Communication

Equation for Calculation of Critical Current Density Using the Bean's Model with Self-Consistent Magnetic Units to Prevent Unit Conversion Errors

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Equation for Calculation of Critical Current Density Using the Bean's Model with Self-Consistent Magnetic Units to Prevent Unit Conversion Errors

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This study analyzes the calculation of the critical current density $J_{c,mag}$ by means of the Bean's critical state model, using the equation formulated by Gyorgy et al. and other similar equations derived from Gyorgy et al. reported in literature. While estimations of $J_{c,mag}$ using the Bean's model are widely done, improper use of different equations with different magnetic units and pre-factors, leads to confusion and to significant errors in the reported values of $J_{c,mag}$. We propose a SINGLE general equation for calculation of $J_{c,mag}$ for a rectangular parallelepiped sample in perpendicular field using the Bean's critical state model, underlying how the simple conversion of magnetic units can lead to a $J_{c,mag}$ in the desired units, without the need to introduce any other correction or use other specific equations depending on the units of $J_{c,mag}$. This resolves all ambiguities and aligns with the correct dimensional analysis, eliminates discrepancies in the calculated $J_{c,mag}$ and will avoid further propagation of errors in the literature. In this equation, the numerical pre-factor is dimensionless, independent of the unit system used. There is tremendous excitement for many large-scale applications of superconductors as reported in Goyal et al. [1]. In particular, energy generation via commercial nuclear fusion is presently of great interest [1]. There is significant research ongoing world-wide to improve the performance of superconducting wires based on various coated conductor technologies to improve the performance and lower the cost of these wires and these have been previously reported. In such studies, characterization of the critical current densities by transport and magnetization is of paramount interest. However, calculation of critical current densities, $J_{c,mag}$, based on magnetization of a coated conductor represented as a rectangular parallelepiped sample in perpendicular magnetic field poses a challenge in terms of expressions used and self-consistency of magnetic units used in past studies and reported in the literature. A commonly used expression to calculate the critical current density in a majority of these publications is the Bean's Model [2, 3]. However, since Bean did not report a specific expression to calculate $J_{c,mag}$ in the considered geometry, the actual formula used in most prior publications is the equation shown below (at a given field and temperature), reported by Gyorgy *et al.* [4] and derived on the basis of the Bean's model. This formula has been used extensively in numerous publications in the literature, only a few of which are cited here [5-33]:

$$J_{c,mag}(B,T) = \frac{20 \times \Delta M}{w \times \left(1 - \frac{w}{3l}\right)} \qquad (1)$$

where the current density, $J_{c,mag}$, is typically desired in [A/cm^2], and the unit-volume full-width of the hysteresis loop for decreasing and increasing applied magnetic field $\Delta M= \mu(B-,T)- \mu(B+,T)/(w \times l \times t)$ is typically measured in [emu/cm^3] (i.e. in gaussian units). The dimensions for the width w, the length l, and the thickness t are typically measured in [cm]. The calculation of critical current density based on magnetization poses a challenge in terms of self-consistency of magnetic units used in past studies and reported in the literature.

In their equation for the calculation of $J_{c,mag}$ in the general case of anisotropic critical currents, Gyorgy *et al*. [4] explicitly state that ΔM is in gauss, and when the field is applied perpendicular to one surface of dimensions *l* by *t* of a rectangular parallelepiped sample, it is expressed by:

$$\Delta M = \frac{J_{c1}\,t}{20}\left(1 - \frac{t}{3l}\frac{J_{c1}}{J_{c2}}\right), \qquad (1)$$

where ΔM (in gauss) is the width of the hysteresis loop for increasing and decreasing H_a, the current densities are in A/cm² and the dimensions in cm. For $l \gg J_{c1}\,t/3J_{c2}$,

Figure 1. Expression for the calculation of $J_{c,mag}$ as reported in Gyorgy et al. (expression 1, page 283), in the general case for anisotropic critical current densities $J_{c1} \neq J_{c2}$.

Assuming that $J_{c1} = J_{c2} = J_c$ the expression (1) becomes:

$$\Delta M = \frac{J_c t}{20}\left(1 - \frac{t}{3l}\right)$$

From Gyorgy *et al*., it is clear that if ΔM is expressed in **gauss**, and the dimensions in cm, the resulting current density is in A/cm², *without* any other conversion of units. This can be checked also by considering the data and specific examples reported in Gyorgy *et al*.

In practice, magnetometers such the SQUID Magnetometer and the PPMS (VSM and ACMS options) give magnetic moment in emu, and sample dimensions are in cm. The interest is in the estimation of current density, $J_{c,mag}$, in A/cm². The moment is converted to volume magnetization, M, by dividing with the sample volume, giving units of M to be emu/cm³.

In order to use Gyorgy *et al*. with the correct magnetic units, ΔM in emu/cm³ has to be converted to gauss. A standard table of magnetic unit conversions is shown in databases of IEEE and NIST [34, 35], where (Volume) Magnetization M can be in emu/cm³, and when M is multiplied by 4π, the (Volume) Magnetization is in gauss (G). Then the calculation using Gyorgy *et al*. gives $J_{c,mag}$, in A/cm².

Following the equation for $J_{c,mag}$ reported in many works in literature, where the magnetization is expressed in emu/cm³ [5-33] the second route to obtaining the magnetic critical current density in A/cm², is by converting ΔM in emu/cm³ to SI units of A/m with the expression 1 emu/cm³ = 10³ A/m or 1 emu/cm³ = 10 A/cm [34, 35]. With this conversion of ΔM to A/cm, using expression (1), yields $J_{c,mag}$, in A/cm².

The third route is to use the expression (1) of Gyorgy *et al*., have ΔM in emu/cm³, and assume that the number 20 in expression (1) has the units of A·cm²/emu. Although not reported in regularly published works, this procedure can be found on some papers deposited on freely accessible online repositories of electronic preprint manuscripts [36, 37].

There is also a fourth route, using the expression (1) with ΔM in emu/mm³, the sample's dimensions in mm, some unclearness between the use of the magnetic moment or the magnetization, and the obtained $J_{c,mag}$ in A/mm² [38,39]

In the first and second cases above, whether ΔM is expressed in gauss or converted (as it should be done) from emu/cm³ to A/cm, an error is obtained when using Gyorgy *et al*.'s expression (1), which is the commonly cited formula in the literature to calculate $J_{c,mag}$ using the Bean's model. In fact, in the first case when ΔM is converted to gauss, a factor 12.56 higher $J_{c,mag}$ is obtained, whereas in the second case $J_{c,mag}$ results 10 times higher. Although the third route provides the correct numerical value, it is conceptually wrong because it artificially introduces a non-dimensionless pre-factor 20 [A·cm²/emu], not predicted by the theory, just to force the dimensional balance of an equation with non-coherent units (as it will be shown in the following). In addition, if different magnetic and/or dimensional units are used, the pre-factor of 20 needs to have different "assumed" units, as it should be for the equation used in the fourth route above, as example.

The scenarios cited above lead to confusion and errors in the calculated $J_{c,mag}$ using the Bean's model. It is difficult to ascertain how Gyorgy *et al*. has been applied in numerous and extensive previous publications, since in most cases, the measured moment and/or volume magnetization and sample dimensions are typically not reported. However, it is likely that such errors have been made

in previous studies, because it is unusual to expect the pre-factor, an integer number, to have complicated and non-intuitive units of A·cm²/emu (or A·mm²/emu, or A·cm²/Gauss or whatever, depending on the authors), specially, given that these units for 20 are not clearly specified in literature.

The correct formula [40] for calculation of $J_{c,mag}$ starting from the *Magnetization* M, within the Bean's critical state model, for a rectangular parallelepiped sample with isotropic critical current densities and with volume w x b x d, when the magnetic field is applied perpendicular to the largest face with dimension w by b (with $w \leq b$), and with *self-consistent magnetic units* that should be used is:

$$J_c = \frac{2 \times \Delta\mu}{V \times w \times \left(1 - \frac{w}{3b}\right)} \tag{2}$$

If $\Delta\mu$ (where μ is the magnetic moment) is expressed in $[A\ m^2]$, the volume $V = (w\ b\ d)$ is in $[m^3]$, and the sample side lengths w and b are in $[m]$, the ΔM results in $\frac{A}{m}$ and the J_c is expressed in SI units as $\frac{Am^2}{m^3 \times m} = \frac{A}{m^2}$. Here, the number 2 is *dimensionless*. So:

$$\frac{2 \times \Delta M[A/m]}{w[m] \times \left(1 - \frac{w[m]}{3b[m]}\right)} = J_{c,mag}[A/m^2] \tag{3}$$

As indicated in ref. [34, 35], it is possible to convert the magnetic moment from $[A\ m^2]$ to *emu*, and the lengths from m to cm, by means of:

$\Delta\mu[emu] = (10^3\ emu/(Am^2))\Delta\mu[Am^2]$

$V[cm^3] = (10^6\ cm^3/(m^3))V[m^3]$

and $w[cm] = (10^2\ cm/(m))w[m]$

Using these conversions, the J_c in the equation [1] can be expressed as $\frac{emu}{cm^3} \times \frac{1}{cm} = \frac{emu}{cm^4}$ with a conversion factor of

$$\frac{10^3}{10^6} \times \frac{1}{10^2} = 10^{-5}$$

with respect to the equation (2) in SI units.

But $\frac{emu}{cm^3} = 10^3 \frac{A}{m}$ and, the simple final conversion from $\frac{A}{m}$ to $\frac{A}{cm}$ can be considered, and so we can write $\frac{emu}{cm^3} = 10^1 \frac{A}{cm}$.

In this way we obtain the units conversion from cgs emu system to some kind of "hybrid" cgs/SI system (or "practical CGS system"). With this conversion, the J_c in the equation (2) expressed as $\frac{emu}{cm^3} \times \frac{1}{cm}$ with a conversion factor of 10⁻⁵, can be expressed in $\frac{A}{cm} \times \frac{1}{cm} = \frac{A}{cm^2}$ with a conversion factor of $10^{-5} \times 10^1 = 10^{-4}$ with respect to the equation in SI units, which is exactly what is expected if the conversion $\frac{A}{m^2} = 10^{-4} \frac{A}{cm^2}$ is applied to the final result obtained in SI units from the equation (2). So, the conversion from the results of equation (2) expressed in different unit system is simply done by correctly applying the unit conversions [34, 35], as for any formula in science, without the need to create different equations for different unit systems and add dimensional pre-factors to the existing equation.

So, if the magnetic moment is in *emu* and the sample's dimensions in *cm*, the described approach leads to write the equation (2) by making the used units explicit:

$$J_{c,mag} = \frac{2 \times \Delta M[emu/cm^3]}{w[cm] \times \left(1 - \frac{w[cm]}{3b[cm]}\right)} \tag{4}$$

where it is evident that an expression having the dimensions of $([emu/cm^3]/cm)$ gives as result a quantity having the dimensions of $[emu/cm^4]$, definitely not practical for a current density. For this reason, this form of the equation 4 can be re-written in more practical units, by the previously indicated conversion $\frac{emu}{cm^3} = 10^1 \frac{A}{cm}$ which makes the dimensionless number 2 in the equation 4 to become the dimensionless number 20 in the 2ⁿᵈ form of the equation 4 and, therefore, the current density to be expressed in the more practical dimensions of $[A/cm^2]$:

$$\frac{20 \times \Delta\mathrm{M}[A/cm]}{w[cm] \times \left(1 - \frac{w[cm]}{3b[cm]}\right)} = J_{c,mag}[A/cm^2] \tag{5}$$

The above discussion clearly shows that equation (2) is the correct one for the sample geometry described in the text, and it can be used independently of the fact that one has the magnetic moment in [$A\ m^2$] (and dimensions in meters) or in [emu] (and dimensions in centimetres), by using the conversion of units, without adding any non-dimensionless pre-factor:

when the magnetic moment is in [$A\ m^2$] and the dimensions in meters, their use in equation (2) leads to the equation (3) where the current density is directly in [A/m^2]

when the magnetic moment is in [emu] and the dimensions in centimetres, their use in equation (2) leads to the equation (4) where the current density is in the unpractical [emu/cm^4]. In order to have the current density expressed in more practical units, the conversion $\frac{emu}{cm^3} = 10^1 \frac{A}{cm}$ can be used, which introduces a multiplicative dimensionless term 10 that transforms the number 2 into 20, and leads to a current density expressed in [A/cm^2] (equation (5)). On the contrary, the expression $J_c = \frac{20 \times \Delta M}{w \times \left(1 - \frac{w}{3l}\right)}$ where ΔM is stated to be in emu/cm³ and J_c is resulting in A/cm² is misleading, because the dimensions on the right and left-hand side of the equation are not coherent and erroneously suggest the need for a further unit conversion.

This approach is also consistent with Poole et al. [40], where the equation to obtain the critical current density using the Bean's model are reported for several geometries. These equations are obtained by starting from the general equation (13.24) for the magnetic moment of an arbitrary shaped sample, and Poole et al. say that: "Equation (13.24) is an SI formula in which current density j is in amperes per square meter, magnetic field B is in tesla, and lengths are in meters. When practical units are used whereby j is measured in A/cm², magnetic field in Gauss, and length in centimetres, the factor $\frac{1}{2}$ in Eq. (13.24) is replaced by 1/20. To convert the formulae below for the magnetic moment to practical units simply divide by 10."

In fact, let's start from the definition of magnetic moment μ, as reported in ref. [40] -page 396, equation 13.24:

$$\mu = \frac{1}{2}\int_V [r \times j(r)]d^3r \tag{6}$$

and for simplicity in the notation let's denote the integrated quantity as X, so: $X = \int_V [r \times j(r)]d^3r$.

The equation (6) is an SI equation, the pre-factor $\frac{1}{2}$ is dimensionless, and if the current density j is expressed in A/m^2, and the lengths in meters (m), the value of X results to be expressed in:

(m) x (A/m^2) x (m^3) = ($A\ m^4/m^2$) = A m^2, coherently with the dimension of the magnetic moment μ in SI [34, 35].

Following what reported in ref. [40], when practical units are used, the current density j is expressed in A/cm^2, and the lengths in centimetres (cm) the value of X results to be expressed in: (cm) x (A/cm^2) x (cm^3) = ($A\ cm^4/cm^2$) = A cm^2. In this way, the equation (6) in practical units appears as:

$$\mu[emu] = \frac{1}{2}X[A\ cm^2] \tag{7}$$

But: $emu = 10^{-3}$ ($A\ m^2$) = 10 ($A\ cm^2$) [34,35] and so equation (7) is:

$$10\mu[A\ cm^2] = \frac{1}{2}X[A\ cm^2] \tag{8}$$

or

$$\mu[A\ cm^2] = \frac{1}{20}X[A\ cm^2] \tag{9}$$

and for the magnetization $M = \mu/V$:

$$M = \frac{\mu[A\ cm^2]}{V[cm^3]} = \frac{1}{20}\frac{\int_V [r \times j(r)]d^3r}{V}\left[\frac{A}{cm}\right] \tag{10}$$

where the current density j is expressed in A/cm^2, and the lengths in centimetres (cm), and the pre-factor 1/20 is still dimensionless, coherently with the equation (5).   Following this approach, as

also reported by Poole et al. [40] to obtain in particular the equation 13.30 at page 396, the expressions (2-5) can be calculated with the proper units and numerical pre-factors.

Since we have analysed how to perform the calculation of J_c both in SI and in the 'hybrid' cgs-emu/SI system, it is worth to consider also what happens in the emu system, where electric current is not an independent physical quantity and can be expressed in abampere (abA), also called biot (Bi). More precisely [41]:

$$abA = Bi = cm^{1/2}g^{1/2}s^{-1}$$

The conversion to SI electric current unit is [41]: $1 Bi = 10\ A$.

Remembering that the term "emu" used as unit for the magnetic moment μ is not an unit but just indicates electromagnetic units, it is important to specify that in emu system, the unit for the magnetic moment $\mu = \pi I a^2$ of a circular loop of radius a carrying a current I is $[erg/Gauss]$, that, of course, as indicated in the Goldfarb's paper [42] can be expressed in terms of Bi as:

$$[erg/Gauss] = [Bi \cdot cm^2]$$

Therefore, the equation (2) above can be used in emu system by indicating $\Delta\mu$ in $[Bi \cdot cm^2]$, the volume V in $[cm^3]$ and the length in w in $[cm]$, so leading to a current density expressed in $\frac{Bi \cdot cm^2}{cm^3} \times \frac{1}{cm} = \frac{Bi}{cm^2}$, i.e.:

$$J_{c,mag} = \frac{2 \times \Delta M[Bi/cm]}{w[cm] \times \left(1 - \frac{w[cm]}{3b[cm]}\right)} = J_{c,mag}\left(\frac{Bi}{cm^2}\right) \tag{11}$$

that is definitely not practical. Since $1 Bi = 10\ A$, the expression above can be written also as:

$$J_{c,mag} = \frac{2 \times 10 \times \Delta M[A/cm]}{w[cm] \times \left(1 - \frac{w[cm]}{3b[cm]}\right)} = \frac{20 \times \Delta M[A/cm]}{w[cm] \times \left(1 - \frac{w[cm]}{3b[cm]}\right)} = J_{c,mag}[A/cm^2] \tag{12}$$

namely again the expression for $J_{c,mag}$ in practical units, were the pre-factor 20 is still dimensionless.

After proving the non-dimensionality of the pre-factor 20, it is probably easier to interpret the equations (2-5):

Starting from the equation (2) above in this paper, with the use of the SI units system where the magnetic moment μ is expressed in $[Am^2]$ and the dimensions are in meters $[m]$ the calculation is direct and the critical current density $J_{c,mag}$ is direcly obtained in $[A/m^2]$. Since many magnetometers present their output in emu (although in several models it is possible to choose the units for the output), the same equation (2) can be used by inserting the magnetic moment μ in emu and the dimensions in centimetres $[cm]$, leading for $J_{c,mag}$ a certain value (for practical reasons, let's call "Z" this value, simply resulting from the numbers given to the quantities in equation (2)) in $[emu/cm^4]$. If the value "Z" in $[emu/cm^4]$ is multiplied by 10, because of the conversion

$$\frac{emu}{cm^3} = 10\ \frac{A}{cm}$$

the $J_{c,mag}$ results in $[A/cm^2]$.

This makes the equation (4) to be converted into the equation (5), without adding any non-dimensionless pre-factors. This is also what Poole et al. [40] mean when at page 396 they write "To convert the formulae below for the magnetic moment to practical units simply divide by 10".

To conclude, no matter if one has the physical quantities expressed in SI, in practical units or in emu, the equation to be used (for the geometrical conditions already stated above) is equation (2), and depending on the final units that are needed for the current density, one has just to use the standard conversion units [34, 35, 41, 42], avoiding to make the error of mixing non-coherent units, and to compensate that error by artificially introducing dimensional pre-factors to force the dimensional balance of the equation. We hope that this will clarify any doubt about the application of the correct equations and units for the calculation of the critical current density from magnetic measurements, in particular, for a rectangular parallelepiped sample in perpendicular field, although the same approach can be extended also to analogous equations for different geometries [40].

In order to check the validity of the discussed approach, the consistency of the measurement units and the absence of any non-dimensionless pre-factor in the expression for the calculation of the critical current density by the equation (2) above in any unit system, it can be helpful to make an

example of the typical calculation of $J_{c,mag}$, if one has on hand the magnetic moment and the dimensions of the sample.

Let's consider a hypothetical sample with these characteristics:

QUANTITY	VALUE IN EMU-CGS	VALUE IN SI SYSTEM
Thickness d = 200 nm	$2 \cdot 10^{-5}$ (cm)	$2 \cdot 10^{-7}$ (m)
Width w = 4 mm	$4 \cdot 10^{-1}$ (cm)	$4 \cdot 10^{-3}$ (m)
Length b = 4 mm	$4 \cdot 10^{-1}$ (cm)	$4 \cdot 10^{-3}$ (m)
Volume (wbd)	$32 \cdot 10^{-7}$ (cm^3)	$32 \cdot 10^{-13}$ (m^3)
Δ(Magnetic moment) = $\Delta\,\mu$	$3\ emu$	$3 \cdot 10^{-3}$ $(A \cdot m^2)$

Within the Bean's critical state model, for a rectangular parallelepiped sample with isotropic critical current densities and with volume w x b x d, when the magnetic field is applied perpendicular to the largest face with dimension w by b (with $w \leq b$) the equation to be used for $J_{c,mag}$ is the equation (2) above in this paper (or equivalently the equation (13.30) in ref. [40]).

So, by considering μ in *emu* and the dimensions in cm, the following calculations apply:

$$J_c = \frac{2 \times \Delta\mu}{V \times w \times \left(1 - \frac{w}{3b}\right)} = \frac{2 \times 3\ emu}{32 \cdot 10^{-7}\ (cm^3) \times 4 \cdot 10^{-1}\ (cm) \times \left(\frac{2}{3}\right)}$$

$$= \frac{6\ emu}{128 \cdot 10^{-7}\ (cm^3) \times 10^{-1}\ (cm)} \times \frac{3}{2} = 0.0703 \cdot 10^8\ \frac{emu}{cm^4}$$

Using the conversion $\frac{emu}{cm^3} = 10\ \frac{A}{cm}$, this can be written as:

$$J_c = 0.0703 \cdot 10^8\ \frac{emu}{cm^3}\frac{1}{cm} = 0.0703 \cdot 10^8 \cdot 10^1\ \frac{A}{cm}\frac{1}{cm} = 0.0703 \cdot 10^9\ \frac{A}{cm^2} = 7.03 \cdot 10^7\ \frac{A}{cm^2}$$

and, since $1\ \frac{A}{cm^2} = 10^4\ \frac{A}{m^2}$

$$J_c = 7.03 \cdot 10^7\ \frac{A}{cm^2} = 7.03 \cdot 10^7 \cdot 10^4\ \frac{A}{m^2} = 7.03 \cdot 10^{11}\ \frac{A}{m^2}$$

On the other hand, by considering μ in $(A \cdot m^2)$ and the dimensions in m, the following calculations apply:

$$J_c = \frac{2 \times \Delta\mu}{V \times w \times \left(1 - \frac{w}{3b}\right)} = \frac{2 \times 3 \cdot 10^{-3}\ (A \cdot m^2)}{32 \cdot 10^{-13}\ (m^3) \times 4 \cdot 10^{-3}\ (m) \times \left(\frac{2}{3}\right)}$$

$$= \frac{6 \cdot 10^{-3}\ (A \cdot m^2)}{128 \cdot 10^{-13}\ (m^3) \times 10^{-3}\ (m)} \times \frac{3}{2} = 0.0703 \cdot 10^{13}\ \frac{A}{m^2} = 7.03 \cdot 10^{11}\ \frac{A}{m^2}$$

as obtained above, and since $1\ \frac{A}{m^2} = 10^{-4}\ \frac{A}{cm^2}$

$$J_c = 7.03 \cdot 10^{11}\ \frac{A}{m^2} = 7.03 \cdot 10^{11} \cdot 10^{-4}\ \frac{A}{cm^2} = 7.03 \cdot 10^7\ \frac{A}{cm^2}$$

as obtained above, with self-consistency and no ambiguity with respect to the magnetic units on the left and right side of the equation (2), without modifying the equation depending on the used unit system, and without artificially introducing any dimensional pre-factors to force the dimensional balance of the equation.

In summary, the formula for calculation of $J_{c,mag}$ starting from the *Magnetization* M, within the Bean's critical state model, for a rectangular parallelepiped sample with isotropic critical current densities and with volume w x b x d, when the magnetic field is applied perpendicular to the largest face with dimension w by b (with $w \leq b$), and with *self-consistent magnetic units* that should be used is:

$$\frac{2 \times \Delta M}{w \times \left(1 - \frac{w}{3b}\right)} = J_{c,mag}$$

where, ΔM is the width of the volume magnetization loop. In this equation, the pre-factor, number 2, is *dimensionless* similar to numbers in any scientific formula or expression. The units of M can be *modified using appropriate magnetic unit conventions* and so can the units of sample dimensions w, b and d to arrive at $J_{c,mag}$ in the desired units, thereby, removing any ambiguities related to units of the pre-factor. Use of this expression will resolve continued propagation of errors in the literature, is consistent with correct dimensional analysis, and eliminates discrepancies in the calculated $J_{c,mag}$.

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