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A third order Newton-like Method and its applications

D. R. Sahu*[a], R. P. Agarwal†[b], Y. J. Cho‡[c,d] and V. K. Singh§[a]

[a] Department of Mathematics, Banaras Hindu University,
Varanasi-221005, India

[b]Department of Mathematics, Texas A&M University-Kingsville,
Kingsville, Texas 78363-8202, USA

[c]School of Mathematical Sciences,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 611731, P. R. China

[d]Department of Mathematics Education,
Gyeongsang National University, Jinju 660–701, Korea

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Abstract

In this paper, we study the third order semilocal convergence of the Newton-like method for finding the approximate solution of nonlinear operator equations in the setting of Banach spaces. First, we discuss the convergence analysis under ω-continuity condition, which is weaker than the Lipschitz and Hölder continuity conditions. Second, we apply our approach to solve Fredholm integral equations, where the first derivative of involved operator not necessarily satisfy the Hölder and Lipschitz continuity conditions. Finally, we also prove that the R-order of the method is $2p+1$ for any $p \in (0, 1]$.

Keywords: Nonlinear operator equations, Fréchet derivative, ω-continuity condition, the Newton like method, Frédholm integral equation.

*drsahudr@gmail.com
†Ravi.Agarwal@tamuk.edu
‡yjcho@gnu.ac.kr
§vipinkumarsingh666@gmail.com

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1 Introduction

Our main purpose of this paper is to compute solution of nonlinear operator equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : D \subset X \to Y$ is a nonlinear operator defined on an open convex subset D of a Banach space X with values into a Banach space Y.

A lot of challenging problems in physics, numerical analysis, engineering and applied mathematics are formulated in terms of finding roots of the equation of the form (1.1). In order to solve such problems, we often use iterative methods. There are many iterative methods available in literature. One of the central method for solving such problems is the Newton method [5,6] defined by

$$x_{n+1} = x_n - F'^{-1}_{x_n} F(x_n) \tag{1.2}$$

for each $n \geq 0$, where F'_x denotes the Fréchet derivative of F at point $x \in D$.

The Newton method and the Newton-like methods are attractive because it converges rapidly from any sufficient initial guess. A number of researchers [7,10–25] have generalized and established local as well as semilocal convergence analysis of the Newton method (1.2) under the following conditions:

(a) *Lipschitz condition*: $\|F'_x - F'_y\| \leq K\|x - y\|$ for all $x, y \in D$ and for some $K > 0$;

(b) *Hölder Lipschitz condition*: $\|F'_x - F'_y\| \leq K\|x - y\|^p$ for all $x, y \in D$ and for some $p \in (0, 1]$ and $K > 0$;

(c) *ω-continuity condition*: $\|F'_x - F'_y\| \leq \omega(\|x - y\|)$ for all $x, y \in D$ and for some function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$.

On the other hand, many mathematical problems such as differential equations, integral equations, economics theory, game theory and variational inequalities can be formulated into the fixed point problem [3,30]:

$$\text{Find } x \in C \text{ such that } x = G(x), \tag{1.3}$$

where $G : C \to X$ is a nonlinear operator defined on a nonempty subset C of a Banach space X. The easiest iterative method for constructing a sequence is Picard iterative method [27] which is given by

$$x_{n+1} = G(x_n) \tag{1.4}$$

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for each $n \geq 0$. The Banach contraction principle (see [3–5, 30]) provides sufficient conditions for the convergence of the iterative method (1.4) to the fixed point of G. More details for approximation of fixed points of nonlinear operators can be found in [31,33,34].

The Newton method and its variant [8,9] are also used to solve the fixed point problem of the form:

$$(I - G)(x) = 0, \tag{1.5}$$

where I is the identity operator defined on X and $G : D \subset X \to X$ is a nonlinear Fréchet differentiable operator defined on an open convex subset D of a Banach space X. For finding approximate solution of the equation (1.5), Bartle [28] used the Newton-like iterative method of the form

$$x_{n+1} = x_n - (I - G'_{y_n})^{-1}(I - G(x_n)) \tag{1.6}$$

for each $n \geq 0$, where G'_x is Fréchet derivative of G at point $x \in D$ and $\{y_n\}$ is the sequence of arbitrary points in D which are sufficiently closed to the desired solution of the equation (1.5). Bartle [28] has discussed the convergence analysis of the form (1.6) under the assumption that G is Fréchet differentiable at least at desired points and a modulus of continuity is known for G' as a function of x. The Newton method (1.2) and the modified Newton method are the special cases of the form (1.6).

Following the idea of Bartle [28], Rall [29] introduced the following Stirling's method for finding a solution of the fixed point problem (1.5):

$$\begin{cases} y_n = G(x_n), \\ x_{n+1} = x_n - (I - G'_{y_n})^{-1}(x_n - G(x_n)) \end{cases} \tag{1.7}$$

for each $n \geq 0$.

Recently, Parhi and Gupta [1,2] have discussed the semilocal convergence analysis of the following Stirling-like iterative method for computing a solution of the equation (1.5):

$$\begin{cases} z_n = G(x_n), \\ y_n = x_n - (I - G'_{z_n})^{-1}(x_n - G(x_n)), \\ x_{n+1} = y_n - (I - G'_{z_n})^{-1}(y_n - G(y_n)) \end{cases} \tag{1.8}$$

for each $n \geq 0$.

The purpose of this paper is to introduce the Newton-like method for solving the operator equation (1.1) and discuss its convergence analysis under the $\omega-$continuity condition in the sense of Parhi and Gupta [1, 2]. Note that the $\omega-$continuity condition in

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the sense of Parhi and Gupta [1, 2] is weaker than the Hölder and Lipschitz continuity conditions. Our iterative method covers a wide variety of iterative methods and so our results generalize the results of Parhi and Gupta [1,2]. Finally, we apply our approach to solve the Fredholm integral equation, where the first derivative of involved operator not necessarily satisfy the Hölder and Lipschitz continuity conditions.

2 Preliminary

In this section, we discuss some technical results. Throughout the paper, we denote $B(X,Y)$ a collection of bounded linear operators from a Banach space X into a Banach space Y and $B(X) = B(X,X)$. For some $r > 0$, $B_r[x]$ and $B_r(x)$ are the closed and open balls with center x and radius r, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Φ denote the collection of nonnegative, nondecreasing, continuous real valued functions defined on $[0,\infty)$.

Lemma 2.1 (Rall [26], Page 50) *Let $\mathcal{L}$ be a bounded linear operator on a Banach space X. Then $\mathcal{L}^{-1}$ exists if and only if there is a bounded linear operator M in X such that M^{-1} exists and*

$$\|M - \mathcal{L}\| \quad < \quad \frac{1}{\|M^{-1}\|}.$$

If $\mathcal{L}^{-1}$ exists, then

$$\left\|\mathcal{L}^{-1}\right\| \leq \frac{\|M^{-1}\|}{1 - \|1 - M^{-1}\mathcal{L}\|} \leq \frac{\|M^{-1}\|}{1 - \|M^{-1}\|\,\|M - \mathcal{L}\|}.$$

Lemma 2.2 *Let $0 < k \leq \frac{1}{3}$ be a real number. Assume that $q = \frac{1}{p+1} + k^p$ for any $p \in (0,1]$ and the scalar equation*

$$(1 - k^p(1+qt)^p t)^{p+1} - \left(\frac{q^p t^p}{p+1} + k^p\right)^p q^p t^{2p} = 0$$

has a minimum positive root α. Then we have the following:

(1) *$q > k$ for all $p \in (0,1]$.*

(2) *$\alpha \in (0,1)$.*

Proof. (1) This part is obvious. Indeed, we have

$$\frac{1}{p+1} + k^p - \frac{1}{3} = \frac{2-p}{3(1+p)} + k^p > 0$$

for all $p \in (0,1]$ and $0 < k \le \frac{1}{3}$.

(2) Set

$$g(t) = (1 - k^p(1+qt)^p t)^{p+1} - \left(\frac{q^p t^p}{p+1} + k^p\right)^p q^p t^{2p}. \tag{2.1}$$

It is clear from the definition of $g(t)$ that $g(0) > 0, g(1) < 0$ and $g'(t) < 0$ in $(0,1)$. Therefore, $g(t)$ is decreasing in $(0,1)$ and hence the equation (2.1) has a minimum positive root $\alpha \in (0,1)$. This completes the proof.

Lemma 2.3 *Let $b_0 \in (0,\alpha)$ be a number such that $k^p(1+qb_0)^p b_0 < 1$, where k, p, α and q are same as in Lemma 2.2. Define the real sequences $\{b_n\}$, $\{\theta_n\}$ and $\{\gamma_n\}$ by*

$$b_{n+1} = \frac{\left(\frac{q^p b_n^p}{p+1} + k^p\right)^p q^p b_n^{2p}}{(1 - k^p(1+qb_n)^p b_n)^{p+1}} b_n, \tag{2.2}$$

$$\theta_n = \frac{\left(\frac{q^p b_n^p}{p+1} + k^p\right) q b_n^2}{1 - k^p(1+qb_n)^p b_n}, \quad \gamma_n = \frac{1}{1 - k^p(1+qb_n)^p b_n} \tag{2.3}$$

for each $n \in \mathbb{N}_0$. Then we have the following:

(1) $\frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right)^p q^p b_0^{2p}}{(1 - k^p(1+qb_0)^p b_0)^{p+1}} < 1$.

(2) *The sequence $\{b_n\}$ is decreasing, that is $b_{n+1} \le b_n$ for all $n \in \mathbb{N}_0$.*

(3) $k^p(1+qb_n)^p b_n < 1$ *for all $n \in \mathbb{N}_0$.*

(4) $b_{n+1} \le \xi^{(2p+1)^n} b_n$ *for all $n \in \mathbb{N}_0$.*

(5) $\theta_n \le \xi^{\frac{(2p+1)^n - 1}{p}} \theta$ *for all $n \in \mathbb{N}_0$, where $\theta_0 = \theta$ and $\xi = \gamma_0 \theta^p$.*

Proof. (1) Since the scalar equation $g(t) = 0$ defined by (2.1) has a minimum positive root $\alpha \in (0,1)$ and $g(t)$ is decreasing in $(0,1)$ with $g(0) > 0$ and $g(1) < 0$. Therefore, $g(t) > 0$ in the interval $(0,\alpha)$ and hence

$$\frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right)^p q^p b_0^{2p}}{(1 - k^p(1+qb_0)^p b_0)^{p+1}} < 1.$$

(2) From (1) and (2.2), we have $b_1 \le b_0$. This shows that (2) is true for $n = 0$. Let $j \ge 0$ be a fixed positive integer. Assume that (2) is true for $n = 0, 1, 2, \cdots, j$. Now, using

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(2.2), we have

$$b_{j+2} = \frac{\left(\frac{q^p b_{j+1}^p}{p+1} + k^p\right)^p q^p b_{j+1}^{2p}}{(1 - k^p(1+qb_{j+1})^p b_{j+1})^{p+1}} b_{j+1} \le \frac{\left(\frac{q^p b_j^p}{p+1} + k^p\right)^p q^p b_j^{2p}}{(1 - k^p(1+qb_j)^p b_n)^{p+1}} b_j = b_{j+1}.$$

Thus (2) holds for $n = j+1$. Therefore, by induction, (2) holds for all $n \in \mathbb{N}_0$.

(3) Since $b_n < b_{n-1}$ for each $n = 1, 2, 3, \cdots$ and $k^p(1+qb_0)^p b_0 < 1$ for all $p \in (0, 1]$, it follows that

$$k^p(1+qb_n)^p b_n < k^p(1+qb_0)^p b_0 < 1.$$

(4) From (3), one can easily prove that the sequences $\{\gamma_n\}$ and $\{\theta_n\}$ are well defined. Using (2.2) and (2.3), one can easily observe that

$$b_{n+1} = \gamma_n \theta_n^p b_n \tag{2.4}$$

for each $n \in \mathbb{N}_0$. Put $n = 0$ and $n = 1$ in (2.4), we have

$$b_1 = \gamma_0 \theta^p b_0 = \xi^{(2p+1)^0} b_0$$

and

$$\begin{aligned} b_2 &= \frac{\left(\frac{q^p b_1^p}{p+1} + k^p\right)^p q^p b_1^{2p}}{(1 - k^p(1+qb_1)^p b_1)^{p+1}} b_1 \\ &\le \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right)^p q^p (\xi b_0)^{2p}}{(1 - k^p(1+qb_0)^p b_0)^{p+1}} b_1 \\ &\le \xi^{2p} \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right)^p q^p b_0^{2p}}{(1 - k^p(1+qb_0)^p b_0)^{p+1}} b_1 \\ &= \xi^{2p} \gamma_0 \theta^p b_1 = \xi^{2p+1} b_1. \end{aligned}$$

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Hence (4) holds for $n = 0$ and $n = 1$. Let $j > 1$ be a fixed integer. Assume that (4) holds for each $n = 0, 1, 2 \cdots, j$. From (2.3) and (2.4), we have

$$\begin{aligned} b_{j+2} &= \frac{\left(\frac{q^p b_{j+1}^p}{p+1} + k^p\right)^p q^p b_{j+1}^{2p}}{(1 - k^p(1 + qb_{j+1})^p b_{j+1})^{p+1}} b_{j+1} \\ &\le \frac{\left(\frac{q^p b_{j+1}^p}{p+1} + k^p\right)^p q^p (\xi^{(2p+1)^j} b_j)^{2p}}{(1 - k^p(1 + qb_{j+1})^p b_{j+1})^{p+1}} b_{j+1} \\ &\le (\xi^{2p(2p+1)^j}) \frac{\left(\frac{q^p b_j^p}{p+1} + k^p\right)^p q^p b_j^{2p}}{(1 - k^p(1 + qb_j)^p b_j)^{p+1}} b_{j+1} \\ &\le \xi^{2p(2p+1)^j} \xi^{2p(2p+1)^{j-1}} \cdots \xi^{2p(2p+1)} \xi^{(2p+1)} b_{j+1} \\ &= \xi^{(2p+1)^{j+1}} b_{j+1}. \end{aligned}$$

Thus (4) holds for $n = j + 1$. Therefore, by induction, (4) holds for all $n \in \mathbb{N}_0$.

(5) From (2.3) and (4), one can easily observe that

$$\theta_1 = \frac{\left(\frac{q^p b_1^p}{p+1} + k^p\right) q b_1^2}{1 - k^p(1 + qb_1)^p b_1} \le \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right) q (\xi b_0)^2}{1 - k^p(1 + qb_0)^p b_0} \le \xi^{\frac{(2p+1)^1 - 1}{p}} \theta.$$

Hence (5) holds for $n = 1$. Let $j > 1$ be a fixed integer. Assume that (5) holds for each $n = 0, 1, 2 \cdots, j$. From (2.3), we have

$$\begin{aligned} \theta_{j+1} &= \frac{\left(\frac{q^p b_{j+1}^p}{p+1} + k^p\right) q b_{j+1}^2}{1 - k^p(1 + qb_{j+1})^p b_{j+1}} \\ &\le \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right) q \left(\xi^{\frac{(2p+1)^{j+1} - 1}{2p}} b_0\right)^2}{1 - k^p(1 + qb_0)^p b_0} \\ &= \xi^{\frac{(2p+1)^{j+1} - 1}{p}} \theta. \end{aligned}$$

Thus (5) holds for $n = j + 1$. Therefore, by induction, (v) holds for all $n \in \mathbb{N}_0$. This completes the proof. $\square$

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3 Computation of a solution of the operator equation (1.1)

Let X and Y be Banach spaces and D be a nonempty open convex subset of X. Let $F : D \subset X \to Y$ be a nonlinear operator such that F is Fréchet differentiable at each point of D and let $L \in B(Y, X)$ such that $(I - LF)(D) \subseteq D$. Starting with $x_0 \in D$ and after $x_n \in D$ is defined, we define the next iterate x_{n+1} as follows:

$$\begin{cases} z_n = (I - LF)(x_n), \\ y_n = (I - F'^{-1}_{z_n} F)(x_n), \\ x_{n+1} = (I - F'^{-1}_{z_n} F)(y_n) \end{cases} \tag{3.1}$$

for each $n \in \mathbb{N}_0$.

If we take $X = Y, F = I - G$ and $L = I \in B(X)$ in (3.1), then the iteration process (3.1) reduces to the Stirling-like iteration process (1.8).

Before proving the main results, we establish the following:

Proposition 3.1 *Let D be a nonempty open convex subset of a Banach space X, $F : D \subset X \to Y$ be a Fréchet differentiable at each point of D with values in a Banach space Y and $L \in B(Y, X)$ such that $(I - LF)(D) \subseteq D$. Let $\omega : [0, \infty) \to [0, \infty)$ be a nondecreasing and continuous real-valued function. Assume that F satisfies the following conditions:*

(1) $\|F'_x - F'_y\| \leq \omega(\|x - y\|)$ *for all* $x, y \in D$;

(2) $\|I - LF'_x\| \leq c$ *for all* $x \in D$ *and for some* $c \in (0, \infty)$.

Define a mapping $T : D \to D$ by

$$T(x) = (I - LF)(x) \tag{3.2}$$

for all $x \in D$. Then

$$\|I - F'^{-1}_{Tx} F'_{Ty}\| \leq \|F'^{-1}_{Tx}\| \omega(c\|x - y\|)$$

for all $x, y \in D$.

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Proof. For any $x, y \in D$, we have

$$
\begin{aligned}
\|I - F'^{-1}_{Tx}F'_{Ty}\| &\leq \|F'^{-1}_{Tx}\|\|F'_{Tx} - F'_{Ty}\| \\
&\leq \|F'^{-1}_{Tx}\|\omega(\|Tx - Ty\|) \\
&= \|F'^{-1}_{Tx}\|\omega(\|x - y - L(F(x) - F(y))\|) \\
&= \|F'^{-1}_{Tx}\|\omega\left(\|x - y - L\int_0^1 F'_{y+t(x-y)}(x-y)dt\|\right) \\
&\leq \|F'^{-1}_{Tx}\|\omega\left(\int_0^1 \|I - LF'_{y+t(x-y)}\|dt\|x-y\|\right) \\
&\leq \|F'^{-1}_{Tx}\|\omega(c\|x-y\|).
\end{aligned}
$$

This completes the proof. $\square$

Now, we are ready to prove our main results for solving the problem (1.1) in the framework of Banach spaces.

Theorem 3.2 *Let D be a nonempty open convex subset of a Banach space X, $F : D \subset X \to Y$ a Fréchet differentiable at each point of D with values in a Banach space Y and $L \in B(Y, X)$ such that $(I - LF)(D) \subseteq D$. Let $x_0 \in D$ be such that $z_0 = x_0 - LF(x_0)$ and $F'^{-1}_{z_0} \in B(Y, X)$ exist. Let $\omega \in \Phi$ and let α be the solution of the equation (2.1). Assume that the following conditions hold:*

(C1) $\|F'_x - F'_y\| \leq \omega(\|x - y\|)$ *for all* $x, y \in D$*;*
(C2) $\|I - LF'_x\| \leq k$ *for all* $x \in D$ *and for some* $k \in (0, \frac{1}{3}]$*;*
(C3) $\|F'^{-1}_{z_0}\| \leq \beta$ *for some* $\beta > 0$*;*
(C4) $\|F'^{-1}_{z_0}F(x_0)\| \leq \eta$ *for some* $\eta > 0$*;*
(C5) $\omega(ts) \leq t^p\omega(s)$, $s \in [0, \infty), t \in [0, 1]$ *and* $p \in (0, 1]$*;*
(C6) $b_0 = \beta\omega(\eta) < \alpha$, $q = \frac{1}{p+1} + k^p$, $\theta = \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right)qb_0^2}{1 - k^p(1+qb_0)^p b_0}$ *and* $B_r[x_0] \subset D$*, where* $r = \frac{1+q}{1-\theta}\eta$.

Then we have the following:

(1) *The sequence $\{x_n\}$ generated by (3.1) is well defined, remains in $B_r[x_0]$ and satisfies the following estimates:*

$$
\begin{cases}
\|y_{n-1} - z_{n-1}\| \leq k\|y_{n-1} - x_{n-1}\|, \\
\|x_n - y_{n-1}\| \leq qb_{n-1}\|y_{n-1} - x_{n-1}\|, \\
\|x_n - x_{n-1}\| \leq (1 + qb_{n-1})\|y_{n-1} - x_{n-1}\|, \\
F'^{-1}_{z_n} \text{ exists and } \|F'^{-1}_{z_n}\| \leq \gamma_{n-1}\|F'^{-1}_{z_{n-1}}\|, \\
\|y_n - x_n\| \leq \theta_{n-1}\|y_{n-1} - x_{n-1}\| \leq \theta^n\|y_0 - x_0\|, \\
\|F'^{-1}_{z_n}\|\omega(\|y_n - x_n\|) \leq b_n
\end{cases} \tag{3.3}
$$

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for all $n \in \mathbb{N}$, *where* $z_n, y_n \in B_r[x_0]$, *the sequences* $\{b_n\}$, $\{\theta_n\}$ *and* $\{\gamma_n\}$ *are defined by* (2.2) *and* (2.3), *respectively.*

(2) *The sequence* $\{x_n\}$ *converges to the solution* $x^* \in B_r[x_0]$ *of the equation* (1.1).

(3) *The priory error bounds on* x^* *is given by:*

$$\|x_n - x^*\| \leq \frac{(1+qb_0)\eta}{\xi^{1/2p^2}\left(1-\xi^{\frac{(2p+1)^n}{p}}\gamma_0^{-\frac{1}{p}}\right)\gamma_0^{n/p}}\left(\xi^{1/2p^2}\right)^{(2p+1)^n}$$

for each $n \in \mathbb{N}_0$.

(4) *The sequence* $\{x_n\}$ *has R-order of convergence at least* $2p+1$.

Proof. (1) First, we show that (3.3) is true for $n = 1$.
Since $x_0 \in D$, $y_0 = x_0 - F'^{-1}_{z_0}F(x_0)$ is well defined. Note that

$$\|y_0 - x_0\| = \|F'^{-1}_{z_0}F(x_0)\| \leq \eta < r.$$

Hence $y_0 \in B_r[x_0]$. Using (3.1), we have

$$\begin{aligned}\|y_0 - z_0\| &= \|-F'^{-1}_{z_0}F(x_0) + LF(x_0)\| \\ &= \|y_0 - x_0 - LF'_{z_0}(y_0 - x_0)\| \\ &\leq \|I - LF'_{z_0}\|\|y_0 - x_0\| \\ &\leq k\|y_0 - x_0\|.\end{aligned}$$

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By Proposition 3.1 and $(C2)$, we have

$$\begin{aligned}
\|x_1 - y_0\| &= \|F_{z_0}'^{-1}(F(y_0) - F(x_0) - F_{z_0}'(y_0 - x_0))\| \\
&\leq \int_0^1 \|F_{z_0}'^{-1}(F_{x_0+t(y_0-x_0)}' - F_{y_0}' + F_{y_0}' - F_{z_0}')\|\|y_0 - x_0\|dt \\
&\leq \beta\left[\int_0^1 \|(F_{x_0+t(y_0-x_0)}' - F_{y_0}')\|dt + \|F_{y_0}' - F_{z_0}'\|\right]\|y_0 - x_0\| \\
&= \beta\left[\int_0^1 \omega((1-t)\|y_0 - x_0\|)dt + \omega(\|y_0 - z_0\|)\right]\|y_0 - x_0\| \\
&= \beta\left[\int_0^1 (1-t)^p\omega(\|y_0 - x_0\|)dt + \omega(k\|y_0 - x_0\|)\right]\|y_0 - x_0\| \\
&= \beta\left[\int_0^1 (1-t)^p\omega(\|y_0 - x_0\|)dt + k^p\omega(\|y_0 - x_0\|)\right]\|y_0 - x_0\| \\
&\leq \beta\left[\frac{1}{p+1} + k^p\right]\omega(\|y_0 - x_0\|)\|y_0 - x_0\| \\
&\leq q\beta\omega(\eta)\|y_0 - x_0\| \leq qb_0\|y_0 - x_0\|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq qb_0\|y_0 - x_0\| + \|y_0 - x_0\| \\
&\leq (1 + qb_0)\|y_0 - x_0\| < r,
\end{aligned} \tag{3.4}$$

which shows that $x_1 \in B_r[x_0]$. Note that $z_1 = (I - LF)(x_1) \in D$. Using Proposition 3.1 and $(C3)$-$(C5)$, we have

$$\begin{aligned}
\|I - F_{z_0}'^{-1}F_{z_1}'\| &\leq \|F_{z_0}'^{-1}\|\omega(k\|x_1 - x_0\|) \\
&\leq \beta\omega(k(1 + qb_0)\|y_0 - x_0\|) \\
&\leq \beta k^p(1 + qb_0)^p\omega(\|y_0 - x_0\|) \\
&\leq k^p(1 + qb_0)^p\beta\omega(\eta) \\
&\leq (k(1 + qb_0))^p b_0 < 1.
\end{aligned}$$

Therefore, by Lemma 2.1, $F_{z_1}'^{-1}$ exists and

$$\|F_{z_1}'^{-1}\| \leq \frac{\|F_{z_0}'^{-1}\|}{1 - (k(1 + qb_0))^p b_0} = \gamma_0\|F_{z_0}'^{-1}\|. \tag{3.5}$$

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Subsequently, we have

$$\begin{aligned}
\|y_1 - x_1\| &= \|F_{z_1}'^{-1} F(x_1)\| \\
&= \|F_{z_1}'^{-1}(F(x_1) - F(y_0) - F_{z_0}'(x_1 - y_0))\| \\
&\leq \|F_{z_1}'^{-1}\| \left[\int_0^1 \|(F_{y_0+t(x_1-y_0)}' - F_{y_0}')\|dt + \|F_{y_0}' - F_{z_0}'\| \right] \|x_1 - y_0\| \\
&\leq \|F_{z_1}'^{-1}\| \left[\frac{1}{p+1}\omega(\|x_1 - y_0\|) + \omega(k\|y_0 - x_0\|) \right] \|x_1 - y_0\| \\
&\leq \|F_{z_1}'^{-1}\| \left[\frac{1}{p+1}\omega(qb_0\|x_0 - y_0\|) + k^p\omega(\|y_0 - x_0\|) \right] qb_0\|x_0 - y_0\| \\
&\leq \|F_{z_1}'^{-1}\| \left[\frac{q^p b_0^p}{p+1}\omega(\|y_0 - x_0\|) + k^p\omega(\|y_0 - x_0\|) \right] qb_0\|y_0 - x_0\| \\
&\leq \gamma_0 \left[\frac{q^p b_0^p}{p+1} + k^p \right] \beta\omega(\eta) qb_0 \|y_0 - x_0\| \\
&\leq \frac{\left(\frac{q^p b_0^p}{p+1} + k^p \right) qb_0^2}{1 - (k(1+qb_0))^p b_0} \|y_0 - x_0\| \\
&\leq \theta\|y_0 - x_0\|. \qquad (3.6)
\end{aligned}$$

From (3.4) and (3.6), we have

$$\begin{aligned}
\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\
&\leq \theta\|y_0 - x_0\| + (1+qb_0)\|y_0 - x_0\| \\
&\leq (1+qb_0)\theta\|y_0 - x_0\| + (1+qb_0)\|y_0 - x_0\| \\
&\leq (1+qb_0)(1+\theta)\eta < r
\end{aligned}$$

and

$$\begin{aligned}
\|z_1 - x_0| &\leq \|z_1 - y_1\| + \|y_1 - x_1\| + \|x_1 - x_0\| \\
&\leq (1+k)\|y_1 - x_1\| + (1+qb_0)\|y_0 - x_0\| \\
&\leq (1+q)\theta\eta + (1+q)\eta \\
&= (1+q)(1+\theta)\eta < r.
\end{aligned}$$

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This shows that $z_1, y_1 \in B_r[x_0]$. From (3.5) and (3.6), we get

$$\begin{aligned}
\|F'^{-1}_{z_1}\|\omega(\|y_1 - x_1\|) &\leq \gamma_0\|F'^{-1}_{z_0}\|\omega(\theta\|y_0 - x_0\|) \\
&\leq \gamma_0\theta^p\beta\omega(\eta) \\
&\leq \gamma_0\theta^p b_0 = b_1.
\end{aligned}$$

Thus we see that (3.3) holds for $n = 1$.

Let $j > 1$ be a fixed integer. Assume that (3.3) is true for $n = 1, 2, \cdots, j$. Since $x_j \in B_r[x_0]$, it follows $z_j = (I - LF)(x_j) \in D$. Using $(C3), (C4)$, (3.1) and (3.3), we have

$$\begin{aligned}
\|y_j - z_j\| &= \|LF(x_j) - F'^{-1}_{z_j}F(x_j)\| = \|(L - F'^{-1}_{z_j})F(x_j)\| \\
&= \|(L - F'^{-1}_{z_j})F'_{z_j}(x_j - y_j)\| \\
&\leq \|I - LF'_{z_j}\|\|y_j - x_j\| \\
&\leq k\|y_j - x_j\|. \qquad (3.7)
\end{aligned}$$

Using (3.1) and (3.7), we have

$$\begin{aligned}
\|x_{j+1} - y_j\| &= \|F'^{-1}_{z_j}F(y_j)\| \\
&\leq \|F'^{-1}_{z_j}\|\|F(y_j) - F(x_j) - F'_{z_j}(y_j - x_j)\| \\
&\leq \|F'^{-1}_{z_j}\|\left[\int_0^1 \|F'_{x_j+t(y_j-x_j)} - F'_{z_j}\|dt\right]\|y_j - x_j\| \\
&\leq \|F'^{-1}_{z_j}\|\left[\int_0^1 \|F'_{x_j+t(y_j-x_j)} - F'_{y_j}\|dt + \|F'_{y_j} - F'_{z_j}\|\right]\|y_j - x_j\| \\
&= \|F'^{-1}_{z_j}\|\left[\int_0^1 \omega(x_j + t(y_j - x_j) - y_j)dt + \omega(\|y_j - z_j\|)\right]\|y_j - x_j\| \\
&\leq \|F'^{-1}_{z_j}\|\left[\int_0^1 \omega((1-t)\|y_j - x_j\|)dt + \omega(k\|y_j - x_j\|)\right]\|y_j - x_j\| \\
&\leq \|F'^{-1}_{z_j}\|\left[\int_0^1 ((1-t)^p + k^p)\omega(\|y_j - x_j\|)dt\right]\|y_j - x_j\| \\
&\leq \|F'^{-1}_{z_j}\|\left[\frac{1}{p+1} + k^p\right]\omega(\|y_j - x_j\|)\|y_j - x_j\| \\
&= q\|F'^{-1}_{z_j}\|\omega(\|y_j - x_j\|)\|y_j - x_j\| \\
&= qb_j\|y_j - x_j\|. \qquad (3.8)
\end{aligned}$$

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From (3.8), we have

$$\begin{aligned} \|x_{j+1} - x_j\| &\leq \|x_{j+1} - y_j\| + \|y_j - x_j\| \\ &\leq qb_j\|y_j - x_j\| + \|y_j - x_j\| \\ &\leq (1 + qb_j)\|y_j - x_j\|. \end{aligned} \tag{3.9}$$

Using (3.8), (3.9) and the triangular inequality, we have

$$\begin{aligned} \|x_{j+1} - x_0\| &\leq \sum_{s=0}^{j} \|x_{s+1} - x_s\| \\ &\leq \sum_{s=0}^{j} (1 + qb_s)\|y_s - x_s\| \\ &\leq \sum_{s=0}^{j} (1 + qb_0)\theta^s\|y_0 - x_0\| \\ &\leq (1 + qb_0)\frac{1 - \theta^{j+1}}{1 - \theta}\eta \\ &\leq \frac{(1 + q)\eta}{1 - \theta} = r, \end{aligned}$$

which implies that $x_{k+1} \in B_r[x_0]$. Again, by using Proposition 3.1, $(C2)$, $(C5)$ and (3.9), we have

$$\begin{aligned} \|I - F_{z_j}'^{-1}F_{z_{j+1}}'\| &\leq \|F_{z_j}'^{-1}\|\omega(k\|x_{j+1} - x_j\|) \\ &\leq \|F_{z_j}'^{-1}\|k^p(1 + qb_j)^p\omega(\|y_j - x_j\|) \\ &\leq k^p(1 + qb_j)^p b_j < 1. \end{aligned}$$

Therefore, by Lemma 2.1, $F_{z_{j+1}}'^{-1}$ exists and

$$\|F_{z_{j+1}}'^{-1}\| \leq \frac{\|F_{z_j}'^{-1}\|}{1 - k^p(1 + qb_j)^p b_j} = \gamma_j\|F_{z_j}'^{-1}\|.$$

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Using (3.1), $(C2)$ and (3.9), we have

$$\begin{aligned}
\|y_{j+1}-x_{j+1}\| &= \|F'^{-1}_{z_{j+1}}F(x_{j+1})\| \\
&= \|F'^{-1}_{z_{j+1}}(F(x_{j+1})-F(y_j)-F'_{z_j}(x_{j+1}-y_j))\| \\
&\leq \|F'^{-1}_{z_{j+1}}\|\left[\int_0^1 \|F'_{y_j+t(x_{j+1}-y_j)}-F'_{y_j}\|dt+\|F'_{y_j}-F'_{z_j}\|\right]\|x_{j+1}-y_j\| \\
&\leq \|F'^{-1}_{z_{j+1}}\|\left[\int_0^1 \omega(t\|x_{j+1}-y_j\|)dt+\omega(\|y_j-z_j\|)\right]\|x_{j+1}-y_j\| \\
&\leq \|F'^{-1}_{z_{j+1}}\|\left[\int_0^1 \omega(tqb_j\|y_j-x_j\|)dt+\omega(k\|y_j-x_j\|)\right]qb_j\|y_j-x_j\| \\
&\leq \gamma_j\|F'^{-1}_{z_j}\|\left[\frac{q^pb_j^p}{p+1}\omega(\|y_j-x_j\|)+k^p\omega(\|y_j-x_j\|)\right]qb_j\|y_j-x_j\| \\
&= \gamma_j\left[\frac{q^pb_j^p}{p+1}+k^p\right]\|F'^{-1}_{z_j}\|\omega(\|y_j-x_j\|)qb_j\|y_j-x_j\| \\
&\leq \gamma_j\left[\frac{q^pb_j^p}{p+1}+k^p\right]qb_j^2\|y_j-x_j\| \\
&\leq \theta_j\|y_j-x_j\|\leq\theta^{j+1}\|y_0-x_0\|,
\end{aligned}$$

$$\begin{aligned}
\|y_{j+1}-x_0\| &\leq \|y_{j+1}-x_{j+1}\|+\|x_{j+1}-x_0\| \\
&\leq \theta^{j+1}\|y_0-x_0\|+\sum_{s=0}^{j}\|x_{s+1}-x_s\| \\
&\leq \theta^{j+1}\|y_0-x_0\|+\sum_{s=0}^{j}(1+qb_0)\theta^s\|y_0-x_0\| \\
&\leq (1+qb_0)\sum_{s=0}^{j+1}\theta^s\eta \\
&\leq \frac{(1+q)\eta}{1-\theta}=r
\end{aligned}$$

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and

$$\begin{aligned}
\|z_{j+1} - x_0\| &\leq \|z_{j+1} - y_{j+1}\| + \|y_{j+1} - x_{j+1}\| + \|x_{j+1} - x_0\| \\
&\leq (1+k)\|y_{j+1} - x_{j+1}\| + \sum_{s=0}^{j} (1+qb_0)\theta^s \eta \\
&\leq (1+q)\theta^{j+1}\eta + \sum_{s=0}^{j} (1+q)\theta^s \eta \\
&\leq \sum_{s=0}^{j+1} (1+q)\theta^s \eta < r
\end{aligned}$$

which implies that $z_{j+1}, y_{j+1} \in B_r(x_0)$. Also, we have

$$\begin{aligned}
\|F'^{-1}_{z_{j+1}}\|\omega(\|y_{j+1} - x_{j+1}\|) &\leq \gamma_j \|F'^{-1}_{z_j}\|\omega(\theta_j \|y_j - x_j\|) \\
&\leq \gamma_j \theta_j^p \|F'^{-1}_{z_j}\|\omega(\|y_j - x_j\|) \\
&\leq \gamma_j \theta_j^p b_j = b_{j+1}.
\end{aligned}$$

Hence we conclude that (3.3) is true for $n = j+1$. Therefore, by induction, (3.3) is true for all $n \in \mathbb{N}_0$.

(2) First, we show that the sequence $\{x_n\}$ is a Cauchy sequence. For this, letting

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$m, n \in \mathbb{N}_0$ and using Lemma 2.3, we have

$$\begin{aligned}
\|x_{m+n} - x_n\| &\leq \sum_{j=n}^{m+n-1} \|x_{j+1} - x_j\| \\
&\leq \sum_{j=n}^{m+n-1} (1 + qb_j)\|y_j - x_j\| \\
&\leq (1 + qb_0) \sum_{j=n}^{m+n-1} \prod_{i=0}^{j-1} \theta_i \|y_0 - x_0\| \\
&\leq (1 + qb_0) \sum_{j=n}^{m+n-1} \prod_{i=0}^{j-1} \xi^{\frac{(2p+1)^i - 1}{p}} \theta \|y_0 - x_0\| \\
&\leq (1 + qb_0) \sum_{j=n}^{m+n-1} \prod_{i=0}^{j-1} \xi^{\frac{(2p+1)^i}{p}} \gamma_0^{-\frac{1}{p}} \|y_0 - x_0\| \\
&= (1 + qb_0) \sum_{j=n}^{m+n-1} \xi^{\sum_{i=0}^{j-1} \frac{(2p+1)^i}{p}} \gamma_0^{-\frac{1}{p}} \|y_0 - x_0\| \\
&\leq (1 + qb_0) \left(\sum_{j=n}^{m+n-1} \xi^{\frac{(2p+1)^j - 1}{2p^2}} \gamma_0^{-\frac{j}{p}} \right) \|y_0 - x_0\|.
\end{aligned}$$

By Bernoulli's inequality, for each $j \geq 0$ and $y > -1$, we have $(1 + y)^j \geq 1 + jy$. Hence we have

$$\begin{aligned}
&\|x_{m+n} - x_n\| \\
\leq\ & (1 + qb_0)\xi^{-\frac{1}{2p^2}} \gamma_0^{-\frac{n}{p}} \left(\xi^{\frac{(2p+1)^n}{2p^2}} + \xi^{\frac{(2p+1)^n(2p+1)}{2p^2}} \gamma_0^{-\frac{1}{p}} + \cdots + \xi^{\frac{(2p+1)^n(2p+1)^{m-1}}{2p^2}} \gamma_0^{-\frac{(m-1)}{p}} \right) \eta \\
\leq\ & (1 + qb_0)\xi^{-\frac{1}{2p^2}} \gamma_0^{-\frac{n}{p}} \left(\xi^{\frac{(2p+1)^n}{2p^2}} + \xi^{\frac{(2p+1)^n(1+2p)}{2p^2}} \gamma_0^{-\frac{1}{p}} + \cdots + \xi^{\frac{(2p+1)^n(1+2(m-1)p)}{2p^2}} \gamma_0^{-\frac{(m-1)}{p}} \right) \eta \\
=\ & (1 + qb_0)\xi^{-\frac{1}{2p^2}} \gamma_0^{-\frac{n}{p}} \left(\xi^{\frac{(2p+1)^n}{2p^2}} + \xi^{(2p+1)^n\left(\frac{1}{2p^2} + \frac{1}{p}\right)} \gamma_0^{-\frac{1}{p}} + \cdots + \xi^{(2p+1)^n\left(\frac{1}{2p^2} + \frac{m-1}{p}\right)} \gamma_0^{-\frac{(m-1)}{p}} \right) \eta \\
=\ & (1 + qb_0)\xi^{\frac{(2p+1)^n - 1}{2p^2}} \gamma_0^{-\frac{n}{p}} \left(1 + \left(\xi^{(2p+1)^n} \gamma_0^{-1}\right)^{\frac{1}{p}} + \cdots + \left(\xi^{(2p+1)^n} \gamma_0^{-1}\right)^{\frac{m-1}{p}} \right) \eta \\
=\ & (1 + qb_0)\xi^{\frac{(2p+1)^n - 1}{2p^2}} \gamma_0^{-\frac{n}{p}} \left(\frac{1 - \left(\xi^{(2p+1)^n} \gamma_0^{-1}\right)^{\frac{m}{p}}}{1 - \left(\xi^{(2p+1)^n} \gamma_0^{-1}\right)^{\frac{1}{p}}} \right) \eta. \qquad (3.10)
\end{aligned}$$

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Since the sequence $\{x_n\}$ is a Cauchy sequence and hence it converges to some point $x^* \in B_r[x_0]$. From (3.1), $(C2)$ and (3.3), we have

$$\begin{aligned} \|LF(x_n)\| &\leq \|z_n - y_n\| + \|y_n - x_n\| \\ &\leq k\|y_n - x_n\| + \|y_n - x_n\| \\ &\leq (1+k)\theta^n \eta. \end{aligned}$$

Taking the limit as $n \to \infty$ and using the continuity of F and the linearity of L, we have

$$F(x^*) = 0.$$

(3) Taking the limit as $m \to \infty$ in (3.10), we have

$$\|x^* - x_n\| \leq \frac{(1+qb_0)\eta}{\xi^{1/2p^2}\left(1 - \xi^{\frac{(2p+1)^n}{p}}\gamma_0^{-\frac{1}{p}}\right)\gamma_0^{n/p}}\left(\xi^{1/2p^2}\right)^{(2p+1)^n} \tag{3.11}$$

for each $n \in \mathbb{N}_0$.

(4) Here, we prove

$$\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|^{2p+1}} \leq K$$

for all $n \in \mathbb{N}_0$ and for some $K > 0$. One can easily observe that there exists $n_0 > 0$ such that

$$\|x_n - x^*\| < 1 \tag{3.12}$$

whenever $n \geq n_0$. Using (3.1) and (3.12), we have

$$\begin{aligned} \|z_n - x^*\| &= \|x_n - x^* - LF(x_n)\| \\ &= \|x_n - x^* - L(F(x_n) - F(x^*))\| \\ &= \|x_n - x^* - L\int_0^1 F'_{x^*+t(x_n-x^*)}(x_n - x^*)dt\| \\ &\leq \int_0^1 \|I - LF'_{x^*+t(x_n-x^*)}\|\|x_n - x^*\|dt \\ &\leq k\|x_n - x^*\| \end{aligned}$$

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and

$$\begin{aligned}
\|y_n - x^*\| &= \|x_n - x^* - F_{z_n}'^{-1}F(x_n)\| \\
&= \|F_{z_n}'^{-1}[F_{z_n}'(x_n - x^*) - F(x_n)]\| \\
&\leq \|F_{z_n}'^{-1}\|\|F(x_n) - x^* - F_{z_n}'(x_n - x^*)\| \\
&= \|F_{z_n}'^{-1}\|\|\int_0^1 (F'_{x^*+t(x_n-x^*)} - F'_{z_n})(x_n - x^*)\|dt \\
&\leq \|F_{z_n}'^{-1}\| \int_0^1 \|F'_{x^*+t(x_n-x^*)} - F'_{z_n}\|\|x_n - x^*\|dt \\
&\leq \|F_{z_n}'^{-1}\| \int_0^1 (\|F'_{x^*+t(x_n-x^*)} - F'_{x^*}\| + \|F'_{x^*} - F'_{z_n}\|)\|x_n - x^*\|dt \\
&\leq \|F_{z_n}'^{-1}\| \int_0^1 (\omega(t\|x_n - x^*\|) + \omega(\|z_n - x^*\|))\|x_n - x^*\|dt \\
&\leq \|F_{z_n}'^{-1}\| \int_0^1 (t^p\|x_n - x^*\|^p\omega(1) + \omega(k\|x_n - x^*\|))\|x_n - x^*\|dt \\
&\leq \|F_{z_n}'^{-1}\| \left(\frac{1}{p+1} + k^p\right)\omega(1)\|x_n - x^*\|^{p+1} \\
&= \|F_{z_n}'^{-1}\|q\omega(1)\|x_n - x^*\|^{p+1}. \qquad (3.13)
\end{aligned}$$

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Using (3.1), (3.12) and (3.13), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\| \\
= \;& \|y_n - F_{z_n}^{\prime -1}F(y_n) - x^*\| \\
\leq \;& \|F_{z_n}^{\prime -1}\|\|F(y_n) - F(x^*) - F'_{z_n}(y_n - x^*)\| \\
= \;& \|F_{z_n}^{\prime -1}\|\| \int_0^1 (F'_{x^*+t(y_n-x^*)} - F'_{z_n})(y_n - x^*)\|dt \\
\leq \;& \|F_{z_n}^{\prime -1}\| \int_0^1 \|F'_{x^*+t(y_n-x^*)} - F'_{z_n}\|\|y_n - x^*\|dt \\
= \;& \|F_{z_n}^{\prime -1}\| \int_0^1 (\|F'_{x^*+t(y_n-x^*)} - F'_{x^*}\| + \|F'_{x^*} - F'_{z_n}\|)\|y_n - x^*\|dt \\
\leq \;& \|F_{z_n}^{\prime -1}\| \int_0^1 (\omega(t\|y_n - x^*\|) + \omega(k\|x_n - x^*\|))\|y_n - x^*\|dt \\
= \;& \|F_{z_n}^{\prime -1}\| \int_0^1 \left(t^p\omega\left(\|F_{z_n}^{\prime -1}\|q\omega(1)\|x_n - x^*\|^{p+1}\right) + \omega(k\|x_n - x^*\|)\right)dt \\
& \times\|F_{z_n}^{\prime -1}\|q\omega(1)\|x_n - x^*\|^{p+1} \\
\leq \;& \|F_{z_n}^{\prime -1}\|^2 \int_0^1 \left(t^p\|x_n - x^*\|^{p(p+1)}\omega\left(\|F_{z_n}^{\prime -1}\|q\omega(1)\right) + k^p\|x_n - x^*\|^p\omega(1)\right)dt \\
& \times q\omega(1)\|x_n - x^*\|^{p+1} \\
= \;& \|F_{z_n}^{\prime -1}\|^2\left(\frac{\|x_n - x^*\|^{p^2}\omega\left(\|F_{z_n}^{\prime -1}\|q\omega(1)\right)}{p+1} + k^p\right)q\omega(1)\|x_n - x^*\|^{2p+1} \\
= \;& K_n\|x_n - x^*\|^{2p+1},
\end{aligned}$$

where

$$K_n = \|F_{z_n}^{\prime -1}\|^2\left(\frac{\|x_n - x^*\|^{p^2}\omega\left(\|F_{z_n}^{\prime -1}\|q\omega(1)\right)}{p+1} + k^p\right)q\omega(1).$$

Let $\|F_{x^*}^{\prime -1}\| \leq d$ and $0 < d < \omega(\sigma)^{-1}$, where $\sigma > 0$. Then, for all $x \in B_\sigma(x^*)$, we have

$$\|I - F_{x^*}^{\prime -1}F'_x\| \leq \|F_{x^*}^{\prime -1}\|\|F'_{x^*} - F'_x\| \leq d\omega(\sigma) < 1$$

and so, by Lemma 2.1, we have

$$\|F_x^{\prime -1}\| \leq \frac{d}{1 - d\omega(\sigma)} := \lambda.$$

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Since $x_n \to x^*$ and $z_n \to x^*$ as $n \to \infty$, there exists a positive integer N_0 such that

$$\|F_{z_n}'^{-1}\| \leq \frac{d}{1 - d\omega(\sigma)}$$

for all $n \geq N_0$. Thus, for all $n \geq N_0$, one can easily observe that

$$K_n \leq \lambda^2 \left(\frac{\sigma^{p^2} \omega\left(\lambda q \omega(1)\right)}{p+1} + k^p \right) q\omega(1) = K.$$

This shows that the R-order of convergence at least $(2p+1)$.
This completes the proof. $\square$

4 Applications

4.1 Fixed points of smooth operators

For the choice of $X = Y$ and $F = I - G$, Theorem 3.2 reduces to the following:

Theorem 4.1 *Let D be a nonempty open convex subset of a Banach space X, $G : D \to X$ be a Fréchet differentiable at each point of D with values into itself. Let $L \in B(X)$ be such that $(I - L(I - G))(D) \subseteq D$. Let $x_0 \in D$ be such that $z_0 = x_0 - L(x_0 - G(x_0))$ and let $(I - G'_{z_0})^{-1} \in B(X)$ exist. Let $\omega \in \Phi$ and α be a solution of the equation* (2.1)*. Assume that the conditions* $(C5)$*-*$(C6)$ *and the following conditions hold:*

(C7) $\|(I - G'_{z_0})^{-1}\| \leq \beta$ *for some* $\beta > 0$*;*
(C8) $\|(I - G'_{z_0})^{-1}(x_0 - G(x_0))\| \leq \eta$ *for some* $\eta > 0$*;*
(C9) $\|G'_x - G'_y\| \leq \omega(\|x - y\|)$ *for all* $x, y \in D$*;*
(C10) $\|I - L(I - G'_x)\| \leq k$ *for all* $x \in D$ *and for some* $k \in (0, \frac{1}{3}]$.

Then the sequence $\{x_n\}$ generated by

$$\begin{cases} z_n = (I - L(I - G))(x_n), \\ y_n = (I - (I - G'_{z_n})^{-1}(I - G))(x_n), \\ x_{n+1} = (I - (I - G'_{z_n})^{-1}(I - G))(y_n) \end{cases} \tag{4.1}$$

for each $n \in \mathbb{N}_0$ is well defined, remains in $B_r[x_0]$ and converges to the fixed point $x^* \in B_r[x_0]$ of the operator G and the sequence $\{x_n\}$ has R-order of convergence at least $2p+1$.

If we put $L = I$, Theorem 4.1 reduces to the following:

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Corollary 4.2 [2, Theorem 1] *Let D be a nonempty open convex subset of a Banach space X and $G : D \to D$ be a Fréchet differentiable operator and let $x_0 \in D$ with $z_0 = G(x_0)$. Let $(I - G'_{z_0})^{-1} \in B(X)$ exists and $\omega \in \Phi$. Assume that the conditions $(C5)$-$(C9)$ and the following condition holds:*

(C11) $\|G'_x\| \le k$ *for all* $x \in D$ *and for some* $k \in (0, \frac{1}{3}]$.

Then the sequence $\{x_n\}$ generated by (1.8) is well defined, remains in $B_r[x_0]$ and converges to the fixed point $x^* \in B_r[x_0]$ of the operator G with $R-$order of convergence at least $2p+1$.

Now, we give some examples to illustrate the main results in this paper.

Example 4.3 Let $X = Y = \mathbb{R}$, and $D = (-1, 1) \subset X$. Define a mapping $G : D \to \mathbb{R}$ by

$$G(x) = \frac{x^3 - x}{6}$$

for all $x \in D$. Clearly, G is Fréchet differentiable on D and its Fréchet derivative at $x \in D$ is $G'_x = \frac{3x^2-1}{6}$ and G'_x is bounded with $\|G'_x\| \le \frac{1}{3} = k$ for all $x \in D$ and G' satisfies the Lipschitz condition

$$\|G'_x - G'_y\| \le K\|x - y\|$$

for all $x, y \in D$, where $K = 1$. For $x_0 = 0.3$, we have

$$z_0 = G(x_0) = -0.04550, \quad \|(I - G'_{z_0})^{-1}\| \le 0.857904032495689 = \beta,$$

$$\|(I - G'_{z_0})^{-1}(x_0 - G(x_0))\| \le 0.296405843227261 = \eta.$$

For $p = 1$, $q = \frac{5}{6}$ and $\omega(t) = Kt$ for all $t \ge 0$, we have

$$b_0 = \beta K \eta = 0.254287768159952 < 1,$$

$$\theta = \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right) q b_0^2}{1 - k(1 + q b_0) b_0} = 0.006362942974610 < 1$$

and

$$r = 0.546890545940483.$$

Hence all the conditions of Theorem 4.1 with $L = I$ are satisfied. Therefore, the sequence $\{x_n\}$ generated by (1.8) is in $B_r[x_0]$ and it converges to the fixed point $x^* = 0 \in B_r[x_0]$ of G.

Example 4.4 Let $X = Y = \mathbb{R}$ and $D = (-6, 6) \subset X$. Define a mapping $G : D \to \mathbb{R}$ by

$$G(x) = 2 + e^{\frac{\sin x}{5}}$$

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for all $x \in D$. It is obvious that G is Fréchet differentiable on D and its Fréchet derivative at $x \in D$ is $G'_x = \frac{\cos x}{5} e^{\frac{\sin x}{5}}$. Clearly, G'_x is bounded with $\|G'_x\| \leq 0.22 < \frac{1}{3} = k$ and

$$\|G'_x - G'_y\| \leq K\|x - y\|$$

for all $x, y \in D$, where $K = 0.245$. For $x_0 = 0$, we have

$$z_0 = G(x_0) = 3, \quad \|(I - G'_{z_0})^{-1}\| \leq 0.834725586524139 = \beta$$

and

$$\|(I - G'_{z_0})^{-1}(x_0 - G(x_0))\| \leq 2.504176759572418 = \eta.$$

For $p = 1$, $q = \frac{5}{6}$ and $\omega(t) = Kt$ for all $t \geq 0$, we have

$$b_0 = \beta K \eta = 0.512123601526580 < 1,$$

$$\theta = \frac{\left(\frac{q^p b_0^p}{p+1} + k^p\right) q b_0^2}{1 - k(1 + q b_0) b_0} = 0.073280601270728 < 1$$

and

$$r = 5.147038576039456.$$

Hence all the conditions of Theorem 4.1 with $L = I$ are satisfied. Therefore, the sequence $\{x_n\}$ generated by (1.8) is in $B_r[x_0]$ and it converges to the fixed point $x^* = 3.023785446275295 \in B_r[x_0]$ of G.

n	$\|x_n - x^*\|$
0	3.0237854462752
1	$1.7795738211156 \times 10^{-2}$
2	$6.216484249588206 \times 10^{-6}$
3	$2.335501569916687 \times 10^{-9}$
4	$8.775202786637237 \times 10^{-13}$
5	$4.440892098500626 \times 10^{-16}$

Table 1: A priori error bounds

4.2 Fredholm integral equations

Let X be a Banach space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with the norm $\|\cdot\|$ and D be an open convex subset of X. Further, let $B(X)$ be the Banach space of bounded linear operators from X into itself. Let $S \in B(X)$, $u \in X$ and $\lambda \in \mathbb{F}$. We investigate a solution $x \in X$ of the nonlinear Fredholm-type operator equation:

$$x - \lambda SQ(x) = u, \tag{4.2}$$

where $Q : D \to X$ is continuously Fréchet differentiable on D. The operator equation (4.2) has been discussed in [16], [32] and [35]. Define an operator $F : D \to X$ by

$$F(x) = x - \lambda SQ(x) - u \tag{4.3}$$

for all $x \in D$. Then solving the operator equation (4.3) is equivalent to solving the operator equation (1.1). From (4.3), we have

$$F'_x(h) = h - \lambda SQ'_x(h) \tag{4.4}$$

for all $h \in X$. Now, we apply Theorem 3.2 to solve the operator equation (4.2).

Theorem 4.5 *Let X be a Banach space and D an open convex subset of X. Let $Q : D \to X$ be a continuously Fréchet differentiable mapping at each point of D. Let $L, S \in B(X)$ and $u \in X$. Assume that, for any $x_0 \in D$, $z_0 = x_0 - L(x_0 - \lambda SQ(x_0) - u)$ and $(I - \lambda SQ'_{z_0})^{-1}$ exist. Assume that the condition $(C6)$ and the following conditions hold:*

(C12) $(I - L(I - \lambda SQ))(x) - u \in D$ *for all* $x \in D$*;*

(C13) $\|(I - \lambda SQ'_{z_0})^{-1}\| \leq \beta$ *for some* $\beta > 0$*;*

(C14) $\|(I - \lambda SQ'_{z_0})^{-1}(x_0 - \lambda SQ(x_0) - u)\| \leq \eta$ *for some* $\eta > 0$*;*

(C15) $\|Q'_x - Q'_y\| \leq \omega_0(\|x - y\|)$ *for all* $x, y \in D$*, where* $\omega_0 \in \Phi$*;*

(C16) $\omega_0(st) \leq s^p \omega_0(t), s \in [0, 1]$ *and* $t \in [0, \infty)$*;*

(C17) $\|I - L(I - \lambda SQ'_x)\| \leq k, k \leq \frac{1}{3}$ *for all* $x \in D$.

Then we have the following:

(1) *The sequence $\{x_n\}$ generated by*

$$\begin{cases} z_n = x_n - L(x_n - \lambda SQ(x_n) - u), \\ y_n = x_n - (I - \lambda SQ'_{z_n})^{-1}(x_n - \lambda SQ(x_n) - u), \\ x_{n+1} = y_n - (I - \lambda SQ'_{z_n})^{-1}(y_n - \lambda SQ(y_n) - u) \end{cases} \tag{4.5}$$

for each $n \in \mathbb{N}_0$ is well defined, remains in $B_r[x_0]$ and converges to a solution x^* of the equation (4.2).

(2) *The R-order convergence of sequence $\{x_n\}$ is at least $2p + 1$.*

Proof. Let $F : D \to X$ be an operator defined by (4.3). Clearly, F is Fréchet differentiable at each point of D and its Fréchet derivative at $x \in D$ is given by (4.4).

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Now, from (C13) and (4.4), we have $\|F'^{-1}_{z_0}\| \leq \beta$ and so it follows that (C3) holds. From (C14), (4.3) and (4.4), we have $\|F'^{-1}_{z_0}(F(x_0))\| \leq \eta$. Hence (C4) is satisfied. For all $x, y \in D$, using (C15), we have

$$\begin{aligned}\|F'_x - F'_y\| &= \sup\{\|(F'_x - F'_y)z\| : z \in X, \|z\| = 1\}\\ &\leq |\lambda|\|S\| \sup\{\|Q'_x - Q'_y\|\|z\| : z \in X, \|z\| = 1\}\\ &\leq |\lambda|\|S\|\omega_0(\|x - y\|)\\ &= \omega(\|x - y\|),\end{aligned}$$

where $\omega(t) = |\lambda|\|S\|\omega_0(t)$. Clearly, $\omega \in \Phi$ and, from (C16), we have

$$\omega(st) \leq s^p\omega(t)$$

for all $s \in [0, 1]$ and $t \in (0, \infty]$. Thus (C1) and (C5) hold. (C2) follows from (C17) for $c = k \in (0, \frac{1}{3}]$. Hence all the conditions of Theorem 3.2 are satisfied. Therefore, Theorem 4.5 follows from Theorem 3.2. This completes the proof. □

Let $D = X = Y = C[a, b]$ be the space of all continuous real valued functions defined on $[a, b] \subset \mathbb{R}$ with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. Consider, the following nonlinear integral equation:

$$x(s) = g(s) + \lambda \int_a^b K(s, t)(\mu x(t)^{1+p} + \nu x(t)^2)dt \tag{4.6}$$

for all $s \in [a, b]$ and $p \in (0, 1]$, where $g, x \in C[a, b]$ with $g(s) \geq 0$ for all $s \in [a, b]$, $K : [a, b] \times [a, b] \to \mathbb{R}$ is a continuous nonnegative real-valued function and $\mu, \nu, \lambda \in \mathbb{R}$. Define two mappings $S, Q : D \to X$ by

$$Sx(s) = \int_a^b K(s, t)x(t)dt \tag{4.7}$$

for all $s \in [a, b]$ and

$$Qx(s) = \mu x(s)^{1+p} + \nu x(s)^2 \tag{4.8}$$

for all $\mu, \nu \in \mathbb{R}$ and $s \in [a, b]$.

One can easily observe that K is bounded on $[a, b]\times[a, b]$, that is, there exists a number $M \geq 0$ such that $|K(s, t)| \leq M$ for all $s, t \in [a, b]$. Clearly, S is bounded linear operator with $\|S\| \leq M(b - a)$ and Q is Fréchet differentiable and its Fréchet derivative at $x \in D$ is given by

$$Q'_x h(s) = (\mu(1 + p)x^p + 2\nu x)h(s) \tag{4.9}$$

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for all $h \in C[a,b]$. For all $x, y \in D$, we have

$$\begin{aligned} \|Q'_x - Q'_y\| &= \sup\{\|(Q'_x - Q'_y)h\| : h \in C[a,b], \|h\| = 1\} \\ &\leq \sup\{\|(\mu(1+p)(x^p - y^p) + 2\nu(x-y))h\| : h \in C[a,b], \|h\| = 1\} \\ &\leq \sup\{(|\mu|(1+p)\|x^p - y^p\| + 2|\nu|\|x-y\|)\|h\| : h \in C[a,b], \|h\| = 1\} \\ &\leq |\mu|(1+p)\|x-y\|^p + 2|\nu|\|x-y\| \\ &= \omega_0(\|x-y\|), \end{aligned} \tag{4.10}$$

where $\omega_0(t) = |\mu|(1+p)t^p + 2|\nu|t, t \geq 0$ with

$$\omega_0(st) \leq s^p \omega_0(t) \tag{4.11}$$

for all $s \in [0,1]$ and $t \in [0,\infty)$. For any $x \in D$, using (4.7) and (4.9), we have

$$\begin{aligned} &\|SQ'_x\| \\ =\ & \sup\{\|SQ'_x h\| : h \in X, \|h\| = 1\} \\ =\ & \sup\left\{\sup_{s\in[a,b]}\left|\int_a^b K(s,t)(\mu(1+p)x(t)^p + 2\nu x(t))h(t)dt\right| : h \in X, \|h\| = 1\right\} \\ \leq\ & \sup\left\{\int_a^b |K(s,t)|(|\mu|(1+p)|x(t)|^p + 2|\nu||x(t)|)|h(t)|dt : h \in X, \|h\| = 1\right\} \\ \leq\ & (|\mu|(1+p)\|x\|^p + 2|\nu|\|x\|)M(b-a) < 1. \end{aligned} \tag{4.12}$$

We now apply Theorem 4.5 to solve the Fredholm integral equation (4.6).

Theorem 4.6 *Let $D = X = Y = C[a,b]$ and $\mu, \nu, \lambda, M \in \mathbb{R}$. Let $S, Q : D \to X$ be operators defined by (4.7) and (4.8), respectively. Let $L \in B(X)$ and $x_0 \in D$ be such that $z_0 = x_0 - L(x_0 - \lambda SQ(x_0) - g) \in D$. Assume that the condition $(C6)$ and the following conditions hold:*

(C18) $\frac{1}{1-|\lambda|(|\mu|(1+p)\|z_0\|^p+2|\nu|\|z_0\|)M(b-a)} = \beta$ *for some* $\beta > 0$;

(C19) $\frac{\|x_0-g\|+|\lambda|(|\mu|\|x_0\|^{p+1}+2|\nu|\|x_0\|^2)M(b-a)}{1-|\lambda|(|\mu|(1+p)\|z_0\|^p+2|\nu|\|z_0\|)M(b-a)} = \eta$ *for some* $\eta > 0$;

(C20) $\|I - L\| + |\lambda|\|L\|(|\mu|(1+p)\|x\|^p + 2|\nu|\|x\|)M(b-a) \leq \frac{1}{3}$ *for all* $x \in D$.

Then the sequence generated by (4.5) with $u = g \in X$ is well defined, remains in $B_r[x_0]$ and converges to the solution $x^* \in B_r[x_0]$ of the equation (4.6) with R-order convergence at least $(2p+1)$.

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Proof. Note that $D = X = Y = C[a, b]$. Obviously, $(C12)$ holds. Using $(C20)$, (4.7), (4.9) and (4.12), we have

$$\|I - (I - \lambda S Q'_{z_0})\| \leq |\lambda|(|\mu|(1+p)\|z_0\|^p + 2|\nu|\|z_0\|)M(b-a) < 1.$$

Therefore, by Lemma 2.1, $(I - \lambda S Q'_{z_0})^{-1}$ exists and

$$\|(I - \lambda S Q'_{z_0})^{-1}\| \leq \frac{1}{1 - |\lambda|(|\mu|(1+p)\|z_0\|^p + 2|\nu|\|z_0\|)M(b-a)}. \tag{4.13}$$

Hence $(C18)$ and (4.13) implies $(C13)$ holds. Using $(C19)$, (4.12) and (4.13), we have

$$\begin{aligned} & \|(I - \lambda S Q'_{z_0})^{-1}(x_0 - \lambda S Q(x_0) - g)\| \\ \leq\; & \|(I - \lambda S Q'_{z_0})^{-1}\|(\|x_0 - g\| + \|\lambda S Q(x_0)\|) \\ \leq\; & \frac{\|x_0 - g\| + |\lambda|(|\mu|\|x_0\|^{p+1} + 2|\nu|\|x_0\|^2)M(b-a)}{1 - |\lambda|(|\mu|(1+p)\|z_0\|^p + 2|\nu|\|z_0\|)M(b-a)} \\ \leq\; & \eta. \end{aligned}$$

Thus the condition $(C14)$ is satisfied. The conditions $(C15)$ and $(C16)$ follow from (4.10) and (4.11), respectively. Now, from $(C20)$ and (4.12), we have

$$\begin{aligned} \|I - L(I - \lambda S Q'_x)\| &\leq \|I - L\| + \|L\|\|\lambda S Q'_x\| \\ &\leq \|I - L\| + \|L\||\lambda|(|\mu|(1+p)\|x\|^p + 2|\nu|\|x\|)M(b-a) \\ &\leq \frac{1}{3}. \end{aligned}$$

This implies that $(C17)$ holds. Hence all the conditions of Theorem 4.5 are satisfied. Therefore, Theorem 4.6 follows from Theorem 4.5. This completes the proof.

Now, we give one example to illustrate Theorem 4.5.

Example 4.7 Let $X = Y = C[0, 1]$ be the space of all continuous real valued functions defined on $[0, 1]$. Let $D = \{x : x \in C[0, 1], \|x\| < \frac{3}{2}\} \subset C[0, 1]$. Consider the following nonlinear integral equation:

$$x(s) = \sin(\pi s) + \frac{1}{10}\int_0^1 \cos(\pi s)\sin(\pi t)x(t)^2 dt. \tag{4.14}$$

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Define two mappings $S : X \to X$ and $Q : D \to Y$ by

$$S(x)(s) = \int_0^1 K(s,t)x(t)dt, \quad Q(x)(s) = x(s)^2,$$

where $K(s,t) = \cos(\pi s)\sin(\pi t)$. For $u = \sin(\pi s)$, the problem (4.14) is equivalent to the problem (4.2). Here, one can easily observe that S is bounded linear operator with $\|S\| \leq 1$ and Q is Fréchet differentiable with $Q'_x h(s) = 2x(s)h(s)$ for all $h \in X$ and $s \in [0,1]$. For all $x, y \in D$, we have

$$\|Q'_x - Q'_y\| \leq 2\|x - y\| = \omega_0(\|x - y\|),$$

where $\omega_0(t) = 2t$ for any $t \geq 0$. Clearly, $\omega_0 \in \Phi$. Define a mapping $F : D \to X$ by

$$F(x)(s) = x(s) - \frac{1}{10}SQ(x)(s) - \sin(\pi s).$$

Clearly, F is Fréchet differentiable on D. We now show that $(C12)$ holds for $L = I \in B(X)$. Note that

$$\|(I - L(I - \lambda SQ))(x) - u\| = \left\| \frac{1}{10}SQ(x)(s) + \sin(\pi s) \right\| \leq \frac{49}{40} < \frac{3}{2}$$

for all $x \in D$. Thus $(I - L(I - \lambda SQ))(x) - u \in D$ for all $x \in D$. For all $x \in D$, we have

$$\|I - F'_x\| \leq \frac{1}{5}\|x\| \leq \frac{3}{10} = k.$$

Therefore, by Lemma 2.1, ${F'_x}^{-1}$ exists and

$${F'_x}^{-1}U(s) = U(s) + \frac{\cos(\pi s)\int_0^1 \sin(\pi t)x(t)U(t)dt}{5 - \int_0^1 \sin(\pi t)\cos(\pi t)x(t)dt} \tag{4.15}$$

for all $U \in Y$.

Let $x_0(s) = \sin(\pi s), \omega(t) = \frac{1}{10}\omega_0(t) = \frac{t}{5}$ and $p = 1$. Then we have the following:

(a) $x_0 \in X$, $F(x_0(s)) = -\frac{2}{15\pi}\cos(\pi s)$;

(b) $z_0(s) = x_0(s) - F(x_0(s)) = \sin(\pi s) + \frac{2}{15\pi}\cos(\pi s)$;

(c) $\|{F'_{z_0}}^{-1}\| \leq 1.201086631226159 = \beta$;

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(d) $\|{F'_{z_0}}^{-1}F(x_0)\| \leq 0.050975699850996 = \eta$;

(e) $b_0 = K\beta\eta = 0.012245246321686$ and $q = \frac{4}{5}$;

(f) $\theta = \frac{\left(\frac{qb_0}{2}+k\right)q{b_0}^2}{1-k(1+q)b_0} = 5.897481575495280 \times 10^{-7}$ and $r = \frac{(1+qb_0)\eta}{1-\theta} = 0.091756313844910$.

Hence all the conditions of Theorem 4.5 are satisfied. Therefore, the sequence $\{x_n\}$ generated by (4.5) is well defined, remains in $B_r[x_0]$ and converges to a solution of the integral equation (4.14).

n	$x_n(s)$	$z_n(s)$	$y_n(s)$
0	$\sin(\pi s)$	$\sin(\pi s) + 0.042441318157839\cos(\pi s)$	$\sin(\pi s) - 0.053779425942169\cos(\pi s)$
1	$\sin(\pi s) - 0.042505426009095\cos(\pi s)$	$\sin(\pi s) - 0.021258998682202\cos(\pi s)$	$\sin(\pi s) - 0.063732700816574\cos(\pi s)$
2	$\sin(\pi s) + 0.042457470555443\cos(\pi s)$	$\sin(\pi s) + 0.021258912220022\cos(\pi s)$	$\sin(\pi s) + 0.021239768415502\cos(\pi s)$
3	$\sin(\pi s) + 0.021230223695582\cos(\pi s)$	$\sin(\pi s) + 0.021230223705270\cos(\pi s)$	$\sin(\pi s) + 0.021230223705279\cos(\pi s)$
4	$\sin(\pi s) + 0.021230223705279\cos(\pi s)$	$\sin(\pi s) + 0.021230223705279\cos(\pi s)$	$\sin(\pi s) + 0.021230223705279\cos(\pi s)$

Table 2: Convergence behavior of Newton-like iteration process (4.5)

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