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*Article*

# Further Accurate Numerical Radius Inequalities

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**Abstract:** In this work, new refinements of some numerical radius inequalities are proved. Namely, new improvements and refinements purify the recent inequalities of some famous inequalities concerning the numerical radius of Hilbert space operators. The proven inequalities in this work are not only an improvement over old inequalities, but rather they are stronger than them. Several examples that support the validity of our results are established as well.

**Keywords:** numerical radius; norm; inequalities; Hermite-Hadamard inequality

**MSC:** 47A12; 47A30; 47A63

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## 1. Introduction

Let $\mathscr{A}(\mathscr{J})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{J}; \langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{J}}$ in $\mathscr{A}(\mathscr{J})$. For a bounded linear operator $\mathfrak{F}$ on a Hilbert space $\mathscr{J}$, The numerical range $W(\mathfrak{F})$ of a bounded operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$ is defined by $W(\mathfrak{F}) = \{\langle \mathfrak{F}\mu, \mu\rangle : \mu \in \mathscr{J}, \|\mu\| = 1\}$. Also, the numerical radius is defined to be

$$\omega(\mathfrak{F}) = \sup_{\beta \in W(\mathfrak{F})} |\beta| = \sup_{\|\mu\|=1} |\langle \mathfrak{F}\mu, \mu\rangle| .$$

We recall that the usual operator norm of an operator $\mathfrak{F}$ is defined to be

$$\|\mathfrak{F}\| = \sup\{\|\mathfrak{F}\mu\| : \mu \in \mathscr{J}, \|\mu\| = 1\},$$

It's well known that the numerical radius $\omega(\cdot)$ defines an operator norm on $\mathscr{A}(\mathscr{J})$ which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|\mathfrak{F}\| \leq \omega(\mathfrak{F}) \leq \|\mathfrak{F}\| \tag{1}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$.

In 2003, Kittaneh [1] provided a refinement of the right-hand side of (1), by obtaining that

$$\omega(\mathfrak{F}) \leq \frac{1}{2}\||\mathfrak{F}| + |\mathfrak{F}^*|\| \leq \frac{1}{2}\left(\|\mathfrak{F}\| + \|\mathfrak{F}^2\|^{1/2}\right) \tag{2}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$.

Two years after that, Kittaneh [2] proved his celebrated two-sided inequality

$$\frac{1}{4}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| \leq \omega^2(\mathfrak{F}) \leq \frac{1}{2}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| \tag{3}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. These inequalities are sharp.

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In [3], Dragomir established an upper bound for the numerical radius of the product of two Hilbert space operators, as follows:

$$\omega^r\left(\mathfrak{G}^*\mathfrak{H}\right) \leq \frac{1}{2}\left\||\mathfrak{H}|^{2r}+|\mathfrak{G}|^{2r}\right\| \qquad (r \geq 1). \tag{4}$$

In his recent work [4], Alomari provided a generalized refinement of the right-hand side of (3) and the recent result of Kittaneh and Moradi [5], as follow:

$$\begin{aligned}\omega^{2p}\left(\mathfrak{F}\right) &\leq \frac{1}{4}\delta\left\||\mathfrak{F}|^{2p\delta}+|\mathfrak{F}^*|^{2p(1-\delta)}\right\|^2+\frac{1}{2}\left(1-\delta\right)\omega^p\left(\mathfrak{F}\right)\left\||\mathfrak{F}|^{2p\delta}+|\mathfrak{F}^*|^{2p(1-\delta)}\right\| \\ &\leq \frac{1}{2}\delta\left\||\mathfrak{F}|^{4p\delta}+|\mathfrak{F}^*|^{4p(1-\delta)}\right\|+\frac{1}{2}\left(1-\delta\right)\omega^p\left(\mathfrak{F}\right)\left\||\mathfrak{F}|^{2p\delta}+|\mathfrak{F}^*|^{2p(1-\delta)}\right\| \\ &\leq \frac{1}{2}\left\||\mathfrak{F}|^{4p\delta}+|\mathfrak{F}^*|^{4p(1-\delta)}\right\|\end{aligned} \tag{5}$$

for any operator $\mathfrak{F} \in \mathscr{A}\left(\mathscr{J}\right)$, $p \geq 1$, and $\delta \in [0,1]$. In particular, it was shown that

$$\begin{aligned}\omega^2\left(\mathfrak{F}\right) &\leq \frac{1}{12}\left\||\mathfrak{F}|+|\mathfrak{F}^*|\right\|^2+\frac{1}{3}\omega\left(\mathfrak{F}\right)\left\||\mathfrak{F}|+|\mathfrak{F}^*|\right\| && \text{(Alomari [4])} \\ &\leq \frac{1}{6}\left\||\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right\|+\frac{1}{3}\omega\left(\mathfrak{F}\right)\left\||\mathfrak{F}|+|\mathfrak{F}^*|\right\| && \text{(Kittaneh–Moradi [5])} \\ &\leq \frac{1}{4}\left\||\mathfrak{F}|+|\mathfrak{F}^*|\right\|^2\end{aligned} \tag{6}$$

In the same work [4], a refinement of (4) was proved, as follows:

$$\begin{aligned}\omega^{2r}\left(\mathfrak{G}^*\mathfrak{H}\right) &\leq \frac{1}{4}\delta\left\||\mathfrak{H}|^{2r}+|\mathfrak{G}|^{2r}\right\|^2+\frac{1}{2}\left(1-\delta\right)\omega^r\left(\mathfrak{H}\right)\left\||\mathfrak{H}|^{2r}+|\mathfrak{G}|^{2r}\right\| \\ &\leq \frac{1}{2}\delta\left\||\mathfrak{F}|^{4r}+|\mathfrak{G}|^{4r}\right\|+\frac{1}{2}\left(1-\delta\right)\omega^r\left(\mathfrak{H}\right)\left\||\mathfrak{H}|^{2r}+|\mathfrak{G}|^{2r}\right\| \\ &\leq \frac{1}{2}\left\||\mathfrak{H}|^{4r}+|\mathfrak{G}|^{4r}\right\|.\end{aligned} \tag{7}$$

In particular, it was shown that

$$\begin{aligned}\omega^2\left(\mathfrak{G}^*\mathfrak{H}\right) &\leq \frac{1}{12}\left\||\mathfrak{H}|^2+|\mathfrak{G}|^2\right\|^2+\frac{1}{3}\omega\left(\mathfrak{G}^*\mathfrak{H}\right)\left\||\mathfrak{H}|^2+|\mathfrak{G}|^2\right\| && \text{(Alomari [4])} \\ &\leq \frac{1}{6}\left\||\mathfrak{H}|^4+|\mathfrak{G}|^4\right\|+\frac{1}{3}\omega\left(\mathfrak{G}^*\mathfrak{H}\right)\left\||\mathfrak{H}|^2+|\mathfrak{G}|^2\right\| && \text{(Kittaneh–Moradi [5])} \\ &\leq \frac{1}{2}\left\||\mathfrak{H}|^4+|\mathfrak{G}|^4\right\|.\end{aligned} \tag{8}$$

In [6], Sababheh and Moradi presented some new numerical radius inequalities. Among others, the well-known Hermite–Hadamard inequality was used to perform the following result.

$$\varphi\left(\omega\left(\mathfrak{F}\right)\right) \leq \left\|\int_0^1 \varphi\left(\left(1-s\right)|\mathfrak{F}|+s\,|\mathfrak{F}^*|\right)ds\right\| \leq \frac{1}{2}\left\|\varphi\left(|\mathfrak{F}|\right)+\varphi\left(|\mathfrak{F}^*|\right)\right\| \tag{9}$$

for every $\mathfrak{F} \in \mathscr{A}\left(\mathscr{J}\right)$, and increasing operator convex function $\varphi : [0,\infty) \to [0,\infty)$.

On the other hand, Moradi and Sababheh In [7], proved the following refinement of (9).

$$\varphi\left(\omega\left(\mathfrak{F}\right)\right) \leq \frac{1}{2}\left\|\varphi\left(\frac{3\,|\mathfrak{F}|+|\mathfrak{F}^*|}{4}\right)+\varphi\left(\frac{|\mathfrak{F}|+3\,|\mathfrak{F}^*|}{4}\right)\right\| \tag{10}$$

for all increasing convex function $\varphi : [0,\infty) \to [0,\infty)$. In particular, they proved

$$\omega^2\left(\mathfrak{F}\right) \leq \frac{1}{32}\left\|\left(3\,|\mathfrak{F}|+|\mathfrak{F}^*|\right)^2+\left(|\mathfrak{F}|+3\,|\mathfrak{F}^*|\right)^2\right\|. \tag{11}$$

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The constant $\frac{1}{32}$ is the best possible.

For more generalizations, counterparts, and recent related results, the reader may refer to [13]–[41].

In this work, new refinements of the previously mentioned inequalities are proved. Namely, new improvement and refinements that purifies the inequalities (4)–(11) are established. The proven inequalities in this work are not only an improvement over the previous inequalities, but rather they are stronger than them. We presented examples that prove the validity of our words.

## 2. Refinements of the Numerical Radius Inequalities

**Lemma 1.** *[10, Theorem 1.4] Let $\mathfrak{P} \in \mathscr{A}(\mathscr{J})^{+}$, then*

$$\langle \mathfrak{P}c, c\rangle^{p} \leq \langle \mathfrak{P}^{p}c, c\rangle, \qquad p \geq 1 \tag{12}$$

*for any vector $c \in \mathscr{J}$. The inequality (12) is reversed if $0 \leq p \leq 1$.*

**Lemma 2.** *[11] Let $\mathfrak{G} \in \mathscr{A}(\mathscr{J})$. Then,*

$$|\langle \mathfrak{G}\lambda, \mu\rangle|^{2} \leq \left\langle |\mathfrak{G}|^{2\eta}\lambda, \lambda\right\rangle \left\langle |\mathfrak{G}^{*}|^{2(1-\eta)}\mu, \mu\right\rangle, \qquad 0 \leq \eta \leq 1, \tag{13}$$

*for any vectors $\lambda, \mu \in \mathscr{J}$, where $|\mathfrak{G}| = (\mathfrak{G}^{*}\mathfrak{G})^{1/2}$.*

The following lemma is an operator version of the classical Jensen inequality.

**Lemma 3.** ([10], Theorem 1.2)
*Let $\mathfrak{G}$ be a selfadjoint operator in Let $\mathscr{A}(\mathscr{J})$. Then, whose spectrum $\mathfrak{G} \subset [m, M]$ for some scalars $m \leq M$, and let $\mu \in \mathscr{J}$ be a unit vector. If $f(t)$ is a convex function on $[m, M]$, then*

$$\varphi(\langle \mathfrak{G}\mu, \mu\rangle) \leq \langle \varphi(\mathfrak{G})\mu, \mu\rangle. \tag{14}$$

We are in a position to state our main first result.

**Theorem 1.** *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0, \infty) \to [0, \infty)$ is an increasing and convex, then*

$$\varphi(\omega(\mathfrak{F})) \leq \frac{1}{2}\left\|\varphi\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^{*}|}{3}\right) + \varphi\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^{*}|}{3}\right)\right\| \tag{15}$$

**Proof.** Since $\varphi$ is increasing and operator convex, then by Jensen's inequality we have

$$\begin{aligned}
&\varphi(|\langle \mathfrak{F}\mu, \mu\rangle|) \\
&\leq \varphi\left(\sqrt{\langle |\mathfrak{F}|\mu, \mu\rangle \langle |\mathfrak{F}^{*}|\mu, \mu\rangle}\right) \qquad \text{(by (13))} \\
&\leq \varphi\left(\left\langle \frac{|\mathfrak{F}| + |\mathfrak{F}^{*}|}{2}\mu, \mu\right\rangle\right) \qquad \text{(by (13))} \\
&= \varphi\left(\frac{1}{2}\cdot\left[\left\langle\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle + \left\langle\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle\right]\right) \\
&\leq \frac{1}{2}\left[\varphi\left(\left\langle\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle\right) + \varphi\left(\left\langle\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle\right)\right] \\
&\leq \frac{1}{2}\left[\left\langle\varphi\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle + \left\langle\varphi\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^{*}|}{3}\right)\mu, \mu\right\rangle\right] \\
&= \frac{1}{2}\left\langle\left[\varphi\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^{*}|}{3}\right) + \varphi\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^{*}|}{3}\right)\right]\mu, \mu\right\rangle.
\end{aligned}$$

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Taking the supremum over all unit vector $\mu \in \mathscr{J}$ in all previous inequalities we get the required result. □

**Corollary 1.** *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0, \infty) \to [0, \infty)$ is an increasing and convex, then*

$$\omega^p(\mathfrak{F}) \leq \frac{1}{2} \left\| \left( \frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3} \right)^p + \left( \frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3} \right)^p \right\|$$

*In a particular case,*

$$\omega^2(\mathfrak{F}) \leq \frac{1}{18} \left\| (2|\mathfrak{F}| + |\mathfrak{F}^*|)^2 + (|\mathfrak{F}| + 2|\mathfrak{F}^*|)^2 \right\| \tag{16}$$

*The constant $\frac{1}{18}$ is the best possible.*

**Proof.** Consider $f(s) = s^p$, $s \geq 0$ $(p \geq 1)$ in (15) the we get the desired result. The particular case in (16) follows directly by setting $p = 2$. To prove the sharpness of (16), assume that (16) holds with another constant $c > 0$, i.e.,

$$\omega^2(\mathfrak{F}) \leq c \left\| (2|\mathfrak{F}| + |\mathfrak{F}^*|)^2 + (|\mathfrak{F}| + 2|\mathfrak{F}^*|)^2 \right\|. \tag{17}$$

Assume $\mathfrak{F}$ is a normal operator and employ the fact that for normal operators we have $\omega(\mathfrak{F}) = \|\mathfrak{F}\|$, then by (17), we deduce that $\frac{1}{18} \leq c$, and this shows that the constant $\frac{1}{18}$ is the best possible and thus the inequality is sharp. □

A non-trivial refinement of (15) is considered in the following result.

**Theorem 2.** *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0, \infty) \to [0, \infty)$ is increasing and operator convex, then*

$$\begin{aligned} \varphi(\omega(\mathfrak{F})) &\leq \left\| \int_0^1 \varphi\left( (1-s)\left( \frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3} \right) + s\left( \frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3} \right) \right) ds \right\| \\ &\leq \frac{1}{2} \left\| \varphi\left( \frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3} \right) + \varphi\left( \frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3} \right) \right\| \\ &\leq \left\| \frac{\varphi(|\mathfrak{F}|) + \varphi(|\mathfrak{F}^*|)}{2} \right\|. \end{aligned} \tag{18}$$

**Proof.** Since $\varphi$ is increasing and operator convex, then by Jensen's inequality we have

$$\begin{aligned} &\varphi(|\langle \mathfrak{F}\mu, \mu \rangle|) \\ &\leq \varphi\left( \sqrt{\langle |\mathfrak{F}| \mu, \mu \rangle \langle |\mathfrak{F}^*| \mu, \mu \rangle} \right) \\ &\leq \varphi\left( \left\langle \frac{|\mathfrak{F}| + |\mathfrak{F}^*|}{2} \mu, \mu \right\rangle \right) \\ &= \varphi\left( \frac{1}{2} \cdot \left[ \left\langle \left( \frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3} \right) \mu, \mu \right\rangle + \left\langle \left( \frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3} \right) \mu, \mu \right\rangle \right] \right) \\ &\leq \int_0^1 \varphi\left( (1-s) \left\langle \left( \frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3} \right) \mu, \mu \right\rangle + s \left\langle \left( \frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3} \right) \mu, \mu \right\rangle \right) ds \end{aligned}$$

On the other hand

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$$
\begin{aligned}
&\int_0^1 \varphi\left((1-s)\left\langle\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle + s\left\langle\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right) ds \\
&= \int_0^1 \varphi\left(\left\langle(1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle + \left\langle s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right) ds \\
&\le \int_0^1 \varphi\left(\left\langle(1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right) ds \\
&\le \left\langle\left(\int_0^1 \left((1-s)f\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + sf\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right) ds\right)\mu,\mu\right\rangle \\
&\le \frac{1}{2}\left\langle\left[\varphi\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + \varphi\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right]\mu,\mu\right\rangle \\
&\le \left\langle\frac{\varphi(|\mathfrak{F}|)+\varphi(|\mathfrak{F}^*|)}{2}\mu,\mu\right\rangle.
\end{aligned}
$$

Taking the supremum over all unit vector $\mu \in \mathscr{J}$ in all previous inequalities we get the required result. □

**Corollary 2.** *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Then,*

$$
\begin{aligned}
\omega^p(\mathfrak{F}) &\le \left\| \int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^p ds \right\| \\
&\le \frac{1}{2}\left\| \left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)^p + \left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)^p \right\| \\
&\le \left\| \frac{|\mathfrak{F}|^p + |\mathfrak{F}^*|^p}{2} \right\|.
\end{aligned} \tag{19}
$$

*for all $1 \le p \le 2$.*

**Proof.** The result follows by applying the increasing operator function $\varphi(t) = t^p, 1 \le p \le 2$. □

**Corollary 3.** *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Then,*

$$
\begin{aligned}
\omega^2(\mathfrak{F}) &\le \left\| \int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^2 ds \right\| \\
&\le \frac{1}{18}\left\| (2|\mathfrak{F}|+|\mathfrak{F}^*|)^2 + (|\mathfrak{F}|+2|\mathfrak{F}^*|)^2 \right\| \\
&\le \frac{1}{2}\left\| |\mathfrak{F}|^2 + |\mathfrak{F}^*|^2 \right\|.
\end{aligned} \tag{20}
$$

**Example 1.** *Let $\mathfrak{F} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. It is easy to observe that $\omega(\mathfrak{F}) = 1.5$. Applying the inequalities in (20), we get*

$$
\begin{aligned}
2.25 = \omega^2(\mathfrak{F}) &\le \left\| \int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^2 ds \right\| = 2.25 \\
&\le \frac{1}{18}\left\| (2|\mathfrak{F}|+|\mathfrak{F}^*|)^2 + (|\mathfrak{F}|+2|\mathfrak{F}^*|)^2 \right\| = 2.27 \\
&\le \frac{1}{2}\left\| |\mathfrak{F}|^2 + |\mathfrak{F}^*|^2 \right\| = 2.5
\end{aligned}
$$

*As we can see the first inequality turns to an equality for this example; that gives the exact value of the numerical radius. Moreover, the second inequality improves Sabaheh-Mordai inequality (11). Roughly, we have*

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$$2.25 = \omega^2(\mathfrak{F}) \leq \frac{1}{32}\left\|(3|\mathfrak{F}| + |\mathfrak{F}^*|)^2 + (|\mathfrak{F}| + 3|\mathfrak{F}^*|)^2\right\| = 2.3125$$

*and this show that our first two inequalities are much better than (11). Practically and more preciously, the first two inequalities in (20) are stronger than the lower bound in (3), and the inequalities in (9), (10), and (11).*

**Theorem 3.** *Let $\mathfrak{H}, \mathfrak{G} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing and operator convex, then*

$$\begin{aligned}\varphi(\omega(\mathfrak{G}^*\mathfrak{H})) &\leq \left\|\int_0^1 \varphi\left((1-s)\left(\frac{2|\mathfrak{H}|^2 + |\mathfrak{G}|^2}{3}\right) + s\left(\frac{|\mathfrak{H}|^2 + 2|\mathfrak{G}|^2}{3}\right)\right) ds\right\| \\ &\leq \frac{1}{2}\left\|\varphi\left(\frac{2|\mathfrak{H}|^2 + |\mathfrak{G}|^2}{3}\right) + \varphi\left(\frac{|\mathfrak{H}|^2 + 2|\mathfrak{G}|^2}{3}\right)\right\| \\ &\leq \left\|\frac{\varphi\left(|\mathfrak{H}|^2\right) + \varphi\left(|\mathfrak{G}|^2\right)}{2}\right\|.\end{aligned} \tag{21}$$

**Proof.** Let $\mu \in \mathscr{J}$ be a unit vector. Then by Cauchy–Schwarz inequality we have

$$\begin{aligned}\varphi(|\langle\mathfrak{G}^*\mathfrak{H}\mu, \mu\rangle|) = \varphi(|\langle\mathfrak{H}\mu, \mathfrak{G}\mu\rangle|) &\leq \varphi(\|\mathfrak{H}\mu\|\,\|\mathfrak{G}\mu\|) \\ &= \varphi\left(\left\langle|\mathfrak{H}|^2\mu, \mu\right\rangle^{\frac{1}{2}}\left\langle|\mathfrak{G}|^2\mu, \mu\right\rangle^{\frac{1}{2}}\right) \\ &\leq \varphi\left(\frac{\left\langle|\mathfrak{H}|^2\mu, \mu\right\rangle + \left\langle|\mathfrak{G}|^2\mu, \mu\right\rangle}{2}\right).\end{aligned}$$

The rest of the proof goes similar to that one given for the proof of Theorem 1; by replacing $|\mathfrak{F}|$ and $|\mathfrak{F}^*|$ by $|\mathfrak{H}|^2$ and $|\mathfrak{G}|^2$, respectively; we get the required result. □

We finish this work by introducing some refined improvements of numerical radius inequalities. Among others, Sababheh–Moradi in [6] and [7], presented some new general forms of numerical radius inequalities for Hilbert space operators. In fact, Sababheh and Moradi used the classical Hermite–Hadamard inequality and its operator version to prove their results. Our approach is based on refining and extending these inequalities in the lighting of Alomari refinement extension of the Hermite–Hadamard inequality [16].

**Theorem 4.** *Let $\Psi : \mathscr{A}(\mathscr{J}) \to \mathscr{A}(\mathscr{R})$ be a positive unital linear map and let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing and convex function, then*

$$\begin{aligned}&\varphi\left(\omega^2(\Psi(\mathfrak{F}))\right) \\ &\leq \varphi\left(\frac{1}{2}\left\|\Psi\left(|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2\right)\right\|\right) \\ &\leq \frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{3|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2}{4}\right)\right\|\right) + \varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2 + 3|\mathfrak{F}^*|^2}{4}\right)\right\|\right)\right] \\ &\leq \sup_{\substack{\mu\in\mathscr{J} \\ \|\mu\|=1}} \int_0^1 \varphi\left(\left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2 + t|\mathfrak{F}^*|^2\right)\mu\right\|^2\right) dt \\ &\leq \frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2}{2}\right)\right\|\right) + \frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right) + \varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|\right] \\ &\leq \frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right) + \varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|\end{aligned} \tag{22}$$

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*for any Unit vector $\mu \in \mathscr{J}$.*

**Proof.** In [16], Alomari proved the following refinement of the classical Hermite–Hadamard inequality

$$\begin{aligned}\frac{(b-a)}{2}\left[g\left(\frac{3a+b}{4}\right)+g\left(\frac{a+3b}{4}\right)\right] &\leq \int_a^b g(t)\,dt \\ &\leq \frac{(b-a)}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right].\end{aligned} \tag{23}$$

for every convex function $g:[a,b]\to\mathbb{R}$. Moreover, since $g$ is convex, then we may rewrite (23), as follows

$$\begin{aligned} g\left(\frac{a+b}{2}\right)=g\left(\frac{1}{2}\left[\frac{3a+b}{4}+\frac{a+3b}{4}\right]\right) &\leq \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right)+g\left(\frac{a+3b}{4}\right)\right] \\ &\leq \int_0^1 g((1-t)a+tb)\,dt \\ &\leq \frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+\varphi(b)}{2}\right] \\ &\leq \frac{g(a)+g(b)}{2} \qquad (g \text{ is convex})\end{aligned} \tag{24}$$

Let $\mathfrak{F}=K+iL$ be the Cartesian decomposition of $\mathfrak{F}\in\mathscr{A}(\mathscr{J})$. Therefore, we have

$$|\mathfrak{F}|^2+|\mathfrak{F}^*|^2=\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*=2\left(K^2+L^2\right) \tag{25}$$

and

$$|\langle\mathfrak{F}\mu,\mu\rangle|^2=\langle K\mu,\mu\rangle+\langle L\mu,\mu\rangle, \qquad \forall\mu\in\mathscr{J}. \tag{26}$$

Replacing $a$ and $b$ by $\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle$ and $\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle$ in (24), for $\mu\in\mathscr{J}$ such that $\|\mu\|=1$, in (4) we obtain

$$\begin{aligned}&\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)\\ &\leq\frac{1}{2}\left[\varphi\left(\frac{3\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{4}\right)+\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+3\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{4}\right)\right]\\ &\leq\int_0^1\varphi\left(\left\langle\Psi\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)dt\\ &\leq\frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\right]\\ &\leq\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\end{aligned}$$

But since $\varphi$ is convex and $\Psi$ is a positive unital linear map, then the last two inequalities could be refined respectively as follows:

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$$\frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\right]$$
$$\leq\frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)\right)\mu,\mu\right\rangle+\left\langle\Psi\left(\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle}{2}\right]$$

and

$$\begin{aligned}&\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\\&\leq\frac{\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)\right)\mu,\mu\right\rangle+\left\langle\Psi\left(\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle}{2}\\&=\frac{1}{2}\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle.\end{aligned}$$

Combining the above two inequalities together we obtain

$$\begin{aligned}&\sup_{\substack{\mu\in\mathscr{J}\\\|\mu\|=1}}\int_0^1\varphi\left(\left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu\right\|^2\right)dt\\&\leq\frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2+|\mathfrak{F}^*|^2}{2}\right)\right\|\right)+\frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|\right]\\&\leq\frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|.\end{aligned}$$

Now, since $\varphi$ is increasing then we have

$$\begin{aligned}\varphi\left(|\langle\Psi(\mathfrak{F})\mu,\mu\rangle|^2\right)&=\varphi\left(\langle\Psi(K)\mu,\mu\rangle^2+\langle\Psi(L)\mu,\mu\rangle^2\right)\\&\leq\varphi\left(\left\langle\Psi^2(K)\mu,\mu\right\rangle+\left\langle\Psi^2(L)\mu,\mu\right\rangle\right)\\&=\varphi\left(\left\langle\Psi\left(K^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(L^2\right)\mu,\mu\right\rangle\right)\\&=\varphi\left(\left\langle\Psi\left(K^2+L^2\right)\mu,\mu\right\rangle\right)\\&=\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)\\&=\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)\end{aligned}$$

Taking the supremum over all unit vector $\mu\in\mathscr{J}$ in all previous inequalities we get the required result. □

The following example ensures that the inequalities in (22) refine Sababheh–Moradi inequality [6, Theorem 2.2].

**Example 2.** *Let* $\mathfrak{F}=\begin{bmatrix}4&2\\0&1\end{bmatrix}$. *Let* $\varphi$ *be a function defined by* $\varphi(t)=t^2$, $t\in[0,\infty)$. *Define the unital positive linear map* $\Psi:\mathfrak{M}_2(\mathbb{C})\to\mathfrak{M}_2(\mathbb{C})$ *be defined by* $\Psi(\mathfrak{F})=\frac{1}{2}(\operatorname{tr}(\mathfrak{F}))I$, *for all matrices* $\mathfrak{F}\in\mathfrak{M}_2(\mathbb{C})$.

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$$
\begin{aligned}
&\omega^4(\Psi(\mathfrak{F})) = 39.0625 \\
&\le \frac{1}{4}\left\|\Psi\left(|\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right)\right\|^2 = 56.25 \\
&\le \frac{1}{2}\left[\left\|\Psi\left(\frac{3|\mathfrak{F}|^2+|\mathfrak{F}^*|^2}{4}\right)\right\|^2+\left\|\Psi\left(\frac{|\mathfrak{F}|^2+3|\mathfrak{F}^*|^2}{4}\right)\right\|^2\right] = 175.78 \\
&\le \sup_{\substack{\mu\in\mathscr{J}\\ \|\mu\|=1}} \int_0^1 \left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu\right\|^4 dt \\
&\le \frac{1}{8}\left\|\Psi\left(|\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right)\right\|^2+\frac{1}{4}\left\|\Psi\left(|\mathfrak{F}|^4+|\mathfrak{F}^*|^4\right)\right\| = 251.125 \\
&\le \frac{1}{2}\left\|\Psi\left(|\mathfrak{F}|^4+|\mathfrak{F}^*|^4\right)\right\| = 446.
\end{aligned}
$$

The following result gives an alternative extensive proof of [6, Theorem 2.2]. The approach presented in the proof is completely different and motivated by the concept of the Cartesian decomposition of an arbitrary Hilbert space operator. At the same time, a chain of inequalities improves the result in [6] and refines the lower bound of the celebrated Kittaneh inequality (3).

**Theorem 5.** *Let $K+iL$ be the Cartesian decomposition of an operator $\mathfrak{F}\in\mathscr{A}(\mathscr{J})$. If $\varphi:[0,\infty)\to[0,\infty)$ is a non-negative increasing operator convex function, then the following chain of inequalities*

$$
\begin{aligned}
\varphi\left(\omega^2(S)\right) &\ge \frac{\varphi\left(\|K\|^2\right)+\varphi\left(\|L\|^2\right)}{2} \\
&\ge \frac{\varphi\left(\|K^2\|\right)+\varphi\left(\|L^2\|\right)}{2} \\
&\ge \int_0^1 \left\|\delta\varphi\left(K^2\right)+(1-\delta)\varphi\left(L^2\right)\right\| d\delta \\
&\ge \int_0^1 \left\|\varphi\left(\delta K^2+(1-\delta)L^2\right)\right\| d\delta \\
&\ge \left\|\int_0^1 \varphi\left(\delta K^2+(1-\delta)L^2\right) d\delta\right\| \\
&\ge \left\|\varphi\left(\frac{S^*S+SS^*}{4}\right)\right\|,
\end{aligned}
\tag{27}
$$

*are hold.*

**Proof.** Since $\mathfrak{F}=K+iL$, then we have

$$|\langle\mathfrak{F}\mu,\mu\rangle|^2 = \langle K\mu,\mu\rangle^2+\langle L\mu,\mu\rangle^2, \qquad \mu\in\mathscr{J}.$$

The monotonicity of $\varphi$ and the above identity imply that

$$\delta\varphi\left(|\langle\mathfrak{F}\mu,\mu\rangle|^2\right) \ge \delta\varphi\left(\langle K\mu,\mu\rangle^2\right),$$

and

$$(1-\delta)\varphi\left(|\langle\mathfrak{F}\mu,\mu\rangle|^2\right) \ge (1-\delta)\varphi\left(\langle L\mu,\mu\rangle^2\right)$$

for all $\delta\in[0,1]$. Therefore,

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$$\begin{aligned}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) &= \delta\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) + (1-\delta)\,\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) \\ &\geq \delta\varphi\left(\langle K\mu,\mu\rangle^2\right) + (1-\delta)\,\varphi\left(\langle L\mu,\mu\rangle^2\right).\end{aligned}$$

Taking the supremum over all unit vector $\mu \in \mathscr{J}$, since $\varphi$ is increasing we get

$$\begin{aligned}\varphi\left(\omega^2(\mathfrak{F})\right) &\geq \delta\varphi\left(\|K\|^2\right) + (1-\delta)\,\varphi\left(\|L\|^2\right) \\ &\geq \delta\varphi\left(\left\|K^2\right\|\right) + (1-\delta)\,\varphi\left(\left\|L^2\right\|\right) && (\text{since } \|\mathfrak{F}^2\| \leq \|\mathfrak{F}\|^2, \text{ for all } \mathfrak{F} \in \mathscr{A}(\mathscr{J})) \\ &= \delta\left\|\varphi\left(K^2\right)\right\| + (1-\delta)\left\|\varphi\left(L^2\right)\right\| && (\text{since } \varphi(\|\mathfrak{F}\|) = \|\varphi(|\mathfrak{F}|)\|) \\ &\geq \left\|\delta\varphi\left(K^2\right) + (1-\delta)\,\varphi\left(L^2\right)\right\| && (\text{by triangle inequality}) \\ &\geq \left\|\varphi\left(\delta K^2 + (1-\delta)\,L^2\right)\right\| && (\varphi \text{ is operator convex})\end{aligned}$$

Integrating with respect to $\delta$ over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\omega^2(\mathfrak{F})\right) &\geq \frac{\varphi\left(\|K\|^2\right) + \varphi\left(\|L\|^2\right)}{2} \\ &\geq \frac{\varphi\left(\|K^2\|\right) + \varphi\left(\|L^2\|\right)}{2} \\ &= \frac{\|\varphi(K^2)\| + \|\varphi(L^2)\|}{2} \\ &\geq \int_0^1 \left\|\delta\varphi\left(K^2\right) + (1-\delta)\,\varphi\left(L^2\right)\right\| d\delta \\ &\geq \int_0^1 \left\|\varphi\left(\delta K^2 + (1-\delta)\,L^2\right)\right\| d\delta \\ &\geq \left\|\int_0^1 \varphi\left(\delta K^2 + (1-\delta)\,L^2\right) d\delta\right\| && (\text{by triangle inequality}) \\ &\geq \left\|\varphi\left(\frac{K^2+L^2}{2}\right)\right\| && (\varphi \text{ is operator convex}) \\ &= \left\|\varphi\left(\frac{\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*}{4}\right)\right\|,\end{aligned}$$

and this proves the required result. □

The following result refines (27) and gives a better estimate of the numerical radius.

**Theorem 6.** *Let $K + iL$ be the Cartesian decomposition of an operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0,\infty) \to [0,\infty)$ is non-negative increasing operator convex function, then*

$$\varphi\left(\omega^2(\mathfrak{F})\right) \geq \frac{r}{r+s}\varphi\left(\|K\|^2\right) + \frac{s}{r+s}\varphi\left(\|L\|^2\right) \geq \left\|\varphi\left(\frac{rK^2+sL^2}{r+s}\right)\right\| \tag{28}$$

*for all real numbers $r, s > 0$.*

**Proof.** Our proof is similar to that one presented in the proof of Theorem 5. Let $r, s > 0$, since $\mathfrak{F} = K + iL$, then we have

$$|\langle \mathfrak{F}\mu,\mu\rangle|^2 = \langle K\mu,\mu\rangle^2 + \langle L\mu,\mu\rangle^2, \qquad \mu \in \mathscr{J}.$$

The monotonicity of $\varphi$ and the above identity imply that

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$$\frac{r}{r+s}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) \geq \frac{r}{r+s}\varphi\left(\langle K\mu,\mu\rangle^2\right)$$

and

$$\frac{s}{r+s}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) \geq \frac{s}{r+s}\varphi\left(\langle L\mu,\mu\rangle^2\right)$$

for all positive real numbers $r, s > 0$. Therefore,

$$\begin{aligned}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) &= \frac{r}{r+s}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) + \frac{s}{r+s}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) \\ &\geq \frac{r}{r+s}\varphi\left(\langle K\mu,\mu\rangle^2\right) + \frac{s}{r+s}\varphi\left(\langle L\mu,\mu\rangle^2\right).\end{aligned}$$

Taking the supremum over all unit vector $\mu \in \mathscr{J}$, since $\varphi$ is increasing we get

$$\begin{aligned}\varphi\left(\omega^2(\mathfrak{F})\right) &\geq \frac{r}{r+s}\varphi\left(\|K\|^2\right) + \frac{s}{r+s}\varphi\left(\|L\|^2\right) \\ &\geq \frac{r}{r+s}\varphi\left(\left\|K^2\right\|\right) + \frac{s}{r+s}\varphi\left(\left\|L^2\right\|\right) && (\text{since } \|\mathfrak{F}^2\| \leq \|\mathfrak{F}\|^2, \text{ for all } \mathfrak{F} \in \mathscr{A}(\mathscr{J})) \\ &= \frac{r}{r+s}\left\|\varphi\left(K^2\right)\right\| + \frac{s}{r+s}\left\|\varphi\left(L^2\right)\right\| && (\text{since } \varphi(\|\mathfrak{F}\|) = \|\varphi(|\mathfrak{F}|)\|) \\ &\geq \left\|\frac{r}{r+s}\varphi\left(K^2\right) + \frac{s}{r+s}\varphi\left(L^2\right)\right\| && (\text{by triangle inequality}) \\ &\geq \left\|\varphi\left(\frac{rK^2+sL^2}{r+s}\right)\right\| && (\varphi \text{ is operator convex})\end{aligned}$$

which yields the desired result. □

**Example 3.** *Consider* $\mathfrak{F} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. *It is easy to observe that* $\omega(\mathfrak{F}) = 6$. *Define the function* $f(t) = t^2$ $(t > 0)$. *So, by applying the first inequality in* (28) *(which is the same result given in [6, Theorem 2.2]) gives that* $\omega(\mathfrak{F}) \geq 5.04635$ *(the case when* $r = s = 1$*)*

$$\omega(\mathfrak{F}) = 6 \geq \begin{cases} 5.04635, & \text{if } r = 1,\ s = 1; \\ 5.42213, & \text{if } r = 2,\ s = 1; \\ 5.84414, & \text{if } r = 9,\ s = 1; \\ 5.97039, & \text{if } r = 50,\ s = 1; \\ 5.99850, & \text{if } r = 1000,\ s = 1. \end{cases}$$

*While selecting various values for r and s yields better estimations. Indeed, in this example; as the value of r is greater than s we get a better estimation (lower bound) and this improves Mordai–Sabaheh's inequality (the case when* $r = s = 1$*, above). In general, once the values of* $\|K\|$ *and r are large (small) enough and the values of* $\|L\|$ *and s are small (large) enough we get better estimation, and vice versa. Based on that, it is convenient to note that* (30) *always gives a better lower bound.*

In [7], Moradi and Sabaheh used the interesting inequality

$$\left(\frac{\mathfrak{H}+\mathfrak{G}}{2}\right)^2 \leq \left(\frac{\mathfrak{H}+\mathfrak{G}}{2}\right)^2 + \left(\frac{|\mathfrak{H}-\mathfrak{G}|}{2}\right)^2 = \frac{\mathfrak{H}^2+\mathfrak{G}^2}{2} \tag{29}$$

for every selfadjoint operators $\mathfrak{H}, \mathfrak{G} \in \mathscr{A}(\mathscr{J})$, to prove the following refinement of the left-hand-side of (3), as follows:

$$\frac{1}{4}\left\|\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*\right\| \leq \frac{1}{4}\left\|\left(\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*\right)^2+\left|\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2\right|^2\right\|^{\frac{1}{2}} \leq \omega^2\left(\mathfrak{F}\right). \tag{30}$$

By recalling the original result in [7], an interesting improvement of (30) holds. Namely, we have

$$\begin{aligned}\frac{1}{4}\left\|\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*\right\| &\leq \frac{1}{4}\left\|\left(\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*\right)^2+\left(\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2\right)^2\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{2}}\left(\left\|\mathfrak{F}+\mathfrak{F}^*\right\|^4+\left\|\mathfrak{F}-\mathfrak{F}^*\right\|^4\right)^{\frac{1}{2}} \\ &\leq \omega^2\left(\mathfrak{F}\right)\end{aligned} \tag{31}$$

The next result extends and refines the inequality (31) as follows:

**Theorem 7.** *Let $K+iL$ be the Cartesian decomposition of $\mathfrak{F}\in\mathscr{A}\left(\mathscr{J}\right)$. Then,*

$$\begin{aligned}&\frac{1}{4}\left\|\left(\frac{r-s}{r+s}\right)\cdot\left(\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2\right)+\left(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}\right)\right\| \\ &\leq \frac{1}{4}\left\|\left[\left(\frac{r-s}{r+s}\right)\cdot\left(\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2\right)+\left(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}\right)\right]^2\right. \\ &\qquad\left.+\left[\left(\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2\right)+\left(\frac{r-s}{r+s}\right)\cdot\left(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}\right)\right]^2\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2\sqrt{2}}\cdot\left(\frac{r^2\left\|\mathfrak{F}+\mathfrak{F}^*\right\|^4+s^2\left\|\mathfrak{F}-\mathfrak{F}^*\right\|^4}{(r+s)^2}\right)^{\frac{1}{2}} \\ &\leq \omega^2\left(\mathfrak{F}\right)\end{aligned} \tag{32}$$

*for all $\delta\in[0,1]$ and all positive real numbers $r,s>0$.*

**Proof.** Since $K+iL$ be the Cartesian decomposition of $\mathfrak{F}$. Then

$$\frac{rK^2+sL^2}{r+s}=\left(\frac{r-s}{r+s}\right)\cdot\frac{\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2}{4}+\frac{\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}}{4}$$

and

$$\frac{rK^2-sL^2}{r+s}=\frac{\mathfrak{F}^2+\left(\mathfrak{F}^*\right)^2}{4}+\left(\frac{r-s}{r+s}\right)\cdot\frac{\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}}{4}$$

Replacing $\mathfrak{H}$ and $\mathfrak{G}$ by $\frac{2r}{r+s}K^2$ and $\frac{2s}{r+s}L^2$ ($\forall r,s>0$), respectively, in (29), we get

$$\begin{aligned}\left(\frac{rK^2+sL^2}{r+s}\right)^2 &\leq \left(\frac{rK^2+sL^2}{r+s}\right)^2+\left(\frac{\left|rK^2-sL^2\right|}{r+s}\right)^2 \\ &= \frac{2r^2K^4+2s^2L^4}{(r+s)^2}\end{aligned}$$

Consequently,

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$$
\begin{aligned}
\left\|\left(\frac{r-s}{r+s}\right)\cdot\frac{\mathfrak{F}^2+(\mathfrak{F}^*)^2}{4}+\frac{\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F}}{4}\right\|^2 &= \left\|\frac{rK^2+sL^2}{r+s}\right\|^2\\
&= \left\|\left(\frac{rK^2+sL^2}{r+s}\right)^2\right\|\\
&\le \left\|\left(\frac{rK^2+sL^2}{r+s}\right)^2+\left(\frac{|rK^2-sL^2|}{r+s}\right)^2\right\|\\
&= \left\|\frac{2r^2K^4+2s^2L^4}{(r+s)^2}\right\|\\
&\le \frac{2r^2\|K\|^4+2s^2\|L\|^4}{(r+s)^2}\\
&\le \omega^4(\mathfrak{F}),
\end{aligned}
$$

which gives the desired result in (32). □

**Remark 1.** *In particular, choosing $r=s=1$ in (32), then we refer to (31).*

**Remark 2.** *In spite of that, (32) still can give a better estimation that (31). By choosing specific values for $r$ and $s$ we would then get a better lower bound. To check that consider the same example considered in Example 3. We left the investigation of this note to the interested reader. Nevertheless, once the values of $\|K\|$ and $r$ are large (small) enough and the values of $\|L\|$ and $s$ are small (large) enough we get a better estimation than (31).*

## 3. Conclusion

In this work, more accurate numerical radius inequalities refine several well-known and sharp inequalities obtained in the literature. Namely, as it is shown the inequality (12) refines Sababheh–Moradi inequality (9). In fact, (16) is sharper than both (14) and (11). An alternative extensive proof of [6, Theorem 2.2] is provided as well. Among other inequalities, two interesting new results are established. Namely, it is shown that

$$
\varphi\left(\omega^2(\mathfrak{F})\right)\ge\frac{r}{r+s}\varphi\left(\|K\|^2\right)+\frac{s}{r+s}\varphi\left(\|L\|^2\right)\ge\left\|\varphi\left(\frac{rK^2+sL^2}{r+s}\right)\right\|
$$

for every increasing operator convex function $\varphi$ and all real numbers $r,s>0$. Also,

$$
\begin{aligned}
&\frac{1}{4}\left\|\left(\frac{r-s}{r+s}\right)\cdot\left(\mathfrak{F}^2+(\mathfrak{F}^*)^2\right)+(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F})\right\|\\
&\le\frac{1}{4}\left\|\left[\left(\frac{r-s}{r+s}\right)\cdot\left(\mathfrak{F}^2+(\mathfrak{F}^*)^2\right)+(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F})\right]^2\right.\\
&\qquad\left.+\left[\left(\mathfrak{F}^2+(\mathfrak{F}^*)^2\right)+\left(\frac{r-s}{r+s}\right)\cdot(\mathfrak{F}\mathfrak{F}^*+\mathfrak{F}^*\mathfrak{F})\right]^2\right\|^{\frac{1}{2}}\\
&\le\frac{1}{2\sqrt{2}}\cdot\left(\frac{r^2\|\mathfrak{F}+\mathfrak{F}^*\|^4+s^2\|\mathfrak{F}-\mathfrak{F}^*\|^4}{(r+s)^2}\right)^{\frac{1}{2}}\\
&\le\omega^2(\mathfrak{F}).
\end{aligned}
$$

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