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$k-$Generalized Fibonacci Numbers as Concatenation of Three Repdigits

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Abstract: The k-Fibonacci sequence is a generalization of the classic Fibonacci sequence with some fixed integer $k \geq 2$. In this paper, we identify all k-Fibonacci numbers which can be represented as concatenation of three repdigits. This work builds upon and extends the previous research by Erduvan and Keskin, who identified all the Fibonacci numbers with this property. The computations were carried out with the help of a simple computer program in *Mathematica*.

Keywords: $k-$Generalized Fibonacci numbers; concatenation; repdigits; linear forms in logarithms; Baker-Davenport reduction method

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1. Introduction

One characteristic of a palindromic number is that it doesn't alter when its digits are switched. A repdigit is a particular kind of palindromic number that is made up of a single digit that is repeated several times in base 10. The mathematical expression for a repdigit is $a(10^m - 1)/9$ for some $m \geq 1$ and $1 \leq a \leq 9$. Interestingly, in the trivial situation of a repdigit, $m = 1$, the outcome is just the digit itself.

The Fibonacci sequence (F_n) is defined recursively by

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2$$

with initial terms 0 and 1, and each subsequent term is the sum of the two preceding terms. An explicit formula for the n-th Fibonacci number, known as the Binet formula, is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$.

The k-generalized Fibonacci sequence or simply the k-Fibonacci sequence denoted as $(F_n^{(k)})_{n \geq 2-k}$, for an integer $k \geq 2$. The first k terms are equal to 0, and subsequent term is 1. The sum of the preceding k phrases determines each next term in the series. More specifically, for every $n \geq 2$, the recurrence relation for the k-generalized Fibonacci sequence is

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}.$$

The initial values are given by $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. For instance, when $k = 2$, the sequence corresponds to the standard Fibonacci numbers. When $k = 3$, the sequence is

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known as the Tribonacci numbers, and for $k = 4$, it is referred to as the Tetranacci numbers. This pattern continues as k increases.

In recent years, diophantine equations with repdigits and terms from binary recurrence sequences have attracted a lot of interest. Specifically, research on Fibonacci numbers and how they relate to repdigits has produced a number of significant findings. According to Luca [12], the Fibonacci sequence's biggest repdigit is 55. A hypothesis by Marques [13] was later confirmed by Bravo and Luca [3], who showed that no repdigit with two or more digits exists in any k-Fibonacci sequence for $k > 3$. Fibonacci numbers, which may be represented as the concatenation of two repdigits, were studied by Alahmadi et al. [1]. This study was later expanded to include k-Fibonacci numbers in [2]. In 2023, Erduvan and Keskin [10] found all Fibonacci numbers which are concatenation of three repdigits. In this paper, we investigate the representation of k-Fibonacci numbers as concatenation of three repdigits. In 2023, Erduvan and Keskin [10] found all Fibonacci numbers which are concatenation of three repdigits. In this paper, we investigate the representation of k-Fibonacci numbers as concatenation of three repdigits. Specifically, we focus on expressing k-Fibonacci numbers in the form

$$F_n^{(k)} = \overline{\underbrace{a\ldots a}_{m_1 \text{ times}}\underbrace{b\ldots b}_{m_2 \text{ times}}\underbrace{c\ldots c}_{m_3 \text{ times}}}, \tag{1.1}$$

extending the work of Erduvan and Keskin [10]. More specifically, we present the following result.

Theorem 1.1. *For $k \geq 2$ and $n \geq k+2$, the Diophantine equation*

$$\begin{aligned} F_n^{(k)} &= \overline{\underbrace{a\ldots a}_{m_1 \text{ times}}\underbrace{b\ldots b}_{m_2 \text{ times}}\underbrace{c\ldots c}_{m_3 \text{ times}}} = \underbrace{\overline{a\cdots a}}_{m_1 \text{ times}} \times 10^{m_2+m_3} + \underbrace{\overline{b\cdots b}}_{m_2 \text{ times}} \times 10^{m_3} + \underbrace{\overline{c\cdots c}}_{m_3 \text{ times}} \\ &= \frac{1}{9}\left(a10^{m_1+m_2+m_3} - (a-b)10^{m_2+m_3} - (b-c)10^{m_3} - c\right), \end{aligned} \tag{1.2}$$

has exactly 28 positive integer solutions $(n, k, m_1, m_2, m_3, a, b, c)$ with $1 \leq d_1 \leq 9$, $0 \leq d_2, d_3 \leq 9$, and $m_1, m_2, m_3 \geq 1$.

$F_{15}^{(2)} = 610$	$F_{16}^{(2)} = 987$	$F_{22}^{(2)} = 17711$	$F_{10}^{(3)} = 149$
$F_{11}^{(3)} = 274$	$F_{12}^{(3)} = 504$	$F_{13}^{(3)} = 927$	$F_{9}^{(4)} = 108$
$F_{10}^{(4)} = 208$	$F_{11}^{(4)} = 401$	$F_{15}^{(4)} = 5536$	$F_{9}^{(5)} = 120$
$F_{10}^{(5)} = 236$	$F_{11}^{(5)} = 464$	$F_{12}^{(5)} = 912$	$F_{9}^{(6)} = 125$
$F_{10}^{(6)} = 248$	$F_{11}^{(6)} = 492$	$F_{12}^{(6)} = 976$	$F_{9}^{(7)} = 127$
$F_{10}^{(7)} = 253$	$F_{11}^{(7)} = 504$	$F_{12}^{(7)} = 1004$	$F_{20}^{(7)} = 24888$
$F_{11}^{(8)} = 509$	$F_{15}^{(9)} = 8144$	$F_{14}^{(10)} = 4088$	$F_{18}^{(15)} = 65533$

Furthermore, for $n < k+2$, $F_n^{(k)}$ is a power of 2, and the only solutions toEquation (1.2)are

$$F_9^{(k)} = 128, \ (for\ k \geq 8), \ F_{10}^{(k)} = 256, \ (for\ k \geq 9), and\ F_{11}^{(k)} = 512, \ (for\ k \geq 10).$$

2. Auxiliary Results

Matveev's finding on lower bounds for nonzero linear forms of logarithms of algebraic numbers will be used frequently to solve the Diophantine equations. These bounds are crucial for effectively resolving these equations. We will start by going over the main ideas and significant findings from algebraic number theory.

Consider δ is an an algebraic number with minimal polynomial

$$g(X) = r_0(X - \delta^{(1)}) \cdots (X - \delta^{(k)}) \in \mathbb{Z}[X].$$

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The conjugates of δ are $\delta^{(i)}$'s and $r_0 > 0$. Consequently, the *absolute logarithmic height* of δ is given by

$$h(\delta) = \frac{1}{k}\left(\log r_0 + \sum_{j=1}^{k} max\{0, \log|\delta^{(j)}|\}\right).$$

If $\delta = \frac{r}{s}$, $(s \neq 0)$ is a rational number with $\gcd(r,s) = 1$, then $h(\delta) = \log(\max\{|r|, s\})$.

The following are some properties of the absolute logarithmic height, with their proofs available in [4] [Theorem B5]. Let ρ and δ be two algebraic numbers, then

$$(i)\ \ h(\rho \pm \delta) \leq h(\rho) + h(\delta) + \log 2,$$
$$(ii)\ \ h(\rho\delta^{\pm 1}) \leq h(\rho) + h(\delta),$$
$$(iii)\ \ h(\rho^k) = |k|h(\rho).$$

Expanding on the previous notations, we present a theorem that improves upon a result by Matveev [14], as further developed by Bugeaud et al. [5]. This theorem establishes a precise upper bound for the variables in Equation (1.2).

Theorem 2.1. *[14]. Let $\gamma_1, \ldots, \gamma_l \in \mathbb{L}$ be positive real numbers in an algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$ and let $e_1, \ldots, e_l$ be nonzero integers. Consider $\Gamma = \prod_{i=1}^{l} \gamma_i^{e_i} - 1$. If $\Gamma \neq 0$, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} \cdot l^4.5 \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}}^2)(1 + \log(D)) A_1 A_2 \cdots A_l,$$

where $D \geq \max\{|e_1|, \ldots, |e_l|\}$ and $A_1, \cdots, A_l$ are positive integers such that $A_j \geq h'(\gamma_j) = \max\{d_{\mathbb{L}} h(\gamma_j), |\log \gamma_j|, 0.16\}$, for $j = 1, \ldots, l$.

Another approach employed in our proofs is the Baker-Davenport reduction method, introduced by Dujella and Pethő [9]. This method will be utilized to refine the upper bounds on the variables involved.

Lemma 2.1. *[9]Assume that M is the upper bound of u and $q > 6M$ represents the n-th convergent of the continued fraction corresponding to an irrational number τ. Consider some real numbers A, B, μ where $A > 0$ and $B > 1$. Using $\|.\|$ to indicate the distance to the nearest integer, define $\epsilon = \|\mu q\| - M\|\tau q\|$. Then, as long as $\epsilon > 0$ with*

$$w \geq \frac{\log(Aq/\epsilon)}{\log B},$$

the inequality $0 < |u\tau - v + \mu| < AB^{-w}$ has no solution, where $u, v, w \in \mathbb{Z}^+$.

The previously stated lemma is not applicable for $\mu = 0$, since $\epsilon < 0$. Rather, in these cases, we use the following well-known continuing fraction characteristic.

Lemma 2.2. *[15]. Let $r/s = p_i/q_i$ represent the convergences of the continued fraction expansion $[a_0, a_1, \ldots]$ of the irrational number τ. Define*

$$a_M = \max\{a_i \mid 0 \leq i \leq N+1\},$$

where $M, N \in \mathbb{N}$ such that $q_N \leq M < q_{N+1}$. Then, the inequality

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a_M + 2)r^2}$$

holds for all $r < M$ and any pair of positive integers (r, s) with $s > 0$.

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Lemma 2.3. *[11] Assume that $z \geq 1$ and $T > 0$ make $T > (4z^2)^z$ and $T > P/(\log P)^z$. It follows that $P < 2^z T(\log T)^z$.*

2.3.1. k-Generalized FIBONACCI Numbers

The initial $k+1$ nonzero terms of $F_n^{(k)}$ can be explicitly identified as powers of 2, specifically:

$$F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \ldots, \quad F_{k+1}^{(k)} = 2^{k-1}.$$

The subsequent term is given by $F_{k+2}^{(k)} = 2^k - 1$.

The characteristic polynomial of the k–generalized Fibonacci sequence, denoted by $F^{(k)} = \{F_n^{(k)}\}_{n \geq -(k-2)}$, is defined as

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

This polynomial has an unique real root greater than 1, $\alpha = \alpha(k)$, which is located inside the range $(2(1-2^{-k}), 2)$ and is irreducible over $\mathbb{Q}[x]$ (see [8]). Often known as the dominant root of $F^{(k)}$, the root α will now be simply represented as α, with its reliance on k removed for convenience. The roots of $\Psi_k(x)$ are denoted by $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}$, where we adopt the convention

$$\alpha = \alpha^{(1)}.$$

The function $f_k(z)$ for every integer $k \geq 2$ is defined as

$$f_k(z) = \frac{z-1}{2} + (k+1)(z-2), \quad \text{for } z \in \mathbb{C}. \tag{2.1}$$

Using these notations, Dresden and Du introduced in [8] a "Binet-like" formula for the terms of $F^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}) \left(\alpha^{(i)}\right)^{n-1}. \tag{2.2}$$

They also showed that the influence of the roots within the unit circle in (2.2) is minimal. Their approximation, which holds for all $n \geq -(k-2)$, is as follows:

$$\left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2}. \tag{2.3}$$

Moreover, Bravo and Luca concluded in [3] that $F_n^{(k)}$ meets the inequality

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \quad \text{for } n \geq 1 \text{ and } k \geq 2. \tag{2.4}$$

Lemma 2.4. *[3] Let $k \geq 2$, and let α represent the dominant root of the sequence $\{F_n^{(k)}\}_{n \geq -(k-2)}$, with $f_k(z)$ defined as in (2.1). Then*

(i) The inequalities

$$\frac{1}{2} < f_k(\alpha) < \frac{3}{4}, \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \leq i \leq k,$$

are satisfied.

(ii) The logarithmic height of $f_k(\alpha)$ is bounded above by

$$h(f_k(\alpha)) < 3 \log k.$$

Lemma 2.5. *[6] For $1 \leq n < 2^{k/2}$ and $k \geq 10$, we have*

$$F_n^{(k)} = 2^{n-2}(1+\zeta), \quad \text{where } |\zeta| < \frac{5}{2^{k/2}}. \tag{2.5}$$

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Lemma 2.6. *All solutions of Equation (1.2)satisfy*

$$(m_1 + m_2 + m_3) \log 10 - 2 \leq n \log \alpha < (m_1 + m_2 + m_3) \log 10 + 2.$$

Proof. The result follows directly from the fact that $\alpha^{n-2} < F_n^{(k)} < \alpha^{n-1}$. The estimate in (2.4) can be used to determine that

$$\alpha^{n-2} < F_n^{(k)} < 10^{m_1+m_2+m_3}.$$

After rearranging and applying the logarithm to both sides, we get

$$n \log \alpha < (m_1 + m_2 + m_3) \log 10 + 2 \log \alpha.$$

Under the condition $\log \alpha < 1$ for $k \geq 2$, this expression further simplifies to $n \log \alpha < (m_1 + m_2 + m_3) \log 10 + 2$. For the lower bound, the inequality $10^{m_1+m_2+m_3-1} \leq F_n^{(k)} < \alpha^{n-1}$ is employed. We obtain

$$(m_1 + m_2 + m_3 - 1) \log 10 \leq (n-1) \log \alpha,$$

by taking logarithms on both sides and its rearrangement leads to

$$n \log \alpha \geq (m_1 + m_2 + m_3) \log 10 - 2.$$

The ultimate boundaries are determined by combining these findings:

$$(m_1 + m_2 + m_3) \log 10 - 2 \leq n \log \alpha < (m_1 + m_2 + m_3) \log 10 + 2.$$

Lemma 2.7. *There are no powers of two in $F_n^{(k)}$ with more than three digits which are concatenation of three repdigits.*

Proof. We begin with the expression

$$2^n = \overline{\underbrace{a \dots a}_{m_1 \text{ times}} \underbrace{b \dots b}_{m_2 \text{ times}} \underbrace{c \dots c}_{m_3 \text{ times}}} \text{ where } \quad a, b, c \in \{0, 1, \dots, 9\}, \quad a \neq 0.$$

Substituting the mathematical expression of repdigits, this expression can be reformulated as

$$9 \times 2^n = a(10^{m_1} - 1)10^{m_2+m_3} + b(10^{m_2} - 1)10^{m_3} + c(10^{m_3} - 1)$$

If $c = 0$, then $5 \mid 9 \times 2^n$, a contradiction. Consequently, c is a member of the set $\{1, \dots, 9\}$. Specifically, $\nu_2(c) \leq 3$, where $\nu_2(z)$ denotes the exponent of 2 in the factorization of an integer z. Analyzing the 2−adic evaluation on both sides of the equation

$$9 \times 2^n - a(10^{m_1} - 1)10^{m_2+m_3} - b(10^{m_2} - 1)10^{m_3} - c10^{m_3} = c,$$

it is deduced that $m_3 \leq 3$ for $n \geq 4$. Next, consider the expression:

$$9 \times 2^n - a10^{m_1+m_2+m_3} - (b-a)10^{m_2+m_3} = (c-b)10^{m_3} - c.$$

The term $(c-b)10^{m_3} - c \neq 0$, (otherwise $5 \mid 9 \times 2^n$), and it's absolute value is at most $9 \times 10^3 + 9 < 2^{14}$. For $n \geq 14$, we examine the 2−adic valuations of both sides of the inequality

$$9 \times 2^n - a10^{m_1+m_2+m_3} - (b-a)10^{m_2+m_3} = (c-b)10^{m_3} - c$$

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and deduce that $m_2 + m_3 \le 13$. Consequently, the absolute value of $(b-a)10^{m_2+m_3} + (c-b)10^{m_3} - c$ is at most $9 \times 10^{13} + 9 \times 10^3 + 9 < 2^{47}$. For $n \ge 47$, further comparison of the 2−adic valuations in the inequality

$$9 \times 2^n - a10^{m_1+m_2+m_3} = (b-a)10^{m_2+m_3} + (c-b)10^{m_3} - c$$

yields $m_1 + m_2 + m_3 \le 46$. Under these constraints, it follows that

$$2^n < 10^{46} + 10^{13} + 10^3 + 1 < 2^{153}.$$

Therefore, $n < 153$, and a numerical verification concludes the proof.

3. The Proof of Theorem 1.1

We begin by assuming that the Diophantine Equation (1.2) holds. The repdigit cases $a = b = c$, $a = b = 0$, $b = c = 0$, and $a = 0, b = c$ are excluded from consideration, as they have been thoroughly addressed in [3]. The scenario $n \le 1000$ is then examined. It is sufficient to assume that $k \le 1000$ since, if $k > 1000$, then $n < k$, therefore $F_n^{(k)}$ is a power of 2. In this case, Lemma 2.7 establishes that Equation (1.2)) has no solutions with $m_1 + m_2 + m_3 \ge 4$. Furthermore, $F_n^{(k)} \le 2^{n-1} \le 2^{999}$, implying that $F_n^{(k)}$ has at most 300 digits. The list of all $F_n^{(k)}$ for $2 \le k, n \le 1000$ and the list of all numbers that are concatenation of three repdigits with a total of at most 300 digits were created in order to address this situation. These two lists were combined to provide the solutions given in Theorem 1.1. We assume that $n > 1000$ from this point on.

3.1. An Upper Bound on n in Terms of k

As explained below, we will now analyse Equation (1.2) in three different scenarios.
Case 1: From equations (1.2), (2.2), and (2.3),

$$9f_k(\alpha)\alpha^{n-1} - a10^{m_1+m_2+m_3} = 9\zeta_n - \big((a-b)10^{m_2+m_3} - (b-c)10^{m_3} - c\big), \quad |\zeta_n| < 1/2$$

is obtained.

Taking the absolute value on both sides of the equation yields

$$\begin{aligned} \left|9f_k(\alpha)\alpha^{n-1} - a10^{m_1+m_2+m_3}\right| &= \left|9\zeta_n - \big((a-b)10^{m_2+m_3} - (b-c)10^{m_3} - c\big)\right| \\ &\le 9/2 + \big(9 \cdot 10^{m_2+m_3} + 9 \cdot 10^{m_3} + 9\big) \\ &< 9/2 + 10^{m_3}(9 \cdot 10^{m_2} + 9 + 0.9) \\ &< 9/2 + 10^{m_3+m_2}(9 + 0.9 + 0.09) \\ &< 10.035 \times 10^{m_2+m_3}. \end{aligned} \tag{3.1}$$

Dividing both sides of (3.1) by $a10^{m_1+m_2+m_3}$ results in

$$\left|\left(\frac{9f_k(\alpha)}{a}\right)\alpha^{n-1}10^{-m_1-m_2-m_3} - 1\right| < \frac{10.035}{10^{m_1}} \tag{3.2}$$

Define

$$\Gamma_1 = \left(\frac{9f_k(\alpha)}{a}\right)\alpha^{n-1}10^{-m_1-m_2-m_3} - 1.$$

Notably, $\Gamma_1 \ne 0$, as the vanishing of Γ would imply that

$$\frac{9f_k(\alpha)}{a} = \frac{10^{m_1+m_2+m_3}}{\alpha^{n-1}}.$$

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For some $i \in \{2, \dots, k\}$, applying a non-trivial Galois automorphism σ of $\mathbb{Q}[\alpha]$ that maps α to $\alpha^{(i)}$ results in

$$\frac{9f_k\left(\alpha^{(i)}\right)}{a} = \frac{10^{m_1+m_2+m_3}}{\left(\alpha^{(i)}\right)^{n-1}}.$$

Using the estimate from Lemma 2.4 (i) and the fact that $\left|\alpha^{(i)}\right| < 1$ for $i = 2, \dots, k$, and taking absolute values on both sides, we arrive at

$$9 > \frac{\left|9f_k\left(\alpha^{(i)}\right)\right|}{a} = \frac{10^{m_1+m_2+m_3}}{\left|\alpha^{(i)}\right|^{n-1}} > 10^{m_1+m_2+m_3},$$

which leads to a contradiction.
Adopting the notation from Theorem 2.1, we choose $t = 3$ and define the following parameters:

$$\gamma_1 = \frac{9f_k(\alpha)}{a}, \gamma_2 = \alpha, \gamma_3 = 10, e_1 = 1, e_2 = n-1, e_3 = -m_1 - m_2 - m_3.$$

As $10^{m_1+m_2+m_3-1} < F_n^{(k)} < \alpha^{n-1}$, it follows that $m_1 + m_2 + m_3 \le n$. So, we set $D = n$. Additionally, since $\mathbb{L} = \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, we can figure out that $d_{\mathbb{L}} = k$. $h(\gamma_2) = (\log \alpha)/k \le 1/k$ and $h(\gamma_3) = \log 10$ are obtained by using the properties of absolute logarithmic height. Furthermore, using Lemma 2.4 (ii), we establish that

$$h(\gamma_1) \le h(9/a) + h(f_k(\alpha)) \le \log 9 + 3\log k \le 7\log k \quad \text{for all} \quad k \ge 2.$$

Accordingly, we define

$$A_1 = 7k\log k, \quad A_2 = 1, \quad A_3 = k\log 10.$$

A lower bound for $\log|\Gamma_1|$ can be determined by using Theorem 2.1 as follows:

$$\begin{aligned} \log|\Gamma_1| &> -1.4 \cdot 30^6 3^{4.5} k^2 (1+\log k)(1+\log n)(7k\log k)(k\log 10) \\ &> -5.8 \times 10^{12} k^4 (\log k)^2 (1+\log n). \end{aligned} \tag{3.3}$$

In this derivation, we use the relation $1 + \log k \le 2.5\log k$ valid for all $k \ge 2$. A comparison of Equation (3.3) with Equation (3.2) leads to

$$m_1 \log 10 - \log 10.035 < 5.8 \cdot 10^{12} k^4 (\log k)^2 (1+\log n),$$

which simplifies to

$$m_1 \log 10 < 5.9 \cdot 10^{12} k^4 (\log k)^2 (1+\log n). \tag{3.4}$$

Case 2: Proceeding with the second rearrangement of Equation (1.2) as

$$f_k(\alpha)\alpha^{n-1} - \left(\frac{a10^{m_1} - (a-b)}{9}\right)10^{m_2+m_3} = \zeta_n - \frac{(b-c)10^{m_3}}{9} - \frac{c}{9}, \tag{3.5}$$

we get

$$\left|9f_k(\alpha)\alpha^{n-1} - (a10^{m_1} - (a-b))10^{m_2+m_3}\right| = |9\zeta_n - (b-c)10^{m_3} - c| < 10.35 \cdot 10^{m_3}. \tag{3.6}$$

Dividing both sides of (3.6) by $(a10^{m_1} - (a-b))10^{m_2+m_3}$, we arrive at

$$\left|1 - \left(\frac{9f(\alpha)}{a10^{m_1} - (a-b)}\right)\alpha^{n-1}10^{-m_2-m_3}\right| < \frac{1.16}{10^{m_2}}. \tag{3.7}$$

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Let us introduce the following parameters:

$$\left(\gamma_1, \gamma_2, \gamma_3\right) = \left(\frac{9f(\alpha)}{a10^{m_1} - (a-b)}, \alpha, 10\right)$$

and

$$(e_1, e_2, e_3) = (1, n-1, -m_2 - m_3).$$

With these definitions, we proceed to apply Theorem 2.1. In this scenario, $d_{\mathbb{L}} = k$ since $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{L} = \mathbb{Q}(\ngeqslant)$. We define

$$\Gamma_2 = 1 - \left(\frac{9f(\alpha)}{a10^{m_1} - (a-b)}\right)\alpha^{n-1}10^{-m_2-m_3}.$$

Following the same reasoning used previously for Γ_1, we conclude that $\Gamma_2 \neq 0$. Using properties of the absolute logarithmic height, we derive

$$\begin{aligned}
h(\gamma_1) &= h\left(\frac{9f(\alpha)}{a10^{m_1} - (a-b)}\right) \\
&< 3\log 9 + \log 2 + 5.9 \cdot 10^{12} \cdot k^4 (\log k)^2 (1 + \log n) + 3\log k \\
&< 6 \cdot 10^{12} \cdot k^4 (\log k)^2 (1 + \log n). \\
h(\gamma_2) &= h(\alpha) = \frac{\log \alpha}{k} \\
h(\gamma_3) &= h(10) = \log 10 < 2.31.
\end{aligned}$$

So, we can assign $A_1 = 6 \cdot 10^{12} \cdot k^5 (\log k)^2 (1 + \log n)$, $A_2 = 1$, and $A_3 = k \log 10$. Since $m_2 + m_3 < n$ and $D \geq \max\{|1|, |n-1|, |-m_2 - m_3|\}$, we set $D = n$.
By using Theorem 2.1 and inequality (3.7), we arrive at

$$|\Gamma_2| > \exp(-C \cdot (1 + \log n) \cdot (1 + \log k) \cdot k \log 10 \cdot A_1 \cdot A_2 \cdot A_3),$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2$. Thus, $m_2 \log 10 < 6.1 \cdot 10^{24} \cdot k^8 (1 + \log n)^2 (\log k)^3$ is obtained. We may approximate $6.1(1 + \log n)^2$ by $7(\log n)^2$ for $n > 10^{10}$ and by substituting this estimate, we obtain

$$m_2 \log 10 < 7 \cdot 10^{24} \cdot k^8 (\log n)^2 (\log k)^3. \tag{3.8}$$

Case 3: The third rearrangement of Equation (1.2) yields

$$9f(\alpha)\alpha^{n-1} - (a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))10^{m_3} = -\left(\frac{9\zeta_n}{2} + c\right). \tag{3.9}$$

We get the following by taking the absolute values of Equation (3.9)'s both sides:

$$\left|9f(\alpha)\alpha^{n-1} - (a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))10^{m_3}\right| \leq 9\zeta_n + c < \frac{9}{2} + 9 < 13.5$$

This brings us to the key inequality:

$$\left|9f(\alpha)\alpha^{n-1} - (a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))10^{m_3}\right| < 13.5. \tag{3.10}$$

Upon dividing both sides of (3.10) by $9f(\alpha)\alpha^{n-1}$, we obtain

$$\left|1 - \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9f(\alpha)}\right)\alpha^{-(n-1)}10^{m_3}\right| \leq \frac{3}{\alpha^{n-1}}. \tag{3.11}$$

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Define

$$\Gamma_3 = 1 - \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9f(\alpha)}\right)\alpha^{-(n-1)}10^{m_3}.$$

If $\Gamma_3 = 0$, then

$$\frac{\alpha^{n-1}}{10^{m_3}} = \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9f(\alpha)}\right). \tag{3.12}$$

So, $\Gamma_3 \neq 0$. By applying an automorphism of $\mathbb{L}$ that maps α to $\alpha^{(i)}$, where $i \geq 2$, and subsequently implementing the absolute value operation, we derive

$$\frac{|\alpha^{(i)}|^{n-1}}{10^{m_3}} = \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{|9f(\alpha^{(i)})|}\right). \tag{3.13}$$

Taking the ratio of Equations (3.12) and (3.13), and applying Lemma 2.4 (i), leads to

$$1 > |f_k(\alpha^{(i)})||\alpha^{(i)}|^{n-1} = f_k(\alpha)\alpha^{n-1} > \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1},$$

a contradiction since $n > 1000$. Next, we apply Theorem 2.1 with

$$(\gamma_1, e_1) = \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9f(\alpha)}, 1\right),$$
$$(\gamma_2, e_2) = (\alpha, -(n-1)), \text{ and}$$
$$(\gamma_3, e_3) = (10, m_3).$$

In the field $mathbbL = mathbbQ(alpha)$, the parameters $gamma_1$, $gamma_2$, and $gamma_3$ are all positive real values, suggesting that $d_mathbbL = k$. Using the absolute logarithmic height's characteristics, we get

$$\begin{aligned} h(\gamma_1) &= h\left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9f(\alpha)}\right) \\ &\leq h(9f(\alpha)) + h\big(a10^{m_1+m_2}\big) + h((a-b)10^{m_2}) + h((b-c) + 2\log 2 \\ &< 10.9 + (m_1+m_2)\log 10 + m_2 \log 10 \\ h(\gamma_2) &= \frac{\log\alpha}{k} \\ h(\gamma_3) &= \log 10 < 2.31. \end{aligned}$$

We derive $h(\gamma_1) < 12.3 \cdot 10^{24} \cdot k^8 (\log k)^3 (1+\log n)^2$ using (3.4) and (3.8). Consequently, we set $A_1 = 12.3 \cdot 10^{24} \cdot k^9 (\log k)^3 (1+\log n)^2$, $A_2 = 1$, and $A_3 = k\log 10$ instead. This leads us to choose $D = n$ since $D \geq \max\{|1|, |-(n-1)|, |m_3|\}$. Therefore, using Theorem 2.1, we get

$$3 \cdot \alpha^{-(n-1)} > |\Gamma_3| > \exp\Big(-4.1 \cdot 10^{36} \cdot k^{12} \cdot (1+\log k)(\log k)^3 \cdot (1+\log n)^3\Big)$$

or

$$(n-1)\log\alpha - \log 3 < 4.1 \cdot 10^{36} \cdot k^{12} \cdot (1+\log k)(\log k)^3 \cdot (1+\log n)^3.$$

This implies $n < 8.9 \cdot 10^{36} \cdot k^{12} \cdot (1+\log k)(\log k)^3 \cdot (1+\log n)^3$. As we have assumed $n > 10^{10}$, so we have $8.9(1+\log n)^3 \leq 11(\log n)^3$. this implies

$$n < 11 \cdot 10^{36} \cdot k^{12} \cdot (1+\log k)(\log k)^3 \cdot (\log n)^3. \tag{3.14}$$

Now by applying Lemma 2.3, with $(1+\log k) \leq 2.5\log k$, we have

$$\frac{n}{(\log n)^3} < 2.8 \times 10^{37} k^{12} (\log k)^4 = T.$$

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We therefore obtain

$$\begin{aligned} &< 2^3 \cdot T(\log T) \\ &< 2.3 \cdot 10^{38} k^{12} (\log k)^4 \cdot (\log(2.8) + 37 \log 10 + 12 \log k + 4(\log \log k))^3 \\ &< 2.3 \cdot 10^{38} k^{12} (\log k)^4 \cdot (134.3 \log k)^3 \\ &< 5.8 \cdot 10^{44} \cdot k^{12} (\log k)^7. \end{aligned}$$

Lemma 2.6 and the fact that $\alpha < 2$ imply that

$$(m_1 + m_2 + m_3) \log 10 < n \log \alpha + 2.$$

This results in

$$m_1 + m_2 + m_3 < \frac{1}{\log 10} \Big((5.8 \cdot 10^{44} \cdot k^{12} (\log k)^7 (\log 2)) + 2 \Big) < 2 \times 10^{44} k^{12} (\log k)^7.$$

The conclusions derived up to this point are summarized in the following lemma.

Lemma 3.2. *The solutions to Equation (1.2) are constrained by the inequalities*

$$m_1 + m_2 + m_3 < 2 \times 10^{44} k^{12} (\log k)^7 \text{ and } n < 5.8 \times 10^{44} k^{12} (\log k)^7.$$

3.3. An Upper Bound on n in the Case of Large k

In order to apply Lemma 2.5, $n < 2^{k/2}$ must be ensured. By applying Lemma 3.2, this condition is satisfied if

$$5.8 \times 10^{44} k^{12} (\log k)^7 < 2^{k/2}$$

and for $k \geq 554$, this inequality is valid. Assuming $k \geq 554$, we may write:

$$F_n^{(k)} = 2^{n-2}(1+\zeta), \quad |\zeta| < \frac{5}{2^{k/2}}. \tag{3.15}$$

In case of k, we have $|\zeta| < 1/2$, which leads to $2^{n-2} \in \Big[(2/3)F_n^{(k)}/2, 2F_n^{(k)}\Big]$. This yields $2^{n-2} > (2/3)F_n^{(k)} \geq (2/3)10^{m_1+m_2+m_3-1}$. Substituting Equation (3.15) into Equation (1.2) produces

$$2^{n-2}(1+\zeta) = \frac{1}{9}\big(a10^{m_1+m_2+m_3} - (a-b)10^{m_2+m_3} - b\big).$$

This can be rewritten as

$$\Big|2^{n-2} - (a/9)10^{m_1+m_2+m_3}\Big| \leq 2^{n-2}|\zeta| + \frac{|a-b|10^{m_2+m_3} + |b-c|10^{m_3} + c}{9} \leq \frac{5 \times 2^{n-2}}{2^{k/2}} + 9.99 \times 10^{m_2+m_3}.$$

Hence,

$$\begin{aligned} \Big|(a/9)10^{m_1+m_2+m_3}2^{-(n-2)} - 1\Big| &\leq \frac{5}{2^{k/2}} + \frac{9.99 \times 10^{m_2+m_3}}{2^{n-2}} \\ &\leq \frac{5}{2^{k/2}} + \frac{9.99 \times 1.5}{10^{m_1-1}} \\ &\leq 155 \max\left\{\frac{1}{2^{k/2}}, \frac{1}{10^{m_1}}\right\}. \end{aligned} \tag{3.16}$$

Assign

$$\Gamma_4 = (a/9)10^{m_1+m_2+m_3}2^{-(n-2)} - 1.$$

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It is clear that $\Gamma_4 \neq 0$, because if it were zero, the equation $10^{m_1+m_2+m_3}a = 9 \cdot 2^{n-2}$ would arise. This is not possible as $5 \mid 10^{m_1+m_2+m_3}a$, but $5 \nmid 9 \cdot 2^{n-2}$. Here, we have set the parameters

$$(\gamma_1, \gamma_2, \gamma_3, e_1, e_2, e_3, t) = (a/9, 10, 2, 1, m_1 + m_2 + m_3, -(n-2), 3).$$

It can be observed that $\mathbb{L}(\gamma_1, \gamma_2, \gamma_3) = \mathbb{Q}$, so $d_{\mathbb{L}} = 1$. $A_1 = \log 9 \geq h(\gamma_1)$, $A_2 = \log 10$, and $A_3 = \log 2$. Since $m_1 + m_2 + m_3 \leq n$, we have $D = n \geq \max\{1, m_1 + m_2 + m_3, |-(n-2)|\}$. By virtue of Theorem 2.1, we derive

$$\log|\Gamma_4| > -6 \times 10^{11}(1 + \log n),$$

which simplifies to

$$\min\{(k/2)\log 2, m_1 \log 10\} < 6 \times 10^{11}(1 + \log n) + \log 155 < 10^{12} \log n.$$

This inequality is valid for $n > 1000$. Assume that

$$(k/2)\log 2 = \min\{(k/2)\log 2, m_1 \log 10\}.$$

This assumption results in the inequality

$$k < \frac{2}{\log 2}\left(10^{12} \log n\right) < 3 \times 10^{12} \log n.$$

Substituting this result into the bound of n from Lemma 3.2, and taking into account that $n \geq k$ (because otherwise, $F_n^{(k)}$ would be a power of 2, which contradicts Lemma 2.7), we obtain

$$\begin{aligned} n &< 5.8 \times 10^{44} k^{12} (\log k)^7 \\ &< 5.8 \times 10^{44} (3 \times 10^{12})^{12} (\log n)^{19} \\ &< 3.1 \times 10^{194} (\log n)^{19}. \end{aligned}$$

Applying Lemma 2.3, we get

$$n < 2^{19} T (\log T)^{19} < 4 \times 10^{250}, \tag{3.17}$$

where $T = 3.1 \times 10^{194}$. Assume

$$\min\{(k/2)\log 2, m_1 \log 10\} = m_1 \log 10.$$

As a result, we have

$$m_1 \log 10 < 10^{12} \log n. \tag{3.18}$$

Equation (3.5) can be rewritten as

$$2^{n-2}(1 + \zeta) - \left(\frac{a10^{m_1} - (a-b)}{9}\right) 10^{m_2+m_3} = \frac{(c-b)10^{m_3}}{9} - \frac{c}{9}.$$

Rearranging this equation, we get

$$\left| 2^{n-2} - \left(\frac{a10^{m_1} - (a-b)}{9}\right) 10^{m_2+m_3} \right| = \left| -2^{n-2}\zeta - \frac{(b-c)10^{m_3}}{9} - \frac{c}{9} \right|. \tag{3.19}$$

Dividing by 2^{n-2} and implementing Lemma 2.5, we get

$$\begin{aligned} \left| \left(\frac{a10^{m_1} - (a-b)}{9}\right) 10^{m_2+m_3} \cdot 2^{-(n-2)} - 1 \right| &\leq |\zeta| + \frac{|b-c|10^{m_3}}{9 \cdot 2^{n-2}} + \frac{c}{9 \cdot 2^{n-2}} \\ &\leq 22 \max\left\{ \frac{1}{2^{k/2}}, \frac{1}{10^{m_1+m_2}} \right\}. \end{aligned} \tag{3.20}$$

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Let

$$\Gamma_5 = \left(\frac{a10^{m_1} - (a-b)}{9}\right)10^{m_2+m_3} \cdot 2^{-(n-2)} - 1.$$

Here also $\Gamma_5 \neq 0$. If $\Gamma_5 = 0$, then the equation becomes

$$2^{n-2}\zeta = \frac{(c-b)10^{m_3}}{9} - \frac{c}{9}.$$

This leads to

$$\begin{aligned} F_n^{(k)} &= 2^{n-2} + 2^{n-2}\zeta \\ &= 2^{n-2} + \frac{(c-b)10^{m_3}}{9} - \frac{c}{9}. \end{aligned}$$

Since $F_n^{(k)}$ is an integer, it follows that $b, c \in \{0, 9\}$ and $b \neq c$.
The case $(b, c) = (9, 0)$ leads to

$$\begin{aligned} 2^{n-2} - 10^{m_3} = F_n^{(k)} &= a\left(\frac{10^{m_1}-1}{9}\right)10^{m_2+m_3} + 9\left(\frac{10^{m_2}-1}{9}\right)10^{m_3} \\ &= a\left(\frac{10^{m_1}-1}{9}\right)10^{m_2+m_3} + 10^{m_2+m_3} - 10^{m_3}. \end{aligned}$$

This implies $10^{m_2+m_3} \mid 2^{n-2}$, which is a contradiction.
If $(b, c) = (0, 9)$, then

$$\begin{aligned} 2^{n-2} + 10^{m_3} - 1 = F_n^{(k)} &= a\left(\frac{10^{m_1}-1}{9}\right)10^{m_2+m_3} + 9 \times \left(\frac{10^{m_3}-1}{9}\right) \\ &= a\left(\frac{10^{m_1}-1}{9}\right)10^{m_2+m_3} + 10^{m_3} - 1, \end{aligned}$$

and so, $10^{m_2+m_3} \mid 2^{n-2}$, a contradiction. Therefore, $\Gamma_5 \neq 0$. Now we can proceed to apply Theorem 2.1. Here we have

$$(\gamma_1, \gamma_2, \gamma_3, e_1, e_2, e_3, t) = \left(\frac{a10^{m_1} - (a-b)}{9}, 10, 2, 1, m_2, -(n-2), 3\right)$$

Now,

$$\begin{aligned} h(\gamma_1) &\leq \log 9 + \log(a10^{m_1}) + \log(a-b) + \log 2 \\ &\leq 3\log 9 + \log 2 + 10^{12}\log n \\ &< 1.1 \times 10^{12}\log n. \end{aligned}$$

Furthermore, since $d_{\mathbb{L}} = 1$, so we assign $A_1 = 1.1 \times 10^{12}\log n$, $A_2 = h(\gamma_2) = \log 10$ and $A_3 = h(\gamma_3) = \log 2$. By implementing Theorem 2.1, we get

$$\log|\Gamma_5| > -2.6 \times 10^{23}(1 + \log n)\log n > -5 \times 10^{23}(\log n)^2.$$

This bound is compared to equation (3.20), and since the right-hand side of (3.20) has a minimum occurs at $k/2$, we get

$$(k/2)\log 2 - \log 22 < 5 \times 10^{23}(\log n)^2,$$

which leads to

$$k < \frac{2}{\log 2}\left(5 \times 10^{23}(\log n)^2 + \log 22\right) < 1.5 \times 10^{24}(\log n)^2.$$

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Using Lemma 3.2 we derive

$$\begin{aligned} n &< 5.8 \times 10^{44} k^{12} (\log k)^7 \\ &< 5.8 \times 10^{44} (1.5 \times 10^{24} (\log n)^2)^{12} (\log n)^7 \\ &< 7.6 \times 10^{334} (\log n)^{31}, \end{aligned}$$

as $k < n$. Application of Lemma 2.3 yields

$$n < 2^{31} \cdot T (\log T)^{31} < 5.2 \times 10^{433}, \tag{3.21}$$

where $T = 7.6 \times 10^{334}$. Assume that

$$\min\{(k/2)\log 2, (m_1 + m_2)\log 10\} = (m_1 + m_2)\log 10.$$

In this scenario, we conclude

$$(m_1 + m_2)\log 10 < 5 \cdot 10^{23} (\log n)^2. \tag{3.22}$$

Next, by rewriting Equation (3.9), we obtain

$$2^{n-2} - \frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9} 10^{m_3} = -2^{n-2}\zeta - \frac{c}{9}. \tag{3.23}$$

Taking the absolute values on both sides and dividing through by 2^{n-2}, Equation (3.23) becomes

$$\begin{aligned} \left| 1 - \frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9} \cdot 2^{-(n-2)} \cdot 10^{m_3} \right| &\leq |\zeta| + \left| \frac{c}{9 \cdot 2^{n-2}} \right| \\ &\leq 6 \max\left\{ \frac{1}{2^{k/2}}, \frac{1}{2^{n-2}} \right\}. \end{aligned} \tag{3.24}$$

On the right-hand side, $k/2$ is the least power of 2. Assume the reverse, that the minimal exponent exceeds $k/2$. In this case, $k/2 \geq n-2$, resulting in $n < k$. Yet, when $n < k$, we have $F_n^{(k)} = 2^{n-2}$, and by Lemma 2.7, this cannot be a concatenation of two repdigits because $n > 1000$. Therefore, the minimum exponent of 2 must be $k/2$. Let

$$\Gamma_6 = 1 - \frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9} \cdot 2^{-(n-2)} \cdot 10^{m_3}.$$

Here $\Gamma_6 \neq 0$. In fact, if $\Gamma_6 = 0$, we then get $2^{n-2}\zeta = -c/9$. Consequently, $F_n^{(k)} = 2^{n-2} + 2^{n-2}\zeta = 2^{n-2} - c/9$ and since this expression is an integer, it follows that c is either 0 or 9. The case $c = 0$ leads to $F_n^{(k)} = 2^{n-2}$ which we have seen that it is impossible. Thus, if $c = 9$ and $F_n^{(k)} = 2^{n-2} - 1$, then

$$\begin{aligned} 2^{n-2} - 1 = F_n^{(k)} &= a\left(\frac{10^{m_1} - 1}{9}\right)10^{m_2+m_3} + b\left(\frac{10^{m_2} - 1}{9}\right)10^{m_3} + 9 \times \left(\frac{10^{m_3} - 1}{9}\right) \\ &= a\left(\frac{10^{m_1} - 1}{9}\right)10^{m_2+m_3} + b\left(\frac{10^{m_2} - 1}{9}\right)10^{m_3} + 10^{m_3} - 1, \end{aligned}$$

which implies $10^{m_3} \mid 2^{n-2}$, a contradiction.
So, $\Gamma_6 \neq 0$. So, Here we have

$$(\gamma_1, \gamma_2, \gamma_3, e_1, e_2, e_3) = \left(\frac{(a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c))}{9}, 2, 10, 1, -(n-2), m_3 \right).$$

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An analogous calculation to the one carried out for Γ_3 in Subsection 3.1 reveals that

$$\begin{aligned} h(\gamma_1) &\leq \log 9 + \log(a10^{m_1+m_2}) + \log((a-b)10^{m_2}) + \log(b-c) + \log 2 \\ &\leq 4\log 9 + 2\log 2 + (m_1+m_2)\log 10 + m_1 \log 10 \\ &\leq 4\log 9 + 2\log 2 + 5\cdot 10^{23}(\log n)^2 + 10^{12}\log n < 1.2\times 10^{25}\log n, \end{aligned}$$

as $(\log n)^2 < 23.5\log n$ for $n > 10^{10}$. Further, $d_{\mathbb{L}} = 1$, so we assign $A_1 = 1.2\times 10^{25}\log n$, $A_2 = h(\gamma_2) = \log 10$ and $A_3 = h(\gamma_3) = \log 2$. By virtue of Theorem 2.1,

$$\log|\Gamma_6| > -2.8\times 10^{36}(1+\log n)\log n > -5.1\times 10^{36}(\log n)^2.$$

By analyzing the derived bound in relation to Equation (3.24) and noting that the minimum value on the right-hand side of (3.24) occurs at $k/2$, we obtain

$$(k/2)\log 2 - \log 6 < 5.1\times 10^{36}(\log n)^2,$$

which can be further simplified to

$$k < \frac{2}{\log 2}\left(5.1\times 10^{37}(\log n)^2 + \log 6\right) < 1.5\times 10^{37}(\log n)^2.$$

Using Lemma 3.2 and the condition $k < n$, we may deduce:

$$\begin{aligned} n &< 5.8\times 10^{44}k^{12}(\log k)^7 \\ &< 5.8\times 10^{44}(1.5\times 10^{37}(\log n)^2)^{12}(\log n)^7 \\ &< 7.6\times 10^{490}(\log n)^{31}. \end{aligned}$$

Now, by applying Lemma 2.3, we get

$$n < 2^{31}T(\log T)^{31} < 7.3\times 10^{594}, \text{ where } T = 7.6\times 10^{490}. \tag{3.25}$$

After comparing the estimates in (3.25) and (3.21), we may conclude that the bound in (3.25) is valid, with specific confirmation for $k \geq 554$. The lemma 3.2 provides an instantaneous constraint on n for $k < 554$, with the condition $n < 9.4\times 10^{471} < 7.3\times 10^{594}$. Consequently, in all scenarios, the estimate in (3.25) remains valid. We formalize this as a lemma.

Lemma 3.4. *If $F_n^{(k)}$ is a concatenation of three repdigits, then*

$$1000 < n < 7.3\times 10^{594}.$$

3.5. Reducing the Upper Bound

We begin by considering the estimate provided in (3.16). Let

$$\Lambda_4 = (m_1+m_2+m_3)\log 10 - (n-2)\log 2 + \log(a/9).$$

It is worth noting that $\Gamma_4 = e^{\Lambda_4} - 1$ and as $\Gamma_4 \neq 0$, so $\Lambda_4 \neq 0$. Assuming $m_1 \geq 2$, it can be observed that the right-hand side of Equation (3.16) is bounded above by 1/2. The inequality $|e^x - 1| < y$ guarantees that x is smaller than $2y$ for real values x, y. With $w = \min\{(k/2)\log 2, m_1\log 10\}$, we obtain

$$|\Lambda_4| < \frac{310}{e^w},$$

and this leads to

$$|(m_1+m_2+m_3)\log 10 - (n-2)\log 2 + \log(a/9)| < \frac{310}{e^w}.$$

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We obtain

$$|(m_1+m_2+m_3)\frac{\log 10}{\log 2}-(n-2)+\frac{\log(a/9)}{\log 2}|<\frac{(310/\log 2)}{e^w} \tag{3.26}$$

by dividing each side of the aforementioned inequality by $\log 2$. In order to implement Lemma 2.1, let us define

$$u=m_1+m_2+m_3,\ v=n-2,\ \tau=\frac{\log 10}{\log 2},\quad \mu=\frac{\log(a/9)}{\log 2},\quad A=448,\ \text{and}\ B=e.$$

We can set $M=7.3\cdot 10^{594}$ as an upper bound of $m_1+m_2+m_3$. The 1187-th convergent of τ, denoted as q_{1187}, has a denominator that exceeds $6M$. So, for $1\le d\le 9$, a quick computation with *Mathematica* yields the inequality $0<\epsilon=\|\mu q_{1187}\|-M\|\tau q_{1187}\|=0.305954$. Applying Lemma 2.1 to the inequality (3.26), we obtain

$$w<\log(Aq_{1187}/\varepsilon)<1380.$$

Since $(k/2)\log 2>1380$ for $k>4000$, we may conclude that $w=m_1\log 10$. Consequently, $m_1<1380/\log 10<599$. For $a\neq 9$, this is true since $\mu=0$ for $a=9$. In this instance, we have

$$\left|\tau-\frac{n-2}{m_1+m_2+m_3}\right|<\frac{A}{(m_1+m_2+m_3)B^w}.$$

Assuming $B^w>2AM$, the right-hand side of the above inequality is strictly less than $1/\left(2(m_1+m_2+m_3)^2\right)$. Consequently, the ratio $(n-2)/(m_1+m_2+m_3)$ corresponds to a convergent of τ. Therefore, it follows that $(n-2)/(m_1+m_2+m_3)=p_j/q_j$ for some $j\le 1187$, where $p_j=(n-2)/d$ and $q_j=(m_1+m_2+m_3)/d$, with $d=\gcd(n-2,m_1+m_2+m_3)$. According to Lemma 2.2, when $a_M=\max\{a_i:0\le i\le 1187\}=5393$, the left-hand side of the inequality is bigger than $1/\left((a_M+2)q_j^2\right)$. As a result, we get

$$B^w\le 5395A(m_1+m_2+m_3)<5395AM,$$

which implies

$$w<\log(5395AM)<1384.$$

In both circumstances, that is, if $a\in\{1,\ldots,8\}$ or if $a=9$, we thus conclude that $m_1<601$ given $k>4000$. Next, consider

$$\Lambda_5=(m_2+m_3)\log 10-(n-2)\log 2+\log\left(\frac{a10^{m_1}-(a-b)}{9}\right).$$

Observe that $\Gamma_5=e^{\Lambda_5}-1$. Since we have shown that $\Gamma_5\neq 0$, it follows that $\Lambda_5\neq 0$. Referring to inequality (3.20), the right-hand side is smaller than $1/2$. Thus,

$$|\Lambda_5|<\frac{44}{e^w}, \tag{3.27}$$

which leads to

$$\left|(m_2+m_3)\frac{\log 10}{\log 2}-(n-2)-\frac{\log((a10^{m_1}-(a-b))/9)}{\log 2}\right|<\frac{64}{e^w}. \tag{3.28}$$

As per notations of Lemma 2.1, here we have

$$u=m_2+m_3,\ v=n-2,\ \mu=\frac{\log((a10^{m_1}-(a-b))/9)}{\log 2},\ \tau=\frac{\log 10}{\log 2},\ A=64,\ \text{and}\ B=e.$$

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Taking the same $M = 7.3 \cdot 10^{594}$, we get that $q_{1187} > 6M$. A computer calculation for $a, b \in \{0, 1, \ldots, 9\}$, with the conditions $a \neq 0$, $b \neq a$, and $m_1 \in [1, 601]$ indicates that $0 < \epsilon = \|\mu q_{1187}\| - M\|\tau q_{1187}\| = 0.30966404$, implying that

$$w < \frac{\log(Aq_{1187}\varepsilon^{-1})}{\log B} < 1378.$$

Consequently, $k < 4000$ for $w = (k/2)\log 2 < 1378$, which results in a contradiction. The implication is that $(m_1 + m_2) < 1378/\log 10 < 598$ since $w = (m_1 + m_2)\log 10$. Every μ is covered by this, except for the nine triples(a, b, m_1) :

$$\begin{aligned} &(1,0,1),(1,9,1),(2,0,1),(3,9,1),(4,0,1)\\ &(4,9,1),(5,0,1),(7,9,1),(8,0,1). \end{aligned} \tag{3.29}$$

For these triples, it can be readily verified that

$$\mu + 2 = \begin{cases} 2, & \text{if } (a,b,m_1) = (1,0,1);\\ 3, & \text{if } (a,b,m_1) = (1,9,1),(2,0,1);\\ 4, & \text{if } (a,b,m_1) = (3,9,1),(4,0,1);\\ 5, & \text{if } (a,b,m_1) = (7,9,1),(8,0,1).\\ 1 + \frac{\log 10}{\log 2}, & \text{if } (a,b,m_1) = (4,9,1),(5,0,1). \end{cases}$$

Under these circumstances, inequality (3.28) becomes

$$\Lambda_5 = \left| u\left(\frac{\log 10}{\log 2}\right) - (n-i) \right| < 64 \cdot B^{-w}, \quad \text{for } i = 2,3,4,5$$

and

$$\Lambda_5 = \left| (u+1)\left(\frac{\log 10}{\log 2}\right) - (n-1) \right| < 64 \cdot B^{-w}, \quad \text{for } i = 1 + \frac{\log 10}{\log 2}.$$

Both of these inequalities lead us to the conclusion that $B^w/(64 \times 5395) < 7.3 \cdot 10^{594}$ using the continuous fraction expansion of $\log 10/\log 2$. The original assumption that $k > 4000$ is contradicted by this finding, which suggests that $k < 4000$. Consequently, it must hold that $w = (m_1 + m_2)\log 10$. Therefore, we obtain the bound $(m_1 + m_2) < 1382/\log 10 < 600$. In both scenarios—whether $a \in 1, \ldots, 8$ or $a = 9$ —we conclude that $(m_1 + m_2) < 600$ under the condition $k > 4000$.

Next, define

$$\Lambda_6 = m_3 \log 10 - (n-2)\log 2 + \log\left(\frac{a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c)}{9}\right).$$

It follows that $\Gamma_6 = e^{\Lambda_6} - 1$. Since we have established that $\Gamma_6 \neq 0$, it necessarily follows that $\Lambda_6 \neq 0$. Turning our attention to inequality (3.24), we observe that its right-hand side is less than $1/2$. This leads us to the result

$$|\Lambda_6| < \frac{12}{2^{k/2}},$$

which indicates that

$$\left| m_3 \log 10 - (n-2)\log 2 + \log\left(\frac{a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c)}{9}\right) \right| < \frac{12}{2^{k/2}}. \tag{3.30}$$

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By dividing both sides of this inequality by log 2, we obtain

$$\left|(m_3)\frac{\log 10}{\log 2}-(n-2)-\frac{\log\left(\frac{a10^{m_1+m_2}-(a-b)10^{m_2}-(b-c)}{9}\right)}{\log 2}\right|<\frac{18}{2^{k/2}}.$$

In order to implement Lemma 2.1, let us assign

$$u=m_3,\ v=n-2,\ \tau=\frac{\log 10}{\log 2},\ \mu=\frac{\log\left(\frac{a10^{m_1+m_2}-(a-b)10^{m_2}-(b-c)}{9}\right)}{\log 2},$$
$$A=18,\ \text{and}\ B=2.$$

We again set $M=7.3\cdot 10^{594}$ and utilize the same continued fraction approximation p_{1187}/q_{1187}. A computer calculation for $m_1\in[1,602]$, and $m_1+m_2<600$ showed that $0<\epsilon=\|\mu q_{1187}\|-M\|\tau q_{1187}\|=0.2998877$ and

$$k/2<\frac{\log(Aq_{1187}\epsilon^{-1})}{\log B}<1987,$$

implies $k<4000$, a contradiction except for the three triples (a,b,c):

$$(1,5,9),(3,1,9),(6,3,9). \tag{3.31}$$

For these triples, we have

$$\mu+2=\begin{cases}6, & \text{if } (a,b,c)=(1,5,9);\\ 7, & \text{if } (a,b,c)=(3,1,9);\\ 8, & \text{if } (a,b,c)=(6,3,9).\end{cases}$$

In these cases, inequality (3.30) turns into

$$\Lambda_6=\left|m_3\left(\frac{\log 10}{\log 2}\right)-(n-i)\right|<18\cdot B^{-w},\quad \text{for } i=6,7,8.$$

Thus, by using Lemma 2.2, we have $B^w/(18\times 5395)<7.3\cdot 10^{594}$, which gives

$$w<\frac{\log(5395AM)}{\log 2}<1992.$$

We have $k<3984$, which violates our expectation of $k>4000$. Thus, we demonstrated that if $k\geq 554$, then $k\leq 4000$. Next, suppose $k>600$. Using Lemma 3.2, we may conclude that $n<2\cdot 10^{83}$.

Using the same technique for Γ_4 and $M=2\cdot 10^{83}$, we discover that $q_{172}>6M$. Consequently, this implies $\epsilon=0.385393$. Implementation of Lemma 2.1 results $w<201$. If $k>600$, then the inequality $(k/2)\log 2<201$ leads to $k<580$, which results in a contradiction. Thus, we must have $w=m_1\log 10<201$, implying that $m_1<87$.

For the case $a=9$, the previous argument gives $B^w<5395AM$, which implies $w<206$. Since $k>600$, this implies that $m_1<90$. Thus, $m_1<90$ applies to all a values. Moving on to Γ_5, we apply the same value for M with $m_1<90$ to give $\epsilon=0.025490933$. This yields $w<202$, implying $k<582$. This is again a contradiction since $k>600$. So, we must have $w=(m_1+m_2)\log 10<202$, which implies $(m_1+m_2)<87$.

If (a,b,m_1) is one of the 9 triples shown at (3.29), we get $B^w<5395AM$. So, now $w=(k/2)\log 2<204$, implies that $k<588$, which is again a contradiction. We must have $w=(m_1+m_2)\log 10<88$, implying that $m_1+m_2<88$ regardless of μ values. Next, we'll look at Γ_6. Using $m_1<90$ in Γ_6 to calculate M yields $\epsilon=0.0017344$. This indicates that $w<294$, giving

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$k < 588$, resulting in a contradiction because $k > 600$. If (a, b, c) is one of the three triples at (3.31), then $B^w < 5395AM$. Hence, we have the inequality $w = (k/2) < 293$, which implies $k < 586$, leading to a contradiction because $k > 600$.

So, from now on, assume that $k \leq 600$. Now, by Lemma 3.2, we get the same $M = 2 \cdot 10^{83}$ for $m_1 + m_2 + m_3$.

So, we have by (3.2),

$$\Lambda_1 = \left|(m_1 + m_2 + m_3)\log 10 - (n-1)\log\alpha + \log\frac{9f_k(\alpha)}{a}\right| < \frac{10.035}{10^{m_1}}. \tag{3.32}$$

Assuming $m_1 \geq 2$, the right–hand side of the above equation is less than $1/2$. As the inequality $|e^z - 1| < y$ for real values of z and y leads to the conclusion that $z < 2y$, hence, we have $\Lambda_1 < 20.07/10^{m_1}$. Dividing Equation (3.32) by $\log\alpha$, we get

$$\left|\frac{(m_1 + m_2 + m_3)\log 10}{\log\alpha} - (n-1) + \frac{\log\left(\frac{9f_k(\alpha)}{a}\right)}{\log\alpha}\right| < \frac{42}{10^{m_1}}.$$

By applying Lemma 2.1, we have $q_{172} > 6M$ and we get $\epsilon = 0.362058$, which implies $m_1 < 86$. Next, consider the expression in (3.7) with $m_1 < 86$.

$$\Lambda_2 = \left|(m_2 + m_3)\log 10 - (n-1)\log\alpha + \log\left(\frac{9f_k(\alpha)}{a10^{m_1} - (a-b)}\right)\right| < \frac{1.16}{10^{m_2}}. \tag{3.33}$$

Assuming $m_2 \geq 1$, the right-hand side in the above inequality is limited to a maximum of $1/2$. Therefore, by the similar argument done for Λ_1, we can assert $|\Lambda_2| < 2.32/10^{m_1}$. Dividing Equation (3.33) by $\log\alpha$, we get

$$\left|\frac{(m_2 + m_3)\log 10}{\log\alpha} - (n-1) + \frac{\log\left(\frac{9f_k(\alpha)}{a10^{m} - (a-b)}\right)}{\log\alpha}\right| < \frac{5}{10^{m_2}}.$$

Now by applying Lemma2.1, we get $\epsilon = 0.021281$, this implies $m_2 < 86$.

Next move to (3.11). Here we have

$$\Lambda_3 = \left|m_3\log 10 - (n-1)\log\alpha + \log\left(\frac{a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c)}{9f_k(\alpha)}\right)\right| < \frac{3}{\alpha^{n-1}}. \tag{3.34}$$

Since $n > 1000$, the right–hand side of (3.34) is less than $1/2$. So, we have

$$|\Lambda_3| < \frac{6}{\alpha^{n-1}}.$$

Dividing both sides of (3.34) by $\log\alpha$, we get

$$\left|\frac{m_3\log 10}{\log\alpha} - (n-1) + \frac{\log\left(\frac{a10^{m_1+m_2} - (a-b)10^{m_2} - (b-c)}{a10^{m} - (a-b)}\right)}{9f_k(\alpha)}\right| < \frac{13}{10^{n-1}}.$$

Finally, by implementing Lemma 2.1, we get $\epsilon = 0.399699$, this implies $n - 1 < 412$.

So, we have $n < 413$, which contradicts our assumption that $n > 1000$.

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