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On the Extended, Generalized, and Grand Riemann Hypothesis for L-Functions

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Article

On the Extended, Generalized, and Grand Riemann Hypothesis for *L*-Functions

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Abstract: To study the non-trivial zero distribution of different kinds of L-functions, this paper generalizes the key lemma—Lemma 8 in reference [1]. As a result, the Extended Riemann Hypothesis (for Dedekind zeta function), the Generalized Riemann Hypothesis (for Dirichlet L-function), and the Grand Riemann Hypothesis (for modular form L-Function, automorphic L-function, and etc.) are proved.

Keywords: Extended Riemann Hypothesis; Generalized Riemann Hypothesis; Grand Riemann Hypothesis; L-functions; Hadamard product; functional equation

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1. Introduction

In reference [1], the Riemann Hypothesis (RH) is proved based on a new expression of the completed zeta function $\xi(s)$, which was obtained through pairing the conjugate zeros ρ_i and $\bar{\rho}_i$ in the Hadamard product expression, with consideration of the multiplicities of zeros, i.e.

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i}$$

where $\xi(0) = \frac{1}{2}$, $\rho_i = \alpha_i + j\beta_i$, $\bar{\rho}_i = \alpha_i - j\beta_i$, with $0 < \alpha_i < 1$, $\beta_i \neq 0$, $0 < |\beta_1| \leq |\beta_2| \leq \cdots$, and $m_i \geq 1$ is the multiplicity of ρ_i.

It should be noted that in this article and reference [1], j is used to denote the imaginary unit ($j^2 = -1$), while i serves as a natural number index.

Lemma 8 is the key lemma to the proof of the RH in reference [1]. The key points include the divisibility contained in the variant of functional equation $\xi(s) = \xi(1-s)$ and the uniqueness of the multiplicity m_i of zero ρ_i (although it is unknown), we actually obtain

$$\xi(s) = \xi(1-s) \Leftrightarrow \Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} = \Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i}, i = 1,2,3,\ldots$$

which is further equivalent to

$$\alpha_i = \frac{1}{2}, 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots, i = 1,2,3,\ldots$$

Here we give the details of Lemma 8 in [1] as the base of subsequent discussions.

Lemma 8: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$f(s) = \prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{1}$$

and

$$f(1-s) = \prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{2}$$

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where s is the complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of the completed zeta function $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots \end{cases} \tag{3}$$

The other related Lemmas in reference [1] are also provided in the following.

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

1) The number of non-trivial zeroes is infinity;

2) $\beta \neq 0$;

3) $0 < \alpha < 1$;

4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

Lemma 3: Let $m(x), g_1(x), ..., g_n(x) \in \mathbb{R}[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

Lemma 4: Let $f(x), m(x) \in \mathbb{R}[x]$. If $m(x)$ is irreducible and $f(x)$ is any polynomial, then either $m(x)$ divides $f(x)$ or $gcd(m(x), f(x)) = 1$.

Lemma 5: Let $m(x), g_1(x), g_2(x), \ldots \in \mathbb{R}[x]$. If $m(x)$ is irreducible and divides the infinite product $\prod_{i=1}^{\infty} g_i(x)$, then $m(x)$ divides one of the polynomials $g_1(x), g_2(x), \ldots$.

Lemma 6: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is a finite positive integer.

Lemma 7: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is unique.

Lemma 9: The infinite product $\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i}$ converges to a non-zero constant, given the conditions: $0 < \alpha_i < 1, \beta_i \neq 0, \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$, and $m_i \geq 1$ is the multiplicity of zero $\alpha_i + j\beta_i$.

Remark: Lemmas 1-4 are the well-known results summarized from related journal papers, or textbooks/monographs. Lemmas 5-9 are proved in reference [1].

2. Generalization of Lemma 8

As pointed in reference [2] (on page 57), we can enumerate the nontrivial zeros of the zeta function in order of the increasing absolute value of their imaginary parts, where zeros whose imaginary parts have the same absolute value are arranged arbitrarily. Thus we remove the ordering of $|\beta_i|$, $|\beta_1| \leq |\beta_2| \leq \cdots$, as condition hereafter for simplicity.

Generalization 1 of Lemma 8: The following Theorem 1 extends the critical strip from $0 < \Re(s) < 1$ in Lemma 8 to $0 < \Re(s) < k$, $k > 0$ is a constant real number.

Theorem 1: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$F(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{4}$$

and

$$F(k-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{5}$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $F(s)$, $0 < \alpha_i < k$, $k > 0$, and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$.

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Then we have

$$F(s) = F(k-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{6}$$

Proof. We will prove both directions of Eq.(6).

Sufficiency ($\Leftarrow$):

Assume that for all i, $\alpha_i = \frac{k}{2}$ and $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$.

Then we have

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}, i = 1,2,3,\cdots \tag{7}$$

Taking infinite products on both sides of Eq.(7), the sufficiency of Eq.(6) is proven.

Necessity ($\Rightarrow$):

According to the definition of divisibility of entire functions [3,4] (or more specifically the definition that a polynomial divides an entire function expressed as infinite product of polynomial factors [1]), the functional equation $F(s) = F(k-s)$ implies that each polynomial factor on either side must divide the infinite product on the opposite side, i.e.,

$$F(s) = F(k-s) \Rightarrow \begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid F(k-s) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid F(s) \\ i = 1,2,3,\cdots \end{cases} \tag{8}$$

where "$|$" is the divisible sign.

Since $1 + \frac{(s-\alpha_i)^2}{\beta_i^2}$ (with discriminant $\Delta = -4 \cdot \frac{1}{\beta_i^2} < 0$) and $1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}$ (with discriminant $\Delta = -4 \cdot \frac{1}{\beta_i^2} < 0$) are irreducible over the field $\mathbb{R}$, then by Lemma 5, Eq.(8) yields:

$$\begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(s-\alpha_l)^2}{\beta_l^2}\right) \\ i = 1,2,3,\cdots; l = 1,2,3,\cdots \end{cases} \tag{9}$$

For the special kind of polynomials in Eq.(9), "divisible" means "equal", which can be verified by comparing the like terms in equation $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = k'\left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right), k' \neq 0 \in \mathbb{R}$ to get $k' = 1$. Further, due to the uniqueness of the multiplicity m_i, the only solution to Eq.(9) is $i = l$, otherwise, new duplicated zeros with $\alpha_l = k - \alpha_i, k - \alpha_l = \alpha_i, \beta_i^2 = \beta_l^2, l \neq i$ would be generated to change m_i. Therefore we have from Eq.(9):

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right), i = 1,2,3,\cdots \tag{10}$$

By comparing the like terms in the above polynomial equation, we obtain $\alpha_i = \frac{k}{2}$. Further, to ensure the uniqueness of m_i while $\alpha_i = \frac{k}{2}$, we need limit the β_i values to be distinct, i.e., $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$.

Thus the necessity of Eq.(6) is proven.

That completes the proof of Theorem 1. $\square$

Generalization 2 of Lemma 8: The following Theorem 2 will extend the polynomial factor expression in Lemma 8 to several other variants.

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Theorem 2: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$
\begin{aligned}
G(s) &= \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} \\
&= \prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)^{m_i}\left(1-\frac{s}{\bar{\rho}_i}\right)^{m_i} \\
&= \prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
G(k-s) &= \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(k-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} \\
&= \prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)^{m_i}\left(1-\frac{k-s}{\bar{\rho}_i}\right)^{m_i} \\
&= \prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)
\end{aligned}
\tag{12}
$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $G(s)$, $0 < \alpha_i < k$, $k > 0$, and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$.

Then we have

$$
G(s) = G(k-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases}
\tag{13}
$$

Proof. According to Theorem 1 and Lemma 9, we have

$$
\begin{aligned}
& F(s) = F(k-s) \\
& \Leftrightarrow \\
& \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} F(s) = \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} F(k-s) \\
& \Leftrightarrow \\
& \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(k-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i} \\
& \Leftrightarrow \\
& G(s) = G(k-s)
\end{aligned}
\tag{14}
$$

Then we know that

$$
G(s) = G(k-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases}
\tag{15}
$$

That completes the proof of Theorem 2. □

Generalization 3 of Lemma 8: In this case, we actually intend to lay a foundation for the study of zero distribution of completed L-functions.

Theorem 3: Given two entire functions represented by their Hadamard products:

$$
\Lambda(\lambda, s) = e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)e^{\frac{s}{\rho_i}}
\tag{16}
$$

and

$$\Lambda(\lambda, k-s) = e^{A(\lambda)+B(\lambda)(k-s)} \prod_{i=1}^{\infty} \left(1 - \frac{k-s}{\rho_i}\right) e^{\frac{k-s}{\rho_i}} \tag{17}$$

where s is a complex variable, λ denotes a mathematical object (e.g., Dirichlet character, modular form, automorphic representation), $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\Lambda(\lambda, s)$, $0 < \alpha_i < k, k > 0$, and $\beta_i \neq 0$ are real numbers, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$.

If the functional equation $\Lambda(\lambda, s) = \Lambda(\lambda, k-s)$ holds, then we have

$$\begin{cases} \alpha_i = \frac{k}{2}, i = 1, 2, 3, \cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{18}$$

where each zero with due multiplicity that is unique although unknown.

Proof. First, we have

$$\begin{aligned} \Lambda(\lambda, s) &= e^{A(\lambda)+B(\lambda)s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) e^{\frac{s}{\rho_i}} \\ &= e^{A(\lambda)+B(\lambda)s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) e^{\frac{2\alpha_i s}{\alpha_i^2+\beta_i^2}} \\ &= e^{A(\lambda)+B(\lambda)s} e^{s \cdot \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) \end{aligned} \tag{19}$$

Noticing that $\sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2} < 2k \cdot \sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$, then we have $\sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2+\beta_i^2} = c, c \in \mathbb{R}, c \neq 0$. With further consideration of the multiplicity $m_i \geq 1$ of each zero, we obtain

$$\Lambda(\lambda, s) = e^{A(\lambda)+[B(\lambda)+c]s} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)^{m_i} \left(1 - \frac{s}{\bar{\rho}_i}\right)^{m_i} \tag{20}$$

Then we can conclude that Theorem 3 is true according to Theorem 2, considering $e^{A(\lambda)+[B(\lambda)+c]s}$ and $e^{A(\lambda)+[B(\lambda)+c](k-s)}$ have no zeros, thus both of them have no effect on the complex zeros related divisibility in the functional equation $\Lambda(\lambda, s) = \Lambda(\lambda, k-s)$.

That completes the proof of Theorem 3. $\square$

Remark: As pointed out in Ref. [5] (on page 102), if $\rho_i = \alpha_i + j\beta_i$ is a zero of $\Lambda(\lambda, s)$, then $\bar{\rho}_i = \alpha_i - j\beta_i$ is a zero of $\Lambda(\bar{\lambda}, s)$. Therefore, if $\lambda \neq \bar{\lambda}$, then we need to reconstruct the symmetric functional equation to ensure that the conjugate zeros of entire function appear together. For more details, see the proofs of Theorem 5 and Theorem 7.

Generalization 4 of Lemma 8: In this case, we make further efforts to lay a foundation for the study of completed L-functions that possess both real and complex non-trivial zeros, denoted by $\rho \in \mathcal{Z}_{\text{real}}$ ($\Im(\rho) = 0$) and $\rho \in \mathcal{Z}_{\text{complex}}$ ($\Im(\rho) \neq 0$), respectively. When these two zero sets have no common elements, we express their disjointness by: $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$.

The reason we need to consider this case is that, so far, we cannot rule out the existence of exceptional zeros (or Landau-Siegel zeros), although their numbers are very limited even if they do exist.

To be specific, denote the set of real zeros in the (extended) critical strip as $\mathcal{Z}_{\text{real}} = \{a_n \in \mathbb{R} \mid 0 < a_n < k,\ n = 1, 2, \ldots, N\}$, where N is a finite natural number. This finiteness follows from the Identity Theorem, which implies that any non-zero entire function cannot have infinitely many zeros in a bounded region, as zeros of such functions are isolated and cannot accumulate.

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Theorem 4: Given two entire functions represented by their Hadamard products:

$$\begin{aligned}\Lambda(\lambda,s) &= e^{A(\lambda)+B(\lambda)s}\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \\ &= e^{A(\lambda)+B(\lambda)s}\prod_{n=1}^{N}\left(1-\frac{s}{a_n}\right)e^{\frac{s}{a_l}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)e^{\frac{s}{\rho_i}}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\Lambda(\lambda,k-s) &= e^{A(\lambda)+B(\lambda)(k-s)}\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}} \\ &= e^{A(\lambda)+B(\lambda)(k-s)}\prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_l}}\prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)e^{\frac{k-s}{\rho_i}}\end{aligned} \tag{22}$$

where s is a complex variable, λ denotes a mathematical object (e.g., Dirichlet character, modular form, automorphic representation), $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\Lambda(\lambda,s)$, $0<\alpha_i<k$, $0<a_n<k$, $k>0$, and $\beta_i \neq 0$ are real numbers, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$, $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$.

If the functional equation $\Lambda(\lambda,s)=\Lambda(\lambda,k-s)$ holds, then all the zeros (both real and complex) of $\Lambda(\lambda,s)$ lie on the (extended) critical line, i.e.,

$$\begin{cases}\alpha_i = \frac{k}{2}, i=1,2,3,\cdots \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots \\ a_n = \frac{k}{2}, n=1\end{cases} \tag{23}$$

where each zero with due multiplicity that is unique although unknown.

Proof. With reference to Theorem 3, to determine the distribution of the complex zeros of $\Lambda(\lambda,s)$, we only need to show that the newly added parts $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ do not affect the complex zeros related divisibility in the functional equation $\Lambda(\lambda,s)=\Lambda(\lambda,k-s)$, which is an obvious fact according to Ref. [3] (on page 174) with the given condition $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$.

Thus, we conclude by Theorem 3 that

$$\begin{cases}\alpha_i = \frac{k}{2}, i=1,2,3,\cdots \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots\end{cases} \tag{24}$$

where each zero with due multiplicity that is unique although unknown.

Next, we consider the real zeros of $\Lambda(\lambda,s)$.

We have from equalling Eq.(21) and Eq.(22) and canceling the complex non-trivial zeros related polynomial factors

$$e^{(B(\lambda)+c)s}\prod_{n=1}^{N}\left(1-\frac{s}{a_n}\right)e^{\frac{s}{a_n}} = e^{(B(\lambda)+c)(k-s)}\prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_n}} \tag{25}$$

where c is the same constant as in the proof of Theorem3.

Further Eq.(25) is equivalent to

$$e^{(B(\lambda)+c')s}\prod_{n=1}^{N}\left(s-a_n\right) = e^{(B(\lambda)+c')(k-s)}\prod_{n=1}^{N}\left(s-(k-a_n)\right) \tag{26}$$

where $c' = c + \sum_{n=1}^{N}\frac{1}{a_n}$.

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Suppose the multiplicity of zero $s = a_n$ is m_n ($m_n \geq 1$) that is finite and unique although unknown. Then Eq.(26) becomes

$$e^{(B(\lambda)+c')s} \prod_{t=1}^{T} \left(s - a_t\right)^{m_t} = e^{(B(\lambda)+c')(k-s)} \prod_{t=1}^{T} \left(s - (k - a_t)\right)^{m_t} \tag{27}$$

where $\sum_{t=1}^{T} m_t = N$.

Considering $(s - a_t)$ and $(s - (k - a_t))$ are irreducible over $\mathbb{R}$, then by Lemma 3, Eq.(27) means

$$\begin{cases} (s - a_t) \mid (s - (k - a_m) \\ (s - (k - a_t)) \mid (s - a_m) \\ t = 1, 2, 3, \cdots, T; m = 1, 2, 3, \cdots, T \end{cases} \tag{28}$$

The only solution to Eq.(28) is $t = m, a_t = \frac{k}{2}, t = 1, \cdots, T$, otherwise the uniqueness of m_t would be violated with $a_t + a_m = k, t \neq m$. To avoid changing the multiplicity of m_t while $a_t = \frac{k}{2}$, we need to limit $T = 1$. Thus we get

$$a_t = \frac{k}{2}, t = 1 \tag{29}$$

where zero $s = a_1 = \frac{k}{2}$ with multiplicity m_1 .

Putting Eq.(24) and Eq.(29) together, we proved Eq.(23).

That completes the proof of Theorem 4. □

Remark: To use Theorem 4 while $\lambda \neq \bar{\lambda}$, we need to construct a new symmetric functional equation $\Lambda(\bar{\lambda}, s)\Lambda(\lambda, s) = \Lambda(\lambda, k - s)\Lambda(\bar{\lambda}, k - s)$ to ensure that the conjugate zeros of entire function appear together. For more details, see the proofs of Theorem 5 and Theorem 7.

3. The Applications of Theorem 4

We will make use of Theorem 4 to prove the Extended Riemann Hypothesis (for Dedekind zeta function), the Generalized Riemann Hypothesis (for Dirichlet L-function), and the Grand Riemann Hypothesis (for modular form L-Function, automorphic L-function, and etc.).

To facilitate the subsequent discussion, we give the details of the Extended Riemann Hypothesis, the Generalized Riemann Hypothesis, and the Grand Riemann Hypothesis. In the following contents, the critical line means: $\Re(s) = \frac{1}{2}$, or more generally, $\Re(s) = \frac{k}{2}, k > 0$ is a constant real number.

The Generalized Riemann Hypothesis: The nontrivial zeros of Dirichlet L-functions lie on the critical line.

The Extended Riemann Hypothesis: The nontrivial zeros of Dedekind zeta function lie on the critical line.

The Grand Riemann Hypothesis: The nontrivial zeros of all L-functions lie on the critical line.

To begin with, we provide a general property of L-functions, which was labeled Lemma 5.5 in reference [5].

Lemma 5.5 Let $L(f, s)$ be an L-function. All zeros ρ of $\Lambda(f, s)$ are in the critical strip $0 \leq \sigma \leq 1$. For any $\epsilon > 0$, we have

$$\sum_{\rho \neq 0,1} |\rho|^{-1-\epsilon} < +\infty.$$

where, $\Lambda(f, s)$ is the completed L-function corresponding to $L(f, s)$, σ is the real part of ρ, and f is similar to λ in this paper as a symbol representing a mathematical object (e.g., Dirichlet character, modular form, automorphic representation).

Another general property of L-functions is as follows.

All the zeros of $\Lambda(f, s)$ are exactly the non-trivial zeros of $L(f, s)$, since the trivial zeros of $L(f, s)$ are canceled by the poles of Gamma factors in the completion process.

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Thus, we can discuss the non trivial zeros of L-functions based on the zeros of the corresponding completed L-functions.

3.1. Dirichlet L-Function

Definition: The Dirichlet L-function associated with a Dirichlet character χ modulo q is defined for $\Re(s) > 1$ by the series:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{30}$$

For the principal character χ_0 (where $\chi_0(n) = 1$ if $\gcd(n, q) = 1$ and $\chi_0(n) = 0$ otherwise), the L-function is related to the Riemann zeta function by:

$$L(\chi_0, s) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \tag{31}$$

Completed L-function: The completed Dirichlet L-function is defined as:

$$\Lambda(\chi, s) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(\chi, s) \tag{32}$$

where $a = 0$ if $\chi(-1) = 1$ (even character) and $a = 1$ if $\chi(-1) = -1$ (odd character).

Functional Equation: The completed Dirichlet L-function satisfies the functional equation:

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1 - s) \tag{33}$$

where $W(\chi)$ is the Gauss sum:

$$W(\chi) = \frac{\tau(\chi)}{i^a \sqrt{q}} \tag{34}$$

and $\tau(\chi) = \sum_{n=1}^{q} \chi(n) e^{2\pi i n/q}$ is the Gauss sum associated with χ.

Hadamard Product: For non-principal characters χ, the completed L-function $\Lambda(\chi, s)$ is an entire function and has the Hadamard product:

$$\Lambda(\chi, s) = e^{A(\chi)+B(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{35}$$

where the product is over all zeros ρ of $\Lambda(\chi, s)$, and $A(\chi)$ and $B(\chi)$ are constants depending on χ.

Next we prove the Generalized Riemann Hypothesis.

Theorem 5: The nontrivial zeros of Dirichlet L-functions lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $\chi = \bar{\chi}$ (self-dual)

It suffices to verify that the properties of $\Lambda(\chi, s)$ with $\chi = \bar{\chi}$ match the conditions of Theorem 4. In Eq.(33), $W(\chi)$ does not affect the non-trivial zero related divisibility since it has no any zero. Eq.(35) is equivalent to Eq.(21) by separating zeros into two sets $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$. Actually, to restrict $\chi = \bar{\chi}$ is to guarantee that the conjugate zeros of $\Lambda(\chi, s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, 1 - \rho_i, 1 - \bar{\rho}_i)$ with their multiplicities appear together according to Eq.(33). The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ can be assured by Lemma 5.5, considering that $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$ is a subseries of $\sum_{\rho \neq 0,1} \frac{1}{|\rho|^2}$; The condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ holds because $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$ are mutually exclusive sets, i.e., if $\rho \in \mathcal{Z}_{\text{real}}$, then $\Im(\rho) = 0$; if $\rho \in \mathcal{Z}_{\text{complex}}$, then $\Im(\rho) \neq 0$.

Thus, by Theorem 4, we know that both the real (if exists) and the complex non-trivial zeros of $\Lambda(\chi, s)$ with $\chi = \bar{\chi}$ lie on the critical line.

CASE 2: $\chi \neq \bar{\chi}$

In this case, the conjugate non-trivial zeros do not appear together in Eq.(35), because if ρ is a zero of $\Lambda(\chi, s)$, then $\bar{\rho}$ is a zero of $\Lambda(\bar{\chi}, s)$.

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Thus, we need to extend Eq.(33) to another form, i.e.,

$$\Lambda(\bar{\chi}, s) = W(\bar{\chi})\Lambda(\chi, 1-s) \tag{36}$$

Combining (36) with (33), we get a new functional equation

$$\Lambda(\bar{\chi}, s)\Lambda(\chi, s) = W(\bar{\chi})W(\chi)\Lambda(\chi, 1-s)\Lambda(\bar{\chi}, 1-s) \tag{37}$$

Both sides of Eq.(37) are the products of entire functions, thus they are still entire functions. And we know that the conjugate zeros of $\Lambda(\bar{\chi}, s)\Lambda(\chi, s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(37). Further, based on Eq.(35) and the derivation procedures in the proof of Theorem 3, we have

$$\begin{aligned}\Lambda(\chi, s)\Lambda(\bar{\chi}, s) &= e^{A(\chi)+A(\bar{\chi})+(B(\chi)+B(\bar{\chi})+c)s} \prod_{\rho \in \mathcal{Z}_{\text{real}}} \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \prod_{\rho \in \mathcal{Z}_{\text{complex}}} \left(1-\frac{s}{\rho}\right) \\ &= e^{A(\chi)+A(\bar{\chi})+(B(\chi)+B(\bar{\chi})+c)s} \prod_{\rho \in \mathcal{Z}_{\text{real}}} \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \prod_{i=1}^{\infty} \left(1-\frac{s}{\rho_i}\right)\end{aligned} \tag{38}$$

The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ hold for the same reasons as in CASE 1.

Thus, by Theorem 4, we know that both the real (if exists) and the complex non-trivial zeros of $\Lambda(\chi, s)$ with $\chi \neq \bar{\chi}$ lie on the critical line.

Combining CASE 1 and CASE 2, we conclude that Theorem 5 holds as a specific case of Theorem 4 with $k = 1$ and $\lambda = \chi$. □

3.2. *Dedekind Zeta Function*

Definition: For a number field K with ring of integers $\mathcal{O}_K$, the Dedekind zeta function is defined for $\Re(s) > 1$ by:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \tag{39}$$

where the sum is over all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, and $N(\mathfrak{a})$ is the norm of the ideal.

Completed Zeta Function: The completed Dedekind zeta function is defined as:

$$\Lambda_K(s) = |D_K|^{s/2}\left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s) \tag{40}$$

where D_K is the discriminant of K, r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings of K.

Functional Equation: The completed Dedekind zeta function satisfies:

$$\Lambda_K(s) = \varepsilon(K)\Lambda_K(1-s) \tag{41}$$

where $\varepsilon(K) = 1$ for all number fields K, showing the symmetry of the functional equation.

Hadamard Product: The completed Dedekind zeta function has a simple pole at $s = 1$ with residue $\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|D_K|}}$, where h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity in K. The function $s(s-1)\Lambda_K(s)$ is entire and has the Hadamard product:

$$s(s-1)\Lambda_K(s) = e^{A_K+B_K s} \prod_{\rho \neq 0,1} \left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{42}$$

where the product is over all non-trivial zeros ρ of $\Lambda_K(s)$ except $\rho = 0$ and $\rho = 1$, and A_K and B_K are constants depending on K.

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For more details of the completed Dedekind zeta function, please be referred to reference [5] (Chapter 5.10) and reference [6] (Section 10.5.1).

In the following Theorem 6, we prove the Extended Riemann Hypothesis for the Dedekind zeta function.

Theorem 6: The nontrivial zeros of Dedekind zeta function lie on the critical line.

Proof. It suffices to show that the properties of $\Lambda_K(s)$ match the conditions of Theorem 4.

Actually, Eq.(41) and Eq.(42) guarantee that

$$e^{A_K+B_K s}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho}=e^{A_K+B_K(1-s)}\prod_{\rho\neq 0,1}\left(1-\frac{1-s}{\rho}\right)e^{1-s/\rho} \tag{43}$$

where $\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho}=\prod_{\rho\in\mathcal{Z}_{\text{real}}\setminus\{0,1\}}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$.

And we know that the conjugate zeros of $\Lambda_k(s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i,\bar{\rho}_i,1-\rho_i,1-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(41). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$ and condition $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4 we know that both the real (if exists) and the complex non-trivial zeros of $\Lambda_K(s)$ lie on the critical line, i.e., Theorem 6 holds as a specific case of Theorem 4 with $k=1$ and without λ. □

In the following contents, we prove the modified Grand Riemann Hypothesis for modular form L-Functions and automorphic L-functions, respectively. After that we will make a summarization, and conclude that the modified Grand Riemann Hypothesis holds for all L-functions satisfying some general properties.

3.3. *Modular Form L-Function*

Definition: For a modular form $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi i n z}$ of weight k for a congruence subgroup Γ, the associated L-function is defined for $\Re(s)>\frac{k+1}{2}$ by:

$$L(f,s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s} \tag{44}$$

Completed L-function: For a cusp form f of weight k for $\Gamma_0(N)$ (level N is any positive integer) with Nebentypus character χ, the completed modular form L-function is defined as:

$$\Lambda(f,s)=N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s) \tag{45}$$

Functional Equation: For a normalized Hecke eigenform f of weight k for $\Gamma_0(N)$ with Nebentypus character χ, the completed modular form L-function satisfies:

$$\Lambda(f,s)=\varepsilon(f)\Lambda(\overline{f},k-s) \tag{46}$$

where $\varepsilon(f)=\pm 1$ is the epsilon factor, which is the eigenvalue of f under the Atkin-Lehner involution, and $\overline{f}$ is the modular form with Fourier coefficients $\overline{a}_n$.

Hadamard Product: For a cusp form f, the completed L-function $\Lambda(f,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(f,s)=e^{A(f)+B(f)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{47}$$

where the product is over all zeros ρ of $\Lambda(f,s)$, and $A(f)$ and $B(f)$ are constants depending on f.

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For more details of the completed modular form L-function, please be referred to references [5,7].

We have the following result about the non-trivial zero distribution of modular form L-Functions.

Theorem 7: The non-trivial zeros of the above-described modular form L-Functions lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $f = \overline{f}$ (self-dual)

It suffices to show that the properties of $\Lambda(f,s)$ with $f = \overline{f}$ match the conditions of Theorem 4. Eq.(47) is equivalent to Eq.(21) by separating all zeros into two sets $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$. Actually, to restrict $f = \bar{f}$ is to guarantee that the conjugate zeros of $\Lambda(f,s))$ appear in pairs. Then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(46). The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Thus, by Theorem 4, we know that both the real (if exists) and the complex non-trivial zeros of $\Lambda(f,s)$ with $f = \overline{f}$ lie on the critical line.

CASE 2: $f \neq \overline{f}$

To deal with this case $f \neq \bar{f}$, we need first to extend Eq.(46) to another form, i.e.,

$$\Lambda(\bar{f},s) = \varepsilon(\bar{f})\Lambda(f,k-s) \tag{48}$$

Combining (48) with (46), we get a new functional equation

$$\Lambda(\bar{f},s)\Lambda(f,s) = \varepsilon(f)\varepsilon(\bar{f})\Lambda(f,k-s)\Lambda(\bar{f},k-s) \tag{49}$$

Obviously, both sides of Eq.(49) are the products of entire functions, thus they are still entire functions. And we know that the conjugate zeros of $\Lambda(\bar{f},s)\Lambda(f,s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(49). Further, based on Eq.(47), we have

$$\Lambda(f,s)\Lambda(\bar{f},s) = e^{A(f)+A(\bar{f})+(B(f)+B(\bar{f})+c)s} \prod_{\rho \in \mathcal{Z}_{\text{real}}} \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right) \tag{50}$$

The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Thus, by Theorem 4, we know that both the real (if exists) and the complex non-trivial zeros of $\Lambda(f,s)$ with $f \neq \overline{f}$ lie on the critical line.

Combining CASE 1 and CASE 2, we conclude that Theorem 7 holds as a specific case of Theorem 4 with $k = k$ and $\lambda = f$. $\square$

3.4. *Automorphic L-Function*

Definition: For an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$, the associated L-function is defined for $\Re(s) > 1$ by:

$$L(\pi,s) = \prod_p L_p(\pi_p,s) \tag{51}$$

where $L_p(\pi_p,s)$ is the local L-factor at the prime p. For unramified π_p with Satake parameters $\{\alpha_{1,p},\ldots,\alpha_{n,p}\}$,

$$L_p(\pi_p,s) = \prod_{i=1}^{n}\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1} \tag{52}$$

Completed L-function: The completed automorphic L-function is defined as:

$$\Lambda(\pi,s) = Q_\pi^{s/2}\prod_{i=1}^{n}\Gamma_*^{(i)}(s+\mu_{i,\pi})\cdot L(\pi,s) \tag{53}$$

where Q_π is the conductor of π, $\mu_{i,\pi}$ are complex numbers determined by the i-th local component of π_∞, and

$$\Gamma_*^{(i)}(s) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), & \text{for real representations.} \\ \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s), & \text{for complex representations.} \end{cases} \tag{54}$$

Functional Equation: The completed automorphic L-function satisfies:

$$\Lambda(\pi,s) = \varepsilon(\pi)\Lambda(\tilde{\pi},1-s) \tag{55}$$

where $\tilde{\pi}$ is the contragredient representation of π and $\varepsilon(\pi)$ is the epsilon factor, a complex number of absolute value 1.

Hadamard Product: For a cuspidal automorphic representation π, the completed L-function $\Lambda(\pi,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(\pi,s) = e^{A(\pi)+B(\pi)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{56}$$

where the product is over all zeros ρ of $\Lambda(\pi,s)$, and $A(\pi)$ and $B(\pi)$ are constants depending on π.

For more details of the completed automorphic L-function, please be referred to references [5,7]

We have the following result about the non-trivial zero distribution of automorphic L-Functions.

Theorem 8: The non-trivial zeros of the above-described automorphic L-Functions lie on the critical line.

Proof. The proof procedures of Theorem 8 is similar to that of Theorem 7 with $k=1$ and f replaced by π, thus details are omitted for simplicity. □

Actually, from the above proofs of Theorem 5, Theorem 6, and Theorem 7, we can note that each proof does not depend on the specific definition of the L-function $L(\lambda,s)$, but rather relies on the following general properties of the corresponding completed L-function $\Lambda(\lambda,s)$:

P1: Symmetric functional equation between $\Lambda(\lambda,s)$ and $\Lambda(\lambda,k-s)$;

P2: Hadamard product expression of entire function $\Lambda(\lambda,s)$ or $s(1-s)\Lambda(\lambda,s)$;

P3: The zeros of $\Lambda(\lambda,s)$ or $\Lambda(\lambda,s)\Lambda(\bar{\lambda},s)$ are exactly the non-trivial zeros of $L(\lambda,s)$;

P4: Zero distribution related items: 1) the concurrence of quadruplets of complex non-trivial zeros with their multiplicities; 2) the property stated in Lemma 5.5; 3) the disjointness of real and complex non-trivial zero sets.

Therefore we conclude that the Grand Riemann Hypothesis holds for all kinds of L-functions satisfying properties **P1**, **P2**, **P3**, and **P4**, i.e., If only the completed L-function $\Lambda(\lambda,s)$ satisfies the requirements of **P1**, **P2**, **P3**, and **P4**, the non-trivial zeros of the corresponding L-Function $L(\lambda,s)$ lie on the critical line.

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