the *S-line* through these points, the intersection of $\mathcal{H}^+$ with a plane through S. Relative to the Minkowski distance formula (see Section 4), the dilation with hyperbolic factor 2 and center N transforms lines into S-lines. Now, the S-line through P_1 and P_2 (the S-line tangent to F_α if $P_1 = P_2$) intersects F_α at one other point R. Then $P_1 \oplus_\alpha P_2 = R^Z$. We have $P \oplus_\alpha N = P$ for any P on F_α because the S-line through P and N is $\mathbf{v} = \mathbf{i} - (\cot\phi)\mathbf{j}$, so $R = P^Z$ in this case. In general, it is not difficult to obtain the addition formula explicitly. From (**1**), $Y^2 = \frac{Z^2-1}{Z^2\tan^2\alpha+1}$, so it suffices to determine the Z-coordinate of $P_1 \oplus_\alpha P_2$. The vector calculations are straightforward, though characteristically tedious. Later we will derive an alternative formula for addition on F_α from the stereographic projection of $\mathcal{H}$ into $\mathbb{R}^2$, but we note here that the Z-coordinate of $P \oplus_\alpha P$ takes a particularly simple form that we will use in Section 5: *If P is on $\mathcal{H}^+$ with $Y \ge 0$ then the Z-coordinate of $P \oplus_\alpha P$ is*

$$\frac{\cos^2\alpha - \sin^4\phi}{\sin^2\alpha - \cos^4\phi}. \tag{2}$$

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Since $Q_\alpha(\phi)$ is real, it follows from (2) that $R^Z = P \oplus_\alpha P$ is on $\mathcal{H}^+$ provided $\cos^2\phi < \sin\alpha$. On the other hand, Figure 1 shows an example where R (and therefore R^Z) is on $\mathcal{H}^-$.

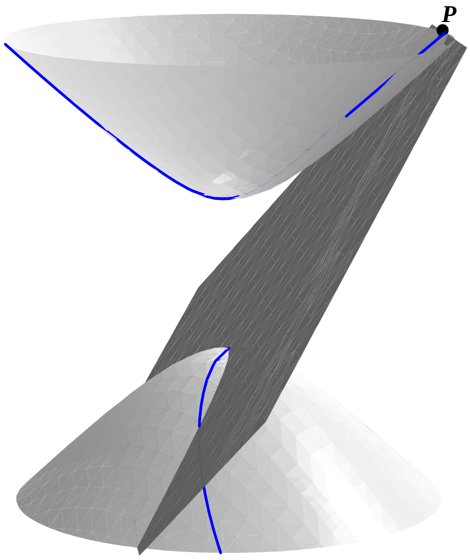

Figure 1. S-line tangent to F_α (blue) at P on $\mathcal{H}^+$ with R on $\mathcal{H}^-$.

Generally, if P_1 and P_2 are distinct points on $\mathcal{H}^+$ we will see (Section 5) that R^Z is on $\mathcal{H}^+$ provided $\cos\phi_1\cos\phi_2 < \sin\alpha$; it is sufficient, but not necessary, that P_1 and P_2 be in the open region $\mathcal{E}$ where $-1 < X < 1$ for this condition to hold. In any case, our constructions will take place within $\mathcal{H}^+$. Figure 2 is an illustration on $\mathcal{H}^+$ of the Main Theorem. The ellipses $E_{\frac{1}{3}}$ and $E_{\frac{2}{3}}$ are shown in red, with $E_{\frac{1}{3}}$ closer to $N = E_0$. Their compositions C for $\iota = \pm 1$ are shown in black. When $\iota = -1$ the conic C intersects $E_{\frac{1}{3}}$ in four points.

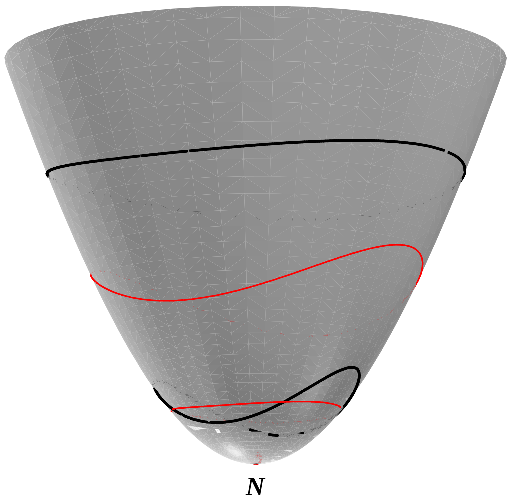

Figure 2. $C = E_{\frac{1}{3}} \circ_\iota E_{\frac{2}{3}}$ for $\iota = \pm 1$.

4. Projection to Plane Curves

Stereographic projection of $\mathcal{H}$, from S into $\mathbb{R}^2$, by

$$(X, Y, Z) \mapsto \left(\frac{X}{1+Z}, \frac{Y}{1+Z}\right) = (x, y)$$

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conformally maps $\mathcal{H}^+$ to the interior of $\mathbb{D}$ and $\mathcal{H}^-$ to the region outside the disk. Thus, N is mapped to $O = (0,0)$, and L' becomes the line $y = 0$ in $\mathbb{D}$. From (**1**), we find that the image of F_α is the polar curve

$$r^2 = \frac{\cos(\alpha + \phi)}{\cos(\alpha - \phi)}. \tag{3}$$

This is the non-singular cubic $F_\alpha(x, y) = 0$, where

$$F_\alpha(x, y) = \left(x^2 + y^2\right)(x + y \tan\alpha) - (x - y \tan\alpha) \tag{3a}$$

which has the single real asymptote $x \cot\alpha + y = 0$. The portion in $\mathbb{D}$ of an F_α is shown in blue in Figure 3, with constant angle α at P.

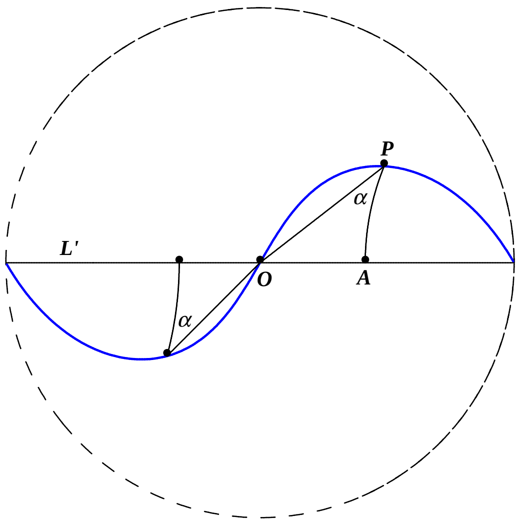

Figure 3. Angle α does not change as P moves on F_α.

This representation of F_α as a cubic curve symmetric about O was chosen to make computation as simple as possible. However, our results are not representation dependent because the image of F_α by a Möbius transformation is either a cubic or an irreducible quartic with two ordinary singularities at non-real points. In particular, incidence is preserved and the elliptic curve addition law we derive would hold in either case.

We now show that the F_α and E_ϵ are orthogonal trajectories of each other. Since stereographic projection is conformal, it follows from (**3**) that the orthogonal trajectories of the F_α are the solutions to the polar equation

$$(\sin 2\phi)d\phi = \frac{1 - r^4}{2r^3}dr.$$

These are the algebraic curves

$$r^4 - 2r^2\left(\cos 2\phi + \frac{2}{\epsilon^2}\right) + 1 = 0 \tag{4}$$

with parameter ϵ, which are the complete forms of the non-trivial E_ϵ [1] (pp. 139-140). Figure 4 shows the portions in $\mathbb{D}$ of F_α (blue) and E_ϵ (red). The curve E_1, shown in bold red, is the projection of the intersections with $\mathcal{H}^+$ of the planes $X = \pm 1$, the boundary of the region $\mathcal{E}$ defined in Section 2. The ellipses are in $\mathcal{E}$.

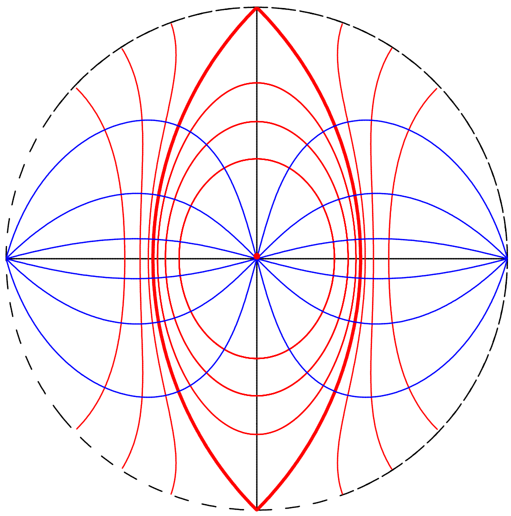

Figure 4. Orthogonal curves F_α (blue) and E_ϵ (red) in $\mathbb{D}$.

We now identify $P = (x, y)$ with the complex number $x + iy$. The image of the transverse axis L' is the real line in $\mathbb{D}$ and the image of the focal axis L is the imaginary line. Writing P^* for $x - iy$, the distance between two points P_1 and P_2 in $\mathbb{D}$ is given by

$$d(P_1, P_2) = \operatorname{arctanh}\left|\frac{P_1 - P_2}{1 - P_1^* P_2}\right| \tag{5}$$

which is equivalent to the *Minkowski distance formula* on $\mathcal{H}^+$ given by

$$\frac{1}{2}\operatorname{arccosh}(Z_1 Z_2 - Y_1 Y_2 - X_1 X_2).$$

Also, the complete curves F_α and E_ϵ are invariant under the transformation $P \mapsto \frac{1}{P}$. For F_α, this is equivalent to replacing ϕ with $-\phi$ in (**1**), whereby $\frac{1}{0}$ is the ideal point for the real asymptote, with projective coordinates $[\tan\alpha : -1 : 0]$. We note that the split-inversion of P in L is

$$\frac{1 \pm P^*}{P^* \mp 1}$$

with the signs determined by the half-plane containing P. The split-inversion of P in L' is

$$\frac{1 \pm iP^*}{P^* \pm i}.$$

Remark 1. The Möbius involution $P \mapsto \frac{P-i}{iP-1}$ maps $\mathbb{D}$ to the upper half-plane, its boundary to the real line, and the remaining points to the lower half-plane. The F_α are mapped to the one-loop Cassini ovals and the E_ϵ to the two-loop ovals. From the algebraic classification of Cassini ovals it follows that the F_α and E_ϵ represent all genus 1 curves with real coefficients, which are algebraically equivalent precisely when they have the same shape invariant, Klein's j-invariant. The invariant j can be computed from the cross-ratio χ of an algebraically equivalent non-singular cubic; the curve is *harmonic* if $j = 1$, suitably normalized. For F_α we can take $\chi = e^{4i\alpha}$ which implies $j \le 1$; for E_ϵ we can take $\chi = \epsilon^4$ which implies $j \ge 1$. Thus, there is no overlap by algebraic equivalence between the F_α and E_ϵ except for the harmonic cases ($\alpha = \pm\frac{\pi}{4}, \epsilon = 2^{\pm\frac{1}{4}}$).

5. Composition of Ellipsesp

In this section we derive an explicit formula for the elliptic curve addition on F_α with points represented as complex numbers. First, we recall the simple form of (**2**) in Section 3 and use it to

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prove a special case of the Main Theorem that motivates the composition of two ellipses in general. Specifically, with $\cos^2\phi < \sin\alpha$, it is easy to find the composition of E_ϵ with itself. We assume $\iota = 1$ since the symmetry of F_α implies $E_\epsilon \circ_{-1} E_\epsilon = E_0 = O$.

Proposition 1. *The composition $E_\epsilon \circ_1 E_\epsilon$ is the circle with center O and radius* $\operatorname{arctanh}\epsilon$.

Proof . Let P be an intersection point of F_α with E_ϵ. Solving (3) and (4) simultaneously, with $\sin^2\alpha \leq \cos^2\phi < \sin\alpha$ because $E_\epsilon \subset \mathcal{E}$, yields

$$\epsilon^2 = 2\frac{\cos 2\alpha + \cos 2\phi}{\sin^2 2\phi}. \tag{6}$$

Thus, from (2) and (6), the Z-coordinate of $P \oplus_\alpha P$ on $\mathcal{H}^+$ is

$$Z = \frac{\cos^2\alpha - \sin^4\phi}{\sin^2\alpha - \cos^4\phi} = \frac{1+\epsilon^2}{1-\epsilon^2}.$$

Since $X^2 + Y^2 = Z^2 - 1$, it follows that

$$x^2 + y^2 = \left(\frac{X}{1+Z}\right)^2 + \left(\frac{Y}{1+Z}\right)^2 = \epsilon^2,$$

the circle in $\mathbb{D}$ with center O and radius $\operatorname{arctanh}\epsilon$. $\square$

If $\epsilon \neq 0$ then the vertices of E_ϵ on its focal axis in $\mathbb{D}$ are $\pm\frac{1}{\epsilon}\left(\sqrt{1-\epsilon^2} - 1\right)i$. It follows from Formula (5) that the radius of the circle in Proposition 1 is the length of the segment between these vertices. Consequently, any circle in $\mathbb{H}^2$ is the composition of a unique ellipse with itself, but no non-trivial circle is an ellipse. We now proceed under the assumptions that $\epsilon_2 \neq \epsilon_1$ and $\epsilon_1\epsilon_2 \neq 0$.

Next, with $P = x + iy$ we obtain a formula for $R^Z = P_1 \oplus_\alpha P_2$ when $P_1 \neq P_2$. This formula applies to the complete F_α. First, the projection into $\mathbb{D}$ of the S-line through P_1 and P_2 is a chord of the unit circle whose points P satisfy

$$(P_1 - P_2)P^* - (P_1^* - P_2^*)P = P_1P_2^* - P_2P_1^*. \tag{7}$$

Since N projects to the identity 0 we assume $P_1P_2 \neq 0$. Next, from (3a), F_α consists of all P such that

$$(PP^* - 1)(P + P^*) = i(PP^* + 1)(P - P^*)\tan\alpha. \tag{8}$$

Solving for P^* in (7) and substituting into (8) produces a cubic polynomial in P with constant term a_0. Since P_1 and P_2 are on F_α, the roots of this polynomial are obtained from the monic factorization $(P - P_1)(P - P_2)(P - R)$. Then $P_1P_2R = -a_0$ and $P_1 \oplus_\alpha P_2 = -R$. It follows after solving for R that

$$P_1 \oplus_\alpha P_2 = \frac{P_1 - P_2}{P_1^* - P_2^*}\left(\frac{P_1^*}{P_1} - \frac{P_2^*}{P_2}\right)\frac{1}{(P_1 - P_2) + e^{2i\alpha}\left(P_1^* - P_2^*\right)}. \tag{9}$$

Formula (9) is undefined when $\frac{P_2 - P_1}{P_1^* - P_2^*} = e^{2i\alpha}$, but this happens only when (7) is parallel to the asymptote of F_α, equivalently, when $P_1P_2 = 1$. In this case the sum is the ideal point $[\tan\alpha : -1 : 0]$. Otherwise, from (3), let $P_j = r_je^{i\phi_j}$. Then $\left|\frac{P_1^*}{P_1} - \frac{P_2^*}{P_2}\right|^2 = 2(1 - \cos(2\phi_1 - 2\phi_2))$. Thus $|P_1 \oplus_\alpha P_2|^2$, which will be used in the proof of the Main Theorem, is determined by the denominator of the third factor in (9). Noting that $\cos 2\alpha + \cos 2\phi_j \geq 0$, there are two cases to consider depending on whether $\cos(\alpha - \phi_1)$ and $\cos(\alpha - \phi_2)$ have the same or opposite signs. For brevity we omit the straightforward calculation that yields

$$\left|(P_1 - P_2) + e^{2i\alpha}(P_1^* - P_2^*)\right|^2 = 2\left(\sqrt{\cos 2\alpha + \cos 2\phi_1} - \iota\sqrt{\cos 2\alpha + \cos 2\phi_2}\right)^2,$$

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where $\iota = 1$ in the same-sign case, and $\iota = -1$ otherwise. Therefore, with $Q_\alpha(\phi)$ as in Section 3,

$$|P_1 \oplus_\alpha P_2|^2 = \frac{1}{\cos 2\alpha + 1} \frac{1 - \cos(2\phi_1 - 2\phi_2)}{(Q_\alpha(\phi_1) - \iota Q_\alpha(\phi_2))^2}. \tag{10}$$

Proposition 2. *If P_1 and P_2 are on F_α in $\mathbb{D}$ then $P_1 \oplus_\alpha P_2$ is in $\mathbb{D}$ if and only if* $\cos\phi_1 \cos\phi_2 < \sin\alpha$.

Proof. Since F_α is symmetric about 0 it suffices to assume P_1 and P_2 are in the first polar quadrant of $\mathbb{D}$. The claim is obvious if P_1 and P_2 are in opposite quadrants because $\cos\phi_1 \cos\phi_2 < 0$ in that case and $P_1 \oplus_\alpha P_2$ is clearly in $\mathbb{D}$. Then $\iota = 1$ in (**10**). Instead of a tedious manipulation with identities, we show that the condition for $|P_1 \oplus_\alpha P_2| < 1$ is determined by the location of P_1 and P_2 relative to the portion of E_1 in this quadrant. First, if P_1 and P_2 are related by inversion in E_1, then

$$P_2 = \frac{1 - P_1^*}{1 + P_1^*}$$

whereby

$$\tan\phi_2 = \frac{2r_1}{1 - r_1^2} \sin\phi_1.$$

Then $\tan\alpha \tan\phi_2 = Q_\alpha(\phi_1)$, equivalently, $\cos\phi_1 \cos\phi_2 = \sin\alpha$. In this case, the S-line through P_1 and P_2 meets the boundary at $R = -1$, as shown in Figure 5 for $P_1 = P_2$ on E_1 and for $P_1 \neq P_2$. Now, if P_2 is replaced by any point $P(\phi)$ on F_α in the first quadrant then the corresponding R is in $\mathbb{D}$ if and only if $\phi_2 > \phi$, that is, $\cos\phi \cos\phi_2 < \sin\alpha$. $\square$

In particular, if P_1 and P_2 are both in $\mathcal{E}$ then $P_1 \oplus_\alpha P_2$ is in $\mathbb{D}$, consistent with Proposition **1** and the statement of the Main Theorem.

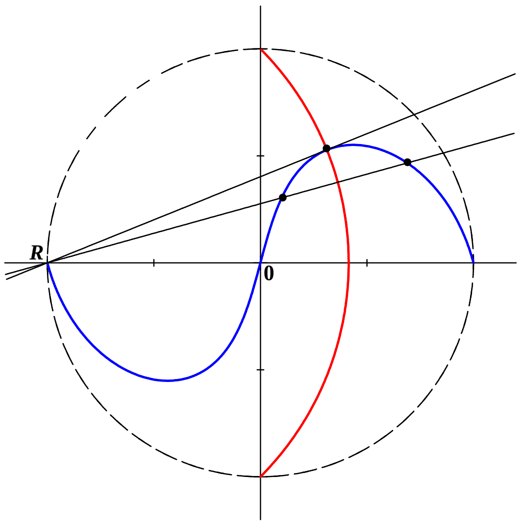

Figure 5. Pairs of points on F_α related by inversion in E_1.

6. The Computation Field

Typical of analytic geometry in the hyperbolic plane, the proof of the Main Theorem requires several algebraic identities that are straightforward but intricate. We establish these in this section. Although the dilation that transforms lines to S-lines also transforms ellipses to quadratic curves in $\mathbb{D}$, it also transforms each F_α to a quartic curve and offers no advantage in the proof. To keep the algebra manageable, we view ϵ_1, ϵ_2, and $\cos 2\alpha$ as formal variables and work within an extension of the *base field* $\mathbb{Q}(\epsilon_1, \epsilon_2, \cos 2\alpha)$. For $j = 1$ or 2, let $\beta_j = \epsilon_j^2 \cos 2\alpha - 1$ and $\gamma_j = \epsilon_j^4 - 2\beta_j - 1$. Adjoining $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ to the base field produces an extension K with basis $B_0 = 1$, $B_1 = \sqrt{\gamma_1}$, $B_2 = \sqrt{\gamma_2}$, $B_3 = B_1 B_2$.

Let $B_4 = \sqrt{B_1+\beta_1}\sqrt{B_2+\beta_2}$. Then each $B_j \geq 0$ if α and ϵ_j are real numbers with $0 \leq \epsilon_j < 1$. We will avoid explicit reference to the parameter α by performing computations in the field $K(B_4)$, where coefficients of the basis elements reduce to polynomials in β_1 and β_2 over the base field.

Certain elements in $\mathbb{Q}(\epsilon_1, \epsilon_2)$ will occur frequently as we manipulate the field relations. These include

$$\begin{aligned}
\delta &= \epsilon_1^2 - \epsilon_2^2 \\
\sigma &= \epsilon_1^2 + \epsilon_2^2 \\
\tau &= \iota\epsilon_1\epsilon_2 \\
\omega &= \tau + \frac{1}{2}\sigma \\
\lambda &= \tau^2 + 1 \\
\mu &= \tau^2 - 1.
\end{aligned}$$

For example, the elements $\beta_0 := 2\tau\cos 2\alpha + \lambda$ and $\gamma_0 := 2\tau^2\cos 2\alpha - \sigma$ can be written as

$$\begin{aligned}
\beta_0 &= \lambda + \frac{2\tau}{\delta}(\beta_1 - \beta_2) \\
\gamma_0 &= \epsilon_2^2\beta_1 + \epsilon_1^2\beta_2.
\end{aligned}$$

Also, the non-zero element δ can be expressed as $\beta_1\epsilon_2^2 - \beta_2\epsilon_1^2$, so the element

$$\Omega = \epsilon_1^2(\beta_2+1) - \epsilon_2^2(\beta_1+1)$$

is zero. To prove the pivotal lemma that concludes this section, we will perform computations mod(Ω), where (Ω) consists of all multiples of Ω. For example,

$$\gamma_0 = \tau(\beta_0 - \mu) - 2\omega \tag{11}$$

because subtracting the RHS from the LHS produces $\frac{\sigma}{\delta}\Omega$. Similarly,

$$\tau\gamma_1\gamma_2 = \beta_0(\gamma_0 + \tau\mu) + 2\omega(\tau\gamma_0 - \mu) \tag{12}$$

because subtraction produces $\frac{1}{\delta}(2\tau(\beta_1+\beta_2) + 4\tau^3 + \sigma\mu - \sigma^2\tau)\Omega$.

Lemma 1. *Let $U_1 = \mu + B_1 + B_2 - B_3$ and $U_2 = \gamma_0 + \epsilon_2^2 B_1 + \epsilon_1^2 B_2$. Then*

$$(\tau\beta_0 - 2\omega + \tau B_3)U_1 = (2\tau\omega - \beta_0 - B_3)U_2.$$

Proof. We have $B_3(b_0 + b_1B_1 + b_2B_2 + b_3B_3) = b_3\gamma_1\gamma_2 + b_2\gamma_2B_1 + b_1\gamma_1B_2 + b_0B_3$, so

$$\begin{aligned}
(\tau\beta_0 - 2\omega + \tau B_3)U_1 &= u_0 + u_1B_1 + u_2B_2 + u_3B_3 \\
(2\tau\omega - \beta_0 - B_3)U_2 &= v_0 + v_1B_1 + v_2B_2 + v_3B_3
\end{aligned}$$

with the coefficients of the B_j as follows.

$$\begin{aligned}
u_0 &= \tau(\mu\beta_0 - \gamma_1\gamma_2) - 2\mu\omega & v_0 &= (2\tau\omega - \beta_0)\gamma_0 \\
u_1 &= \tau\beta_0 + \tau\gamma_2 - 2\omega & v_1 &= \epsilon_2^2(2\tau\omega - \beta_0) - \epsilon_1^2\gamma_2 \\
u_2 &= \tau\beta_0 + \tau\gamma_1 - 2\omega & v_2 &= \epsilon_1^2(2\tau\omega - \beta_0) - \epsilon_2^2\gamma_1 \\
u_3 &= \tau(\mu - \beta_0) + 2\omega & v_3 &= -\gamma_0
\end{aligned}$$

Then $u_3 = v_3$ and $u_0 = v_0$ by (**11**) and (**12**) respectively, whereas $(u_1 - v_1)\delta = 2\epsilon_1(\epsilon_1 + \iota\epsilon_2)\Omega$ and $(u_2 - v_2)\delta = 2\epsilon_2(\epsilon_2 + \iota\epsilon_1)\Omega$. $\square$

Corollary 1. *Let* $\rho = \frac{U_1 - 2B_4}{U_2 - 2\tau B_4}$. *Then* $\rho^2 - \frac{\beta_0 + B_3}{\omega}\rho + 1 = 0$.

Proof. From Lemma 1 we have $\frac{\beta_0 + B_3}{\omega} = 2\frac{U_1 + \tau U_2}{U_2 + \tau U_1}$. Then

$$(U_2 - 2\tau B_4)^2\left(\rho^2 - 2\frac{U_1 + \tau U_2}{U_2 + \tau U_1}\rho + 1\right) = \frac{\tau U_1 - U_2}{U_2 + \tau U_1}\left(U_1^2 - U_2^2 + 4\mu B_4^2\right).$$

Now $B_4^2 = (\beta_1\beta_2 + \beta_2 B_1 + \beta_1 B_2)$, and

$$U_1^2 = \left(\mu^2 + \gamma_1\gamma_2 + \gamma_1 + \gamma_2\right) + 2(\mu - \gamma_2)B_1 + 2(\mu - \gamma_1)B_2 + 2(1 - \mu)B_3$$
$$U_2^2 = \left(\gamma_0^2 + \epsilon_2^4\gamma_1 + \epsilon_1^4\gamma_2\right) + 2\gamma_0\epsilon_2^2 B_1 + 2\gamma_0\epsilon_1^2 B_2 + 2\tau^2 B_3$$

whereby $U_1^2 - U_2^2 = c_0 + 2c_1 B_1 - 2c_2 B_2 - 4\mu B_3$ with

$$c_0 = \delta^2 - \left(\beta_1^2\epsilon_2^4 + \beta_2^2\epsilon_1^4\right) - 2(\mu - 1)\beta_1\beta_2$$
$$c_1 = \epsilon_2^2\delta - \beta_1\epsilon_2^4 - (\mu - 1)\beta_2$$
$$c_2 = \epsilon_1^2\delta + (\mu - 1)\beta_1 + \beta_2\epsilon_1^4.$$

Since $\delta = \beta_1\epsilon_2^2 - \beta_2\epsilon_1^2$, these coefficients reduce to $c_0 = -4\mu\beta_1\beta_2$, $c_1 = -2\mu\beta_2$, $c_2 = 2\mu\beta_1$. It follows that $U_1^2 - U_2^2 + 4\mu B_4^2 = 0$. $\square$

7. Proof of the Main Theorem

In this section we assume $0 \leq \epsilon_1 \leq \epsilon_2$ and begin with an illustration in $\mathbb{D}$ of $C = C_\iota(\kappa, \eta)$ for the two cases $\iota = \pm 1$. Figure 6 shows ellipses E_{ϵ_1} and E_{ϵ_2} (red) and S-lines through $P_1 = E_{\epsilon_1} \cap F_\alpha$ and $P_2 = E_{\epsilon_2} \cap F_\alpha$, for various F_α (blue) when P_1 and P_2 are in the same half-plane of the focal axis L. The third intersection with each F_α is shown on C (black) in the opposite half-plane, so $P_1 \oplus_\alpha P_2$, the reflection in O of the third intersection, is also on C.

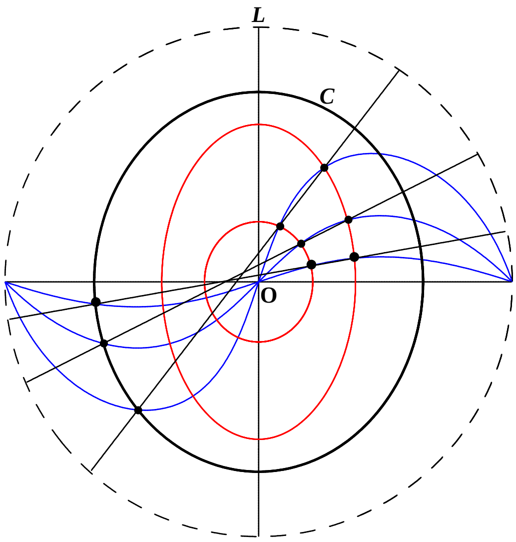

Figure 6. Composition C (black) of two ellipses (red) for $\iota = 1$.

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If $\iota = -1$ then C intersects E_{ϵ_1} at four points, as shown in Figure 7 for the same two ellipses and a typical F_α when P_1 and P_2 are in opposite half-planes of L.

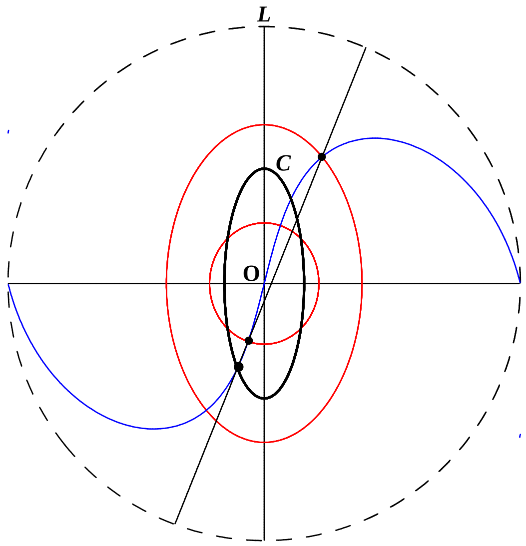

Figure 7. Composition C (black) for $\iota = -1$.

Remark 2. By symmetry, C is produced without reflecting the third intersection with each F_α in O. We have followed the definition of elliptic curve addition to ensure associativity in **Section 9** where the ellipses will be viewed as oriented figures with a group structure.

Proof of the MainTheorem. Let $\pm i \tanh \frac{\eta}{2}$ be the foci of $C_\iota(\kappa, \eta)$ in the theorem, and let $\varepsilon = \frac{\tanh \eta}{\tanh \kappa}$. A direct calculation using (5) shows that $C_\iota(\kappa, \eta)$ is described by the polar equation

$$\frac{1 + r^4}{2r^2} = \frac{2 - \varepsilon^2 - \tanh^2 \kappa + \left(\varepsilon^2 - \tanh^2 \eta\right) \cos 2\theta}{\tanh^2 \kappa - \tanh^2 \eta}. \tag{13}$$

The strategy is as follows. The theorem asserts that

$$\tanh^2 \kappa = \frac{\sigma + 2\tau}{\lambda + 2\tau}$$
$$\tanh^2 \eta = \frac{\delta^2}{\mu^2}$$

whereby (13) becomes

$$\cos 2\theta = \frac{\omega r^4 - 2\lambda r^2 + \omega}{(\sigma - 2\tau) r^2}. \tag{14}$$

Now let φ be the polar angle for $P_1 \oplus_\alpha P_2$ and let $\rho = |P_1 \oplus_\alpha P_2|^2$. Since P_1 and P_2 are on F_α it follows from (3), with $\rho = \frac{\cos(\alpha+\varphi)}{\cos(\alpha-\varphi)}$, that

$$\cos 2\varphi = \frac{\left(1 + \rho^2\right) \cos 2\alpha - 2\rho}{2\rho \cos 2\alpha - \left(1 + \rho^2\right)}. \tag{15}$$

We must show that $\cos 2\varphi$ in (15) is equal to $\cos 2\theta$ in (14) when $r^2 = \rho$. That is, since $\beta_0 = \lambda + 2\tau \cos 2\alpha$, we will show that ρ is a root of the quartic polynomial

$$W^4 - 2aW^3 + 2(b+1)W^2 - 2aW + 1$$

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with $a = \frac{1}{\omega}\beta_0$ and $b = \frac{1}{2\omega^2}(\beta_0^2 - B_3^2)$. The roots occur in reciprocal pairs and the polynomial factors as

$$\left(W^2 - \frac{\beta_0 + B_3}{\omega}W + 1\right)\left(W^2 - \frac{\beta_0 - B_3}{\omega}W + 1\right).$$

We will show that ρ is a root of the first factor.

First we write ρ as an element of $K(B_4)$. Let U_1 and U_2 be as in Lemma 1. From (**10**),

$$\rho = \frac{1}{2\cos^2\alpha}\frac{1 - \cos(2\phi_1 - 2\phi_2)}{(Q_\alpha(\phi_1) - Q_\alpha(\phi_2))^2}$$

and from (**6**),

$$\cos 2\phi_1 = \frac{B_1 - 1}{\epsilon_1^2}, \quad \cos 2\phi_2 = \frac{B_2 - 1}{\epsilon_2^2}.$$

Then $\sin 2\phi_j = \frac{\sqrt{2}}{\epsilon_j^2}\sqrt{\beta_j + B_j}$, so $\tau^2 \sin 2\phi_1 \sin 2\phi_2 = 2B_4$ and

$$1 - \cos(2\phi_1 - 2\phi_2) = \frac{1}{\tau^2}(U_1 - 2B_4).$$

Further,

$$\sqrt{\cos 2\alpha + \cos 2\phi_j} = \frac{1}{\epsilon_j}\sqrt{B_j + \beta_j}.$$

Recall $\gamma_0 = \beta_1\epsilon_2^2 + \beta_2\epsilon_1^2$. Then

$$(Q_\alpha(\phi_1) - Q_\alpha(\phi_2))^2 = \frac{1}{\gamma_0 + \sigma}(U_2 - 2\tau B_4).$$

Thus,

$$\rho = \frac{U_1 - 2B_4}{U_2 - 2\tau B_4}.$$

By Corollary 1, ρ is a root of $W^2 - \frac{\beta_0 + B_3}{\omega}W + 1$ as claimed. □

Remark 3. Though not needed, it can be shown that the roots of the second factor are conjugate points on the unit circle. Note that $\rho = \rho(\iota)$ and $\rho(1) > \rho(-1)$, as shown in Figures 6 and 7.

Corollary 2. *$C_\iota(\kappa, \eta)$ is the composition of a unique pair of ellipses. An ellipse is the composition only of itself with E_0.*

Proof. Let $C_\iota(\kappa, \eta) = E_{\epsilon_1} \circ_\iota E_{\epsilon_2}$ and let $\varepsilon = \frac{\tanh\eta}{\tanh\kappa}$. Then the theorem implies $\varepsilon = \tanh(\kappa_2 - \iota\kappa_1)$. Further, if $\tanh^2\kappa > \tanh\eta$ then $\iota = 1$, equivalently, $\kappa > \operatorname{arctanh}\varepsilon$; if $\tanh^2\kappa < \tanh\eta$ then $\iota = -1$, equivalently, $\kappa < \operatorname{arctanh}\varepsilon$. Thus $\kappa = \kappa_2 + \iota\kappa_1$ has the unique solution

$$\kappa_1 = \frac{\iota}{2}(\kappa - \operatorname{arctanh}\varepsilon)$$
$$\kappa_2 = \frac{1}{2}(\kappa + \operatorname{arctanh}\varepsilon)$$

which shows, in particular, that $\kappa_1 = 0$ and $\kappa_2 = \kappa$ if $\tanh^2\kappa = \tanh\eta$, that is, if $C_\iota(\kappa, \eta)$ is an ellipse. □

8. Split-inversion and Central Conics

In this section, corollaries to the Main Theorem will produce representatives for the central conics in the remaining categories listed in Section 2. We denote the split-inversion of a conic C in its focal axis

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by $C^{\ddagger}$, and in its transverse axis by $C_{\ddagger}$. We will see that the categories are related by these inversions, as summarized in Table 1. Since we reserved the terms *ellipse*, *hyper-ellipse*, and *hyperbola* for Steiner conics, we will add the prefix KC to the descriptions of categories C1, C2, and C4; for example, the conics in C4 are the KC-ellipses $C = C_\iota(\kappa, \eta)$, each of which is the composition of a unique pair of ellipses (Corollary 2). For the C in C4, it was shown in [1] (pp. 140-142) that the $C^{\ddagger}$ comprise C2 (KC-concave hyperbolas) and the $C_{\ddagger}$ comprise C1 (KC-convex hyperbolas). It was also shown that each $C_{\ddagger}$ is the locus of points $\bar{C}_\iota(\kappa', \eta')$ the *difference* of whose distances from two foci is $\kappa' \neq 0$ with $\eta' \neq 0$ the distance between the foci. This locus is related to $C_\iota(\kappa, \eta)$ by

$$\begin{aligned} \cosh \kappa' &= \coth \kappa \\ \cosh \eta' &= \coth \eta. \end{aligned} \tag{16}$$

In particular, if $\epsilon \neq 0$ then $(E_\epsilon)_{\ddagger}$ is a hyperbola, distinguished among the $\bar{C}_\iota(\kappa', \eta')$ by $\cosh \eta' = \cosh^2 \kappa'$, equivalently, $\frac{\tanh \eta'}{\tanh \kappa'} = \sqrt{1+\epsilon^2}$.

Remark 4. The KC-concave hyperbolas are also the split-inversions of the KC-convex hyperbolas in their focal axes. Specifically, $\left((E_{\epsilon_1} \circ_\iota E_{\epsilon_2})_{\ddagger}\right)^{\ddagger}$ is congruent to $(E_{\epsilon_1} \circ_\iota E_{\epsilon_2})^{\ddagger}$ by reflection in the asymptotes of the hyperbolas $(E_\epsilon)_{\ddagger}$.

Corollary 3. *The locus $(E_\epsilon \circ_1 E_\epsilon)_{\ddagger}$ is a pair of ultra-parallel lines.*

Proof. Let $(E_\epsilon)_{\ddagger}$ represent the hyperbola, for some $0 < \epsilon < 1$. Proposition 1 shows that that $E_\epsilon \circ_1 E_\epsilon$ is a circle of radius $\operatorname{arctanh} \epsilon$, so category C10 consists of compositions of ellipses with themselves. However, split-inversion of a circle in a diameter line is a pair of lines perpendicular to the diameter since the inversion is conformal. Thus, $(E_\epsilon \circ_1 E_\epsilon)_{\ddagger}$ is a pair of ultra-parallel lines with the focal axis of the hyperbola as the common perpendicular. □

Remark 5. With $\epsilon = \tanh \kappa$, the distance between the lines $(E_\epsilon \circ_1 E_\epsilon)_{\ddagger}$ is $\operatorname{arctanh}(\operatorname{sech} 2\kappa)$. This accounts for category C9B.

The remaining categories, C9 and C9A, are obtained from the Main Theorem by continuity, as compositions of ellipses with the equidistant curves E_1, the boundary of $\mathcal{E}$. The proof of the following corollary is straightforward.

Corollary 4. *The composition $E_\epsilon \circ_\iota E_1$ is the reducible curve represented in $\mathbb{D}$ by*

$$x^2 + y^2 = 1 \pm 2x\frac{1-\iota\epsilon}{1+\iota\epsilon},$$

which is contained in $\mathcal{E}$ if $\iota = -1$, and outside of $\mathcal{E}$ if $\iota = 1$. This is a pair of equidistant curves that meet the focal axis at angle $\arctan \frac{1+\iota\epsilon}{1-\iota\epsilon}$, *from which it follows that $(E_\epsilon \circ_\iota E_1)_{\ddagger}$ is the pair of intersecting lines*

$$y = \pm\frac{1-\iota\epsilon}{1+\iota\epsilon}x.$$

The constructions that produce each of the central conics are summarized in Table 1.

Table 1. Construction of Central Conics

Construction	Category	KC-Description
$E_{\epsilon_1} \circ_\iota E_{\epsilon_2}$	C4	ellipse
$E_\epsilon \circ_\iota E_\epsilon$	C10	circle
$(E_{\epsilon_1} \circ_\iota E_{\epsilon_2})^{\ddagger}$	C2	concave hyperbola
$(E_{\epsilon_1} \circ_\iota E_{\epsilon_2})_{\ddagger}$	C1	convex hyperbola
$E_\epsilon \circ_\iota E_1$	C9	equidistant curves
$(E_\epsilon \circ_\iota E_1)_{\ddagger}$	C9A	intersecting lines
$(E_\epsilon \circ_1 E_\epsilon)_{\ddagger} : \epsilon \neq 0$	C9B	ultra-parallel lines

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9. The Group of Steiner Conics

The lines perpendicular to a given line L in $\mathbb{H}^2$ comprise an ultra-parallel pencil. Reflections in the lines of this pencil generate a subgroup $G(L)$ of collineations. Denote the *identity component* of this group by $G_0(L)$. The identity component consists of translations, which are products of an even number of reflections. In this section we use the Main Theorem to identify $G(L)$ with $\mathcal{D}_0 \cup \mathcal{O}$ (see Section 1) and $G_0(L)$ with $\mathcal{D}_0$. In light of Remark 4, we restrict our attention to the conics with locus descriptions $C_\iota(\kappa, \eta)$ and $\bar{C}_\iota(\kappa', \eta')$.

To obtain this representation we identify each ellipse in $\mathcal{D}_0$ with a translation in $G_0(L)$, and each hyperbola in $\mathcal{O}$ with a single reflection. First, with $\kappa \in \mathbb{R}$ the group of translations along L consists of the Möbius transformations

$$g(\kappa, Z) = \frac{Z + i \tanh \kappa}{1 - Zi \tanh \kappa}.$$

With $\epsilon = \tanh \kappa$, we want E_ϵ to represent $g(\kappa, Z)$. Then E_0 would represent $g(0, Z)$ and the inverse of E_ϵ must be defined to represent $g(-\kappa, Z)$. We define it in accordance with $E_\epsilon \circ_{-1} E_\epsilon = E_0$ by giving an ellipse two orientations, E_ϵ and $E_{-\epsilon}$. With this convention, explicit reference to $\iota = \pm 1$ is no longer needed. Accordingly, $E_{\epsilon_1} \circ E_{\epsilon_2}$ will denote the composition C of $E_{\epsilon_1}$and E_{ϵ_2}, with ϵ_1 and ϵ_2 taking independent values in $(-1, 1)$. Since $g(\kappa_2, g(\kappa_1, Z)) = g(\kappa_1 + \kappa_2, Z)$, we represent this translation by E_ϵ with $\epsilon = \frac{\epsilon_1 + \epsilon_2}{1 + \epsilon_1 \epsilon_2}$. Note that E_ϵ is the unique ellipse containing the vertices of $C = E_{\epsilon_1} \circ E_{\epsilon_2}$ on L. We denote it by $\pi(C)$. It is implicit from the polar coordinate calculations in the proof of the Main Theorem that $\pi(C)$ is tangent to C at these points. Considering E_ϵ to be tangent to itself, we summarize this paragraph as follows.

Proposition 3. *Let $C = E_{\epsilon_1} \circ E_{\epsilon_2}$ and let $\pi(C)$ be the unique (oriented) ellipse tangent to C at its intersections with L. Then $\pi(C)$ represents*
$g(\kappa_2, g(\kappa_1, Z))$ in the commutative group $G_0(L)$.

The elements of $G(L)$ not in the identity component are reflections. For $\kappa > 0$, the involution

$$h(\kappa, Z) = \frac{Z^* \cosh \kappa + i}{\cosh \kappa - Z^* i}$$

is reflection in the line perpendicular to L at $ie^{-\kappa}$. Reflection in the line perpendicular to L at $-ie^{-\kappa}$ is

$$\bar{h}(\kappa, Z) = \frac{Z^* \cosh \kappa - i}{\cosh \kappa + Z^* i} = -h(\kappa, -Z).$$

Since $Z \mapsto Z^*$ is reflection in L' we will denote it by $h(\infty) = \bar{h}(\infty)$ and refer to it as the *absolute reflection*. For brevity, we now express compositions in $G(L)$ in product form without Z in the notation. Thus, $g(\kappa_2)g(\kappa_1) := g(\kappa_2, g(\kappa_1, Z))$. In particular, let $\kappa'_j = \operatorname{arctanh} \operatorname{sech} \kappa$. Then $h(\kappa)h(\infty) = g(\kappa') = h(\infty)\bar{h}(\kappa)$, and so $h(\infty)h(\kappa) = g(-\kappa') = \bar{h}(\kappa)h(\infty)$.

To extend Proposition 3 we identify each reflection with a hyperbola $(E_\epsilon)_\ddagger$. With $\kappa > 0$ and $\epsilon = \tanh \kappa$, let $h(\kappa)$ and $\bar{h}(\kappa)$ be represented respectively by the oppositely oriented hyperbolas $(E_\epsilon)_\ddagger$ and $(E_{-\epsilon})_\ddagger$. Equivalently, from **(16)**, this determines a correspondence between $h(\kappa)$ and $g(\kappa')$ with $\tanh \kappa' = \operatorname{sech} \kappa$, and between $\bar{h}(\kappa)$ and $g(\kappa')$ with $\tanh \kappa' = -\operatorname{sech} \kappa$. Previously, we did not define $(E_0)_\ddagger$ because it would consist of the absolute points $\pm i$, but here it represents $h(\infty)$ as the degenerate *absolute hyperbola*.

With these assignments we complete the representation of $G(L)$, beginning with products of non-absolute reflections. These are straightforward. With $\tanh \kappa'_j = \epsilon'_j = \operatorname{sech} \kappa_j$ in all cases, they are listed in Table 2 along with the ellipses that represent them.

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Table 2. Products of Two Reflections

Reflections	Product	Representative
$h(\kappa_2)h(\kappa_1)$	$g(\kappa_2' - \kappa_1')$	$\pi\left(E_{-\epsilon_1'} \circ E_{\epsilon_2'}\right)$
$\overline{h}(\kappa_2)\overline{h}(\kappa_1)$	$g(\kappa_1' - \kappa_2')$	$\pi\left(E_{\epsilon_1'} \circ E_{-\epsilon_2'}\right)$
$h(\kappa_2)\overline{h}(\kappa_1)$	$g(\kappa_1' + \kappa_2')$	$\pi\left(E_{\epsilon_1'} \circ E_{\epsilon_2'}\right)$
$\overline{h}(\kappa_2)h(\kappa_1)$	$g(-\kappa_2' - \kappa_1', Z)$	$\pi\left(E_{-\epsilon_1'} \circ E_{-\epsilon_2'}\right)$

Representing the product of a translation and a non-absolute reflection is more intricate, as indicated by the following proposition. Since split-inversion respects tangency we set $\pi(C_\ddagger) := \pi(C)_\ddagger$ for C in category C4. This is the unique (oriented) hyperbola tangent to $C_\ddagger$ at its intersections with L.

Proposition 4. *If* $\cosh\kappa_2 \tanh\kappa_1 < 1$ *then the reflection* $h(\kappa_2)g(\kappa_1)$ *is represented by* $\pi\left(E_{\epsilon_1'} \circ E_{\epsilon_2'}\right)_\ddagger$ *with* $\epsilon_1' = \tanh\ln(e^\kappa + 1)$ *and* $\epsilon_2' = -\tanh\ln(e^\kappa - 1)$, *where*

$$\kappa = \ln\frac{\cosh\kappa_1\cosh\kappa_2 - \sinh\kappa_1 + \sinh\kappa_2}{\cosh\kappa_1 - \cosh\kappa_2\sinh\kappa_1}.$$

Proof. First, $e^\kappa > 1$ because $\kappa_2 > 0$. The product $h(\kappa_2)g(\kappa_1) = h(t)$ with $\cosh t = \frac{\cosh\kappa_2 - \tanh\kappa_1}{1 - \cosh\kappa_2\tanh\kappa_1}$. But this is equal to $\cosh\kappa$, so $h(\kappa_2)g(\kappa_1) = h(\kappa)$. We need to find κ' so that $\tanh\kappa' = \operatorname{sech}\kappa$. Since $\operatorname{sech}\kappa_2 > \tanh\kappa_1$, we have $\kappa' = \ln\frac{e^\kappa+1}{e^\kappa-1}$. Then $\kappa' = \kappa_1' + \kappa_2'$ with $\tanh\kappa_1' = \epsilon_1'$ and $\tanh\kappa_2' = \epsilon_2'$, whereby $\pi\left(E_{\epsilon_1'} \circ E_{\epsilon_2'}\right)_\ddagger$ represents the product reflection. □

If $\cosh\kappa_2\tanh\kappa_1 > 1$ then $h(\kappa_2)g(\kappa_1) = \overline{h}(\kappa)$ with e^κ replaced by $-e^\kappa$. All other products are listed in Table 3. The condition that determines whether the product is $h(\kappa)$ or $\overline{h}(\kappa)$ uses the functions $p(j) = \cosh\kappa_1\tanh\kappa_2 + j$ and $\tilde{p}(j) = \cosh\kappa_2\tanh\kappa_1 + j$. The formula for κ uses

$$\begin{aligned}
\xi_1(j,k) &= \ln(\cosh\kappa_1\cosh\kappa_2 + j\sinh\kappa_1 + k\sinh\kappa_2) \\
\xi_2(j,k) &= -\ln(j\cosh\kappa_1\sinh\kappa_2 + k\cosh\kappa_2) \\
\tilde{\xi}_2(j,k) &= -\ln(j\cosh\kappa_2\sinh\kappa_1 + k\cosh\kappa_1).
\end{aligned}$$

Each product is represented by $\pi\left(E_{\epsilon_1'} \circ E_{\epsilon_2'}\right)_\ddagger$ with $\epsilon_1' = \tanh\ln(e^\kappa + 1)$ in all cases and $\epsilon_2' = q(j) = \tanh\ln\frac{j}{e^\kappa - 1}$.

Table 3. Product of Reflection and Translation

Product	Constraint	κ	ϵ_2'
$g(\kappa_2)h(\kappa_1) = h(\kappa)$	$p(1) > 0$	$\xi_1(1,1) + \xi_2(1,1)$	$q(1)$
$g(\kappa_2)h(\kappa_1) = \overline{h}(\kappa)$	$p(1) < 0$	$\xi_1(1,1) + \xi_2(-1,-1)$	$q(1)$
$g(\kappa_2)\overline{h}(\kappa_1) = h(\kappa)$	$p(-1) > 0$	$\xi_1(-1,-1) + \xi_2(1,-1)$	$q(-1)$
$g(\kappa_2)\overline{h}(\kappa_1) = \overline{h}(\kappa)$	$p(-1) < 0$	$\xi_1(-1,-1) + \xi_2(-1,1)$	$q(-1)$
$h(\kappa_2)g(\kappa_1) = h(\kappa)$	$\tilde{p}(-1) < 0$	$\xi_1(-1,1) + \tilde{\xi}_2(-1,1)$	$q(1)$
$h(\kappa_2)g(\kappa_1) = \overline{h}(\kappa)$	$\tilde{p}(-1) > 0$	$\xi_1(-1,1) + \tilde{\xi}_2(1,-1)$	$q(1)$
$\overline{h}(\kappa_2)g(\kappa_1) = h(\kappa)$	$\tilde{p}(1) < 0$	$\xi_1(1,-1) + \tilde{\xi}_2(-1,-1)$	$q(-1)$
$\overline{h}(\kappa_2)g(\kappa_1) = \overline{h}(\kappa)$	$\tilde{p}(1) > 0$	$\xi_1(1,-1) + \tilde{\xi}_2(1,1)$	$q(-1)$

A fiber structure over $\mathcal{D}_0 \cup \mathcal{O}$ will be defined in the next section. We have avoided Lie notation, such as for orthogonal groups and their indefinite forms, in our description of $G(L)$ since there will be no discussion of topology.

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10. Central Conics as Fiber Elements

In this section we partition the conics in categories C4 and C1 by the Steiner conics tangent to them. Recall that every C in category C4 is the composition of a unique pair of ellipses (Corollary 2) and that the $C_{\ddagger}$ comprise category C1.

Definition 3. The *fiber* over a given ellipse E_ϵ is the collection of C in C4 such that $\pi(C) = E_\epsilon$. Equivalently, the set of $E_{\epsilon_1} \circ E_{\epsilon_2}$ with $\kappa_1 + \kappa_2 = \kappa$, where $\epsilon = \tanh\kappa$ and $\epsilon_j = \tanh\kappa_j$ (so the fiber over E_0 is just E_0).

Each element of the fiber over E_ϵ is in the closed disk bounded by the circle $E_{\epsilon'} \circ E_{\epsilon'}$, where $\epsilon' = \tanh\frac{1}{2}\kappa$. The fiber over $E_{\frac{4}{5}}$ is shown in Figure 8, with the ellipse in red and the circle in blue.

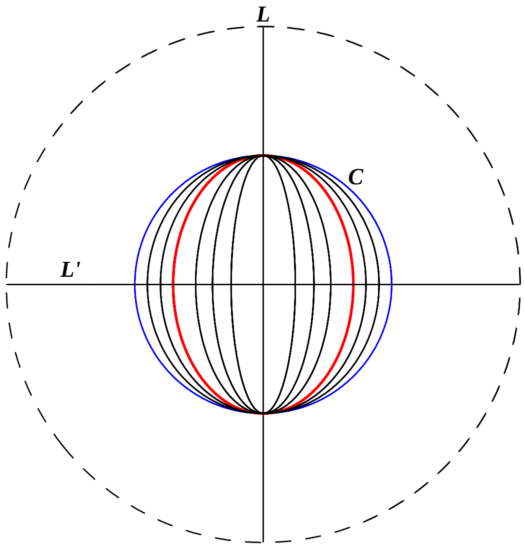

Figure 8. Ellipse $E_{\frac{4}{5}}$ (red) with fiber elements inside C (blue).

In the fiber over E_ϵ let $C(\varepsilon) = E_{\epsilon_1} \circ E_{\epsilon_2}$, where $\varepsilon = \tanh(\kappa_2 - \kappa_1)$. As with ellipses, we consider $C(\varepsilon)$ and $C(-\varepsilon)$ to be oppositely oriented fiber elements and define ε to be the (oriented) *eccentricity* of $C(\varepsilon)$. The fiber elements are parameterized by $\varepsilon_t = \tanh(\kappa + t)$, with $t = -2\kappa_1$ because $\kappa_1 + \kappa_2 = \kappa$. Define $C(\varepsilon_{t_1}) + C(\varepsilon_{t_2}) := C(\varepsilon_t)$, where $\varepsilon_t = \tanh(2\kappa + t_1 + t_2)$.Then the fiber is a group isomorphic to $G_0(L)$. The identity element is the bounding circle $C(0)$. The fiber is preserved by $G_0(L)$ with the action defined by $g \cdot C(\varepsilon_t) := C(\varepsilon_t) + C(\gamma)$, where g is represented by the ellipse E_γ. Further, $G_0(L)$ acts transitively on the set of fibers, with g sending the fiber over E_ϵ to the fiber over $\pi(E_\epsilon \circ E_\gamma)$.

Proposition 5. *The conics in category* C4 *are partitioned into fibers over the ellipses. The group $G_0(L)$ acts transitively on the fibers, and each fiber is a group isomorphic to $G_0(L)$.*

The conics can also be partitioned by eccentricity. Define the *ε-section* by selecting the (unique) element in each fiber with eccentricity ε. Then E_ε is the unique ellipse in the section, as shown in Figure 9 for $\varepsilon = \frac{1}{2}$.

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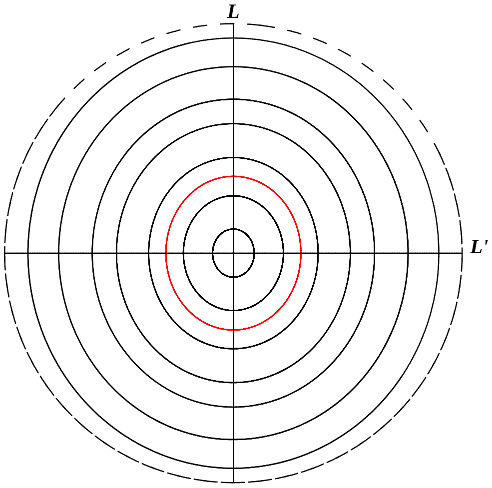

Figure 9. Section containing the ellipse (red) $E_{\frac{1}{2}}$.

For $\epsilon \neq 0$, the fiber over a hyperbola $(E_\epsilon)_\ddagger$ is the split-inversion of the fiber over E_ϵ. Figure 10 shows the fiber over $\left(E_{\frac{4}{5}}\right)_\ddagger$, with the hyperbola in red. The elements are bounded by $\left(E_{\frac{1}{2}} \circ E_{\frac{1}{2}}\right)_\ddagger$, which is in category C9B by Corollary 3. In this example, the distance between these ultra-parallel lines (blue) is $\ln 2$ (see Remark 5).

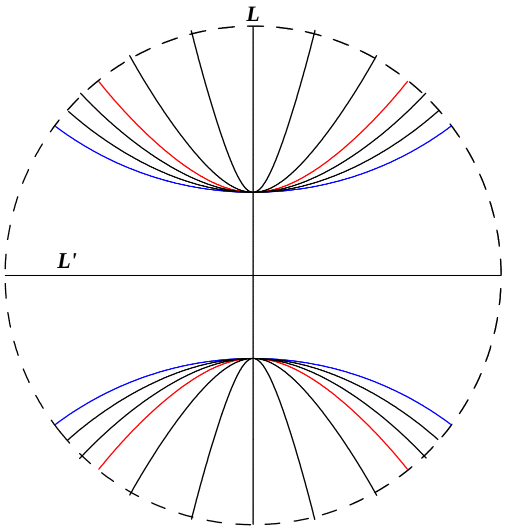

Figure 10. Fiber over $\left(E_{\frac{4}{5}}\right)_\ddagger$ (red) with ultra-parallel lines (blue).

The KC-convex hyperbolas in the fiber over $(E_\epsilon)_\ddagger$ can be assigned eccentricities greater than 1 in accordance with (**16**) and consistent with the fiber over E_ϵ. A section for a given eccentricity would then be the split inversion of an ε-section of KC-ellipses.

11. Conclusions

In Jakob Steiner's preface to [2] he states

> *Here the main thing is neither the synthetic nor the analytic method, but the discovery of the mutual dependence of the figures and of the way in which their properties are carried over from the simpler to the more complex ones.*

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The source of this quote has been the inspiration for the constructions in this article. These constructions were obtained intrinsically, within conformal models of the hyperbolic plane, from the incidence properties of $\mathbb{H}^2$ and the action of its collineation group. This is in the spirit of Steiner's construction of conics. We used elliptic curve addition on the orthogonal trajectories of the central Steiner conics to generate all central conics, which in turn imposed a group structure on the Steiner conics themselves.

The underlying symmetries of the central conics made these constructions possible. Though we previously classified all Steiner conics in $\mathbb{H}^2$ we are not aware of analogous results for the categories of non-central conics. Many of these are of so-called parabolic type, but there does not appear to be an inversive representation of the Steiner parabolas for which their orthogonal trajectories have a simple addition structure. Perhaps this work will encourage investigation in that direction.

Conflicts of Interest: The author declares no conflicts of interest.

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