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Degeneracy of the Operator-Valued Poisson Kernel Near the Numerical Range Boundary

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Degeneracy of the Operator-Valued Poisson Kernel Near the Numerical Range Boundary

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Abstract

Let $A \in \mathbb{C}^{d\times d}$ and let $W(A)$ denote its numerical range. In the convex-domain functional calculus of Delyon–Delyon and Crouzeix, a central role is played by the boundary kernel $P_\Omega(\sigma, A) = \mathrm{Re}\big(n_\Omega(\sigma)(\sigma I - A)^{-1}\big)$ on $\partial\Omega$, which is positive definite whenever $W(A) \subset \Omega$. We study the loss of pointwise coercivity as $\Omega \downarrow W(A)$. Along any C^1 convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if boundary data $(\sigma_\varepsilon, n_{\Omega_\varepsilon}(\sigma_\varepsilon))$ converge to a supporting pair (σ_0, n) with $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the near-kernel aligns with $(\sigma_0 I - A)\mathcal{M}(n)$, where $\mathcal{M}(n)$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$. Quantitatively, the collapse is governed by the support gap $\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n))$: under a spectral-gap hypothesis for $H(n)$ we obtain a full collapsing eigenvalue cluster with a computable slope spectrum given by an explicit Gram matrix, and show that these slopes are intrinsic after rescaling by δ. This yields a rigorous face detector and explains a mechanism for ill-conditioning in boundary-integral discretizations as Ω approaches $W(A)$. At spectral support points $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ we obtain a three-scale splitting ($1/\varepsilon$ blow-up, $O(\varepsilon)$ cluster, and $O(1)$ bulk) under non-tangential offsets; for defective eigenvalues, higher-order blow-up related to Jordan structure may occur. In the normal case we give a complete description in terms of the supporting face. Numerical experiments validate the predicted slopes and splittings.

Keywords: numerical range; operator-valued Poisson kernel; convex domains; functional calculus; boundary integral methods

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1. Introduction

Let $A \in \mathbb{C}^{d\times d}$ and define its numerical range by

$$W(A) \;:=\; \{x^*Ax : \; x \in \mathbb{C}^d, \; \|x\| = 1\}.$$

The Toeplitz–Hausdorff theorem asserts that $W(A)$ is a compact convex subset of $\mathbb{C}$ (see, e.g., [1,2]). Crouzeix conjectured that $W(A)$ is a 2-spectral set for A, i.e.,

$$\|p(A)\| \;\leq\; 2 \max_{z\in W(A)} |p(z)| \qquad \text{for every polynomial } p. \tag{1}$$

See [3,4] for the formulation and [5] for the best known universal constant $1+\sqrt{2}$.

Background and relation to the convex-domain functional calculus. Up to normalization conventions, a central tool in the convex-domain approach of Delyon–Delyon and Crouzeix is the operator-valued boundary kernel

$$P_\Omega(\sigma, A) \;:=\; \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big), \qquad \sigma \in \partial\Omega, \tag{2}$$

defined for a bounded convex domain $\Omega \subset \mathbb{C}$ with C^1 boundary and $\mathrm{spec}(A) \subset \Omega$. Here $n_\Omega(\sigma)$ denotes the outward unit normal at σ. This kernel appears in double-layer potential representations and boundary integral operators used to obtain functional calculus bounds on convex domains [4–8]. Following this convex-domain Carl Neumann/double-layer terminology (see, e.g., [9]), we refer to P_Ω as an operator-valued Poisson kernel; see Remark 3 for the distinction from classical disk Poisson kernels in Sz.-Nagy–Foiaş theory [10]. For convex Ω with $W(A) \subset \Omega$, positivity/coercivity of $\sigma \mapsto P_\Omega(\sigma, A)$ on $\partial\Omega$ encodes strict separation of supporting half-planes and serves as a key structural input in such estimates [7,8,11].

Motivation: loss of coercivity near $\partial W(A)$. In applications and numerical implementations of boundary-integral calculi, one often approximates $W(A)$ by C^1 convex supersets $\Omega_\varepsilon \downarrow W(A)$. It is therefore natural to ask whether coercivity of the pointwise kernel $P_{\Omega_\varepsilon}(\sigma, A)$ can remain uniform as $\varepsilon \to 0$. The results below show that this is impossible in general: even when the resolvent stays bounded (i.e. at non-spectral boundary points $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$), the smallest eigenvalue of $P_{\Omega_\varepsilon}(\sigma, A)$ must collapse to 0 at boundary points $\sigma \in \partial\Omega_\varepsilon$ approaching $\partial W(A)$ in a fixed supporting direction.

Coercivity, conditioning, and boundary-integral estimates. The convex-domain functional calculus can be phrased in boundary-integral terms: one combines resolvent evaluations with boundary integral operators (e.g. in the Carl Neumann double-layer potential framework [9]) to control $\|f(A)\|$ by $\|f\|_\Omega := \sup_{z\in\Omega} |f(z)|$. In this setting, pointwise coercivity of $\sigma \mapsto P_\Omega(\sigma, A)$ enters both theory and numerics: if $c_\Omega := \inf_{\sigma\in\partial\Omega} \lambda_{\min}(P_\Omega(\sigma, A))$ is small, then $P_\Omega(\sigma, A)$ becomes nearly singular at some boundary points and $\|P_\Omega(\sigma, A)^{-1}\|$ must be large, which degrades constants in coercivity-based estimates and typically leads to ill-conditioned discretizations of the associated boundary-integral operators. The collapse results below therefore explain a concrete mechanism behind the loss of stability as Ω approaches $W(A)$.

What is new in this paper. The existing convex-domain literature primarily exploits positivity of (2) for fixed domains $\Omega \supsetneq W(A)$ [4,7,8,11]. Here we analyze the complementary limiting regime in which Ω *shrinks* to $W(A)$, and we make explicit the resulting loss of coercivity of the pointwise kernel. The analysis is driven by a congruence identity and by a scalar *support gap* $\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n))$ (where $H(n) = \mathrm{Re}(\overline{n}A)$), which admits a support-function interpretation in standard convex-geometry terminology.

- We prove a qualitative degeneracy theorem (Theorem 1): along any C^1 convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if $\sigma_\varepsilon \in \partial\Omega_\varepsilon$ approaches a non-spectral boundary point $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$ with convergent outward normals $n_{\Omega_\varepsilon}(\sigma_\varepsilon) \to n$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the limiting min-eigenvector directions lie in $(\sigma_0 I - A)\mathcal{M}(n)$, where $\mathcal{M}(n)$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$.
- We establish two-sided bounds for $P(\sigma, n)$ in terms of the support gap $\delta(\sigma, n)$, yielding a linear degeneracy rate under bounded-resolvent hypotheses (Lemma 3 and Corollary 3), and compute δ explicitly for standard outer offsets $W(A) + \varepsilon\mathbb{D}$ (Proposition 2).
- Under a spectral-isolation hypothesis for $\lambda_{\max}(H(n))$, we quantify the *entire collapsing eigenvalue cluster*: exactly $m = \dim \mathcal{M}(n)$ eigenvalues collapse linearly with an explicit *slope spectrum* given by the eigenvalues of a computable $m \times m$ Gram matrix $G(n, \sigma_0)^{-1}$ (Proposition 7), while the remaining $d - m$ eigenvalues stay uniformly bounded away from 0 (Proposition 8). This yields a rigorous "face detector" based on counting eigenvalues below a threshold proportional to ε (Corollary 5). The same slope spectrum is shown to be *intrinsic* under arbitrary C^1 convex exhaustions after normalization by the support gap (Proposition 9).
- We analyze the contrasting *spectral-support* regime $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$. For general matrices we obtain a three-scale splitting under non-tangential offsets: an exact $1/\varepsilon$ blow-up on $\mathrm{Ker}(\sigma_0 I - A)$, an $O(\varepsilon)$ collapsing cluster on $\mathcal{M}(n) \ominus \mathrm{Ker}(\sigma_0 I - A)$ with an explicit slope spectrum, and an $O(1)$ bulk separated from 0 (Proposition 12). For defective eigenvalues, additional higher-order blow-up may occur; see Remark 11. For normal matrices we recover a simple degeneracy dichotomy at spectral support points in terms of whether the supporting face contains multiple eigenvalues (Proposition 11 and Corollary 7).

- We include reproducible numerical experiments (Python) validating the predicted slopes, splittings, and direction-dependent sensitivity profiles (Section 4.9).

Organization. Section 2 fixes notation and recalls support-function identities. Section 3 introduces $P_\Omega(\sigma, A)$, proves the key congruence identity, and establishes quantitative support-gap bounds together with a geometric interpretation of δ. Section 4 contains the degeneracy theorem, quantitative corollaries, slope spectra and two-/three-scale spectral splittings (including the spectral-support regime), subspace convergence, explicit examples, and reproducible numerical tests. It concludes with a brief discussion of remaining open questions.

2. Preliminaries

We use the standard notation for disks:

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}, \qquad \overline{\mathbb{D}} := \{z \in \mathbb{C} : \ |z| \le 1\}.$$

Throughout, $A \in \mathbb{C}^{d\times d}$ is fixed. For vectors $x \in \mathbb{C}^d$ we use $\|x\| := (x^*x)^{1/2}$. For matrices $B \in \mathbb{C}^{d\times d}$ we use the induced operator norm $\|B\| := \sup_{\|x\|=1}\|Bx\|$. We write B^* for the conjugate transpose. We use Re both for scalars and matrices: for $z \in \mathbb{C}$ it denotes the real part, and for $B \in \mathbb{C}^{d\times d}$ we set $\operatorname{Re}(B) := (B + B^*)/2$ (the Hermitian part).

For a Hermitian matrix B, we write its eigenvalues in nondecreasing order as

$$\lambda_1^{\uparrow}(B) \le \cdots \le \lambda_d^{\uparrow}(B),$$

and in nonincreasing order as $\lambda_1^{\downarrow}(B) \ge \cdots \ge \lambda_d^{\downarrow}(B)$. In particular, $\lambda_{\min}(B) = \lambda_1^{\uparrow}(B)$ and $\lambda_{\max}(B) = \lambda_1^{\downarrow}(B)$.

Remark 1 (Spectrum is contained in the numerical range)**.** *One has* $\operatorname{spec}(A) \subset W(A)$*. Indeed, if* $Ax = \lambda x$ *with* $\|x\| = 1$*, then* $x^*Ax = \lambda \in W(A)$*. Consequently,* $W(A) \subset \Omega$ *implies* $\operatorname{spec}(A) \subset \Omega$ *for any open set* $\Omega \subset \mathbb{C}$*.*

2.1. Support functions and the Hermitian pencil

For unimodular $\omega \in \mathbb{C}$ (i.e. $|\omega| = 1$), define the Hermitian matrix

$$H(\omega) \ := \ \operatorname{Re}(\overline{\omega}\, A) \ = \ \tfrac{1}{2}(\overline{\omega}\, A + \omega A^*). \tag{3}$$

We will later write $n \in \mathbb{C}$ (with $|n| = 1$) for outward unit normals on $\partial\Omega$; in the support-function identities below and throughout, such an n simply plays the role of the unimodular direction ω.

Let $\lambda_{\max}(H(\omega))$ denote its largest eigenvalue and let

$$\mathcal{M}(\omega) \ := \ \operatorname{Ker}\big(\lambda_{\max}(H(\omega))I - H(\omega)\big)$$

denote the corresponding maximal eigenspace.

Lemma 1 (Support function of the numerical range)**.** *For every unimodular* $\omega \in \mathbb{C}$*,*

$$\max_{z\in W(A)} \operatorname{Re}(\overline{\omega}\, z) \ = \ \lambda_{\max}(H(\omega)). \tag{4}$$

Moreover, if $x \in \mathbb{C}^d$ *is a unit eigenvector of* $H(\omega)$ *associated with* $\lambda_{\max}(H(\omega))$*, then* $x^*Ax \in \partial W(A)$ *and*

$$\operatorname{Re}\big(\overline{\omega}\, x^*Ax\big) = \lambda_{\max}(H(\omega)).$$

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Proof. For $\|x\| = 1$,

$$\mathrm{Re}\big(\overline{\omega}\, x^* A x\big) = \mathrm{Re}\big(x^*(\overline{\omega}A)x\big) = x^* \ \mathrm{Re}(\overline{\omega}A)\, x = x^* H(\omega) x.$$

Taking the maximum over $\|x\| = 1$ yields (4) by Rayleigh–Ritz (see, e.g., [12]). If x is a maximizing unit vector, then $x^* A x \in W(A)$ attains the support functional in direction ω, hence lies on $\partial W(A)$ and satisfies the stated identity. □

Definition 1 (Support function and exposed face)**.** *Let $K \subset \mathbb{C}$ be compact and convex and let $n \in \mathbb{C}$ be unimodular. Define the support function and the exposed face of K in direction n (see, e.g., [13]) by*

$$h_K(n) := \max_{z \in K} \mathrm{Re}(\overline{n}\, z), \qquad F_K(n) := \{z \in K : \ \mathrm{Re}(\overline{n}\, z) = h_K(n)\}.$$

We call any element of $F_K(n)$ a support point *of K in direction n; the face $F_K(n)$ is a singleton precisely when K has an* exposed point *in direction n. Equivalently, n is a supporting direction for K at every $\sigma \in F_K(n)$, and (σ, n) may be viewed as a supporting pair.*

2.2. Convex Domains with C^1 Boundary and Normals

We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with C^1 boundary. Then for each $\sigma \in \partial\Omega$ there is a unique outward unit normal vector. This C^1 assumption is used only to guarantee that the outward unit normal $n_\Omega(\sigma)$ exists and is unique at every boundary point, ensuring that $P_\Omega(\sigma, A)$ is well-defined; no higher regularity (e.g. curvature bounds) is used. We represent the normal as a unimodular complex number $n_\Omega(\sigma) \in \mathbb{C}$ with $|n_\Omega(\sigma)| = 1$ so that the supporting half-plane at σ is

$$\mathcal{H}_\Omega(\sigma) \ = \ \Big\{z \in \mathbb{C} : \ \mathrm{Re}\big(\overline{n_\Omega(\sigma)}(z - \sigma)\big) \le 0\Big\}. \tag{5}$$

Equivalently, by convexity one has $\overline{\Omega} \subseteq \mathcal{H}_\Omega(\sigma)$ and $\Omega \subset \{z \in \mathbb{C} : \ \mathrm{Re}(\overline{n_\Omega(\sigma)}(z - \sigma)) < 0\}$. Under the identification $\mathbb{C} \simeq \mathbb{R}^2$, the functional $z \mapsto \mathrm{Re}(\overline{n}z)$ is the Euclidean inner product with the unit vector corresponding to n.

Definition 2 (C^1 convex exhaustion)**.** *A family $\{\Omega_\varepsilon\}_{\varepsilon > 0}$ is called a C^1* convex exhaustion *of a compact convex set $K \subset \mathbb{C}$ if:*

(i) each $\Omega_\varepsilon \subset \mathbb{C}$ is a bounded open convex set with C^1 boundary;
(ii) $\Omega_{\varepsilon'} \subset \Omega_\varepsilon$ for $0 < \varepsilon' < \varepsilon$;
(iii) $K \subset \Omega_\varepsilon$ for all $\varepsilon > 0$;
(iv) $\bigcap_{\varepsilon > 0} \overline{\Omega_\varepsilon} = K$.

Remark 2 (Subsequence selection for convergent normals)**.** *Let $\varepsilon_k \downarrow 0$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ be any sequence. Since each outward normal $n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k)$ is unimodular, the sequence $\{n_k\} \subset \{z \in \mathbb{C} : |z| = 1\}$ lies in a compact set. Hence there is always a subsequence (not relabeled) such that $n_k \to n$ for some unimodular n. In particular, the normal convergence hypothesis in Theorem 1 can always be arranged by passing to a subsequence.*

3. The Operator-Valued Poisson Kernel

Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with C^1 boundary and assume $\mathrm{spec}(A) \subset \Omega$. Then $(\sigma I - A)^{-1}$ exists for all $\sigma \in \partial\Omega$.

Definition 3 (Operator-valued Poisson kernel)**.** *For $\sigma \in \partial\Omega$, define*

$$P_\Omega(\sigma, A) \ := \ \mathrm{Re}\Big(n_\Omega(\sigma)\, (\sigma I - A)^{-1}\Big). \tag{6}$$

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Remark 3 (Terminology: "Poisson kernel"). *The boundary kernel $P_\Omega(\sigma, A)$ is the geometric matrix-valued kernel used in the convex-domain functional calculus and depends on the supporting data $(\sigma, n_\Omega(\sigma))$. It should not be confused with the classical operator-valued Poisson kernels for contractions on the disk in Sz.-Nagy–Foiaş theory [10], which are defined on $\mathbb{D}$ and arise from harmonic extensions/functional models.*

3.1. A Congruence Identity

Lemma 2 (Congruence identity). *Let $\sigma \notin \operatorname{spec}(A)$ and let $n \in \mathbb{C}$ be unimodular. Then*

$$(\sigma I - A)^* \operatorname{Re}\big(n(\sigma I - A)^{-1}\big)(\sigma I - A) \;=\; \operatorname{Re}\big(\overline{n}(\sigma I - A)\big) \;=\; \operatorname{Re}(\overline{n}\,\sigma)\, I - \operatorname{Re}(\overline{n}A). \tag{7}$$

Proof. Write $R := (\sigma I - A)^{-1}$. Then $R(\sigma I - A) = I$ and $(\sigma I - A)^* R^* = I$. Using $\operatorname{Re}(X) = \frac{1}{2}(X + X^*)$,

$$\begin{aligned}(\sigma I - A)^* \operatorname{Re}(nR)(\sigma I - A) &= \tfrac{1}{2}\Big((\sigma I - A)^*(nR)(\sigma I - A) + (\sigma I - A)^*(\overline{n}\,R^*)(\sigma I - A)\Big)\\ &= \operatorname{Re}\big(\overline{n}(\sigma I - A)\big).\end{aligned}$$

Expanding gives (7). □

3.2. Support-Gap Bounds

For unimodular $n \in \mathbb{C}$ define the *support gap*

$$\delta(\sigma, n) \;:=\; \operatorname{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)), \qquad H(n) = \operatorname{Re}(\overline{n}A).$$

Lemma 3 (Support-gap characterization and quantitative bounds). *Let $A \in \mathbb{C}^{d\times d}$, let $\sigma \notin \operatorname{spec}(A)$, and let $n \in \mathbb{C}$ be unimodular. Set*

$$P(\sigma, n) \;:=\; \operatorname{Re}\big(n(\sigma I - A)^{-1}\big), \qquad \alpha \;:=\; \operatorname{Re}(\overline{n}\,\sigma), \qquad \delta \;:=\; \alpha - \lambda_{\max}(H(n)).$$

(This notation emphasizes dependence on the prescribed direction n; when $n = n_\Omega(\sigma)$ one has $P(\sigma, n) = P_\Omega(\sigma, A)$.) Then:

- *(a) $P(\sigma, n) \succeq 0$ if and only if $\delta \geq 0$, and $P(\sigma, n) \succ 0$ if and only if $\delta > 0$.*
- *(b) If $\delta = 0$, then $P(\sigma, n)$ is singular and*

$$\operatorname{Ker}(P(\sigma, n)) \;=\; (\sigma I - A)\,\mathcal{M}(n), \qquad \mathcal{M}(n) = \operatorname{Ker}(\lambda_{\max}(H(n))I - H(n)).$$

- *(c) If $\delta > 0$, then*

$$\frac{\delta}{\|\sigma I - A\|^2} \;\leq\; \lambda_{\min}\big(P(\sigma, n)\big) \;\leq\; \delta\,\|(\sigma I - A)^{-1}\|^2. \tag{8}$$

Proof. Let $B := \sigma I - A$ and $P := P(\sigma, n)$. By Lemma 2,

$$B^* P B = \operatorname{Re}(\overline{n}B) = \alpha I - \operatorname{Re}(\overline{n}A) = \alpha I - H(n) =: Q.$$

Since B is invertible, congruence by B preserves inertia (Sylvester's law of inertia [14]), hence in particular (semi)definiteness, so $P \succeq 0 \iff Q \succeq 0$ and $P \succ 0 \iff Q \succ 0$. As Q is Hermitian with $\lambda_{\min}(Q) = \alpha - \lambda_{\max}(H(n)) = \delta$, this proves (a).

If $\delta = 0$, then $Q \succeq 0$ is singular with $\operatorname{Ker}(Q) = \mathcal{M}(n)$, and $P \succeq 0$ by (a). For $P \succeq 0$, $x \in \operatorname{Ker}(P) \iff x^* P x = 0$. Writing $x = By$,

$$x^* P x = y^* Q y,$$

so $x \in \operatorname{Ker}(P) \iff y \in \operatorname{Ker}(Q) = \mathcal{M}(n)$, proving (b).

If $\delta > 0$, then $Q \succ 0$ and $P = B^{-*}QB^{-1} \succ 0$. For $\|x\| = 1$ and $y = B^{-1}x$, one has $x = By$ and hence $\|y\| \geq 1/\|B\|$; thus

$$x^*Px = y^*Qy \geq \lambda_{\min}(Q)\|y\|^2 = \delta\|y\|^2 \geq \delta/\|B\|^2,$$

giving the lower bound in (8). For the upper bound, take y a unit eigenvector of Q for $\lambda_{\min}(Q) = \delta$ and set $x = By/\|By\|$; then

$$x^*Px = \frac{y^*Qy}{\|By\|^2} = \frac{\delta}{\|By\|^2} \leq \delta\,\|B^{-1}\|^2.$$

□

Remark 4 (Connection with the convex-domain Poisson kernel literature). *Up to normalization conventions, $P_\Omega(\sigma, A)$ is the operator-valued boundary kernel appearing in the Carl Neumann double-layer potential framework for convex domains; see, e.g., [7–9,11]. Lemma 3 isolates the dependence of $\lambda_{\min}(P(\sigma, n))$ on the scalar support gap $\delta(\sigma, n)$.*

3.3. Strict Positivity When $W(A) \subset \Omega$

Lemma 4 (Strict separation at a supporting line). *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with C^1 boundary and let $K \subset \Omega$ be compact. Fix $\sigma \in \partial\Omega$ and let $n = n_\Omega(\sigma)$ be the outward unit normal. Then*

$$\max_{z \in K} \operatorname{Re}(\overline{n}\, z) \;<\; \operatorname{Re}(\overline{n}\, \sigma).$$

Proof. By (5), $\Omega \subset \{z : \operatorname{Re}(\overline{n}(z - \sigma)) < 0\}$, hence $K \subset \{z : \operatorname{Re}(\overline{n}(z - \sigma)) < 0\}$. The continuous function $z \mapsto \operatorname{Re}(\overline{n}(z - \sigma))$ attains its maximum on compact K, and this maximum is strictly negative (if it were 0 at some $z \in K$, then the segment $[z, \sigma] \subset \overline{\Omega}$ would lie on the supporting line, forcing $z \in \partial\Omega$, contradicting $K \subset \Omega$). Rearranging yields the claim. □

Proposition 1 (Positivity of the Poisson kernel). *Assume $W(A) \subset \Omega$. Then for every $\sigma \in \partial\Omega$,*

$$P_\Omega(\sigma, A) \;\succ\; 0.$$

Proof. Fix $\sigma \in \partial\Omega$ and set $n := n_\Omega(\sigma)$ and $\alpha := \operatorname{Re}(\overline{n}\, \sigma)$. By Lemma 4 with $K = W(A)$ and Lemma 1,

$$\lambda_{\max}(H(n)) = \max_{z \in W(A)} \operatorname{Re}(\overline{n} z) \;<\; \alpha,$$

so $\delta(\sigma, n) = \alpha - \lambda_{\max}(H(n)) > 0$. Now apply Lemma 3 (a). □

3.4. Geometric Meaning of the Support Gap and Offset Exhaustions

Recall that for a compact convex set $K \subset \mathbb{C}$ and unimodular $n \in \mathbb{C}$, its support function is $h_K(n) = \max_{z \in K} \operatorname{Re}(\overline{n}\, z)$ (Definition 1). If $\Omega \subset \mathbb{C}$ is a bounded open convex set with C^1 boundary and $\sigma \in \partial\Omega$ has outward normal $n = n_\Omega(\sigma)$, then necessarily $\operatorname{Re}(\overline{n}\, \sigma) = h_{\overline{\Omega}}(n)$, i.e. the boundary point lies on the supporting line in direction n.

Lemma 5 (Support gap as a support-function difference). *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with C^1 boundary and $\sigma \in \partial\Omega$. Let $n := n_\Omega(\sigma)$. Then*

$$\delta(\sigma, n) = \operatorname{Re}(\overline{n}\, \sigma) - \lambda_{\max}(H(n)) \;=\; h_{\overline{\Omega}}(n) - h_{W(A)}(n).$$

In particular, $\delta(\sigma, n)$ measures the separation between the supporting line of $\overline{\Omega}$ in direction n and the corresponding supporting line of $W(A)$.

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Proof. Since n is the outward unit normal at $\sigma \in \partial\Omega$, the supporting half-plane characterization implies $\mathrm{Re}(\overline{n}\,z) \le \mathrm{Re}(\overline{n}\,\sigma)$ for all $z \in \overline{\Omega}$, hence $h_{\overline{\Omega}}(n) = \mathrm{Re}(\overline{n}\,\sigma)$. By Lemma 1, $h_{W(A)}(n) = \lambda_{\max}(H(n))$. Combining gives the claim. □

Figure 1 illustrates this interpretation schematically.

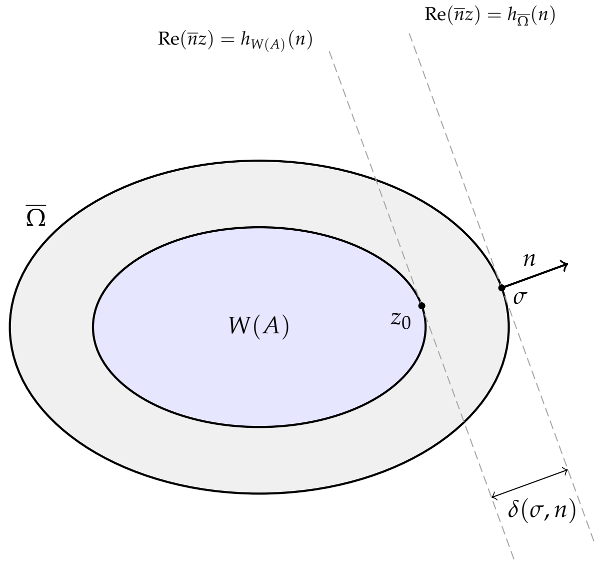

Figure 1. Geometric meaning of the support gap: the supporting lines of $\overline{\Omega}$ and $W(A)$ in direction n are given by $\mathrm{Re}(\overline{n}z) = h_{\overline{\Omega}}(n)$ and $\mathrm{Re}(\overline{n}z) = h_{W(A)}(n)$, and their separation equals $\delta(\sigma, n) = h_{\overline{\Omega}}(n) - h_{W(A)}(n)$. Here $z_0 \in F_{W(A)}(n)$ denotes a support point of $W(A)$ in direction n.

Proposition 2 (Outer offsets: δ is explicit)**.** *Let $K \subset \mathbb{C}$ be compact and convex and fix $\varepsilon > 0$. Define the outer offset (outer parallel set)*

$$K_\varepsilon \;:=\; K + \varepsilon\mathbb{D} \;=\; \{z + w :\; z \in K,\; w \in \mathbb{C},\; |w| \le \varepsilon\}.$$

Then for every unimodular $n \in \mathbb{C}$,

$$h_{K_\varepsilon}(n) = h_K(n) + \varepsilon.$$

In particular, taking $K = W(A)$ and $\Omega_\varepsilon := W(A) + \varepsilon\mathbb{D}$ (so that $\overline{\Omega_\varepsilon} = W(A) + \varepsilon\overline{\mathbb{D}}$), for any boundary point $\sigma \in \partial\Omega_\varepsilon$ with outward normal $n = n_{\Omega_\varepsilon}(\sigma)$ (whenever defined) one has

$$\delta(\sigma, n) = \varepsilon.$$

Consequently, since $\sigma \in \partial\Omega_\varepsilon$ implies $\sigma \notin W(A)$ and $\mathrm{spec}(A) \subset W(A)$, we have $\sigma \notin \mathrm{spec}(A)$ and Lemma 3 (c) yields

$$\frac{\varepsilon}{\|\sigma I - A\|^2} \;\le\; \lambda_{\min}\big(P_{\Omega_\varepsilon}(\sigma, A)\big) \;\le\; \varepsilon\,\|(\sigma I - A)^{-1}\|^2.$$

Proof. Fix unimodular n. For any $z \in K$ and $w \in \mathbb{C}$ with $|w| \le \varepsilon$,

$$\mathrm{Re}(\overline{n}(z + w)) = \mathrm{Re}(\overline{n}z) + \mathrm{Re}(\overline{n}w) \le h_K(n) + |w| \le h_K(n) + \varepsilon,$$

so $h_{K_\varepsilon}(n) \le h_K(n) + \varepsilon$. On the other hand, choosing $z_\star \in K$ with $\mathrm{Re}(\overline{n}z_\star) = h_K(n)$ and $w_\star = \varepsilon n$ gives $|w_\star| = \varepsilon$ and

$$\mathrm{Re}(\overline{n}(z_\star + w_\star)) = h_K(n) + \varepsilon,$$

so $h_{K_\varepsilon}(n) \ge h_K(n) + \varepsilon$. This proves the support-function identity and hence the displayed formula for δ follows from Lemma 5.

The final eigenvalue bounds are an immediate substitution of $\delta = \varepsilon$ into (8). □

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Remark 5 (Smoothness versus offsets). *If K has flat faces, then $\partial(K+\varepsilon\overline{\mathbb{D}})$ is typically $C^{1,1}$ but generally not C^2 (curvature may jump at transitions between translated faces and rounded arcs). Proposition 2 is therefore best viewed as a geometric model illustrating how the support gap scales with the outer distance parameter ε. All the offset-based computations in this paper are formulated at the level of the support function; consequently, the main theorems apply equally to any C^1 smoothing that produces the same first-order support-gap. For the purposes of Definition 2, one may replace $K+\varepsilon\overline{\mathbb{D}}$ by any convex domain with C^1 boundary whose support function differs from h_K by a quantity comparable to ε; the same interpretation of δ then applies. For example, one may take Minkowski sums (see, e.g., [13]) with a fixed smooth strictly convex unit ball (instead of $\overline{\mathbb{D}}$) or smooth the support function to obtain a genuine C^1 (indeed smooth) convex exhaustion with the same first-order support-gap scaling.*

3.5. Hausdorff Distance and Support-Function Control of the Support Gap

For a nonempty compact set $K\subset\mathbb{C}$ and $z\in\mathbb{C}$, write

$$\operatorname{dist}(z,K):=\inf_{w\in K}|z-w|.$$

For nonempty compact sets $K,L\subset\mathbb{C}$, define the (Euclidean) Hausdorff distance

$$d_{\mathrm{H}}(K,L):=\max\Big\{\sup_{z\in K}\operatorname{dist}(z,L),\ \sup_{w\in L}\operatorname{dist}(w,K)\Big\}.$$

Lemma 6 (Hausdorff distance via support functions). *Let $K,L\subset\mathbb{C}$ be nonempty compact convex sets. Then*

$$d_{\mathrm{H}}(K,L)\ =\ \sup_{|n|=1}\big|h_K(n)-h_L(n)\big|.$$

If moreover $K\subseteq L$, then $h_K(n)\le h_L(n)$ for all $|n|=1$ and hence

$$d_{\mathrm{H}}(L,K)\ =\ \sup_{|n|=1}\big(h_L(n)-h_K(n)\big).$$

Proof. For $t\ge 0$ and a nonempty compact set K, the Minkowski sum $K+t\overline{\mathbb{D}}$ [13] is the closed t-neighborhood of K, i.e. $K+t\overline{\mathbb{D}}=\{u\in\mathbb{C}:\operatorname{dist}(u,K)\le t\}$. Consequently,

$$d_{\mathrm{H}}(K,L)=\inf\Big\{t\ge 0:\ K\subseteq L+t\overline{\mathbb{D}}\ \text{ and }\ L\subseteq K+t\overline{\mathbb{D}}\Big\}.$$

For compact convex sets $M,N\subset\mathbb{C}$ one has $M\subseteq N$ if and only if $h_M(n)\le h_N(n)$ for all $|n|=1$. (Indeed, the forward direction is immediate; conversely, if $x\in M\setminus N$, a separating supporting line for the convex compact set N yields a unimodular n with $\operatorname{Re}(\overline{n}\,x)>\max_{z\in N}\operatorname{Re}(\overline{n}\,z)=h_N(n)$, hence $h_M(n)\ge\operatorname{Re}(\overline{n}\,x)>h_N(n)$.)

Moreover, for $|n|=1$, Proposition 2 gives $h_{K+t\overline{\mathbb{D}}}(n)=h_K(n)+t$. Therefore, $K\subseteq L+t\overline{\mathbb{D}}$ is equivalent to $h_K(n)\le h_L(n)+t$ for all $|n|=1$, and similarly $L\subseteq K+t\overline{\mathbb{D}}$ is equivalent to $h_L(n)\le h_K(n)+t$ for all $|n|=1$. Thus $d_{\mathrm{H}}(K,L)$ is the smallest t such that $|h_K(n)-h_L(n)|\le t$ for all $|n|=1$, i.e.

$$d_{\mathrm{H}}(K,L)=\sup_{|n|=1}|h_K(n)-h_L(n)|.$$

If $K\subseteq L$, then $h_K\le h_L$, so the absolute value may be dropped, giving the second identity. □

Corollary 1 (Support gap bounded by the Hausdorff approximation error). *Assume $W(A)\subset\Omega$, and set*

$$\Delta(\Omega)\ :=\ d_{\mathrm{H}}(\overline{\Omega},W(A))\ =\ \sup_{|n|=1}\big(h_{\overline{\Omega}}(n)-h_{W(A)}(n)\big).$$

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Then for every $\sigma \in \partial\Omega$ with outward normal $n = n_\Omega(\sigma)$,

$$\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)) = h_{\overline{\Omega}}(n) - h_{W(A)}(n) \;\le\; \Delta(\Omega).$$

Consequently, since $\sigma \notin \mathrm{spec}(A)$ for $\sigma \in \partial\Omega$, Lemma 3 (c) yields

$$\lambda_{\min}\big(P_\Omega(\sigma, A)\big) \;\le\; \delta(\sigma, n)\,\|(\sigma I - A)^{-1}\|^2 \;\le\; \Delta(\Omega)\,\|(\sigma I - A)^{-1}\|^2.$$

Moreover, there exists $\sigma_\star \in \partial\Omega$ such that

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = \Delta(\Omega),$$

and for this point one has the two-sided estimate

$$\frac{\Delta(\Omega)}{\|\sigma_\star I - A\|^2} \;\le\; \lambda_{\min}\big(P_\Omega(\sigma_\star, A)\big) \;\le\; \Delta(\Omega)\,\|(\sigma_\star I - A)^{-1}\|^2.$$

Proof. The identity $\delta(\sigma, n) = h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is Lemma 5, and the bound $\delta(\sigma, n) \le \Delta(\Omega)$ follows from the definition of $\Delta(\Omega)$. The eigenvalue bounds are then immediate from Lemma 3 (c).

Finally, the function $n \mapsto h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is continuous on the unit circle, so it attains its maximum at some unimodular $n_\star$. Choose $\sigma_\star \in \overline{\Omega}$ such that $\mathrm{Re}(\overline{n_\star}\,\sigma_\star) = h_{\overline{\Omega}}(n_\star)$; then $\sigma_\star \in \partial\Omega$ and the supporting line $\{z : \mathrm{Re}(\overline{n_\star} z) = \mathrm{Re}(\overline{n_\star}\sigma_\star)\}$ is a supporting line for $\overline{\Omega}$ at $\sigma_\star$. Since $\partial\Omega$ is C^1, the outward unit normal at $\sigma_\star$ is uniquely defined and equals $n_\star$, and hence

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = h_{\overline{\Omega}}(n_\star) - h_{W(A)}(n_\star) = \Delta(\Omega).$$

□

4. Degeneracy Along a C^1 Convex Exhaustion

Regime map. Fix a limiting supporting pair (σ_0, n) for $W(A)$. We distinguish the *non-spectral* case $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$ (bounded resolvent; Theorem 1 and Corollary 3) from the *spectral-support* case $\sigma_0 \in \partial W(A) \cap \mathrm{spec}(A)$ (resolvent blow-up; Proposition 11 and Proposition 12). Within either case, if the supporting pencil $H(n) = \mathrm{Re}(\overline{n}A)$ has an isolated top eigenvalue (gap $\gamma_H(n) > 0$) we obtain explicit slope spectra and two-/three-scale splittings (Section 4.4); without this isolation we retain qualitative collapse but not a full cluster description.

4.1. Qualitative Degeneracy and Limiting Kernel Directions

Theorem 1 (Degeneracy of the operator-valued Poisson kernel). *Let $A \in \mathbb{C}^{d\times d}$ and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a C^1 convex exhaustion of $W(A)$ (Definition 2). For $\sigma \in \partial\Omega_\varepsilon$, set*

$$P_{\Omega_\varepsilon}(\sigma, A) \;:=\; \mathrm{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

Fix any sequence $\varepsilon_k \downarrow 0$ and points $\sigma_k \in \partial\Omega_{\varepsilon_k}$ such that

$$\sigma_k \to \sigma_0 \in \partial W(A), \qquad n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n \in \mathbb{C}, \quad |n| = 1.$$

(After passing to a subsequence, the convergence $n_k \to n$ is automatic; see Remark 2.) Assume $\sigma_0 \notin \mathrm{spec}(A)$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and $\mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n))$. Note that the supporting half-plane property for Ω_{ε_k} and passage to the limit imply that n is a supporting direction for $W(A)$ at σ_0, i.e. $\mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n))$.

Then:

(1) (Vanishing) $\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \to 0$ *as $k \to \infty$.*

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(2) (Limiting directions) *If u_k is any unit eigenvector of $P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ for $\lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A))$, then every accumulation point u_0 of $\{u_k\}$ satisfies*

$$u_0 \in (\sigma_0 I - A)\, \mathcal{M}(n).$$

(3) (One-dimensional case) *If $\dim \mathcal{M}(n) = 1$, then there exist phases $\theta_k \in \mathbb{R}$ such that*

$$e^{i\theta_k} u_k \longrightarrow \frac{(\sigma_0 I - A)v}{\|(\sigma_0 I - A)v\|} \qquad (k \to \infty),$$

where v is any unit vector spanning $\mathcal{M}(n)$.

Proof. Set $B_k := \sigma_k I - A$ and $R_k := B_k^{-1}$, and define

$$P_k := \mathrm{Re}(n_k R_k), \qquad \alpha_k := \mathrm{Re}(\overline{n_k} \sigma_k).$$

Define also $B_0 := \sigma_0 I - A$, $R_0 := B_0^{-1}$, $P_0 := \mathrm{Re}(n R_0)$, $\alpha_0 := \mathrm{Re}(\overline{n}\, \sigma_0)$.
Step 1: Congruence identities. By Lemma 2,

$$B_k^* P_k B_k = \alpha_k I - H(n_k), \qquad B_0^* P_0 B_0 = \alpha_0 I - H(n), \tag{9}$$

where $H(n_k) = \mathrm{Re}(\overline{n_k} A)$.
Step 2: $\alpha_0 = \lambda_{\max}(H(n))$. Since n_k is the outward normal at $\sigma_k \in \partial\Omega_{\varepsilon_k}$, the supporting half-plane property gives $\mathrm{Re}(\overline{n_k} z) \le \alpha_k$ for all $z \in \Omega_{\varepsilon_k}$ and hence for all $z \in W(A)$. Passing to the limit yields $\mathrm{Re}(\overline{n}\, z) \le \alpha_0$ for all $z \in W(A)$. Because $\sigma_0 \in W(A)$ and $\alpha_0 = \mathrm{Re}(\overline{n}\, \sigma_0)$, equality holds in the last inequality at $z = \sigma_0$, so $\alpha_0 = \max_{z \in W(A)} \mathrm{Re}(\overline{n} z)$. Lemma 1 now gives

$$\alpha_0 = \lambda_{\max}(H(n)), \qquad \mathrm{Ker}(\alpha_0 I - H(n)) = \mathcal{M}(n),$$

so $\alpha_0 I - H(n) \succeq 0$ is singular.
Step 3: $P_k \to P_0$ in operator norm. Since $\sigma_0 \notin \mathrm{spec}(A)$, B_0 is invertible. Write

$$B_k = B_0 + (\sigma_k - \sigma_0) I = B_0 (I + E_k), \qquad E_k := (\sigma_k - \sigma_0) R_0.$$

Then $\|E_k\| \to 0$, so for large k, $I + E_k$ is invertible and

$$R_k = B_k^{-1} = (I + E_k)^{-1} R_0, \qquad \|R_k - R_0\| \to 0.$$

Therefore,

$$\|P_k - P_0\| = \|\mathrm{Re}(n_k R_k - n R_0)\| \le |n_k - n|\, \|R_k\| + \|R_k - R_0\| \to 0.$$

Step 4: $\lambda_{\min}(P_0) = 0$ and $\lambda_{\min}(P_k) \to 0$. Since P_k, P_0 are Hermitian, Weyl's inequality (see, e.g., [12]) yields

$$\big|\lambda_{\min}(P_k) - \lambda_{\min}(P_0)\big| \le \|P_k - P_0\| \to 0,$$

so $\lambda_{\min}(P_k) \to \lambda_{\min}(P_0)$. By (9) and Step 2,

$$B_0^* P_0 B_0 = \alpha_0 I - H(n) \succeq 0 \ \text{ is singular.}$$

Geometrically, this singularity reflects the supporting-line identity $\alpha_0 = \lambda_{\max}(H(n))$ (i.e. the support gap $\delta(\sigma_0, n) = 0$). Since B_0 is invertible, $P_0 \succeq 0$ is singular, hence $\lambda_{\min}(P_0) = 0$, proving (1). Moreover,

$$\mathrm{Ker}(P_0) = B_0\ \mathrm{Ker}(\alpha_0 I - H(n)) = (\sigma_0 I - A)\mathcal{M}(n)$$

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by Lemma 3 (b) (with $\delta = 0$).

Step 5: Limiting eigenvectors. Let u_k be unit min-eigenvectors: $P_k u_k = \lambda_{\min}(P_k) u_k$. Along a convergent subsequence, $u_k \to u_0$. Then

$$u_0^* P_0 u_0 = \lim_{k\to\infty} u_k^* P_0 u_k = \lim_{k\to\infty} \left(u_k^* P_k u_k + u_k^*(P_0 - P_k) u_k\right) = \lim_{k\to\infty} \left(\lambda_{\min}(P_k) + o(1)\right) = 0.$$

Since $P_0 \succeq 0$, this implies $u_0 \in \mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$, proving (2).

Step 6: One-dimensional case. If $\dim \mathcal{M}(n) = 1$, then $\dim \mathrm{Ker}(P_0) = 1$, so the smallest eigenvalue of P_0 is simple and $\lambda_2^{\uparrow}(P_0) > 0$. By Weyl's inequality [12] and $P_k \to P_0$, for sufficiently large k one has $\lambda_2^{\uparrow}(P_k) \geq \lambda_2^{\uparrow}(P_0)/2$, hence $\lambda_{\min}(P_k)$ is simple for large k. By the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces (see [15]), the corresponding one-dimensional eigenspaces of P_k converge to $\mathrm{Ker}(P_0)$ in gap metric, hence there exist phases θ_k such that $e^{i\theta_k} u_k \to u_\star$, where $u_\star$ spans $\mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$. This gives (3). □

Remark 6 (Why $\sigma_0 \notin \mathrm{spec}(A)$ is essential)**.** *The hypothesis $\sigma_0 \notin \mathrm{spec}(A)$ ensures that $(\sigma I - A)^{-1}$ remains bounded near σ_0, so P_0 is a finite Hermitian matrix. When $\sigma_0 \in \mathrm{spec}(A)$, the resolvent diverges and the behavior of $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A))$ depends on the spectral geometry; see Proposition 11 below and Section 4.11.*

Corollary 2 (Global coercivity collapse along a C^1 convex exhaustion)**.** *Assume that A is not a scalar multiple of the identity (equivalently, $W(A)$ is not a singleton). Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a C^1 convex exhaustion of $W(A)$ and define the global coercivity constant*

$$c(\varepsilon) := \inf_{\sigma\in\partial\Omega_\varepsilon} \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A)), \qquad P_{\Omega_\varepsilon}(\sigma, A) = \mathrm{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

Then

$$\liminf_{\varepsilon\downarrow 0} c(\varepsilon) = 0.$$

In particular, there do not exist $\varepsilon_0 > 0$ and $c_0 > 0$ such that $P_{\Omega_\varepsilon}(\sigma, A) \succeq c_0 I$ for all $0 < \varepsilon < \varepsilon_0$ and all $\sigma \in \partial\Omega_\varepsilon$.

Proof. Since A is not scalar, the compact convex set $W(A)$ contains more than one point, hence $\partial W(A)$ is infinite, whereas $\mathrm{spec}(A)$ is finite. Choose $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$.

Fix any sequence $\varepsilon_k \downarrow 0$. We claim that $\mathrm{dist}(\sigma_0, \partial\Omega_{\varepsilon_k}) \to 0$. Indeed, if not, then there exist $r > 0$ and a subsequence (not relabeled) such that $\mathrm{dist}(\sigma_0, \partial\Omega_{\varepsilon_k}) \geq r$ for all k, hence the open ball $B(\sigma_0, r)$ is contained in Ω_{ε_k} for all k. Taking closures and intersecting over k yields $B(\sigma_0, r) \subset \bigcap_k \overline{\Omega_{\varepsilon_k}} = W(A)$, contradicting $\sigma_0 \in \partial W(A)$.

Therefore we may choose $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with $\sigma_k \to \sigma_0$. By compactness of the unit circle, after passing to a subsequence we have $n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n$ for some unimodular n. Theorem 1 then gives

$$\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \longrightarrow 0.$$

Since $c(\varepsilon_k) \leq \lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A))$, it follows that $\liminf_{\varepsilon\downarrow 0} c(\varepsilon) = 0$. □

4.2. Quantitative Degeneracy Rate

Corollary 3 (Linear rate in terms of the support gap)**.** *In the setting of Theorem 1, define*

$$\delta_k := \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k)), \qquad H(n_k) = \mathrm{Re}(\overline{n_k} A).$$

Then $\delta_k > 0$ for each k and $\delta_k \to 0$. Moreover, for all sufficiently large k,

$$\frac{\delta_k}{4\|\sigma_0 I - A\|^2} \leq \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \leq 4\,\delta_k\,\|(\sigma_0 I - A)^{-1}\|^2. \tag{10}$$

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In particular, $\lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) = \Theta(\delta_k)$.

Proof. Since $W(A) \subset \Omega_{\varepsilon_k}$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with normal n_k, Lemma 4 and Lemma 1 imply $\lambda_{\max}(H(n_k)) < \mathrm{Re}(\overline{n_k}\sigma_k)$, so $\delta_k > 0$.

As $n_k \to n$ and $\sigma_k \to \sigma_0$, $\mathrm{Re}(\overline{n_k}\sigma_k) \to \mathrm{Re}(\overline{n}\,\sigma_0)$. Also $H(n_k) \to H(n)$ in operator norm, hence $\lambda_{\max}(H(n_k)) \to \lambda_{\max}(H(n))$. By Step 2 in the proof of Theorem 1, $\mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n))$, so $\delta_k \to 0$.

Set $B_k = \sigma_k I - A$ and $B_0 = \sigma_0 I - A$. Since $B_k \to B_0$ and B_0 is invertible, for large k one has $\|B_k\| \le 2\|B_0\|$ and $\|B_k^{-1}\| \le 2\|B_0^{-1}\|$. Applying Lemma 3 (c) to $(\sigma, n) = (\sigma_k, n_k)$ gives

$$\frac{\delta_k}{\|B_k\|^2} \le \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \le \delta_k \|B_k^{-1}\|^2,$$

and the stated constants follow. □

4.3. Sharpness and Refined Local/Global Bounds

Lemma 3 (c) bounds $\lambda_{\min}(P(\sigma,n))$ in terms of the scalar support gap $\delta(\sigma,n)$ and the global operator norms $\|\sigma I - A\|$, $\|(\sigma I - A)^{-1}\|$. We first record that these norm-based bounds are optimal in the strongest possible sense, and then derive refinements that capture the correct constant in the degeneracy regime $\delta \downarrow 0$.

Proposition 3 (Optimality of the norm-based support-gap bounds)**.** *The constants in the two-sided inequality* (8) *are sharp and cannot be improved uniformly. More precisely, for $d = 1$ both inequalities in* (8) *hold with equality.*

Proof. Let $d = 1$ and write $A = [a]$ with $a \in \mathbb{C}$. Then $B = \sigma - a$ is a nonzero scalar and

$$P(\sigma,n) = \mathrm{Re}\Big(\frac{n}{\sigma - a}\Big) = \frac{\mathrm{Re}(\overline{n}(\sigma - a))}{|\sigma - a|^2}.$$

Moreover $\delta(\sigma,n) = \mathrm{Re}(\overline{n}\sigma) - \mathrm{Re}(\overline{n}a) = \mathrm{Re}(\overline{n}(\sigma - a))$, and $\|B\| = |\sigma - a|$, $\|B^{-1}\| = 1/|\sigma - a|$. Therefore

$$\lambda_{\min}(P(\sigma,n)) = \frac{\delta(\sigma,n)}{\|B\|^2} = \delta(\sigma,n)\,\|B^{-1}\|^2,$$

so (8) is an equality in both directions. □

Proposition 4 (Generalized eigenvalue characterization)**.** *Let $\sigma \notin \mathrm{spec}(A)$ and $|n| = 1$. Set $B := \sigma I - A$, $\alpha := \mathrm{Re}(\overline{n}\sigma)$, $H(n) = \mathrm{Re}(\overline{n}A)$ and $Q := \alpha I - H(n)$. Then*

$$\lambda_{\min}(P(\sigma,n)) = \min_{y \neq 0} \frac{y^* Q y}{\|By\|^2} = \lambda_{\min}(Q,\, B^*B), \tag{11}$$

where $\lambda_{\min}(Q, B^*B)$ denotes the smallest generalized eigenvalue of the Hermitian definite pencil (Q, B^*B).

Proof. By Lemma 2, $P(\sigma,n) = B^{-*}QB^{-1}$. For any $x \neq 0$ write $x = By$ (bijective since B is invertible). Then

$$\frac{x^* P(\sigma,n) x}{\|x\|^2} = \frac{y^* Q y}{\|By\|^2}.$$

Taking the minimum over $x \neq 0$ is equivalent to taking the minimum over $y \neq 0$, which gives the first equality in (11); the second is the standard variational characterization of generalized eigenvalues. □

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Proposition 5 (Refined bounds via restriction to the maximal eigenspace)**.** *Let $\sigma \notin \operatorname{spec}(A)$ and $|n| = 1$, and define B, Q as in Proposition 4. Let $\lambda_{\max}(H(n))$ have maximal eigenspace $\mathcal{M}(n)$ and set*

$$\delta := \lambda_{\min}(Q) = \alpha - \lambda_{\max}(H(n)), \qquad \beta(\sigma, n) := \|B|_{\mathcal{M}(n)}\| = \max_{\substack{y \in \mathcal{M}(n) \\ \|y\|=1}} \|By\|.$$

Assume $\delta > 0$ (equivalently $P(\sigma, n) \succ 0$). Then:

(a) (Refined upper bound)

$$\lambda_{\min}(P(\sigma, n)) \;\le\; \frac{\delta}{\beta(\sigma, n)^2}. \tag{12}$$

(b) (Asymptotically sharp two-sided bound) *Assume in addition that $\lambda_{\max}(H(n))$ is spectrally isolated with gap*

$$\gamma_H(n) := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) \;>\; 0, \qquad m := \dim \mathcal{M}(n) \;(< d). \tag{13}$$

Then

$$\frac{\delta}{\beta(\sigma, n)^2 + \frac{\|B\|^2}{\gamma_H(n)}\,\delta} \;\le\; \lambda_{\min}(P(\sigma, n)) \;\le\; \frac{\delta}{\beta(\sigma, n)^2}. \tag{14}$$

In particular, as $\delta \downarrow 0$ in a regime where $\|B^{-1}\|$ remains bounded (e.g. $\|B^{-1}\| \le C$), $\lambda_{\min}(P(\sigma, n)) = \frac{\delta}{\beta(\sigma,n)^2} + O(\delta^2)$.

Proof. By Proposition 4,

$$\lambda_{\min}(P(\sigma, n)) = \min_{y \neq 0} \frac{y^* Q y}{\|By\|^2}.$$

(a) On $\mathcal{M}(n)$ one has $H(n)y = \lambda_{\max}(H(n))y$, hence $Qy = (\alpha - \lambda_{\max}(H(n)))y = \delta y$. Choose $y \in \mathcal{M}(n)$ with $\|y\| = 1$ attaining $\|By\| = \beta(\sigma, n)$. Then

$$\lambda_{\min}(P(\sigma, n)) \le \frac{y^* Q y}{\|By\|^2} = \frac{\delta}{\beta(\sigma, n)^2},$$

which is (12).

(b) Decompose $y = y_M + y_\perp$ with $y_M \in \mathcal{M}(n)$ and $y_\perp \perp \mathcal{M}(n)$. Writing $Q = Q_0 + \delta I$ with $Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0$, one has $Q_0 y_M = 0$ and, by the gap hypothesis (13), $Q_0 \succeq \gamma_H(n) I$ on $\mathcal{M}(n)^\perp$. Hence

$$y^* Q y \;\ge\; \delta \|y_M\|^2 + \gamma_H(n) \|y_\perp\|^2. \tag{15}$$

Moreover, using $\|By_M\| \le \beta(\sigma, n)\|y_M\|$ and $\|By_\perp\| \le \|B\|\,\|y_\perp\|$ gives

$$\|By\| \le \beta(\sigma, n)\|y_M\| + \|B\|\,\|y_\perp\|.$$

By Cauchy–Schwarz, for $a = \|y_M\|$ and $b = \|y_\perp\|$,

$$\big(\beta(\sigma, n)a + \|B\|b\big)^2 \le \left(\frac{\beta(\sigma, n)^2}{\delta} + \frac{\|B\|^2}{\gamma_H(n)}\right)(\delta a^2 + \gamma_H(n) b^2),$$

so combining with the two displays above gives

$$\frac{y^* Q y}{\|By\|^2} \;\ge\; \frac{\delta}{\beta(\sigma, n)^2 + \frac{\|B\|^2}{\gamma_H(n)}\,\delta}.$$

Taking the minimum over $y \neq 0$ yields the lower bound in (14); the upper bound is part (a). Dividing (14) by δ and letting $\delta \downarrow 0$ gives the stated first-order expansion. □

Corollary 4 (Sharp first-order constant for outer offsets). *Fix $|n| = 1$ and let $z_0 \in \partial W(A)$ be a support point in direction n, i.e.* $\mathrm{Re}(\bar{n} z_0) = \lambda_{\max}(H(n))$. *Assume $z_0 \notin \mathrm{spec}(A)$ and $\gamma_H(n) > 0$ as in (13). For $\varepsilon > 0$ set $\sigma_\varepsilon := z_0 + \varepsilon n$ and*

$$P_\varepsilon(n) := \mathrm{Re}\Big(n(\sigma_\varepsilon I - A)^{-1}\Big).$$

Then

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda_{\min}(P_\varepsilon(n))}{\varepsilon} = \big\|(z_0 I - A)|_{\mathcal{M}(n)}\big\|^{-2}. \tag{16}$$

In particular, $\lambda_{\min}(P_\varepsilon(n)) = \varepsilon \, \|(z_0 I - A)|_{\mathcal{M}(n)}\|^{-2} + O(\varepsilon^2)$.

Proof. For $\sigma_\varepsilon = z_0 + \varepsilon n$ one has $\delta(\sigma_\varepsilon, n) = \varepsilon$. Applying Proposition 5 (b) to $(\sigma, n) = (\sigma_\varepsilon, n)$ gives a two-sided bound of the form

$$\frac{\varepsilon}{\beta(\sigma_\varepsilon, n)^2 + O(\varepsilon)} \;\leq\; \lambda_{\min}(P_\varepsilon(n)) \;\leq\; \frac{\varepsilon}{\beta(\sigma_\varepsilon, n)^2}.$$

Since $z_0 \notin \mathrm{spec}(A)$, $B_\varepsilon := \sigma_\varepsilon I - A \to B_0 := z_0 I - A$ in operator norm and B_0 is invertible. Hence $\beta(\sigma_\varepsilon, n) = \|B_\varepsilon|_{\mathcal{M}(n)}\| \to \|B_0|_{\mathcal{M}(n)}\|$, and dividing by ε and letting $\varepsilon \downarrow 0$ yields (16). ☐

Proposition 6 (Global first-order slope for direction-sampled offsets). *Assume that one can choose support points* continuously, *i.e. there exists a continuous map $z_0 : \{n \in \mathbb{C} : |n| = 1\} \to \partial W(A)$ with* $\mathrm{Re}(\bar{n} z_0(n)) = \lambda_{\max}(H(n))$ *and such that:*

(i) $z_0(n) \notin \mathrm{spec}(A)$ *(so $(\sigma I - A)^{-1}$ stays bounded as $\sigma \to z_0(n)$), and*
(ii) the top eigenspace $\mathcal{M}(n)$ is spectrally isolated with a uniform gap

$$\inf_{|n|=1} \gamma_H(n) \;>\; 0. \tag{17}$$

For each $\varepsilon > 0$ define the direction-sampled coercivity constant

$$\tilde{c}(\varepsilon) := \inf_{|n|=1} \lambda_{\min}(P_\varepsilon(n)), \qquad P_\varepsilon(n) = \mathrm{Re}\Big(n\big((z_0(n) + \varepsilon n)I - A\big)^{-1}\Big).$$

Set

$$\beta_0(n) := \big\|(z_0(n) I - A)|_{\mathcal{M}(n)}\big\|, \qquad \beta_{\max} := \sup_{|n|=1} \beta_0(n).$$

Then

$$\lim_{\varepsilon \downarrow 0} \frac{\tilde{c}(\varepsilon)}{\varepsilon} = \frac{1}{\beta_{\max}^2}. \tag{18}$$

Proof. Under the uniform gap assumption (17), the spectral projector onto $\mathcal{M}(n)$ depends continuously on n. Since $n \mapsto z_0(n)$ is continuous by assumption, so is $B_0(n) := z_0(n) I - A$, and hence

$$\beta_0(n) = \big\|B_0(n)|_{\mathcal{M}(n)}\big\| = \|B_0(n)\Pi(n)\|$$

is continuous (where $\Pi(n)$ is the orthogonal projector onto $\mathcal{M}(n)$). In particular, β_0 attains its maximum $\beta_{\max}$.

For $\varepsilon > 0$ set $\sigma_\varepsilon(n) := z_0(n) + \varepsilon n$ and $B_\varepsilon(n) := \sigma_\varepsilon(n) I - A$. Since $z_0(\cdot)$ is continuous on the compact unit circle and avoids $\mathrm{spec}(A)$, the continuous function $n \mapsto \mathrm{dist}(z_0(n), \mathrm{spec}(A))$ attains a positive minimum $d_\star > 0$; choosing $\varepsilon_0 := d_\star/2$ ensures $\sigma_\varepsilon(n) \notin \mathrm{spec}(A)$ for all $|n| = 1$ and all $0 \leq \varepsilon \leq \varepsilon_0$. Proposition 5 (b) applied at $(\sigma, n) = (\sigma_\varepsilon(n), n)$ (where $\delta(\sigma_\varepsilon(n), n) = \varepsilon$) gives, for all $0 < \varepsilon \leq \varepsilon_0$ and all $|n| = 1$,

$$\frac{1}{\beta_\varepsilon(n)^2 + C\varepsilon} \;\leq\; \frac{\lambda_{\min}(P_\varepsilon(n))}{\varepsilon} \;\leq\; \frac{1}{\beta_\varepsilon(n)^2}, \qquad \beta_\varepsilon(n) := \big\|B_\varepsilon(n)|_{\mathcal{M}(n)}\big\|,$$

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with a finite constant

$$C := \sup_{\substack{|n|=1 \\ 0\le\varepsilon\le\varepsilon_0}} \frac{\|B_\varepsilon(n)\|^2}{\gamma_H(n)} < \infty$$

(using (17) and boundedness of z_0 on the unit circle). Since $(\varepsilon, n) \mapsto \beta_\varepsilon(n)$ is continuous on $[0, \varepsilon_0] \times \{n : |n| = 1\}$, it is uniformly continuous, and hence $\beta_\varepsilon(n) \to \beta_0(n)$ uniformly in n as $\varepsilon \downarrow 0$. Moreover, $\beta_0(n) > 0$ for all n, so by continuity on the compact unit circle there exists $\beta_{\min} > 0$ with $\beta_0(n) \ge \beta_{\min}$ for all $|n| = 1$, and therefore the same uniform convergence holds for the reciprocals. Therefore $\lambda_{\min}(P_\varepsilon(n))/\varepsilon \to 1/\beta_0(n)^2$ uniformly in n, and taking infima over $|n| = 1$ yields

$$\lim_{\varepsilon\downarrow 0} \frac{\tilde{c}(\varepsilon)}{\varepsilon} = \inf_{|n|=1} \frac{1}{\beta_0(n)^2} = \frac{1}{\sup_{|n|=1} \beta_0(n)^2} = \frac{1}{\beta_{\max}^2},$$

which is (18). □

Remark 7 (On continuous selections of support points)**.** *The hypothesis that $z_0(\cdot)$ can be chosen continuously is automatic, for example, if the exposed face $F_{W(A)}(n)$ (Definition 1) is a singleton for every $|n| = 1$ (e.g. when $\partial W(A)$ is strictly convex). More generally, Proposition 6 applies on any arc of directions for which the exposed face is a singleton and the maximal eigenspace $\mathcal{M}(n)$ has constant multiplicity. At directions corresponding to flat portions of $\partial W(A)$ or to multiplicity changes of $\lambda_{\max}(H(n))$, a global continuous selection need not exist.*

4.4. Slope Spectra, Two-Scale Splitting, and Face Detection at Non-Spectral Support Points

The bounds and slope constants in Section 4.3 focus on $\lambda_{\min}$. Under a spectral-gap hypothesis for the supporting pencil $H(n)$, one can quantify the *entire* cluster of eigenvalues that collapses to 0 as the support gap closes. This yields a two-scale spectral splitting (an $O(\varepsilon)$ cluster plus an $O(1)$ bulk) and leads to a simple computational “face detector” based on counting small eigenvalues.

4.4.1. A Rigid Non-Tangential Offset Model

Fix a unimodular direction $n \in \mathbb{C}$ and assume that $\lambda_{\max}(H(n))$ is isolated with multiplicity

$$m := \dim \mathcal{M}(n), \qquad \gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) > 0 \qquad (m < d), \tag{19}$$

where $\mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n))$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$. Choose any unit vector $v \in \mathcal{M}(n)$ and define the associated numerical-range support point

$$\sigma_0 := v^*Av \in \partial W(A), \qquad \mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n)) \tag{20}$$

(Lemma 1). We impose the bounded-resolvent hypothesis

$$\sigma_0 \notin \mathrm{spec}(A). \tag{21}$$

In finite dimensions, (21) is equivalent to local boundedness of the resolvent: $(\sigma I - A)^{-1}$ stays bounded as $\sigma \to \sigma_0$.

For $\varepsilon > 0$, define the outer offset point and the offset kernel

$$\sigma_\varepsilon := \sigma_0 + \varepsilon n, \qquad P_\varepsilon := P(\sigma_\varepsilon, n) = \mathrm{Re}\Big(n\big(\sigma_\varepsilon I - A\big)^{-1}\Big). \tag{22}$$

Let $V \in \mathbb{C}^{d\times m}$ have orthonormal columns spanning $\mathcal{M}(n)$ and set

$$B_0 := \sigma_0 I - A, \qquad G := V^* B_0^* B_0 V \in \mathbb{C}^{m\times m}. \tag{23}$$

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Equivalently, B_0V consists of the vectors $(\sigma_0 I - A)w$ for an orthonormal basis w of $\mathcal{M}(n)$, so $G = (B_0V)^*(B_0V)$ is the Gram matrix of the image subspace $(\sigma_0 I - A)\mathcal{M}(n)$. In particular, the slope spectrum below is determined by the restriction of B_0 to $\mathcal{M}(n)$.

Proposition 7 (Offset slope spectrum for the collapsing eigenvalue cluster). *Under* (19)–(21), $G \succ 0$ *and*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\,\lambda_j^{\uparrow}(P_\varepsilon) \;=\; \lambda_j^{\uparrow}(G^{-1}), \qquad j = 1,\dots,m. \tag{24}$$

Equivalently, if $0 < g_1 \le \cdots \le g_m$ *are the eigenvalues of* G*, then*

$$\lambda_j^{\uparrow}(P_\varepsilon) \;=\; \frac{\varepsilon}{g_{m+1-j}} + o(\varepsilon), \qquad j = 1,\dots,m.$$

In particular, if $m = 1$ *and* $\mathcal{M}(n) = \operatorname{span}\{v\}$*, then*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\lambda_{\min}(P_\varepsilon) \;=\; \frac{1}{\|(\sigma_0 I - A)v\|^2},$$

which recovers Corollary 4 for this rigid offset family.

Proof. Step 1: $G \succ 0$**.** Since $\sigma_0 \notin \operatorname{spec}(A)$, the matrix $B_0 = \sigma_0 I - A$ is invertible. The $d \times m$ matrix V has full column rank (orthonormal columns), hence B_0V has full column rank. Therefore the Gram matrix

$$G = (B_0V)^*(B_0V) = V^*B_0^*B_0V$$

is Hermitian positive definite.

Step 2: Congruence for the offset family. Set $B_\varepsilon := \sigma_\varepsilon I - A = B_0 + \varepsilon n I$ and

$$Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0.$$

By Lemma 2 (applied at $(\sigma, n) = (\sigma_\varepsilon, n)$) and $\operatorname{Re}(\overline{n}\sigma_\varepsilon) = \operatorname{Re}(\overline{n}\sigma_0) + \varepsilon = \lambda_{\max}(H(n)) + \varepsilon$, we obtain

$$B_\varepsilon^* P_\varepsilon B_\varepsilon = \operatorname{Re}(\overline{n}\sigma_\varepsilon)I - H(n) = Q_0 + \varepsilon I. \tag{25}$$

Since $\operatorname{dist}(\sigma_0, \operatorname{spec}(A)) > 0$, choose $0 < \varepsilon_0 < \operatorname{dist}(\sigma_0, \operatorname{spec}(A))$ so that $\sigma_\varepsilon = \sigma_0 + \varepsilon n \notin \operatorname{spec}(A)$ and hence B_ε is invertible for all $0 < \varepsilon \le \varepsilon_0$. In particular, $P_\varepsilon \succ 0$ for such ε.

Step 3: Work with the inverse. From (25),

$$P_\varepsilon^{-1} = B_\varepsilon\,(Q_0 + \varepsilon I)^{-1}\,B_\varepsilon^*. \tag{26}$$

Let $\Pi := VV^*$ be the orthogonal projector onto $\mathcal{M}(n) = \operatorname{Ker}(Q_0)$. On $\mathcal{M}(n)$ one has $(Q_0 + \varepsilon I)^{-1} = (1/\varepsilon)I$. On $\mathcal{M}(n)^{\perp}$ the gap assumption gives $Q_0 \succeq \gamma_H(I - \Pi)$ and hence

$$\|(Q_0 + \varepsilon I)^{-1}(I - \Pi)\| \le \frac{1}{\gamma_H}.$$

Thus

$$\varepsilon(Q_0 + \varepsilon I)^{-1} = \Pi + \varepsilon R_\varepsilon, \qquad R_\varepsilon := (Q_0 + \varepsilon I)^{-1}(I - \Pi), \quad \|R_\varepsilon\| \le \gamma_H^{-1}.$$

Multiplying (26) by ε yields

$$\varepsilon\,P_\varepsilon^{-1} = B_\varepsilon \Pi B_\varepsilon^* + \varepsilon\,B_\varepsilon R_\varepsilon B_\varepsilon^*. \tag{27}$$

Since $B_\varepsilon \to B_0$ and $\{R_\varepsilon\}$ is uniformly bounded, we conclude that

$$\varepsilon\,P_\varepsilon^{-1} \;\longrightarrow\; S := B_0 \Pi B_0^* \qquad (\varepsilon \downarrow 0) \tag{28}$$

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in operator norm.

Step 4: Identify the nonzero spectrum of the limit. Since $S = B_0 V V^* B_0^* = (B_0 V)(B_0 V)^*$, $S \succeq 0$ has rank m. Its nonzero eigenvalues coincide with the eigenvalues of

$$(B_0 V)^*(B_0 V) = V^* B_0^* B_0 V = G.$$

Writing eigenvalues in nondecreasing order,

$$\lambda^{\uparrow}_{d-m+i}(S) = \lambda^{\uparrow}_i(G), \qquad i = 1, \dots, m, \qquad \lambda^{\uparrow}_j(S) = 0 \ \ (j \le d-m).$$

Step 5: Pass to eigenvalues and invert. By Weyl's inequality [12] and (28),

$$\lambda^{\uparrow}_{d-m+i}(\varepsilon P_\varepsilon^{-1}) \to \lambda^{\uparrow}_i(G), \qquad i = 1, \dots, m.$$

Since $P_\varepsilon \succ 0$, the eigenvalues of P_ε and P_ε^{-1} are reciprocal and reversed:

$$\lambda^{\uparrow}_j(P_\varepsilon) = \frac{1}{\lambda^{\uparrow}_{d+1-j}(P_\varepsilon^{-1})}.$$

Hence for $j = 1, \dots, m$,

$$\frac{1}{\varepsilon}\lambda^{\uparrow}_j(P_\varepsilon) = \frac{1}{\lambda^{\uparrow}_{d+1-j}(\varepsilon P_\varepsilon^{-1})} \longrightarrow \frac{1}{\lambda^{\uparrow}_{d+1-j}(S)}.$$

Now $d+1-j = d-m+(m+1-j)$, so $\lambda^{\uparrow}_{d+1-j}(S) = \lambda^{\uparrow}_{m+1-j}(G)$ and therefore

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\lambda^{\uparrow}_j(P_\varepsilon) = \frac{1}{\lambda^{\uparrow}_{m+1-j}(G)} = \lambda^{\uparrow}_j(G^{-1}),$$

which is (24). The case $m = 1$ follows from $G = \|B_0 v\|^2$. □

Remark 8 (Basis invariance)**.** *Although G is defined using a particular orthonormal basis V of $\mathcal{M}(n)$, its eigenvalues (and hence the slope spectrum $\lambda^{\uparrow}_j(G^{-1})$) depend only on the subspace $\mathcal{M}(n)$: if $\widetilde{V} = VU$ for unitary $U \in \mathbb{C}^{m\times m}$, then $\widetilde{G} = U^* G U$ has the same spectrum.*

Proposition 8 (Two-scale spectral splitting and a uniform $O(1)$ lower bound)**.** *Assume the hypotheses of Proposition 7. Let $\mu_1 \le \cdots \le \mu_m$ be the eigenvalues of G^{-1} (the slope spectrum). Then:*

(i) $\lambda^{\uparrow}_j(P_\varepsilon) = \mu_j\,\varepsilon + o(\varepsilon)$ *for $j = 1, \dots, m$.*

(ii) The remaining eigenvalues stay bounded away from 0: there exist $\varepsilon_0 > 0$ and $c_{\star,\mathrm{off}} > 0$ such that

$$\lambda^{\uparrow}_{m+1}(P_\varepsilon) \ \ge\ c_{\star,\mathrm{off}} \qquad \text{for all } 0 < \varepsilon \le \varepsilon_0.$$

A concrete bound is

$$c_{\star,\mathrm{off}} := \frac{\gamma_H}{2\|B_0\|^2}, \qquad \varepsilon_0 := \min\left\{\frac{1}{2\|B_0^{-1}\|}, \frac{\gamma_H}{4\|B_0\|^2\|B_0^{-1}\|^2}\right\}. \tag{29}$$

Proof. Part (i) is exactly Proposition 7.

For (ii), define the $\varepsilon = 0$ kernel

$$P_0 := P(\sigma_0, n) = \mathrm{Re}\big(n(\sigma_0 I - A)^{-1}\big),$$

which is well-defined since $\sigma_0 \notin \mathrm{spec}(A)$. By Lemma 2 at $(\sigma, n) = (\sigma_0, n)$ and the support identity $\mathrm{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n))$,

$$B_0^* P_0 B_0 = \lambda_{\max}(H(n))I - H(n) = Q_0 \succeq 0.$$

The eigenvalues of Q_0 are 0 (multiplicity m) and at least γ_H on $\mathcal{M}(n)^{\perp}$ by the gap assumption. By the Courant–Fischer min–max principle (see, e.g., [12]) with the change of variables $x = B_0 y$,

$$\lambda^{\uparrow}_{m+1}(P_0) = \min_{\dim S = m+1} \max_{\substack{x\in S\\ \|x\|=1}} x^* P_0 x = \min_{\dim S = m+1} \max_{\substack{y\in B_0^{-1}S\\ y\neq 0}} \frac{y^* Q_0 y}{\|B_0 y\|^2} \geq \frac{1}{\|B_0\|^2} \lambda^{\uparrow}_{m+1}(Q_0) = \frac{\gamma_H}{\|B_0\|^2}.$$

Now compare P_ε to P_0 in operator norm. For $\varepsilon \leq 1/(2\|B_0^{-1}\|)$, the Neumann series gives $B_\varepsilon^{-1} = (B_0 + \varepsilon n I)^{-1}$ well-defined and

$$\|B_\varepsilon^{-1}\| \leq 2\|B_0^{-1}\|, \qquad \|B_\varepsilon^{-1} - B_0^{-1}\| = \|B_\varepsilon^{-1}(B_0 - B_\varepsilon)B_0^{-1}\| \leq 2\varepsilon\, \|B_0^{-1}\|^2.$$

Therefore

$$\|P_\varepsilon - P_0\| = \|\,\mathrm{Re}(n(B_\varepsilon^{-1} - B_0^{-1}))\| \leq \|B_\varepsilon^{-1} - B_0^{-1}\| \leq 2\varepsilon\, \|B_0^{-1}\|^2.$$

Weyl's inequality [12] gives

$$\lambda^{\uparrow}_{m+1}(P_\varepsilon) \geq \lambda^{\uparrow}_{m+1}(P_0) - \|P_\varepsilon - P_0\| \geq \frac{\gamma_H}{\|B_0\|^2} - 2\varepsilon\|B_0^{-1}\|^2.$$

If $\varepsilon \leq \gamma_H/(4\|B_0\|^2\|B_0^{-1}\|^2)$, then the last term is $\leq \gamma_H/(2\|B_0\|^2)$, and we obtain

$$\lambda^{\uparrow}_{m+1}(P_\varepsilon) \geq \frac{\gamma_H}{2\|B_0\|^2} = c_{\star,\mathrm{off}},$$

with the explicit choices (29). □

Corollary 5 (A rigorous "face detector" threshold)**.** *Assume the hypotheses of Proposition 7. Let* $\mu_{\max} := \mu_m = \lambda_{\max}(G^{-1})$. *Fix any* $\tau > \mu_{\max}$. *Then there exists* $\varepsilon_\tau > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_\tau$,

$$\#\Big\{j : \lambda^{\uparrow}_j(P_\varepsilon) \;\leq\; \tau\,\varepsilon\Big\} = m.$$

Proof. By Proposition 7, for each $j \leq m$, $\lambda^{\uparrow}_j(P_\varepsilon)/\varepsilon \to \mu_j \leq \mu_{\max} < \tau$, so for sufficiently small ε one has $\lambda^{\uparrow}_j(P_\varepsilon) \leq \tau\varepsilon$ for all $j \leq m$. By Proposition 8 (ii), $\lambda^{\uparrow}_{m+1}(P_\varepsilon) \geq c_{\star,\mathrm{off}} > 0$ for small ε, so $\lambda^{\uparrow}_{m+1}(P_\varepsilon) > \tau\varepsilon$ for all sufficiently small ε. This implies the stated count. □

4.4.2. Intrinsic rescaled clusters under general C^1 convex exhaustions

The offset analysis above uses the rigid non-tangential approach $\sigma_\varepsilon = \sigma_0 + \varepsilon n$. Along a general C^1 convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, the natural small parameter is the support gap $\delta = \mathrm{Re}(\overline{n_\Omega(\sigma)}\,\sigma) - \lambda_{\max}(H(n_\Omega(\sigma)))$. After rescaling by δ, the slope spectrum is intrinsic (independent of the exhaustion).

Proposition 9 (Intrinsic rescaled collapsing cluster under arbitrary C^1 exhaustions)**.** *Let* $\{\Omega_\varepsilon\}$ *be a* C^1 *convex exhaustion of* $W(A)$. *Let* $\varepsilon_k \downarrow 0$ *and let* $\sigma_k \in \partial\Omega_{\varepsilon_k}$ *satisfy*

$$\sigma_k \to \sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A), \qquad n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n, \qquad |n_k| = |n| = 1.$$

In particular, n *is a supporting direction for* $W(A)$ *at* σ_0, *i.e.* $\mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n))$. *Assume that* $\lambda_{\max}(H(n))$ *is isolated with multiplicity*

$$m := \dim \mathcal{M}(n)\ (< d), \qquad \mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n)),$$

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and gap

$$\gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) > 0.$$

Define the support gap

$$\delta_k := \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k)) \ \geq \ 0,$$

and assume $\delta_k > 0$ for all sufficiently large k.

Let $V \in \mathbb{C}^{d\times m}$ have orthonormal columns spanning $\mathcal{M}(n)$ and set

$$B_0 := \sigma_0 I - A, \qquad G := V^* B_0^* B_0 V.$$

Then $G \succ 0$ and for each $j = 1, \dots, m$,

$$\lim_{k\to\infty} \frac{1}{\delta_k} \lambda_j^{\uparrow}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) = \lambda_j^{\uparrow}\big(G^{-1}\big). \tag{30}$$

Moreover, the remaining eigenvalues stay uniformly bounded away from 0: there exist $c_{\star,\mathrm{exh}} > 0$ and k_0 such that

$$\lambda^{\uparrow}_{m+1}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \geq c_{\star,\mathrm{exh}} \qquad \textit{for all } k \geq k_0.$$

Proof. Set $P_k := P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$. Since $\sigma_0 \notin \mathrm{spec}(A)$ and $\mathrm{spec}(A)$ is finite, $\mathrm{dist}(\sigma_0, \mathrm{spec}(A)) > 0$ and thus $\sigma_k \notin \mathrm{spec}(A)$ for all sufficiently large k. In particular, $B_k := \sigma_k I - A$ is invertible for large k.

Step 1: Positivity of G. As in Proposition 7, B_0 is invertible and V has full column rank, hence $B_0 V$ has full column rank and $G = (B_0V)^*(B_0V) \succ 0$.

Step 2: Congruence at (σ_k, n_k) and a useful bound $\delta_k \to 0$. Because Ω_{ε_k} is C^1 convex and n_k is the outward unit normal at $\sigma_k \in \partial\Omega_{\varepsilon_k}$,

$$P_k = \mathrm{Re}\big(n_k(\sigma_k I - A)^{-1}\big) =: P(\sigma_k, n_k).$$

Let $H_k := H(n_k) = \mathrm{Re}(\overline{n_k}A)$ and define

$$Q_k := \lambda_{\max}(H_k)I - H_k \succeq 0.$$

Applying Lemma 2 with $(\sigma, n) = (\sigma_k, n_k)$ yields

$$B_k^* P_k B_k = \mathrm{Re}(\overline{n_k}\sigma_k)I - H_k = Q_k + \delta_k I. \tag{31}$$

Since $\sigma_k \to \sigma_0 \in W(A)$, we have the estimate

$$0 \leq \delta_k = \mathrm{Re}(\overline{n_k}\sigma_k) - \max_{z\in W(A)} \mathrm{Re}(\overline{n_k}z) \leq \mathrm{Re}(\overline{n_k}\sigma_k) - \mathrm{Re}(\overline{n_k}\sigma_0) \leq |\sigma_k - \sigma_0|, \tag{32}$$

hence $\delta_k \to 0$.

Step 3: Uniform gap persistence and convergence of spectral projectors. Since $n_k \to n$ and $H(\cdot)$ is continuous, $H_k \to H := H(n)$ in operator norm. By Weyl's inequality for Hermitian matrices [12],

$$|\lambda_j(H_k) - \lambda_j(H)| \leq \|H_k - H\| \qquad (j = 1, \dots, d),$$

so for all sufficiently large k the top eigenvalue cluster of H_k has the same multiplicity m and a uniform gap

$$\gamma_k := \lambda_{\max}(H_k) - \lambda^{\downarrow}_{m+1}(H_k) \ \geq \ \gamma_H/2. \tag{33}$$

Let Π_k be the orthogonal projector onto $\mathcal{M}(n_k) = \mathrm{Ker}(Q_k)$ and $\Pi := VV^*$ the orthogonal projector onto $\mathcal{M}(n)$. Standard Riesz projector arguments (cf. [12]) yield $\|\Pi_k - \Pi\| \to 0$.

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Step 4: Work with the inverse and rescale by the support gap. From (31) and invertibility of B_k,

$$P_k^{-1} = B_k\,(Q_k + \delta_k I)^{-1}\,B_k^*. \tag{34}$$

On $\mathrm{Ker}(Q_k) = \mathrm{ran}(\Pi_k)$ one has $(Q_k + \delta_k I)^{-1} = (1/\delta_k)I$, so

$$\delta_k(Q_k + \delta_k I)^{-1}\Pi_k = \Pi_k.$$

On $\mathrm{ran}(I - \Pi_k)$, the gap bound (33) implies $Q_k \succeq \gamma_k(I - \Pi_k)$ and hence

$$\|(Q_k + \delta_k I)^{-1}(I - \Pi_k)\| \le \frac{1}{\gamma_k} \le \frac{2}{\gamma_H}.$$

Therefore

$$\delta_k(Q_k + \delta_k I)^{-1} = \Pi_k + \delta_k R_k, \qquad R_k := (Q_k + \delta_k I)^{-1}(I - \Pi_k), \qquad \|R_k\| \le \frac{2}{\gamma_H}.$$

Multiplying (34) by δ_k gives

$$\delta_k\,P_k^{-1} = B_k\Pi_k B_k^* + \delta_k\,B_k R_k B_k^*. \tag{35}$$

Step 5: Take the limit $k \to \infty$ and identify the spectrum. Since $B_k \to B_0$ and $\Pi_k \to \Pi$ in operator norm, we have

$$\delta_k\,P_k^{-1} \longrightarrow S := B_0 \Pi B_0^* \qquad \text{in operator norm.}$$

As in Proposition 7, $S = (B_0V)(B_0V)^*$ has rank m and its nonzero eigenvalues coincide with those of $G = (B_0V)^*(B_0V)$.

Step 6: Invert the corresponding eigenvalues. By Weyl's inequality [12] and the reciprocity of eigenvalues of P_k and P_k^{-1}, one obtains (30) exactly as in the offset proof. To obtain the uniform $O(1)$ lower bound, apply Courant–Fischer [12] to (31). For k large, B_k is invertible and, with the change of variables $x = B_k y$,

$$\lambda_{m+1}^{\uparrow}(P_k) \;\ge\; \frac{1}{\|B_k\|^2}\,\lambda_{m+1}^{\uparrow}(Q_k + \delta_k I) = \frac{\gamma_k + \delta_k}{\|B_k\|^2} \;\ge\; \frac{\gamma_k}{\|B_k\|^2}.$$

For k large, $\gamma_k \ge \gamma_H/2$ by (33) and $\|B_k\| \to \|B_0\|$, so choosing k_0 with $\gamma_k \ge \gamma_H/2$ and $\|B_k\| \le 2\|B_0\|$ for all $k \ge k_0$ gives the explicit bound

$$\lambda_{m+1}^{\uparrow}(P_k) \;\ge\; \frac{\gamma_H}{8\|B_0\|^2} =: c_{\star,\mathrm{exh}} \qquad (k \ge k_0).$$

□

Corollary 6 (Exhaustion-invariant face-detector threshold)**.** *Assume the hypotheses of Proposition 9 and let $\mu_{\max} := \lambda_{\max}(G^{-1})$. Fix any $\tau > \mu_{\max}$. Then there exists k_τ such that for all $k \ge k_\tau$,*

$$\#\Big\{j:\ \lambda_j^{\uparrow}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big)\ \le\ \tau\,\delta_k\Big\} = m.$$

Proof. By (30), for each $j \le m$, $\lambda_j^{\uparrow}(P_k)/\delta_k \to \lambda_j^{\uparrow}(G^{-1}) \le \mu_{\max} < \tau$, hence $\lambda_j^{\uparrow}(P_k) \le \tau\delta_k$ for all sufficiently large k. The uniform bound in Proposition 9 gives $\lambda_{m+1}^{\uparrow}(P_k) \ge c_{\star,\mathrm{exh}} > 0$ for large k. Since $\delta_k \to 0$, eventually $\tau\delta_k < c_{\star,\mathrm{exh}}$, hence $\lambda_{m+1}^{\uparrow}(P_k) > \tau\delta_k$ for large k. This yields the count. □

Remark 9 (Geometric meaning of the support gap)**.** *For a convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, the scalar*

$$\delta_k = \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k))$$

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is the support-function mismatch between Ω_{ε_k} and $W(A)$ in direction n_k. Support functions and their stability properties are classical in convex geometry; see, e.g., [13].

4.5. Convergence of the Near-Kernel Subspace

Proposition 10 (Convergence of the near-kernel spectral projector). *Assume the setting of Theorem 1 and set $m := \dim \mathcal{M}(n)$. Assume that $\lambda_{\max}(H(n))$ is isolated with multiplicity $m < d$, i.e.*

$$\gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) \; > \; 0. \tag{36}$$

Let

$$P_0 := \mathrm{Re}\big(n(\sigma_0 I - A)^{-1}\big), \qquad \mathcal{K}_0 := \mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n), \qquad \Pi_0 : \mathbb{C}^d \to \mathcal{K}_0$$

be the orthogonal projector onto $\mathcal{K}_0$.

For each k, let $P_k := P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ and let Π_k be the orthogonal projector onto the direct sum of the eigenspaces of P_k corresponding to its m smallest eigenvalues. Then $\|\Pi_k - \Pi_0\| \to 0$ as $k \to \infty$.

Moreover, writing $B_0 = \sigma_0 I - A$, one has the explicit spectral-gap bound

$$\lambda^{\uparrow}_{m+1}(P_0) \; \geq \; \frac{\gamma_H}{\|B_0\|^2}, \tag{37}$$

and consequently, for all sufficiently large k,

$$\|\Pi_k - \Pi_0\| \; \leq \; \frac{2\|P_k - P_0\|}{\lambda^{\uparrow}_{m+1}(P_0)} \; \leq \; \frac{2\|B_0\|^2}{\gamma_H} \, \|P_k - P_0\|. \tag{38}$$

Proof. By Lemma 2 and Step 2 of Theorem 1,

$$B_0^* P_0 B_0 = Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0.$$

The eigenvalues of Q_0 are 0 with multiplicity m and at least γ_H on $\mathcal{M}(n)^{\perp}$, so $\lambda^{\uparrow}_{m+1}(Q_0) = \gamma_H$.

Using the Courant–Fischer characterization [12] with the change of variables $x = B_0 y$, one obtains for every j

$$\lambda^{\uparrow}_j(P_0) = \min_{\dim S = j} \max_{\substack{x \in S \\ \|x\|=1}} x^* P_0 x = \min_{\dim S = j} \max_{\substack{y \in B_0^{-1} S \\ y \neq 0}} \frac{y^* Q_0 y}{\|B_0 y\|^2} \; \geq \; \frac{1}{\|B_0\|^2} \lambda^{\uparrow}_j(Q_0).$$

Taking $j = m+1$ gives (37).

Next, Theorem 1 gives $\|P_k - P_0\| \to 0$. Since P_0 has an isolated cluster of m eigenvalues at 0 separated by the gap $\lambda^{\uparrow}_{m+1}(P_0) > 0$, the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces [15] yields (38), and hence $\|\Pi_k - \Pi_0\| \to 0$. □

4.6. The Spectral-Support Regime for Normal Matrices

Proposition 11 (Normal matrices: explicit eigenvalues near a spectral support point). *Let A be normal with eigenvalues $\lambda_1, \ldots, \lambda_d$ (listed with algebraic multiplicity). Fix $\sigma \notin \mathrm{spec}(A)$ and unimodular $n \in \mathbb{C}$. Then*

$$P(\sigma, n) := \mathrm{Re}\big(n(\sigma I - A)^{-1}\big)$$

is unitarily diagonalizable and its eigenvalues are the scalars

$$p_j(\sigma, n) := \mathrm{Re}\left(\frac{n}{\sigma - \lambda_j}\right) = \frac{\mathrm{Re}(\overline{n}(\sigma - \lambda_j))}{|\sigma - \lambda_j|^2}, \qquad j = 1, \ldots, d. \tag{39}$$

Now fix $\sigma_0 \in \mathrm{spec}(A)$ and let

$$I_0 := \{j \in \{1, \ldots, d\} : \; \lambda_j = \sigma_0\}.$$

Let $\sigma_k \notin \mathrm{spec}(A)$ and unimodular n_k satisfy

$$\sigma_k \to \sigma_0, \qquad n_k \to n.$$

Write $p_{k,j} := p_j(\sigma_k, n_k)$. Then:

(i) For every $j \notin I_0$,

$$p_{k,j} \longrightarrow p_j(\sigma_0, n) = \mathrm{Re}\Big(\frac{n}{\sigma_0 - \lambda_j}\Big).$$

(ii) For every $j \in I_0$ one has the exact *identity*

$$p_{k,j} = \mathrm{Re}\Big(\frac{n_k}{\sigma_k - \sigma_0}\Big) = \frac{\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0))}{|\sigma_k - \sigma_0|^2}.$$

In particular, if there exists $c > 0$ such that

$$\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0)) \geq c\,|\sigma_k - \sigma_0| \qquad \textit{for all sufficiently large } k, \tag{40}$$

which geometrically means that $\sigma_k - \sigma_0$ has a uniformly positive component in the direction n_k (i.e. the approach is bounded away from tangential). then $p_{k,j} \to +\infty$ for every $j \in I_0$.

(iii) Assume in addition that n is a supporting direction for $W(A) = \mathrm{conv}\{\lambda_1, \ldots, \lambda_d\}$ at σ_0, i.e.

$$\mathrm{Re}(\overline{n}\,\lambda_j) \leq \mathrm{Re}(\overline{n}\,\sigma_0), \qquad j = 1, \ldots, d. \tag{41}$$

Then for every $j \notin I_0$,

$$p_j(\sigma_0, n) = \frac{\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j))}{|\sigma_0 - \lambda_j|^2} = \frac{\mathrm{Re}(\overline{n}\,\sigma_0) - \mathrm{Re}(\overline{n}\,\lambda_j)}{|\sigma_0 - \lambda_j|^2} \geq 0,$$

and $p_j(\sigma_0, n) = 0$ if and only if λ_j lies on the same supporting line $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$. If moreover (40) holds (so that all $p_{k,j} \to +\infty$ for $j \in I_0$), then

$$\lambda_{\min}(P(\sigma_k, n_k)) \longrightarrow \min_{j \notin I_0} p_j(\sigma_0, n),$$

which is strictly positive if and only if no eigenvalue $\lambda_j \neq \sigma_0$ lies on the supporting line $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$.

Proof. Since A is normal, $A = U\,\mathrm{diag}(\lambda_1, \ldots, \lambda_d)\,U^*$ for some unitary U, hence

$$(\sigma I - A)^{-1} = U\,\mathrm{diag}\Big(\frac{1}{\sigma - \lambda_1}, \ldots, \frac{1}{\sigma - \lambda_d}\Big)U^*.$$

Therefore,

$$\begin{aligned} P(\sigma, n) &= \mathrm{Re}\big(n(\sigma I - A)^{-1}\big) \\ &= U\ \mathrm{Re}\Big(\mathrm{diag}\Big(\frac{n}{\sigma - \lambda_1}, \ldots, \frac{n}{\sigma - \lambda_d}\Big)\Big)U^* \\ &= U\,\mathrm{diag}\Big(\mathrm{Re}\Big(\frac{n}{\sigma - \lambda_1}\Big), \ldots, \mathrm{Re}\Big(\frac{n}{\sigma - \lambda_d}\Big)\Big)U^*, \end{aligned}$$

which proves (39). The limit in (i) follows by continuity of the map $(\sigma, n) \mapsto \mathrm{Re}\big(n/(\sigma - \lambda_j)\big)$ when $\sigma_0 \neq \lambda_j$. For (ii), if $\lambda_j = \sigma_0$ then

$$\mathrm{Re}\Big(\frac{n_k}{\sigma_k - \sigma_0}\Big) = \mathrm{Re}\Big(\frac{n_k\overline{\sigma_k - \sigma_0}}{|\sigma_k - \sigma_0|^2}\Big) = \frac{\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0))}{|\sigma_k - \sigma_0|^2},$$

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and (40) implies $p_{k,j} \geq c/|\sigma_k - \sigma_0| \to +\infty$.

Finally, (41) implies $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) \geq 0$ for all j, giving the nonnegativity (and the characterization of equality) in (iii). If additionally (40) holds, then $p_{k,j} \to +\infty$ for all $j \in I_0$ while $p_{k,j} \to p_j(\sigma_0, n) \in [0, \infty)$ for $j \notin I_0$, so for large k the minimum eigenvalue is attained among indices $j \notin I_0$, yielding the stated limit and positivity criterion. □

Definition 4 (Poisson degeneracy exponent for normal matrices). *Let A be normal and let $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ be a spectral support point with supporting direction n, i.e. $|n| = 1$ and $\mathrm{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n))$. For the non-tangential offset family $\sigma_\varepsilon = \sigma_0 + \varepsilon n$ ($\varepsilon > 0$), define the* Poisson degeneracy exponent

$$\alpha(\sigma_0, n) := \lim_{\varepsilon \downarrow 0} \frac{\log \lambda_{\min}\big(P(\sigma_\varepsilon, n)\big)}{\log \varepsilon},$$

whenever the limit exists (note that $\log \varepsilon \to -\infty$).

Corollary 7 (Normal matrices: a dichotomy for the exponent). *Let A be normal with eigenvalues $\{\lambda_j\}_{j=1}^d$. Assume that $\mathrm{spec}(A)$ contains at least two distinct points (equivalently, A is not a scalar multiple of the identity). Fix a spectral support point $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ and a supporting direction n. Let $J_0 := \{j : \mathrm{Re}(\overline{n}\lambda_j) = \lambda_{\max}(H(n))\}$ be the set of eigenvalues on the supporting line, and let*

$$\Lambda_0 := \{\lambda_j : j \in J_0\}$$

denote the corresponding set of distinct points (i.e. ignoring algebraic multiplicity). Then:

(i) If $\Lambda_0 = \{\sigma_0\}$ (possibly with multiplicity), then $\lambda_{\min}(P(\sigma_0 + \varepsilon n, n)) \to c_0 > 0$ as $\varepsilon \downarrow 0$ and $\alpha(\sigma_0, n) = 0$.

(ii) If Λ_0 contains a point different from σ_0 (equivalently, there exists $j \in J_0$ with $\lambda_j \neq \sigma_0$), then

$$\lambda_{\min}\big(P(\sigma_0 + \varepsilon n, n)\big) = C\,\varepsilon + o(\varepsilon), \qquad C = \min_{\substack{j \in J_0 \\ \lambda_j \neq \sigma_0}} \frac{1}{|\sigma_0 - \lambda_j|^2},$$

and $\alpha(\sigma_0, n) = 1$.

Proof. By Proposition 11, for $\sigma = \sigma_0 + \varepsilon n$ the eigenvalues of $P(\sigma, n)$ are the scalars

$$p_j(\sigma, n) = \frac{\mathrm{Re}(\overline{n}(\sigma - \lambda_j))}{|\sigma - \lambda_j|^2}.$$

If $\Lambda_0 = \{\sigma_0\}$, then for every j with $\lambda_j \neq \sigma_0$ one has $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) > 0$, hence $p_j(\sigma_0, n) > 0$. Continuity gives $\min_{\lambda_j \neq \sigma_0} p_j(\sigma_0 + \varepsilon n, n) \to \min_{\lambda_j \neq \sigma_0} p_j(\sigma_0, n) =: c_0 > 0$, while for every j with $\lambda_j = \sigma_0$ one has $p_j(\sigma_0 + \varepsilon n, n) = 1/\varepsilon \to \infty$. Thus $\lambda_{\min}(P(\sigma_0 + \varepsilon n, n)) \to c_0$ and $\alpha(\sigma_0, n) = 0$.

If there exists $j \in J_0$ with $\lambda_j \neq \sigma_0$, then $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) = 0$ and

$$p_j(\sigma_0 + \varepsilon n, n) = \frac{\varepsilon}{|\sigma_0 - \lambda_j + \varepsilon n|^2} = \frac{\varepsilon}{|\sigma_0 - \lambda_j|^2} + o(\varepsilon).$$

All $j \notin J_0$ give $p_j(\sigma_0, n) > 0$, hence contribute $O(1)$ values as $\varepsilon \downarrow 0$. Therefore the minimum is attained among $j \in J_0$ with $\lambda_j \neq \sigma_0$, yielding the stated C and $\alpha(\sigma_0, n) = 1$. □

Example 1 (Nondegeneracy at a spectral support point). *Let $A = \mathrm{diag}(0, 1)$, so $W(A) = [0, 1]$. Take $\sigma = 1 + \varepsilon$ with $\varepsilon > 0$ and $n = 1$. Then*

$$P(\sigma, n) = \mathrm{Re}\big((\sigma I - A)^{-1}\big) = \mathrm{diag}\Big(\frac{1}{1+\varepsilon}, \frac{1}{\varepsilon}\Big),$$

so $\lambda_{\min}(P(\sigma,n)) = \frac{1}{1+\varepsilon} \to 1$ *as* $\varepsilon \downarrow 0$. *Thus the smallest eigenvalue does* not *degenerate when the limiting support point is spectral and unique on the support face.*

Example 2 (Degeneracy at a spectral point with a flat support face)**.** *Let* $A = \operatorname{diag}(1, 1+i)$ *and take* $\sigma = 1+\varepsilon$, $n = 1$. *Then*

$$P(\sigma,n) = \operatorname{diag}\left(\frac{1}{\varepsilon}, \operatorname{Re}\left(\frac{1}{\varepsilon - i}\right)\right) = \operatorname{diag}\left(\frac{1}{\varepsilon}, \frac{\varepsilon}{\varepsilon^2+1}\right),$$

so $\lambda_{\min}(P(\sigma,n)) = \frac{\varepsilon}{\varepsilon^2+1} \sim \varepsilon \downarrow 0$. *Here the supporting functional* $\operatorname{Re}(z)$ *is maximized by more than one eigenvalue, and degeneracy persists at* $\sigma_0 = 1 \in \operatorname{spec}(A)$.

4.7. Spectral Support Points for Nonnormal Matrices: A Three-Scale Splitting

We now turn to the *spectral-support* regime, where the boundary point is itself an eigenvalue:

$$\sigma_0 \in \operatorname{spec}(A) \cap \partial W(A).$$

In contrast to the non-spectral case, $P(\sigma,n)$ may develop both a collapsing cluster and an *exact blow-up* as $\sigma \to \sigma_0$ along non-tangential offsets.

Fix a supporting direction n with $|n| = 1$ and

$$\operatorname{Re}(\overline{n}\sigma_0) = \lambda_{\max}(H(n)), \qquad H(n) = \operatorname{Re}(\overline{n}A), \tag{42}$$

and assume the same spectral-isolation hypothesis (19) for $\lambda_{\max}(H(n))$ with multiplicity $m = \dim \mathcal{M}(n)$ and gap $\gamma_H > 0$. For $\varepsilon > 0$ define $\sigma_\varepsilon = \sigma_0 + \varepsilon n$ and $P_\varepsilon = P(\sigma_\varepsilon, n)$ as in (22).

Let

$$E := \operatorname{Ker}(\sigma_0 I - A), \qquad r := \dim E \geq 1.$$

Lemma 7 (Spectral eigenspace inclusion)**.** *Under* (42)*, one has* $E \subseteq \mathcal{M}(n)$.

Proof. Let $0 \neq v \in E$, so $Av = \sigma_0 v$. Then

$$v^* H(n) v = \operatorname{Re}(\overline{n}\, v^* A v) = \operatorname{Re}(\overline{n}\sigma_0)\|v\|^2 = \lambda_{\max}(H(n))\|v\|^2.$$

Since $\lambda_{\max}(H(n))$ is the maximal Rayleigh quotient of $H(n)$, this forces $v \in \mathcal{M}(n)$. □

Set $F := \mathcal{M}(n) \ominus E$, so $\dim F = m - r$ (possibly 0), and choose matrices with orthonormal columns $V_E \in \mathbb{C}^{d \times r}$ spanning E and $V_F \in \mathbb{C}^{d \times (m-r)}$ spanning F. Define

$$B_0 := \sigma_0 I - A, \qquad G_F := V_F^* B_0^* B_0 V_F \in \mathbb{C}^{(m-r)\times(m-r)}. \tag{43}$$

Proposition 12 (Three-scale splitting at a spectral support point)**.** *Assume* (42) *and the gap hypothesis* (19) *for* $H(n)$. *For* $\varepsilon > 0$ *sufficiently small,* $P_\varepsilon \succ 0$ *and:*

(i) ***Exact blow-up on the geometric eigenspace.*** *For every* $v \in E$,

$$P_\varepsilon v = \frac{1}{\varepsilon} v. \tag{44}$$

In particular, P_ε *has an eigenvalue* $1/\varepsilon$ *with multiplicity at least* r.

(ii) $O(\varepsilon)$ ***collapsing cluster on*** $\mathcal{M}(n) \ominus E$. *If* $m > r$*, then* $G_F \succ 0$ *and*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \lambda_j^{\uparrow}(P_\varepsilon) = \lambda_j^{\uparrow}(G_F^{-1}), \qquad j = 1, \ldots, m-r. \tag{45}$$

Equivalently, the $m - r$ *smallest eigenvalues collapse linearly with slopes given by the eigenvalues of* G_F^{-1}.

(iii) $O(1)$ ***bulk separated from*** 0*. There exist* $\varepsilon_0 > 0$ *and* $c_{\star,\mathrm{ss}} > 0$ *such that*

$$\lambda^{\uparrow}_{m-r+1}(P_\varepsilon) \ \geq\ c_{\star,\mathrm{ss}} \qquad \text{for all } 0 < \varepsilon \leq \varepsilon_0.$$

Proof. **(i)** Let $v \in E$. By Lemma 7 one has $v \in \mathcal{M}(n)$, hence $H(n)v = \lambda_{\max}(H(n))v = \mathrm{Re}(\overline{n}\sigma_0)\,v$. Writing $H(n) = \frac{1}{2}(\overline{n}A + nA^*)$ and using $Av = \sigma_0 v$, this identity yields

$$\frac{1}{2}\big(\overline{n}\sigma_0\, v + nA^* v\big) = \frac{1}{2}\big(\overline{n}\sigma_0 + n\overline{\sigma_0}\big)v,$$

so $A^*v = \overline{\sigma_0}\,v$. Therefore $(\sigma_\varepsilon I - A)v = \varepsilon n\, v$ and $(\overline{\sigma_\varepsilon} I - A^*)v = \varepsilon\overline{n}\, v$, and hence

$$P_\varepsilon v = \frac{1}{2}\Big(n(\sigma_\varepsilon I - A)^{-1} + \overline{n}(\overline{\sigma_\varepsilon} I - A^*)^{-1}\Big)v = \frac{1}{2}\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\right)v = \frac{1}{\varepsilon}v,$$

which is (44).

(ii) Assume $m > r$. Since $F \cap \mathrm{Ker}(B_0) = \{0\}$, the restriction of B_0 to F is injective and $B_0 V_F$ has full column rank, hence $G_F \succ 0$.

Let $\Pi := V_E V_E^* + V_F V_F^*$ be the orthogonal projector onto $\mathcal{M}(n) = E \oplus F$. The congruence identity (Lemma 2) at $(\sigma, n) = (\sigma_\varepsilon, n)$ yields

$$B_\varepsilon^* P_\varepsilon B_\varepsilon = \mathrm{Re}(\overline{n}\sigma_\varepsilon)I - H(n) = (\lambda_{\max}(H(n))I - H(n)) + \varepsilon I = Q_0 + \varepsilon I,$$

where $B_\varepsilon = \sigma_\varepsilon I - A$ and $Q_0 \succeq 0$ has kernel $\mathcal{M}(n)$. Exactly as in (27)–(28), one obtains the operator-norm limit

$$\varepsilon\, P_\varepsilon^{-1} \ \longrightarrow\ S := B_0 \Pi B_0^* \qquad (\varepsilon \downarrow 0).$$

Since B_0 annihilates E and is injective on F, $\mathrm{rank}(S) = m - r$ and the nonzero eigenvalues of S coincide with the eigenvalues of

$$(B_0 V_F)^*(B_0 V_F) = V_F^* B_0^* B_0 V_F = G_F.$$

The eigenvalue convergence (45) then follows by the same inversion/reversal argument as in Proposition 7.

(iii) Set

$$Y := E \oplus \mathcal{M}(n)^{\perp}, \qquad S_\varepsilon := B_\varepsilon Y.$$

Since B_ε is invertible for $\varepsilon > 0$ sufficiently small, $\dim S_\varepsilon = \dim Y = d - m + r$. Let $0 \neq x \in S_\varepsilon$ and write $x = B_\varepsilon y$ with $y = y_E + y_\perp$, $y_E \in E$ and $y_\perp \in \mathcal{M}(n)^\perp$. By the congruence identity $B_\varepsilon^* P_\varepsilon B_\varepsilon = Q_0 + \varepsilon I$ and the gap bound $y_\perp^* Q_0 y_\perp \geq \gamma_H \|y_\perp\|^2$ (compare (15)), we have

$$x^* P_\varepsilon x = y^*(Q_0 + \varepsilon I)y \geq \varepsilon\|y_E\|^2 + \gamma_H\|y_\perp\|^2.$$

Moreover, $B_\varepsilon y_E = \varepsilon n\, y_E$ and $|n| = 1$, so

$$\|x\| = \|B_\varepsilon y\| \leq \varepsilon\|y_E\| + \|B_\varepsilon\|\,\|y_\perp\|.$$

By Cauchy–Schwarz,

$$(\varepsilon\|y_E\| + \|B_\varepsilon\|\|y_\perp\|)^2 \leq \left(\varepsilon + \frac{\|B_\varepsilon\|^2}{\gamma_H}\right)(\varepsilon\|y_E\|^2 + \gamma_H\|y_\perp\|^2),$$

and hence, for every unit $x \in S_\varepsilon$,

$$x^* P_\varepsilon x \geq \frac{\gamma_H}{\|B_\varepsilon\|^2 + \varepsilon\gamma_H}.$$

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By Courant–Fischer [12],

$$\lambda^{\uparrow}_{m-r+1}(P_\varepsilon) = \max_{\dim S = d-m+r} \min_{\substack{x\in S\\ \|x\|=1}} x^* P_\varepsilon x \geq \min_{\substack{x\in S_\varepsilon\\ \|x\|=1}} x^* P_\varepsilon x \geq \frac{\gamma_H}{\|B_\varepsilon\|^2 + \varepsilon\gamma_H}.$$

Choosing $\varepsilon_0 > 0$ such that $\sigma_\varepsilon \notin \mathrm{spec}(A)$ for all $0 < \varepsilon \leq \varepsilon_0$ and using $\|B_\varepsilon\| \leq \|B_0\| + \varepsilon$, we obtain the uniform bound

$$\lambda^{\uparrow}_{m-r+1}(P_\varepsilon) \geq c_{\star,\mathrm{ss}} := \frac{\gamma_H}{(\|B_0\| + \varepsilon_0)^2 + \varepsilon_0\gamma_H} \qquad \text{for all } 0 < \varepsilon \leq \varepsilon_0.$$

□

Remark 10 (Defective eigenvalues and higher-order blow-up)**.** *Proposition 12 isolates the contribution of the* geometric *eigenspace $E = \mathrm{Ker}(\sigma_0 I - A)$, producing an exact $1/\varepsilon$ blow-up along non-tangential offsets. If σ_0 is* defective *(nontrivial Jordan chains), generalized eigenvectors can lead to stronger growth (typically $1/\varepsilon^p$ along a Jordan block of length p), and the interaction between Jordan structure and the Hermitian support pencil is a pseudospectral phenomenon; see, e.g., [16].*

4.8. A Fully Explicit 2×2 Example: A Nilpotent Jordan Block

Example 3 (Exact Poisson kernel and exact degeneracy rate for a disk exhaustion)**.** *Let*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $W(A) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. For $r > \frac{1}{2}$, let $\Omega_r := \{z \in \mathbb{C} : |z| < r\}$ and choose $\sigma = re^{it} \in \partial\Omega_r$. The outward normal at σ is $n_{\Omega_r}(\sigma) = e^{it}$ and

$$(\sigma I - A)^{-1} = \begin{pmatrix} \frac{1}{\sigma} & \frac{1}{\sigma^2} \\ 0 & \frac{1}{\sigma} \end{pmatrix}.$$

Hence

$$P_{\Omega_r}(\sigma, A) = \mathrm{Re}\big(e^{it}(\sigma I - A)^{-1}\big) = \begin{pmatrix} \frac{1}{r} & \frac{e^{-it}}{2r^2} \\ \frac{e^{it}}{2r^2} & \frac{1}{r} \end{pmatrix},$$

whose eigenvalues are $\lambda_\pm(r) = \frac{2r\pm1}{2r^2}$. In particular,

$$\lambda_{\min}\big(P_{\Omega_r}(\sigma, A)\big) = \frac{2r-1}{2r^2} = \frac{r - \frac{1}{2}}{r^2},$$

so the degeneracy is linear *as $r \downarrow \frac{1}{2}$.*

Moreover, a min-eigenvector is $u(r,t) \propto (-e^{-it}, 1)^\top$ (independent of r). For the support direction $n = e^{it}$,

$$H(n) = \mathrm{Re}(\overline{n}A) = \frac{1}{2}\begin{pmatrix} 0 & e^{-it} \\ e^{it} & 0 \end{pmatrix}, \quad \mathcal{M}(n) = \mathrm{span}\{(e^{-it}, 1)^\top\}.$$

At $\sigma_0 = \frac{1}{2}e^{it} \in \partial W(A)$,

$$(\sigma_0 I - A)(e^{-it}, 1)^\top = \tfrac{1}{2}(-1, e^{it})^\top \propto (-e^{-it}, 1)^\top,$$

in agreement with Theorem 1.

4.9. Numerical Experiments

This section provides numerical illustrations of: (i) the linear degeneracy predicted by Corollary 3 (and, in offset form, Proposition 2), (ii) local sharpness and improved bounds (Proposition 5 and Corollary 4), (iii) global coercivity collapse and its first-order slope under direction sampling (Corollary 2 and Proposition 6), and (iv) the contrasting behavior at spectral support points for normal matrices (Proposition 11 and Examples 1–2).

Sampling model for an "outer offset" exhaustion. We adopt the rigid non-tangential offset model $\sigma_\varepsilon = \sigma_0 + \varepsilon n$ from (22) (with σ_0 a support point of $W(A)$ in direction n). Fix a unimodular direction $n \in \mathbb{C}$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and let $v \in \mathcal{M}(n)$ be a unit vector in the maximal eigenspace of $H(n)$ (Lemma 1). The corresponding numerical-range support point is

$$z_0(n) \ := \ v^* A v \ \in \ \partial W(A), \qquad \mathrm{Re}(\overline{n}\, z_0(n)) = \lambda_{\max}(H(n)).$$

If $\dim \mathcal{M}(n) > 1$, then the exposed face $F_{W(A)}(n)$ is not a singleton and $z_0(n)$ depends on the chosen $v \in \mathcal{M}(n)$; in that case we fix a representative v (and hence a specific support point) for the experiment. For $\varepsilon > 0$ we define the offset boundary point

$$\sigma_\varepsilon(n) \ := \ z_0(n) + \varepsilon n.$$

By Proposition 2, the support gap equals $\delta(\sigma_\varepsilon(n), n) = \varepsilon$. This rigid offset model enforces $\delta = \varepsilon$ and isolates the predicted scaling without requiring explicit parametrizations of $\partial W(A)$. Moreover, $\sigma_\varepsilon(n) \notin W(A)$, hence $\sigma_\varepsilon(n) \notin \mathrm{spec}(A)$ (since $\mathrm{spec}(A) \subset W(A)$), so the resolvent is well-defined.

We evaluate the pointwise kernel

$$P_\varepsilon(n) \ := \ P(\sigma_\varepsilon(n), n) = \mathrm{Re}\Big(n\big(\sigma_\varepsilon(n)I - A\big)^{-1}\Big),$$

and track $\lambda_{\min}(P_\varepsilon(n))$ as $\varepsilon \downarrow 0$. In the generic (bounded-resolvent) regime $z_0(n) \notin \mathrm{spec}(A)$, Corollary 3 predicts the linear scaling $\lambda_{\min}(P_\varepsilon(n)) = \Theta(\varepsilon)$ and convergence of min-eigenvectors to $(z_0(n)I - A)\mathcal{M}(n)$ (Theorem 1).

Experiment 1: exact linear rate for the nilpotent Jordan block. We revisit Example 3 with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and the disk exhaustion $\Omega_r = \{z : |z| < r\}$, $r > \frac{1}{2}$. Writing $\varepsilon = r - \frac{1}{2}$, one has the exact formula $\lambda_{\min}(P_{\Omega_r}(\sigma, A)) = \frac{\varepsilon}{r^2}$, hence linear degeneracy as $\varepsilon \downarrow 0$. Figure 2 compares the computed smallest eigenvalue to the exact expression.

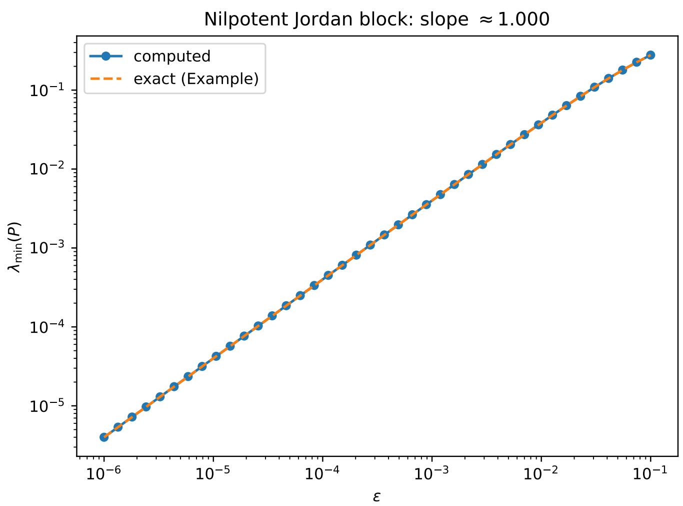

Figure 2. Nilpotent Jordan block (Example 3): log–log plot of $\lambda_{\min}$ versus $\varepsilon = r - \frac{1}{2}$, illustrating the predicted linear scaling.

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Experiment 2: generic nonnormal matrix—linear degeneracy and eigenvector convergence. We generate a fixed random complex matrix $A \in \mathbb{C}^{5\times5}$ (seeded for reproducibility), fix one direction $n = e^{i\theta}$, and form $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$ as above. Figure 3 shows $\lambda_{\min}(P_\varepsilon(n))$ against ε on a log–log scale, together with a reference $\propto \varepsilon$ line; the observed slope is ≈ 1 on the plotted range. Figure 4 tracks the distance of a min-eigenvector u_ε of $P_\varepsilon(n)$ to the predicted limiting subspace $(z_0(n)I - A)\mathcal{M}(n)$, quantified by $\|(I - \Pi)\, u_\varepsilon\|$ where Π is the orthogonal projector onto $(z_0(n)I - A)\mathcal{M}(n)$ (consistent with Theorem 1).

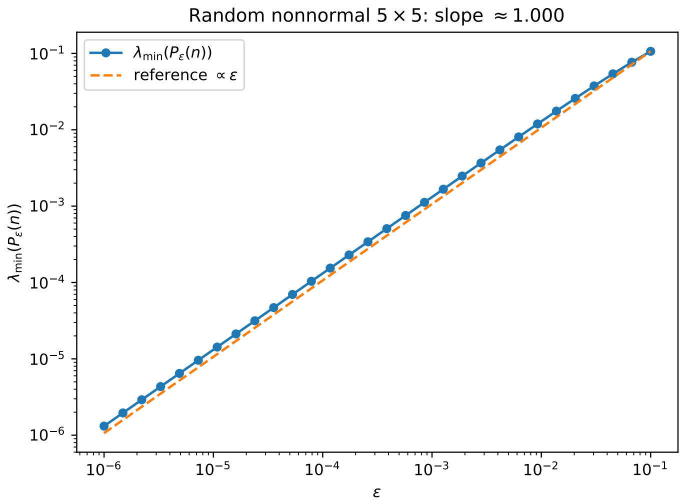

Figure 3. Random nonnormal $A \in \mathbb{C}^{5\times5}$ (fixed seed) and fixed direction n: $\lambda_{\min}(P_\varepsilon(n))$ scales linearly with ε (Corollary 3).

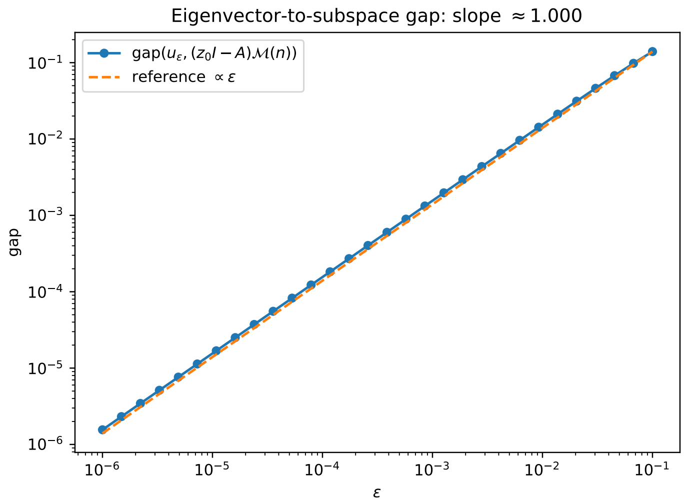

Figure 4. Same setup as Figure 3: the direction of a min-eigenvector u_ε converges to $(z_0(n)I - A)\mathcal{M}(n)$ as $\varepsilon \downarrow 0$ (Theorem 1). The plotted "gap" is $\|(I - \Pi)u_\varepsilon\|$.

Experiment 2b: local sharpness of norm-based versus refined bounds. We compare the norm-based bounds from Lemma 3 (c) with the refined bounds of Proposition 5. For the Jordan block, Figure 5 shows that the refined bounds track the exact eigenvalue closely and are asymptotically sharp, while the norm-based upper bound is typically much looser. For the random nonnormal matrix, Figure 6 shows the same qualitative behavior.

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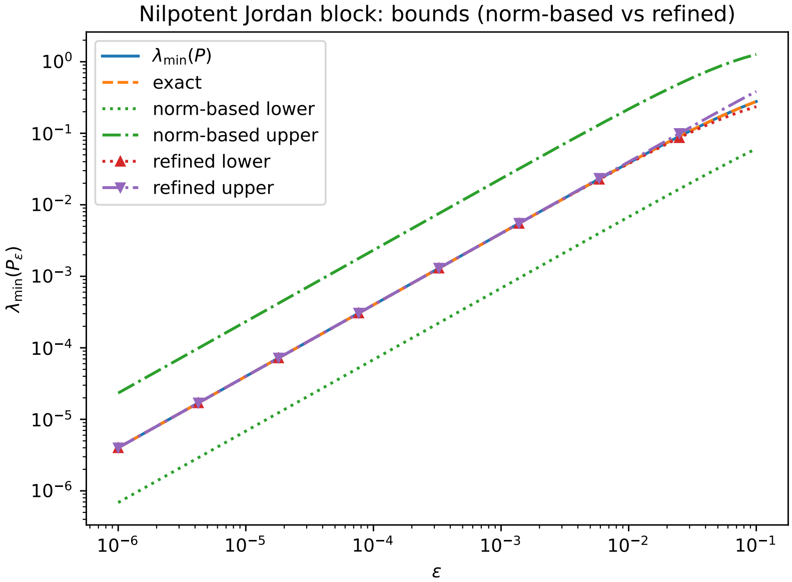

Figure 5. Nilpotent Jordan block: comparison of $\lambda_{\min}(P)$ with the norm-based bounds from Lemma 3 (c) and the refined bounds from Proposition 5.

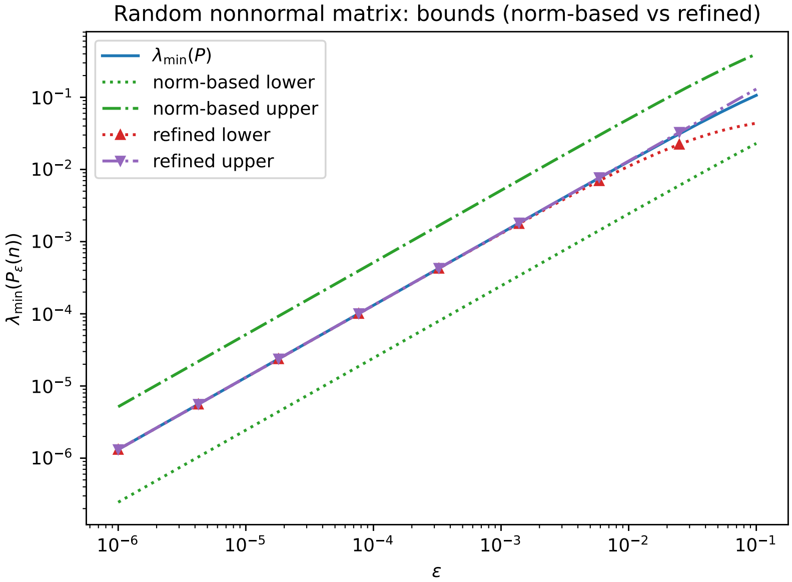

Figure 6. Random nonnormal $A \in \mathbb{C}^{5\times5}$, fixed direction n: comparison of $\lambda_{\min}(P_\varepsilon(n))$ with the norm-based and refined bounds. The refined bounds recover the correct first-order constant as $\varepsilon \downarrow 0$.

Experiment 2c: full slope spectrum at a flat face ($m = 2$). Proposition 7 predicts a *two-dimensional* $O(\varepsilon)$ collapsing cluster when the supporting pencil $H(n)$ has a two-dimensional maximal eigenspace and the chosen support point σ_0 lies in the interior of the corresponding face. To isolate this mechanism in a fully explicit setting, we take a normal diagonal matrix

$$A = \mathrm{diag}(1,\ 1+2i,\ 0,\ -1), \qquad n = 1,$$

so that $H(n) = \mathrm{Re}(A)$ has $\lambda_{\max} = 1$ with multiplicity $m = 2$ and $\mathcal{M}(n) = \mathrm{span}\{e_1, e_2\}$. With $v = (e_1 + e_2)/\sqrt{2}$, the associated support point is $\sigma_0 = v^* A v = 1 + i \notin \mathrm{spec}(A)$, lying in the relative interior of the face joining 1 and $1 + 2i$. In this basis, $B_0 = \sigma_0 I - A$ restricts to $\mathrm{diag}(i, -i)$ on $\mathcal{M}(n)$, hence $G = I_2$ and the predicted slope spectrum is $\{1, 1\}$. Numerically we observe $\lambda_1^{\uparrow}(P(\sigma_0 + \varepsilon n, n))/\varepsilon \to 1$

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and $\lambda_2^{\uparrow}(P(\sigma_0 + \varepsilon n, n))/\varepsilon \to 1$ as $\varepsilon \downarrow 0$, and the face-detector count in Corollary 5 stabilizes at $m = 2$. Figure 7 plots the two collapsing eigenvalues and their rescaled slopes.

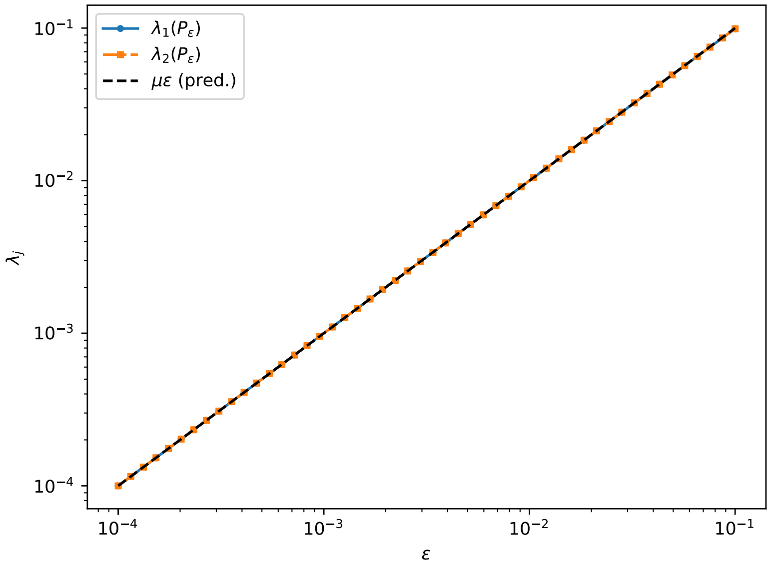

Figure 7. Experiment 2c: two collapsing eigenvalues and their rescaled slopes in the $m = 2$ flat-face normal example.

Experiment 3: approximate global coercivity collapse. For the same random nonnormal matrix A as in Experiment 2, we approximate the global coercivity constant

$$c(\varepsilon) = \inf_{\sigma \in \partial\Omega_\varepsilon} \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A))$$

by sampling a fine grid of directions $\{n_j\}$ and using the offset model $\sigma_\varepsilon(n_j) = z_0(n_j) + \varepsilon n_j$. Figure 8 plots the sampled minimum $\min_j \lambda_{\min}(P_\varepsilon(n_j))$ versus ε, illustrating the collapse asserted by Corollary 2. (Here the offset model has $\Delta(\Omega_\varepsilon) = \varepsilon$, so Corollary 1 also predicts that uniform coercivity cannot persist as $\varepsilon \downarrow 0$.)

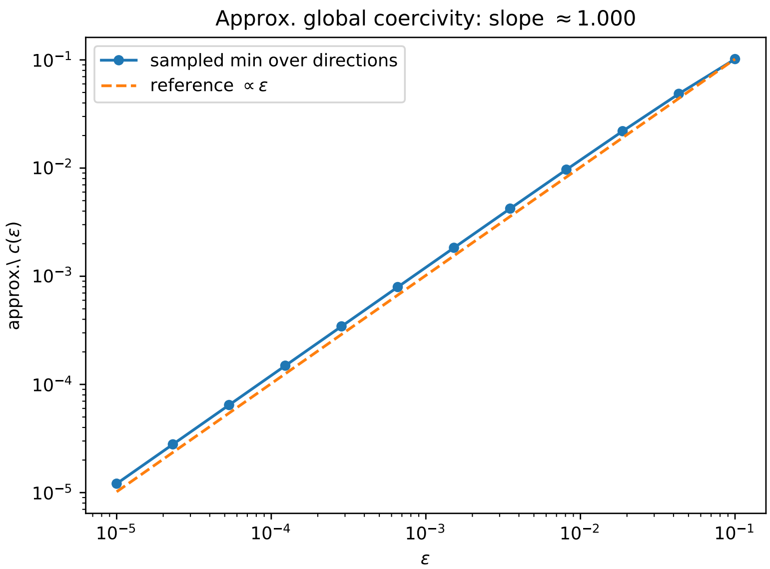

Figure 8. Approximate global coercivity constant $c(\varepsilon)$ computed by sampling directions and using the offset model $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$. The sampled minimum tends to 0 as $\varepsilon \downarrow 0$ (Corollary 2).

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Experiment 3b: local slope profile across directions. For the same random nonnormal matrix and the offset model, Corollary 4 predicts the local first-order constant $\lambda_{\min}(P_\varepsilon(n))/\varepsilon \to 1/\beta_0(n)^2$ as $\varepsilon \downarrow 0$. Figure 9 compares the empirically observed slope $\lambda_{\min}(P_\varepsilon(n))/\varepsilon$ at a small fixed ε to the prediction $1/\beta_0(n)^2$ across directions $n = e^{i\theta}$, while Figure 10 plots the relative error.

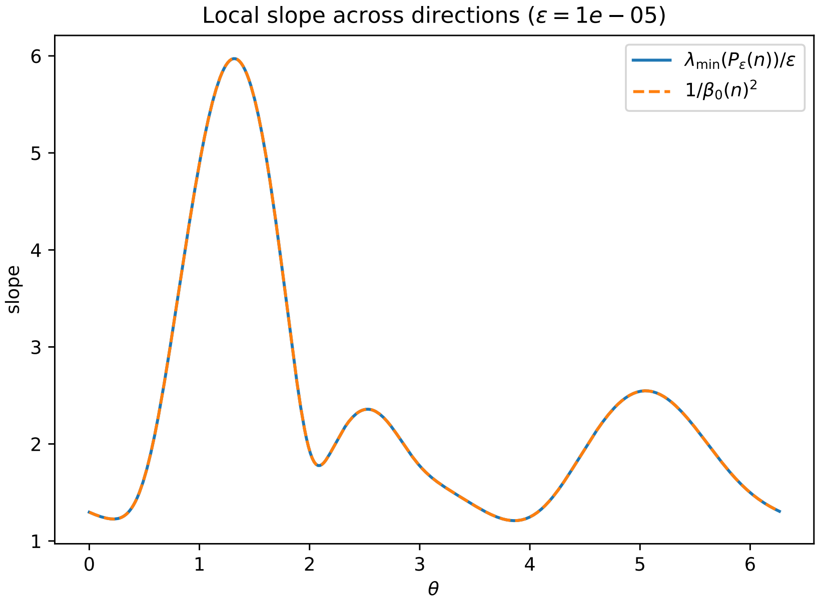

Figure 9. Random nonnormal matrix: direction-wise comparison of the empirical slope $\lambda_{\min}(P_\varepsilon(n))/\varepsilon$ with the predicted limit $1/\beta_0(n)^2$ (Corollary 4).

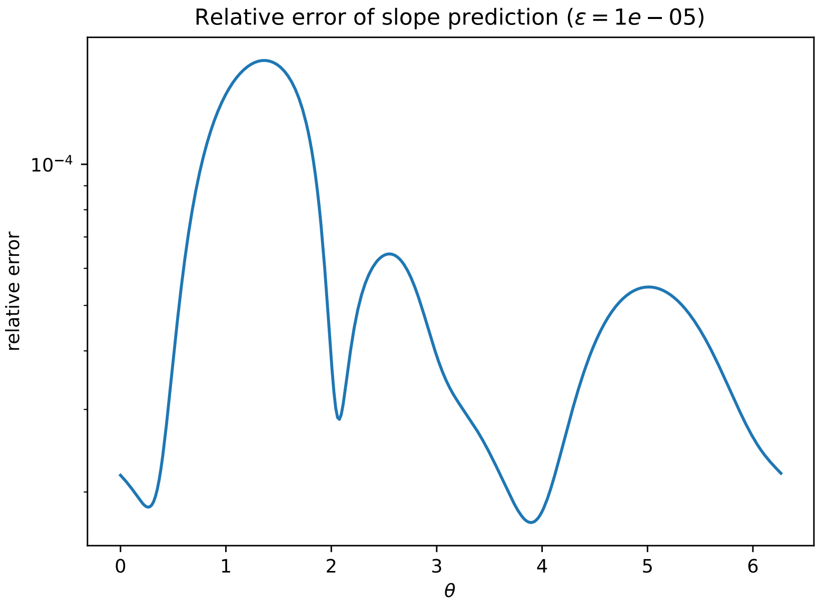

Figure 10. Random nonnormal matrix: relative error of the prediction in Figure 9. The error decreases as the test value of ε is reduced.

Experiment 3c: global slope constant and the β_0 profile. Proposition 6 predicts that the direction-sampled coercivity constant satisfies $\tilde{c}(\varepsilon)/\varepsilon \to 1/\max_\theta \beta_0(\theta)^2$. Figure 11 plots the profile of $\beta_0(\theta)^2$, and Figure 12 compares the sampled ratio $\tilde{c}(\varepsilon)/\varepsilon$ to the predicted limit.

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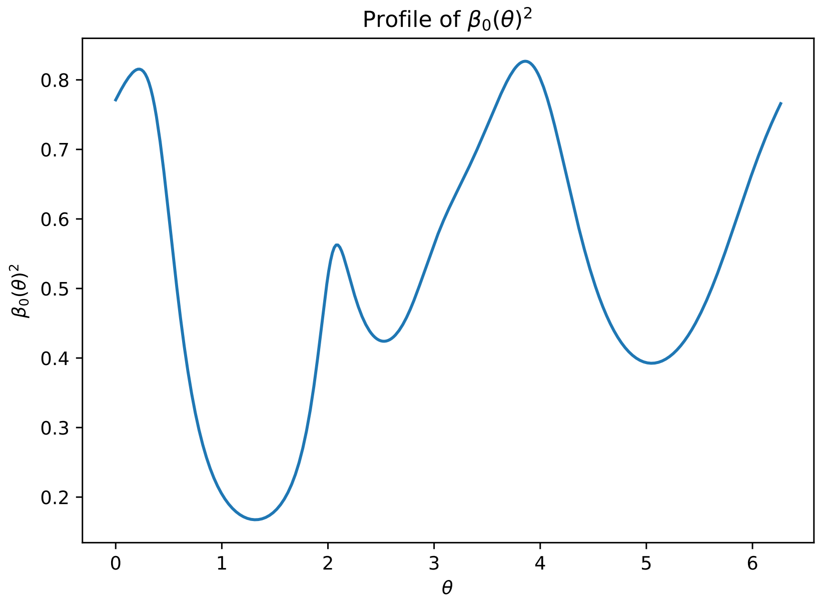

Figure 11. Random nonnormal matrix: profile of $\beta_0(\theta)^2 = \|(z_0(\theta)I - A)|_{\mathcal{M}(e^{i\theta})}\|^2$. The global slope constant is set by the maximum of this profile (Proposition 6).

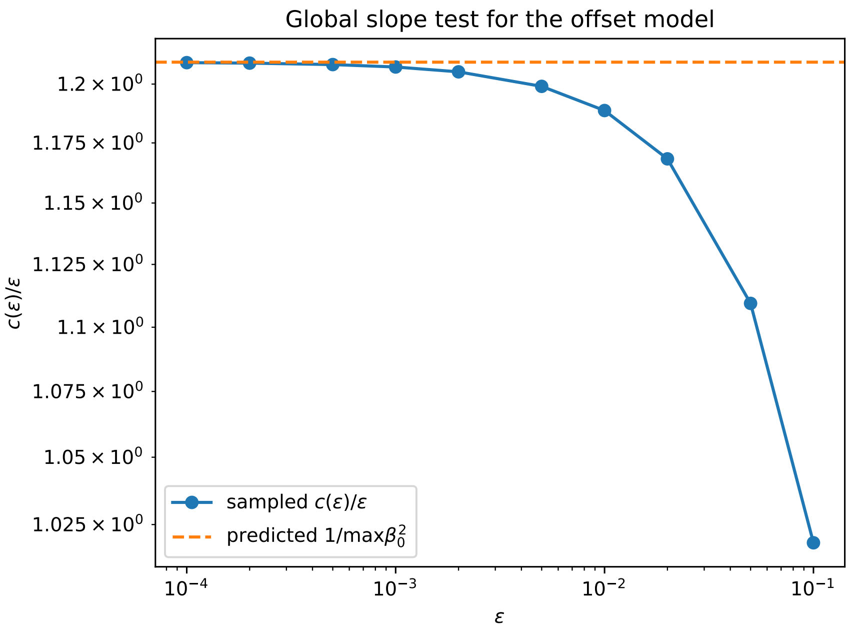

Figure 12. Random nonnormal matrix: the sampled ratio $\tilde{c}(\varepsilon)/\varepsilon$ versus ε, together with the predicted limit $1/\max_\theta \beta_0(\theta)^2$ (Proposition 6).

Experiment 4: normal matrices at spectral support points. We reproduce the contrasting behavior in Examples 1–2 by evaluating $P(1+\varepsilon, 1) = \mathrm{Re}\big(((1+\varepsilon)I - A)^{-1}\big)$ for two diagonal (hence normal) matrices: $A = \mathrm{diag}(0, 1)$ and $A = \mathrm{diag}(1, 1+i)$. Figure 13 shows that the former remains bounded away from 0 as $\varepsilon \downarrow 0$, while the latter degenerates linearly, consistent with Proposition 11.

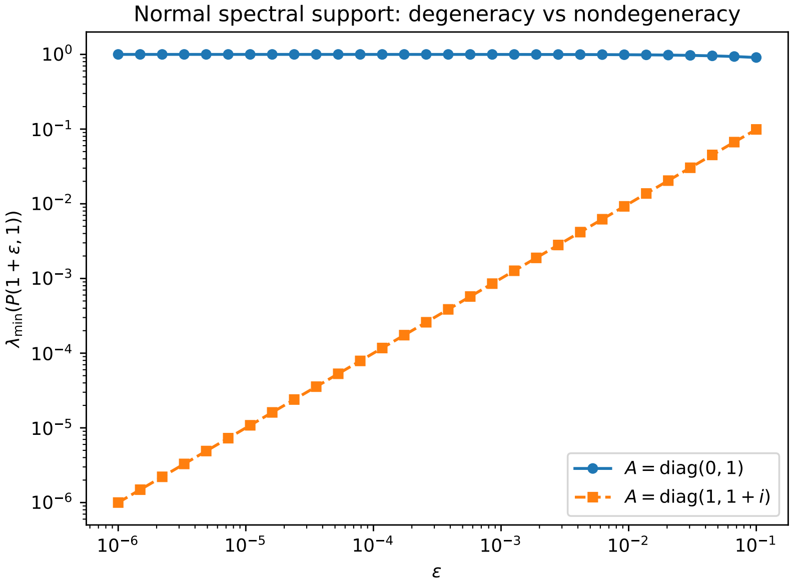

Figure 13. Normal matrices at a spectral support point: $A = \mathrm{diag}(0,1)$ remains bounded away from 0, whereas $A = \mathrm{diag}(1,1+i)$ degenerates linearly, consistent with Proposition 11.

Experiment 4b: three-scale splitting at spectral support points (nonnormal). Proposition 12 predicts a three-scale structure at $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ under non-tangential offsets $\sigma_\varepsilon = \sigma_0 + \varepsilon n$: an exact $1/\varepsilon$ blow-up on the geometric eigenspace $E = \mathrm{Ker}(\sigma_0 I - A)$, an $O(\varepsilon)$ collapsing cluster of dimension $m - r$ on $\mathcal{M}(n) \ominus E$, and an $O(1)$ bulk bounded away from 0. We test this on three small nonnormal examples with $n = 1$ and $\sigma_0 = 1$:

(SS1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, for which $m = r = 1$ (blow-up only; no $O(\varepsilon)$ cluster);

(SS2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 0 \end{bmatrix}$, for which $m = 2$ and $r = 1$ (one collapsing slope);

(SS3) $A = H + iK$ with $H = \mathrm{diag}(1,1,1,0,0)$ and $K_{24} = K_{42} = 1$, $K_{35} = K_{53} = 2$, for which $m = 3$ and $r = 1$ (two collapsing slopes).

In each case we observe the predicted behavior: the largest eigenvalue scales as $1/\varepsilon$, the smallest $m - r$ eigenvalues scale linearly in ε with slopes given by G_F^{-1}, and the next eigenvalue remains separated from 0 uniformly in ε. Figure 14 shows the observed three-scale splitting.

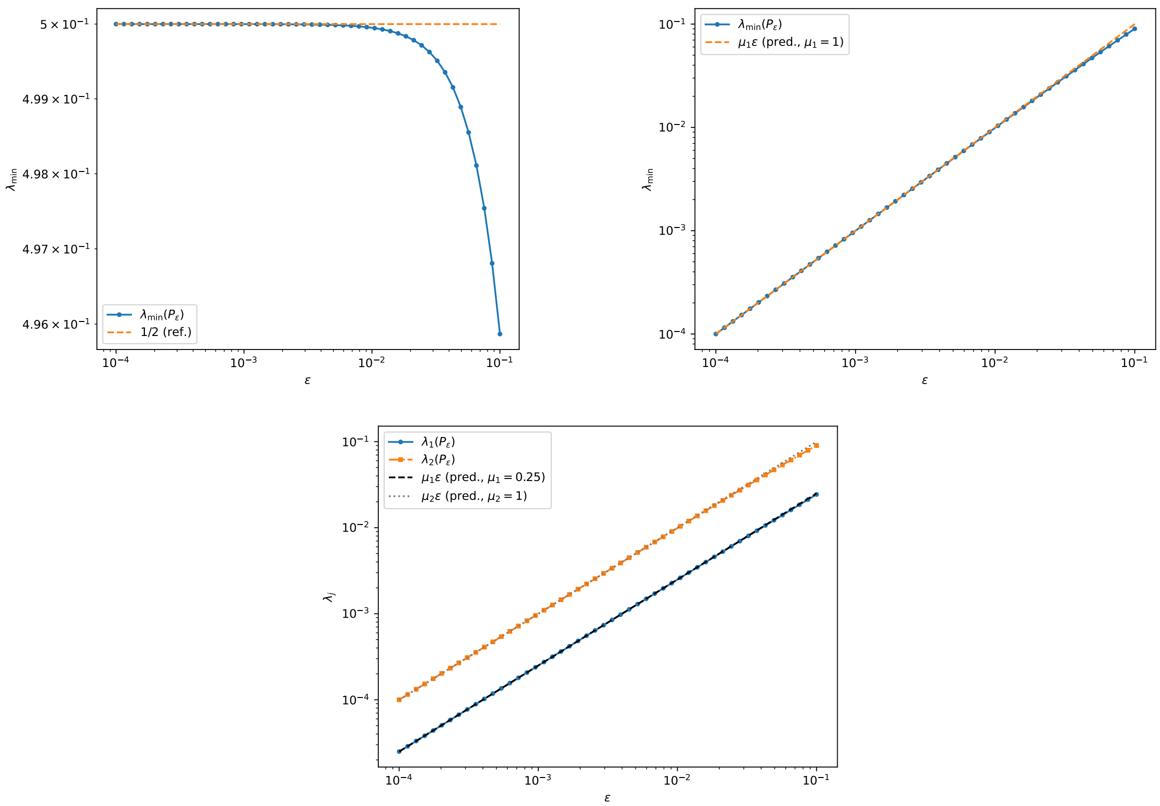

Figure 14. Experiment 4b: three-scale splitting at spectral support points (SS0–SS2). Each panel shows the $1/\varepsilon$ blow-up eigenvalue(s), the collapsing $O(\varepsilon)$ cluster, and the $O(1)$ bulk.

Reproducibility. All figures are generated by the accompanying scripts `poisson_utils.py` and `run_numerical_experiments.py`, which require only NumPy and Matplotlib and save PDF figures into a `figs/` folder.

4.10. A Curvature Surrogate Question at Smooth Exposed Points

The direction-dependent slope data in Corollary 4 and Proposition 7 suggest that the rescaled Poisson degeneracy rate can be viewed as a quantitative "stiffness" of the numerical range boundary in a given direction.

Let $n = e^{i\theta}$ and consider the support function

$$h(\theta) := \max_{z \in W(A)} \mathrm{Re}(e^{-i\theta} z) = \lambda_{\max}\big(H(e^{i\theta})\big),$$

where $H(e^{i\theta}) = \mathrm{Re}(e^{-i\theta} A)$. At directions where the maximal eigenvalue is simple, choose the corresponding unit eigenvector $v(\theta)$ and set the exposed support point

$$\sigma_0(\theta) := v(\theta)^* A v(\theta) \in \partial W(A).$$

For the non-tangential offset family $\sigma_\varepsilon(\theta) = \sigma_0(\theta) + \varepsilon e^{i\theta}$, define $P_\varepsilon(\theta) = P(\sigma_\varepsilon(\theta), e^{i\theta})$. When $\dim \mathcal{M}(e^{i\theta}) = 1$, Proposition 7 gives the explicit slope

$$s(\theta) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \lambda_{\min}\big(P_\varepsilon(\theta)\big) = \frac{1}{\|(\sigma_0(\theta) I - A) v(\theta)\|^2}. \tag{46}$$

A natural question is how $s(\theta)$ compares to geometric curvature data of $\partial W(A)$. In convex geometry, the support function determines the boundary, and for C^2 strictly convex curves the radius of curvature can be expressed in terms of h and its second derivative (see, e.g., [13]). For numerical ranges, $\partial W(A)$ is an algebraic curve determined by the Kippenhahn polynomial [17], with corners and flat portions corresponding to eigenvalue multiplicities and supporting-face degeneracies.

Question. At smooth exposed boundary points where $\dim \mathcal{M}(e^{i\theta}) = 1$, is the slope $s(\theta)$ in (46) comparable (in an appropriate quantitative sense) to the curvature (or radius of curvature) of $\partial W(A)$ at $\sigma_0(\theta)$? Numerically, plots of $s(\theta)$ (Experiment 3b) show sharp spikes near directions associated with nearly-flat faces, suggesting that $s(\theta)$ may serve as an easily computable curvature surrogate.

4.11. Discussion and Remaining Open Problems

Remark 11 (Beyond the geometric three-scale picture)**.** *The bounded-resolvent regime $\sigma_0 \notin \mathrm{spec}(A)$ is treated in Theorem 1 and quantified by the slope spectra in Section 4.4. At spectral support points $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$, Proposition 11 and Corollary 7 give a complete normal-matrix description, and Proposition 12 provides a three-scale splitting for general matrices under a gap hypothesis for the supporting pencil $H(n)$.*

Several aspects of the nonnormal spectral-support regime remain open:

(i) ***Defective eigenvalues.*** *When σ_0 has nontrivial Jordan chains, generalized eigenvectors may exhibit higher-order resolvent blow-up (typically $1/\varepsilon^p$ for a length-p Jordan block). A systematic description of how Jordan structure and the support pencil interact to determine the full singular-value/eigenvalue profile of $P(\sigma_0 + \varepsilon n, n)$ is closely related to pseudospectral growth; see [16].*

(ii) ***Tangential approach geometry.*** *Our slope spectra are derived for non-tangential offsets and for general C^1 exhaustions after normalization by the scalar support gap δ. Understanding when the $O(\delta)$ scaling persists under more tangential approach, or when higher-order scalings appear, is largely open.*

(iii) ***Loss of spectral isolation.*** *The explicit slope spectra rely on a gap separating $\lambda_{\max}(H(n))$ from the rest of the spectrum. When this gap closes, the associated projectors become ill-conditioned and new multi-scale behaviors may appear.*

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org.

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