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Notes on Prerequisites of Loop Quantum Gravity

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Abstract

Loop quantum gravity (LQG) quantizes gravity through the quantization of the space-time. It begins with general relativity (GR) and brings some concepts from quantum field t heory t o q uantize t he s pace-time. T he c onceptual b ackground o f G R i s necessary to comprehend the LQG framework. Therefore, in this paper, prerequisite concepts such as special relativity, general relativity, covariant electromagnetism, tensor analysis, Lagrangian formulation, Hamiltonian formulation and basics of quantum mechanics are briefly introduced; that, are needed to study loop quantum gravity(LQG).

Keywords: special relativity; general relativity; covariant electromagnetism; Lagrangian formulation; Hamiltonian; loop quantum gravity

Contents

1 Introduction

Loop quantum gravity (LQG) is one of the supposed theory that quantizes gravity successfully. In literature, there are many introductory notes or reviews are available that address prerequisites concepts to study loop quantum gravity. But, very few provides prerequisites of those theories that are needed to study the LQG. This article complements those all classical text [1-10] and review articles [11-22] of the LQG. There are many mathematical theories that are required to study the LQG. The LQG mostly uses the differential geometry and the functional analysis; however other mathematical theories are also essential such as (1) topology, (2) global analysis (3) algebraic geometry, (4) group theory, (5) knot theory, (6) index theorem, (7) infinite dimensional analysis, (8) non-commutative algebra. In this article, basic concepts of relativity (with necessary mathematical background), background of classical field theory (Lagrangian and Hamiltonian formulation), covariant formulation of electromagnetism and basics necessary equations of quantum physics are described up to necessary extent.

2 List of Prerequisites concepts

Here, all prerequisites theories that is needed to study LQG are given up to relevant extent.

2.1 Special relativity and general relativity

In this section, basics of special and general relativity is given up to relevant extent in context with LQG.

2.1.1 Special Relativity

Special theory of relativity (SR) was discovered by modifying Galilean relativity by Albert Einstein in 1905. According to Galilean relativity, law of physics are same in all inertial frames and speed of light is variable and can take any value. According to Maxwells electromagnetic theory (EMT), the value of speed of light is $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 m/s$. By unifying Galilean relativity with EMT, speed of light is invariant; namely, speed of light is constant in all reference frames. Therefore, Galilean transformation is replaced by Lorentz transformation. Lorentz transformation equation can be given as [23]

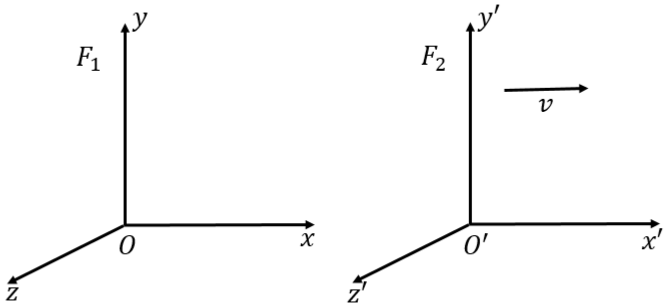

Fig. 1: Stationary (left side) and constant velocity (Right side) inertial reference frames.

$$t^{'} = \gamma\left(t - \frac{vx}{c^2}\right), x^{'} = \gamma(x - vt), y = y^{'}, z = z^{'} \tag{1}$$

Where, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is Lorentz factor.

These equations have remarkable consequences in contemporary physics such as time dilation, length contraction, relativity of simultaneity etc. According to SR, every motion is relative and it correlates mass with energy, space with time. In essence, in the all inertial frame of reference the postulates of SR are [23]

1. The law of physics are invariant.
2. The speed of light is constant.

2.1.2 Terminology

In this section, framework of special relativity such as novel notations, conventions, and terminology are given to follow next section. Reference frame is the place of observer from where, phenomenon is being observed. Constant velocity or rest position is said to be inertial frame and accelerating frame is said to be non-inertial [23].

In SR, space and time are at equal footing. Trajectory of particle in spacetime diagram i.e., world line is given in $t - x$ spacetime diagram. Distance between two points in spacetime is given by metric or line element or interval [23],

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

In $t - x$ spacetime diagram, light always travel at 45^o. Light travels 1 meter distance in 1 meter of time, in geometrical units; thus, slope of such line is given by $\frac{dt}{dx} = \frac{1m}{1m} = 1, \therefore c = 1$[23].

$$\therefore c^2dt^2 = dt^2 \tag{3}$$

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{4}$$

The proper time is defined as the time measured by the clock of the observer. It is given by [23]

$$d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{5}$$

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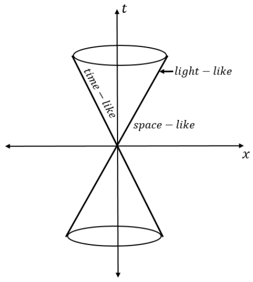

Fig. 2: Schematic diagram of light cone with different interval.

By suppressing 2 space dimensions into 1 space dimensions world of line of particle is drawn by light cone in $t-x$ spacetime diagram. Due to the invariance of light speed, motion of any object is confined within light cone. Interval inside the light cone is timelike ($\Delta s^2 < 0$) interval outside the light cone is spacelike ($\Delta s^2 > 0$), interval on light cone is null or light-like ($\Delta s^2 = 0$)[23].

Four velocity and four momentum: since, special relativity unifies space with time (space-time continuum); there are some four vector that are defined with respect to proper time. Four velocity is defined as $\vec{v} = (v_t, v_x, v_y, v_z) = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau})$. Four momentum is defined as $\vec{p} = (p_t, p_x, p_y, p_z) = (\frac{E}{c}, p_x, p_y, p_z)$: where, $p_t = \frac{mc^2}{c} = \frac{E}{c}$ [23].

2.2 General relativity

In this section, basics of general relativity is given in context with LQG up to relevant extent.

2.2.1 Newtons gravity

Before 1915, gravity was understood by Newtonian gravity that solely depends on Euclidean geometry (flat). In his book Principia, Newton explained universal law of gravitation force by inverse square law in the following ways [23],

$$F = \frac{Gm_1m_2}{r^2} \tag{6}$$

Where, G is universal constant, m_1 and m_2 are two masses, r is the distance between two masses and F is gravitation force between two given objects.

Newton could determine the value of gravitational force; but, he failed to interpret to gravitational force. It was Einstein who interpreted gravity successfully with the aid of his general relativity [23].

2.2.2 Einstein's General relativity

Introduction: in 1915, Albert Einstein discovered general theory of relativity (GR). According to GR, gravity and spacetime has concrete relationship and deformed spacetime is the proof of present of gravity. In John Wheeler's word, mass tells spacetime how to curve, and spacetime tells mass how to move [23].

In general relativity, there are three sort of masses (1) inertial mass - due to which body resists to change it position in the motion (2) active mass - that behaves as the origin or source of gravitational field (3) passive gravitational mass that describes interaction of body with considered gravitational field. Though, these three masses are distinct to each other, these three masses are equivalent to each other. GR has two principles: principle of equivalence and principle of covariance [23]

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Equivalence principle: there are two types of equivalence principle, i.e.

Weak equivalence principle: due to the universal nature of gravitational field, coupling of gravitational field with all mass and energy is same and hence equivalence.

Strong equivalence principle: the law of physics is invariant in accelerated as well as uniform or static frame of reference.

Due to tidal nature of gravitational field for large spatial or temporal region, equivalence principle is no longer applied.

General covariance: law of physics are invariant under coordinate transformation or same for all coordinate system that involve tensorial transformation and tensor equations.

2.2.3 Introduction to Tensor analysis and differential geometry

In this section, mathematical framework of tensor analysis and differential geometry is given up to relevant extent in context with LQG. As it is originated from GR; these mathematical theories are prerequisite.

2.2.4 Novel notations and definitions

The spacetime metric can also be written in the following way [23],

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{7}$$

where, $(x^0, x^1, x^2, x^3) = (t, x, y, z)$

In general [23],

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{8}$$

Where, $g_{\mu\nu}$ is second rank tensor components (4×4 matrix , $\mu = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$) [23].

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{9}$$

The Einstein summation convention: index with once raised and lowered in the term appeared to be summed in the following way [23]

$$\sum_{i=1}^{3} x_i y^i \rightarrow x_i y^i = x_1 y^1 + x_2 y^2 + x_3 y^3 \tag{10}$$

Such summed over index is said to be dummy index and can be replaced by any other index. Index that appear in both side of expression is free index [23].

$$X^a Y_{ab} = X^d Y_{db} \tag{11}$$

Where, a is dummy index therefore replaced by d, $\therefore a \leftrightarrow d$. And b is free index [23].

Vector A can be defined in terms of Einstein summation convention in the following way $A = A^c e_c$, Where c is dummy index, A^c (with superscript) is contra variant component of vector A and $e_c = \partial_c = \frac{\partial}{\partial c}$ is basis vector or coordinate basis [23].

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In dual vector space, vector X can be represented as its covariant component or one form in the following way, $\tilde{X} = X_a \omega^a$. Where X_a (with subscript) is covariant component of one form $\tilde{X}$ and $\omega^a = dx^a$ is basis one form [23].

The mapping of basis vector by basis one form to a number is defined by Kronecker delta function as [23]

$$\omega^x e_y = \delta^x_y \rightarrow \delta^x_y = \begin{cases} 0, x \neq y \\ 1, x = y. \end{cases} \tag{12}$$

Inverse metric: product of metric with its inverse is represented as [23]

$$g_{bd} g^{de} = \delta^e_b \tag{13}$$

Determinant of the metric can be written as [23]

$$g = det(g_{xy}) \tag{14}$$

Raising and lowering of index: raising and lowering index can be done in the following way [23]

$$A_x = g_{xy} A^y, A^x = g^{xy} A_y \tag{15}$$

2.2.5 Coordinate transformation and Jacobian matrix

Transformation from one coordinate system to another can be done using transformation matrix i.e. Cartesian to polar coordinate system. For basis vector [23],

$$e_{b'} = \Lambda^a_{b'} e_a = \frac{\partial x^a}{\partial x^{b'}} e_a \tag{16}$$

Where, $e_{b'}$ is new basis. e_a is old basis and $\Lambda^a_{b'}$ is transformation matrix. Such transformation matrix is also said to be Jacobian matrix. Any Λ^b_a Jacobian matrix can also be represented as [23]

$$J = \Lambda^b_a = \frac{\partial x^b}{\partial x^a} = \begin{pmatrix} \frac{\partial x^{b^1}}{\partial x^{a^1}} & \cdots & \frac{\partial x^{b^1}}{\partial x^{a^n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^{b^n}}{\partial x^{a^1}} & \cdots & \frac{\partial x^{b^n}}{\partial x^{a^n}} \end{pmatrix} \tag{17}$$

Square root of determinant of metric: it is very useful in determining the area and volume of object; as, the metric involves the calculation of angles and lengths in geometry. i.e. area of parallelogram. The factor $\sqrt{-g}$ is very useful in such calculations, coordinate transformation and in the action that involves Lagrangian. If vector $\vec{a}$ and $\vec{b}$ are sides of parallelogram; then area of parallelogram is given in the following way [23],

$$\text{A} = \sqrt{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})}$$

$$= \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})} \sin^2 \theta$$

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$$\therefore A = \sqrt{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})(1-\cos^2\theta)} \tag{18}$$

In terms of metric, any metric M_{pq} where,$(p, q \in \{\vec{a}, \vec{b}\})$ is given as [23]

$$M_{pq} = \begin{pmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{pmatrix} \tag{19}$$

By comparison between equation (2.18) and (2.19), the area of parallelogram [23],

$$A = \sqrt{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2}$$

$$\therefore A = \sqrt{\det M_{pq}} \tag{20}$$

Transformation between metric g_{ab} and Minkowski metric η_{cd} can be given as [23]

$$g_{ab} = \Lambda_a^c \Lambda_b^d \eta_{cd} \tag{21}$$

Using the absolute value of determinant of above equation if $\eta = \det \eta_{cd} = -1$,[23]

$$|J| = \sqrt{\frac{g}{\eta}} = \sqrt{-g} \tag{22}$$

This value is used in the integral that is given as [23]

$$\int f(a^1, a^2, ..., a^n) da^1 ... da^n = \int f(b^1, b^2, ..., b^n) |detJ| db^1 ... db^n \tag{23}$$

For covariant vector, contravariant vector and basis one form coordinate transformation equations are [23]

$$V_c = \Lambda_c^d V_d, V^c = \Lambda_d^c V^d, \omega^c = dx^c = \Lambda_d^c \omega^d \tag{24}$$

2.2.6 Introduction to Tensor

Tensor is the generalization of vector or array of number. In general, any tensor can be defined as [23]

$$T = T^{abc...}_{xyz...} \omega^x \otimes \omega^y \otimes \omega^z ... e_a \otimes e_b \otimes e_c ... \tag{25}$$

Where, $\otimes$ is the sign of tensor product. In other way, tensor is a kind of function that map vector or one form into real number or vector into one form or vice versa. It transforms according to certain rules under the change of coordinates [23]. For example,

$$T(\omega^c, \omega^d) = T^{cd} = T^{cd} e_c \otimes e_d \tag{26}$$

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Symmetric tensor: for any tensor, symmetric part can be written as [23]

$$T_{(cd)} = \frac{1}{2}(T_{cd} + T_{dc}) \tag{27}$$

Antisymmetric tensor: for any tensor, antisymmetric part can be written as [23]

$$T_{[cd]} = \frac{1}{2}(T_{cd} - T_{dc}) \tag{28}$$

Transformation of metric tensor: it is defined as in the following way [23]

$$g_{c'd'} = \Lambda^a_{c'}\Lambda^b_{d'}g_{ab} \tag{29}$$

Wedge product: wedge product is used in exterior derivative calculation. Product of any two one form A and B is two form [23],

$$A \wedge B = A \otimes B - B \otimes A \tag{30}$$

Wedge product is anti-commutative; hence [23], $A \wedge B = -B \wedge A$ and $A \wedge A = 0$

2.2.7 The covariant derivative and Christoffel symbol

In curved geometry, vector or one form vary from point to point. Therefore, partial derivative of vector or one form is cumbersome to calculate; therefore one need to define new form of derivative and that is covariant derivative. For any vector $\vec{V}$, [23]

$$\frac{d\vec{V}}{dx^a} = \frac{\partial}{\partial x^a}(V^b e_b) = \frac{\partial V^b}{\partial x^a}e_b + V^b\frac{\partial e_b}{\partial x^a} \tag{31}$$

Using $\omega^c(e_b) = \delta^c_b$. By multiplying both side of the equation by basis one form ω^c, one can have relation $\omega^c(e_b) = \delta^c_b$ [23]

$$\frac{dV^c}{dx^a} = \frac{\partial V^c}{\partial x^a} + V^b\frac{\partial e_b}{\partial x^a} \cdot \omega^c \tag{32}$$

The above equation can be written in compact form in the following way [23],

$$\nabla_a V^c = \partial_a V^c + V^b \Gamma^c_{ab} \tag{33}$$

Where, $\Gamma^c_{ab} = \frac{\partial e_b}{\partial x^a} \cdot \omega^c$ is said to be Christoffel symbol or affine connection and whole equation is known as covariant derivative. For upper indices Christoffel is positive and for lower indices it is negative, i.e., [23]

$$\nabla_a V_c = \partial_a V_c - V_b \Gamma^b_{ac} \tag{34}$$

Christoffel symbol can also be given in the following way [23],

$$\partial_a g_{bc} = (\partial_a e_b)e_c + e_b(\partial_a e_c)$$

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$$= \Gamma^d_{ab} e_d e_c + \Gamma^d_{ac} e_d e_b$$

$$\therefore \partial_a g_{bc} = \Gamma^d_{ab} g_{dc} + \Gamma^d_{ac} g_{db} \tag{35}$$

Two more equation can be obtained by changing the order of a, b, c. Hence, [23],

$$\therefore \partial_b g_{ca} = \Gamma^d_{bc} g_{da} + \Gamma^d_{ba} g_{dc} \tag{36}$$

$$\therefore \partial_c g_{ab} = \Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{da} \tag{37}$$

Christoffel symbols are symmetric in lower indices i.e., $\Gamma^d_{cb} = \Gamma^d_{bc}$. Now,

$$\partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc}$$

$$\Gamma^d_{bc} g_{da} + \Gamma^d_{ab} g_{dc} + \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{da} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{db}$$

$$\Gamma^d_{bc} g_{da} + \Gamma^d_{bc} g_{da} = 2\Gamma^d_{bc} g_{da}$$

Multiplying the above equation by the inverse metric g^{da}, for d and a, putting $a \leftrightarrow d$ [23]

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{38}$$

2.2.8 Lie derivative and other important tensor

Lie derivative can be defined as [23]

$$L_X Y = X^a \nabla_a Y^b - Y^b \nabla_b X^a \tag{39}$$

Lie derivative of any tensor [23]

$$L_X T_{bc} = X^a \nabla_a T_{bc} + T_{ac} \nabla_b X^a + T_{ba} \nabla_c X^a \tag{40}$$

Killing vector: a vector that satisfies an equation $\nabla_d X_c + \nabla_c X_d = 0$ is killing vector. Killing vector also satisfies [23]

$$L_K g_{cd} = 0 \tag{41}$$

Cross product of vector in terms of index notation: for any vectors $\vec{P}$ and $\vec{Q}$ cross product using Levi Civita symbol ε^{ijk} in index notation can be given as [23]

$$(\vec{P} \times \vec{Q})^i = \varepsilon^{ijk} P_j Q_k, \varepsilon^{ijk} \varepsilon_{jkm} = \delta^i_m \tag{42}$$

Where,

$$\varepsilon^{ijk} = \begin{cases} +1, & \text{for even permutation of indices 123} \\ -1, & \text{for odd permutation of indices 123} \\ 0, & \text{otherwise} \end{cases}$$

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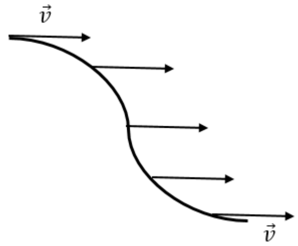

Fig. 3: Parallel transport of any vector in flat geometry.

2.2.9 Parallel transport in flat geometry and geodesics

In flat geometry, parallel transport of vectors is possible. For any curve, initial and final vector are parallel to each other. In flat geometry, shortest distance between two points is known as geodesics [23].

2.2.10 Parallel transport of vector in curved geometry and geodesic deviation

In flat geometry, those vectors that begins parallel with each other from one point; ends parallel to each other at another point without deviation or no change of normal distance between them. Therefore, parallel transport of two vector or more than two vector are possible in flat geometry. But in curved geometry or Riemannian geometry, parallel transport involves geodesic deviation. Vectors that begin parallel may not end parallel due to curved geometry. Consider two adjacent tall building with same height which are separated by few miles. If two ball are thrown from the terrace of building simultaneously; they are parallel at the outset, but as the earth surface is reached they are no longer parallel due to congruence and curvature of earth surface. Due to tidal force of gravity of earth, both balls are attracted towards earth's center and converges. Therefore, two objects that starts parallel to each other in gravitational field or curved geometry will not end parallel to each other but rather with some deviation; namely, geodesic deviation [23].

In other words, to verify whether given geometry is curved or flat consider tangent vector of given curve in closed loop path. If orientation of tangent vector at the initial and final point are same (angle between initial and final tangent vector is zero) then given geometry is flat. In curved geometry, orientation of tangent vector in closed loop between initial point and final point is not preserved; namely, initial tangent vector and final tangent vector makes certain angle between them (in case of fig. 2.4, it is 90^o). A quantity that measures the extent of orientation between given initial and final tangent vector on closed loop or a kind of connection on manifold of differential geometry that cannot preserve data of geometry during parallel transport of vectors is holonomy [23].

The Riemann tensor or the curvature tensor: to understand parallel transport in curved geometry, consider two different path on same closed path on spherical surface. Then, the magnitude of rotation is different. By considering covariant derivative one can differentiate between two paths. Consider covariant component V_b of $\vec{V}$. Consider two paths i.e. (1) V_b along e_c first and e_d second (2) V_b along e_d first and e_c second. The covariant derivative can be given by $\nabla_d \nabla_c \nabla_b$ and $\nabla_c \nabla_d \nabla_b$ respectively [23].

Both covariant derivatives are not same; since parallel transport on curved surface is not same as flat surface. The difference between both covariant derivative depends of background curvature i.e., the Riemann tensor or the curvature tensor. Therefore, the commutator [23] i.e.

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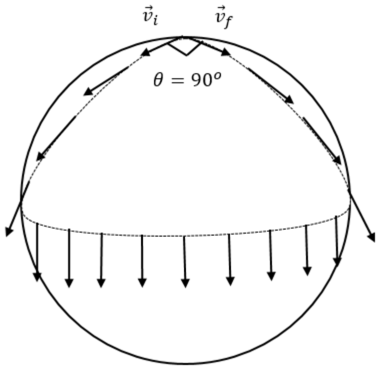

Fig. 4: Parallel transport of vector in curved geometry 90^o angle is made between initial and final vector due to curved geometry.

$$\nabla_d \nabla_c V_b - \nabla_c \nabla_d V_b = R^e_{bcd} V_e \tag{43}$$

The above equation can be derived in following way; as, it is necessary in the next section [23].

$$\nabla_c V_b = \partial_c V_b - \Gamma^e_{bc} V_e$$

$$\nabla_d \nabla_c V_b = \partial_d(\partial_c V_b) - \Gamma^f_{bd} \nabla_c V_f - \Gamma^f_{cd} \nabla_f V_b$$

$$\partial_d(\partial_c V_b) = \partial_d \partial_c \partial_b - \partial_d \Gamma^e_{bc} V_e$$

$$\partial_d(\partial_c V_b) = \partial_d \partial_c \partial_b - \partial_d(\Gamma^e_{bc}) V_e - \partial_d \Gamma^e_{bc}(V_e)$$

$$\Gamma^f_{bd} \nabla_c V_f = \Gamma^f_{bd}(\partial_c V_f - \Gamma^e_{cf} V_e)$$

$$\Gamma^f_{cd} \nabla_f V_b = \Gamma^f_{cd}(\partial_f V_b - \Gamma^e_{bf} V_e)$$

$$\begin{aligned} \therefore \nabla_d \nabla_c V_b = \partial_d \partial_c \partial_b - \partial_d(\Gamma^e_{bc}) V_e - \partial_d \Gamma^e_{bc}(V_e) \\ - \Gamma^f_{bd}(\partial_c V_f - \Gamma^e_{cf} V_e) - \Gamma^f_{cd}(\partial_f V_b - \Gamma^e_{bf} V_e) \end{aligned} \tag{44}$$

Same as [23],

$$\begin{aligned} \therefore \nabla_c \nabla_d V_b = \partial_c \partial_d \partial_b - \partial_c(\Gamma^e_{bd}) V_e - \partial_c \Gamma^e_{bd}(V_e) \\ - \Gamma^f_{bc}(\partial_d V_f - \Gamma^e_{df} V_e) - \Gamma^f_{dc}(\partial_f V_b - \Gamma^e_{bf} V_e) \end{aligned} \tag{45}$$

By subtracting, doing some simple operations with indices and simplifying [23],

$$\therefore \nabla_d \nabla_c V_b - \nabla_c \nabla_d V_b = R^e_{bcd} V_e \tag{46}$$

Where, $\partial_c(\Gamma^e_{bd}) - \partial_d(\Gamma^e_{bc}) + \Gamma^f_{bd}\Gamma^e_{cf} + \Gamma^f_{bc}\Gamma^e_{df}$
Symmetry of Riemann tensor in lower indices can be given as in the following way [23],

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$$R_{bcde} = R_{debc} = -R_{bced} = -R_{cbde} \tag{47}$$

$$R_{bcde} + R_{bdec} + R_{becd} = 0 \tag{48}$$

Bianchi identities of Riemann tensor can be written in the following ways [23],

$$\nabla_b R_{efcd} + \nabla_d R_{efcb} + \nabla_c R_{efdb} = 0 \tag{49}$$

Ricci Tensor and Ricci scalar: Ricci tensor is obtained by contraction from Riemann tensor and Ricci scalar is also obtained by contraction in the following way, $R_{cd} = R^e_{ced}$ and $R = g^{cd}R_{cd} = R^c_c$ respectively [23].

Einstein tensor: Einstein tensor can be obtained using Ricci tensor and Ricci scalar [23],

$$G_{cd} = R_{cd} - \frac{1}{2}Rg_{cd} \tag{50}$$

Energy-momentum tensor: stress energy or energy momentum tensor T_{bc} is gravitational field source. It is symmetric tensor. There are four component of energy momentum tensor (1) time-time component - energy density($T_{bc} = \rho = T_{tt}$), (2) time-space energy flux($T_{bc} = T_{ti}$), (3) space-time component - momentum density($T_{bc} = T_{it}$), and (4) space-space component - stress($T_{bc} = \pi_i = T_{ij}$). Where i and j are spatial indices and t is time. If one impose constraint on energy momentum tensor, one gets $T_{bc} = 0$ [23].

2.2.11 Einsteins field equation

According to GR, curvature tensor is directly related to energy momentum tensor through famous tensor equation; namely, Einsteins field equation (EFE) [23].

In the standard form, Einstein field equation can be given as [23],

$$G_{\mu\nu} \propto T_{\mu\nu} \tag{51}$$

$$G_{\mu\nu} = kT_{\mu\nu} \tag{52}$$

Where, $k = \frac{8\pi G}{c^4}$

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{53}$$

To make the universe static, Einstein had added one term; namely, cosmological constant and which was according to him, biggest blunder of his life after the discovery of expanding universe. But today, cosmological constant is very important quantity. Einstein field equation with cosmological constant can be written in the following way [23],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{54}$$

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Where, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Therefore,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{55}$$

Where,$\mu, \nu = 0, 1, 2, 3$, $R_{\mu\nu}$ is Ricci tensor R is Ricci Scalar, $g_{\mu\nu}$ is metric tensor, Λ is cosmological constant $T_{\mu\nu}$ is energy momentum tensor, G, is gravitational constant c is speed of light.

2.2.12 Introduction to manifold

Manifold is the mathematical object that is very important in mathematics of curved geometry. Manifold is points with continuous space and it looks locally flat but, globally it is curved i.e., any spherical planet or star. Manifold can be understood using coordinate patches that are locally flat and can be mapped to Euclidean space but globally curved. Manifold needs more than one coordinate system at a time [23].

Mathematically, manifold is a topological set that is mappable to flat space $\mathbb{R}^n$ by considering local patches. One of the important manifold i.e., differentiable manifold is very important in GR and LQG. It is continuous as well as differentiable manifold [23].

2.2.13 Diffeomorphism invariance

Diffeomorphism invariance is a sort of symmetry in which law of physics are remained invariant under differentiable coordinate transformation or under various choice of coordinate system. This symmetry is key result behind the idea of general covariance [23].

Almost, all fundamental physics theory are invariant under passive diffeomorphism. But general relativity involves active as well as passive view of diffeomorphism [23].

Passive diffeomorphism gives new basis in terms of old by keeping given object fixed; while, active diffeomorphism keeps basis unchanged and changes the position of object. In other words, active diffeomorphism generates new metrics, while passive merely express again the original metric in new terms (coordinate transformation). i.e., manifold is topological space which appears locally same as of Euclidean space at each point. In terms of manifold, change of coordinates means by changing the mapping to $\mathbb{R}^n$ space with keeping point of manifold fixed and that is passive diffeomorphism [23].

In the opposite picture, by changing the point of move around the point of manifold with keeping the map $\mathbb{R}^n$ fixed. Such notion has one to one correspondence in mapping. It means mapping between two points i.e., x and $f(x)$ (point and its derivative) does not violate topological neighborhood. And gives invariance even after mapping. In active diffeomorphism, there is no prior geometry or no preferred point of object until one introduces it. This is the elegance of background independence picture emerged from diffeomorphism symmetry [23].

Consider two sets A and B. Any map Ω has one corresponding point of Set A for each and every point of set B. If any point m belong to set A, in set B must have any a point n [23]

$$n = \Omega(m) \tag{56}$$

It is said to be push forward operation of point m into B. One can also have a function F that defines a real number for each and every point of set B. Using the mapping between set A and B one can define another function $\Omega^*(F)$ [23].

$$\Omega^*(F)(m) \equiv F(\Omega(m)) = \Omega(n) \tag{57}$$

It is said to be pull back operation of function F from B to A [23].

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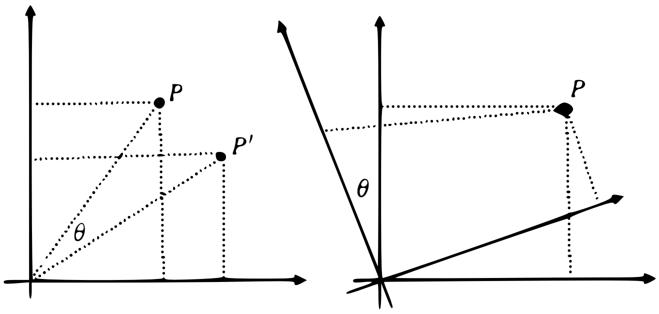

Fig. 5: Active (left side) and passive (right side) diffeomorphism invariance.

2.3 Electromagnetism

In this section, electromagnetism with its covariant formulation is given up to relevant extent in context with LQG.

2.3.1 Standard form of Maxwells equation

The standard form of these four equations can be given as [23]

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{58}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{59}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{60}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{61}$$

Where, electric field $\vec{E}$ is vector and magnetic field $\vec{B}$ is pseudo vector. $\vec{J}$ is current density. ρ is charge density. μ_0 and ε_0 are permeability and permittivity of free space respectively. $\vec{\nabla}$ is Laplacian operator.

2.3.2 Essential quantities of covariant EM

The equation of continuity in electromagnetism can be given as [23]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{62}$$

The electrostatic potential ϕ and vector potential A can be defined as [23],

$$E = -\nabla\phi - \frac{\partial A}{\partial t}, B = \nabla \wedge A \tag{63}$$

Gauge transformation for the electrostatic potential ϕ and vector potential A using any differentiable function $\xi(r,t)$ can be given as [23]

$$A^{'} = A + \nabla\xi, \phi^{'} = \phi - \frac{\partial\xi}{\partial t} \tag{64}$$

The four current vector can be obtained using charge density and current density i.e., $J^{\mu} = (c\rho, j)$. The equation of continuity in covariant formulation is [23]

$$\partial_{\mu} j^{\mu} = 0 \tag{65}$$

Hence, The Lorentz gauge condition can be given as $\partial_{\mu} A^{\mu} = 0$; and Covariant gauge transformation can be given as [23]

$$(A^{'})^{\mu} = A^{\mu} - \partial^{\mu}\xi \tag{66}$$

The field tensor is defined as [23],

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{67}$$

Where, $F^{0i} = -\frac{E_i}{c}$

The field tensor (contravariant form and covariant form) in the matrix form can be defined as [23],

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{68}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} \tag{69}$$

Transformation equation of field tensor upper to lower indices is given by [23]

$$F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho} F_{\sigma\rho} \tag{70}$$

The field tensor is anti-symmetric; therefore [23], $F_{\mu\nu} = -F_{\nu\mu}$.

2.3.3 Covariant form of Maxwells equation

Using equation $(\vec{P} \times \vec{Q})^{i} = \varepsilon^{ijk} P_j Q_k$ where $\forall i, j, k = 1, 2, 3$ Maxwell equation with defined constant can be written as [23]

$$\varepsilon^{ijk}\partial_j B_k - \partial_0 E^{i} = 4\pi j^{i} \tag{71}$$

$$\partial_i E^i = \partial_0 E^i = 4\pi j^0 \tag{72}$$

$$\varepsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{73}$$

$$\partial_i B^i = 0 \tag{74}$$

From matrix one can write $F^{0i} = E^i$ and $F^{ij} = \varepsilon^{ijk} B_k$, maxwell equation can be written as [23],

$$\partial_j F^{ij} + \partial_0 F^{i0} = 4\pi j^i \tag{75}$$

$$\partial_i F^{0i} = 4\pi j^0 \tag{76}$$

In general, Maxwell equations are written in shorter form [23],

$$\partial_\mu F^{\nu\mu} = 4\pi j^\nu \tag{77}$$

Using Levi-Civita symbol, four Maxwell equation can be written as [23]

$$\varepsilon^{\sigma\mu\nu\kappa}\partial_\mu F_{\mu\kappa} = 0 \tag{78}$$

Field tensor is Lorentz invariant and thus [23],

$$F^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\rho F^{\sigma\rho} \tag{79}$$

2.4 Classical mechanics

In this section, basics of quantum field theory are given up to relevant extent in context with LQG.

2.4.1 An overview of Lagrangian and Hamiltonian formulation

LQG follows both Lagrangian as well as Hamiltonian mechanics to give theory of quantum gravity. The former is useful in covariant approach of LQG i.e., spin foam; and the latter is useful in canonical approach of LQG i.e., canonical quantization of gravity. Here, both mechanics are introduced up to relevant extent. Since, QFT quantize any field using Lagrangian and Hamiltonian formulation; firstly, both are outlined [1-5].

 doi:10.20944/preprints202209.0163.v1

2.4.2 Lagrangian formulation

Lagrangian L can be defined as [24]

$$L = L(q_j, \dot{q}_j, t), L = T - V \tag{80}$$

Where, $j = 0, 1, 2, , n$ and $q_j, \dot{q}_j, t$ is generalize coordinates, generalized velocity and time respectively. Lagrangian equation of motion can be given as [24].

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \tag{81}$$

Where, $j = 0, 1, 2, , n$ and $\frac{\partial L}{\partial q_j}$ is canonical or conjugate momentum.

Techniques of calculus of variation: for any function $f(y, y^{'}, x)$, dependent variable $y(x)$ is functional. The integral in term of functional for any function $f(y, y^{'}, x)$ can be given as [24],

$$I = \int_{x_1}^{x_2} = f(y, y^{'}, x)dx \tag{82}$$

2.4.3 Hamiltonian formulation

Hamiltonian formulation is the modification of Lagrangian formulation that uses phase space to define dynamics of given object [24].

Hamiltonian H can be defined as [24]

$$H = H(q_j, p_j, t), H = T + V \tag{83}$$

Hamiltonian H using Legendre transform can also be defined as [24]

$$H = \Sigma_j p_j \dot{q}_j - L \tag{84}$$

Using calculus of variation, Hamilton principle in terms of Lagrangian can be written as [24]

$$H \to \delta J = \delta \int_{t_1}^{t_2} Ldt = 0 \tag{85}$$

Hamilton's principle can be used as dynamical action in classical mechanics, quantum mechanics, quantum field theory and any theory of quantum gravity also [24].

Hamilton's canonical equation of motion can be written as [24]

$$\dot{q_j} = \frac{\partial H}{\partial p_j}, -\dot{p_j} = \frac{\partial H}{\partial q_j} \tag{86}$$

Poisson bracket: in phase space, Poisson bracket on any two function f and g can be given as [24],

$$[f, g]_{q_i, p_i} = \Sigma_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{87}$$

2.5 Preliminaries of quantum mechanics

In quantum mechanics, time dependent three-dimensional Schrodinger equation can be given as [25]

$$H\psi = E\psi \tag{88}$$

Where at left hand side of equation $H \to -\frac{\hbar^2}{2m}\nabla^2 + V(x)$ and ψ are Quantum mechanical Hamiltonian and wave function respectively; at right hand side, $E \to -i\hbar\frac{\partial}{\partial t}$ and ψ are energy eigen value and eigen function respectively. Therefore, Schrodinger equation is when $V(x, y, z) = 0$ [25].

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -i\hbar\frac{\partial\psi}{\partial t} \tag{89}$$

The Schrodinger equation in Diracs notation can be given as [25]

$$\hat{H}|\psi\rangle = \hat{E}|\psi\rangle \tag{90}$$

In quantum mechanics, commutator is defined as [25]

$$[A, B] = AB - BA \tag{91}$$

For instance, two operators $\hat{x}$ and $\hat{p}$ commutator relation is [25]

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \tag{92}$$

Using Hamiltonian of classical mechanics if quantization is performed then it is said to be canonical quantization with canonical variable q and p. In this procedure phase space is replaced by Hilbert space that exhibits inner products [25].

The result of inner product is finite and function between this inner product exhibits an integration over the independent variable. A square integrable function $(\int dx|f(x)|^2)$ of $f(x)$ makes such Hilbert space. In this procedure canonical variable and Poisson bracket of classical mechanics are replaced by quantum operator and commutator respectively $(q, p) \to [\hat{q}, \hat{p}]$ [25].

The Poisson bracket of two observables is replaced by commutator when one goes from classical mechanics to quantum mechanics; where, the scalar value of the commutator is $i\hbar$ times the scalar value of the equivalent Poisson bracket [25], i.e.

$$f, g = 1, [\hat{f}, \hat{g}] = i\hbar \tag{93}$$

3 Concluding remarks

In this notes, all necessary prerequisite concepts that are required to begin the study of LQG are given up to relevant extent. This concepts provides a step for beginners who want to study LQG. Since, the LQG originates from the GR; firstly, it is explained with necessary mathematical background. Thereafter, basics of covariant formulation of the electromagnetism, basics (Lagrangian and Hamiltonian formulation) that is required for classical field theory and some necessary equations of the quantum mechanics are elaborated.

4 Acknowledgement

The authors are thankful to Physics Department, Saurashtra University, Rajkot, India.

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