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On Morgado and Sette's Implicative Hyperlattices as Models of da Costa Logic C_ω

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Abstract: José Morgado introduced in 1962 an original and interesting notion of hyperlattices, that he called *reticuloides*. In his Master's thesis defended in 1971 (and supervised by Newton da Costa), Antonio M. Sette proposed a novel notion of implicative hyperlattices (here called SIHLs) based on Morgado's hyperlattices. He also extended SIHLs by adding an unary hyperoperator, obtaining a class of hyperalgebras (here called SHC$_\omega$s) which correspond to da Costa algebras for C_ω, being so a suitable semantics for the logic C_ω. In this paper, after characterizing Sette's implicative hyperlattices in (hyper)lattice-theoretic terms, and proving some basic results on SIHLs, we introduce a class of swap structures, a special class of hyperalgebras over the signature of C_ω naturally induced by implicative lattices. It is proven that these swap structures are indeed SHC$_\omega$s. Finally, it is proven that the class of SHC$_\omega$s, as well as the above mentioned class of swap structures, characterize the logic C_ω.

Keywords: hyperlattices; hyperalgebras; da Costa logics; paraconsistent logics; implicative lattices; non-deterministic semantics; non-deterministic matrices; swap structures

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1. Introduction

Hyperstructures are algebraic systems characterized by at least one multioperation, also known as a hyperoperation, which produces, for a given input, a set of possible outcomes rather than a single one. The concept of hyperstructures, also referred to as multialgebras, was introduced by Frédéric Marty in 1934 during the 8th Congress of Scandinavian Mathematicians [1]. Marty's pioneering work laid the foundation for the development of hypergroup theory and the broader study of hyperstructures. In the context of non-classical logics, hyperstructures play a pivotal role by providing a flexible semantical framework which intends to expand the horizons of Abstract Algebraic Logic (AAL). Indeed, hyperalgebras (and non-deterministic semantics in general) enable the representation of logical systems that cannot be adequately captured by traditional algebraic methods, thereby facilitating the study and application of a broader range of logical systems.

In the general context of ordered structures, the concept of hyperlattices was first introduced by Benado in 1953 [2]. Since then, alternative axiomatizations have been proposed, such as those by D. J. Hansen [3], who aimed to avoid the partial associativity property present in Benado's formulation, and by other researchers seeking to refine and generalize the original concept. Notably, José Morgado, in his book "Introdução à Teoria dos Reticulados" [4], provides a definition of hyperlattice that, to the best of our knowledge, is entirely original.[1] He introduced the term "reticuloide" (reticuloid) to denote

[1] Presumably, Morgado's ideas are fully original, but verifying the originality of his "reticuloides" to the theory of hyperlattices is a challenging task. This difficulty arises from the fact that the available literature does not always explicitly clarify the novelty of specific results. In particular, in the book [4], which, according to [5], is based on lecture notes from a seminar

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hyperlattices, and introduced the concepts of “supremoide” and “infimoide” as the corresponding hyperoperations for supremum and infimum. His two equivalent definitions of hyperlattices are more intuitive than Benado’s, as they resemble the usual lattice definitions while generalizing supremum and infimum properties in quasi-ordered structures. Morgado’s formulation allows for multiple suprema and infima, leading to a natural extension of classical lattice structures.

José Morgado (1921–2003) was a distinguished Portuguese mathematician whose scientific contributions significantly advanced the field of lattice theory. His work encompassed books and numerous papers, with a focus on the structure and properties of lattices. In [4], Morgado offered a comprehensive introduction to lattice theory, presenting original definitions and concepts that have influenced subsequent research. His innovative approach to hyperlattices, particularly through the introduction of "reticuloides," has been instrumental in extending classical lattice structures to more generalized settings. Morgado’s work continues to be a valuable reference for researchers in the field.

The Brazilian mathematician Antonio Antunes Mario Sette (1939-1999) also made a significant contribution to the study of hyperlattices (and also to the field of non-classical logics). Motivated by the original notion of C_n*-algebras* introduced by Newton da Costa in [6] for his paraconsistent calculi C_n (for $1 \leq n < \omega$), Sette and da Costa propose in [7] the C_ω*-algebras* for the limiting paraconsistent logic system C_ω. In 1971, Sette introduced in his master’s dissertation under da Costa’s supervision [8] the concept of C_ω*-hyperlattices*, proving that they are in correspondence with C_ω-algebras. His approach built upon the hyperlattice definition given by José Morgado. Sette, working under da Costa’s guidance, extended Morgado’s framework to introduce implicative hyperlattices, a natural generalization of implicative lattices within Morgado’s hyperlattice context. This innovation proved to be particularly relevant to the semantics of logic C_ω, as we shall see.

In this paper, we bring to the broader research community the concepts of m-hyperlattices and SIHLs (Sette Implicative Hyperlattices), which were previously available only in Portuguese. We establish fundamental properties of these structures, which in turn allow us to prove the soundness and completeness of C_ω with respect to SHC$_\omega$ (Sette hyperalgebras for C_ω) semantics. Furthermore, we introduce a natural subclass of SIHLs defined by *swap structures*, which also provide an adequate semantics for C_ω.

2. Morgado Reticuloids

There are several definitions of hyperalgebra in the literature, considering that each hyperalgebra application in a specific area of Mathematics (mainly Algebra and Logic) requires a particular adaptation. Here, we adapt the notion of hyperalgebra used in [9].

Definition 1. *A* ***hyperalgebraic signature*** *is a sequence of pairwise disjoint sets*

$$\Sigma = (\Sigma_n)_{n \in \mathbb{N}},$$

where $\Sigma_n = S_n \sqcup M_n$*, which* S_n *is the set of strict hyperoperation symbols and* M_n *is the set of hyperoperation symbols. In particular,* $\Sigma_0 = S_0 \sqcup M_0$*,* F_0 *is the set of symbols for constants and* M_0 *is the set of symbols for hyperconstants. A hyperalgebraic signature can also be denoted by*

$$\Sigma = ((S_n)_{n \geq 0}, (M_n)_{n \geq 0}).$$

Definition 2. *Let* A *be any set, and* $P^*(A) := \mathcal{P}(A) \setminus \{\emptyset\}$*.*

1. *A hyperoperation of arity* $n \in \mathbb{N}$ *over a set* A *is a function* $\# : A^n \to \mathcal{P}^*(A)$*.*

delivered in 1960 at the University of Ceará (Brazil), some original results are presented. However, it remains unclear which of these results are truly novel and which may have been previously established in other works. Although this kind of investigation lies beyond the scope of the present work, it remains an interesting topic for future research.

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2. *A hyperoperation # of arity $n \in \mathbb{N}$ over a set A is* strict *whenever it factors through the singleton function $s_A : A \to \mathcal{P}^*(A)$, $a \mapsto \{a\}$. Thus, it can be naturally identified with an ordinary n-ary operation $\# : A^n \to A$.*

A 0-ary hyperoperation (respectively *strict* hyperoperation) on A can be identified with a non-empty subset of A (respectively a singleton subset of A).

Definition 3. *A* ***hyperalgebra*** *over a signature $\Sigma = ((S_n)_{n\geq 0}, (M_n)_{n\geq 0})$, is a set A endowed with a family of n-ary hyperoperations*

$$\sigma_n^A : A^n \to \mathcal{P}^*(A),\ \sigma_n \in S_n \sqcup M_n,\ n \in \mathbb{N},$$

such that: if $\sigma_n \in S_n$, then $\sigma_n^A : A^n \to \mathcal{P}^*(A)$ is a strict *n-ary hyperoperation.*

Remark 1.

1. *Every algebraic signature $\Sigma = (F_n)_{n\in\mathbb{N}}$ is a hyperalgebraic signature where $M_n = \emptyset$, for every $n \in \mathbb{N}$. Each algebra*

$$(A, ((A^n \stackrel{f^A}{\to} A)_{f\in F_n})_{n\in\mathbb{N}})$$

over the algebraic signature Σ can be naturally identified with a hyperalgebra

$$(A, ((A^n \stackrel{f^A}{\to} A \stackrel{s_A}{\to} \mathcal{P}^*(A))_{f\in F_n})_{n\in\mathbb{N}})$$

over the same signature.

2. *Every hyperalgebraic signature $\Sigma = ((S_n)_{n\in\mathbb{N}}, (M_n)_{n\in\mathbb{N}})$ induces naturally a first-order language*

$$L(\Sigma) = ((F_n)_{n\in\mathbb{N}}, (R_{n+1})_{n\in\mathbb{N}})$$

where $F_n := S_n$ is the set of n-ary operation symbols and $R_{n+1} := M_n$ is the set of (n+1)-ary relation symbols. In this way, hyper-algebras

$$(A, ((A^n \stackrel{\sigma^A}{\to} \mathcal{P}^*(A))_{\sigma\in S_n\sqcup M_n})_{n\in\mathbb{N}})$$

over a hyperalgebraic signature $\Sigma = (S_n \sqcup M_n)_{n\in\mathbb{N}}$ can be naturally identified with the first-order structures over the language $L(\Sigma)$ that satisfies the $L(\Sigma)$-sentences:

$$\forall x_0 \cdots \forall x_{n-1} \exists x_n (\sigma_n(x_0, \ldots, x_{n-1}, x_n)),\ \textit{for each}\ \ \sigma_n \in R_{n+1},\ n \in \mathbb{N}.$$

Definition 4 (Prosets)**.** *A preordered set (Proset) is a pair $\mathsf{P} = \langle P, \preceq \rangle$ such that P is a non-empty set and $\preceq$ is a reflexive and transitive relation on P (i.e., a preorder). That is: $x \preceq x$, for every $x \in P$; and $x \preceq y$, $y \preceq z$ implies $x \preceq z$, for every $x, y, z \in P$.*

If $x \preceq y$ and $y \preceq x$ we say that x and y are similar*, and we write $x \equiv y$. Given $B, C \subseteq P$, $B \preceq C$ means that $x \preceq y$ for every $x \in B$ and every $y \in C$, and so $x \preceq B$ denotes that $x \preceq y$ for every $y \in B$, and $B \preceq x$ denotes that $y \preceq x$ for every $y \in B$.*

Observe that $\emptyset \preceq B$ and $B \preceq \emptyset$ for every $B \subseteq P$. Analogously, $x \preceq \emptyset$ and $\emptyset \preceq x$ for every $x \in P$.

Definition 5. *Let P be a proset, and let $B \subseteq P$.*

1. *The set of* minima *of B is $\mathsf{Min}(B) = \{x \in B : x \preceq B\}$, and the set of* maxima *of B is $\mathsf{Max}(B) = \{x \in B : B \preceq x\}$.*
2. *The set of* upper bounds *of B is $Ub(B) = \{z \in P : B \preceq z\}$. The set of lower bounds of B is $Lb(B) = \{z \in P : z \preceq B\}$.*

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Observe that $\mathsf{Min}(\emptyset) = \mathsf{Max}(\emptyset) = \emptyset$, and $Ub(\emptyset) = Lb(\emptyset) = P$.

Definition 6 (Morgado hyperlattices, [4, Ch. II, §2, p. 122])**.** *Let* P *be a proset, and let* $x, y \in P$*.*

1. *The* Morgado hypersupremum *(or* supremoid*) of* x *and* y *is the set* $x \curlyvee y = \mathsf{Min}(Ub(\{x,y\}))$*.*
2. *The* Morgado hyperinfimum *(or* infimoid*) of* x *and* y *is the set* $x \curlywedge y = \mathsf{Max}(Lb(\{x,y\}))$*.*
3. P *is said to be a* Morgado hyperlattice *(or an* m-hyperlattice*) if* $x \curlyvee y$ *and* $x \curlywedge y$ *are nonempty sets for every* $x, y \in P$*.*

Remark 2. *As observed in [4, Ch. II, §2, pp. 129–132], m-hyperlattices can be alternatively defined as hyperstructures* $\mathsf{L} = \langle L, \curlywedge, \curlyvee \rangle$ *satisfying the following properties:*

R1 - $a \curlyvee b = b \curlyvee a$

R2 - *if* $x \in a \curlyvee b$ *and* $y \in b \curlyvee c$ *then* $x \curlyvee c = a \curlyvee y$

R3 - *if* $x \in a \curlyvee b$ *then* $a \in a \curlywedge x$

R4 - $a \curlywedge b = b \curlywedge a$

R5 - *if* $x \in a \curlywedge b$ *and* $y \in b \curlywedge c$ *then* $x \curlywedge c = a \curlywedge y$

R6 - *if* $x \in a \curlywedge b$ *then* $a \in a \curlyvee x$*.*

In this case, the preorder $\preceq$ *is defined as follows:* $x \preceq y$ *iff* $x \in x \curlywedge y$ *(iff* $y \in x \curlyvee y$*).*

It is quite straightforward to deduce Axioms R1-R6 from Definition 6. But the converse is not so trivial. For instance, let us extract some basic properties from Axioms R1-R6 (we leave to the reader to check the main details of the equivalence between Axioms R1-R6 and Definition 6):

Lemma 1. *Let* $\mathsf{L} = \langle L, \curlywedge, \curlyvee \rangle$ *be a hyperalgebra satisfying Axioms R1-R6, and let* $x, y \in L$*. Then:*

1. *Let* $\preceq$ *be the relation defined as follows:* $x \preceq y$ *iff* $x \in x \curlywedge y$ *(iff* $y \in x \curlyvee y$*). Then* $\preceq$ *is a preorder.*
2. $x \curlywedge y \preceq x$ *and* $x \curlywedge y \preceq y$*.*
3. $x \preceq x \curlyvee y$ *and* $y \preceq x \curlyvee y$*.*
4. $x \curlywedge y \preceq x \curlyvee y$*.*

Proof.

1. Since L is a hyperalgebra, there exists some $z \in x \curlyvee x$ which implies (by R3) that $x \in x \curlywedge z$. By R6 we get $x \in x \curlyvee x$. Similarly we conclude $x \in x \curlywedge x$. In particular, $x \preceq x$.
   Now, let $x \preceq y$ and $y \preceq z$. Then $x \in x \curlyvee y$ and $y \in y \curlyvee z$ which implies (by R2) that $x \curlyvee z = x \curlyvee y$. In particular, $x \in x \curlyvee z$ and $x \preceq z$.
2. Let $z \in x \curlywedge y$. Using R1 and R6 we get $x \in x \curlyvee z = z \curlyvee x$. By R3 we get $z \in z \curlywedge x$, which means that $z \preceq x$. Similarly we get $z \preceq y$.
3. Let $z \in x \curlyvee y$. Using R1 and R3 we get $x \in x \curlywedge z = z \curlywedge x$. By R6 we get $z \in z \curlyvee x$, which means that $x \preceq z$. Similarly we get $y \preceq z$.
4. Just combine items (1)-(3).

□

Example 1.

1. *Every lattice is a m-hyperlattice.*
2. *Let* L *be the set of formulas of propositional classical logic over signature* $\{\wedge, \vee, \rightarrow, \neg\}$*. It is well-known (from the fact that propositional classical logic is a Tarskian logic) that the relation in* L

$$\alpha \preceq \beta \ \textit{iff} \ \alpha \vdash \beta$$

defines a preorder and so $\langle P, \preceq \rangle$ *is a proset. Moreover, the sets* $\alpha \curlywedge \beta$ *and* $\alpha \curlyvee \beta$ *are nonempty for every* $\alpha, \beta \in L$*: indeed,* $\alpha \wedge \beta \in \alpha \curlywedge \beta$ *and* $\alpha \vee \beta \in \alpha \curlyvee \beta$*. This is a consequence of the following facts:*

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(i) $\alpha \wedge \beta \vdash \alpha$ and $\alpha \wedge \beta \vdash \beta$; and: $\gamma \vdash \alpha$ and $\gamma \vdash \beta$ implies that $\gamma \vdash \alpha \wedge \beta$.
(ii) $\alpha \vdash \alpha \vee \beta$ and $\beta \vdash \alpha \vee \beta$; and: $\alpha \vdash \gamma$ and $\beta \vdash \gamma$ implies that $\alpha \vee \beta \vdash \gamma$.
Then, L is a m-hyperlattice.

3. *The previous example can be adapted to any propositional Tarskian logic containing (standard) conjunction and disjunction as, for instance, (positive) intuitionistic logic.*
4. *([4, Ch. II, §2, p. 124]) Let V be a vector space (over a field F) and for $A, B \subseteq V$ consider*

$$A \curlyvee B := \{E \subseteq V : \langle E \rangle = \langle A \cup B \rangle\}$$
$$A \curlywedge B := \{E \subseteq V : \langle E \rangle = \langle A \rangle \cap \langle B \rangle\},$$

where $\langle E \rangle$ denotes the vector subspace of V generated by $E \subseteq V$. Then $\langle \mathcal{P}(V), \curlyvee, \curlywedge \rangle$ is a m-hyperlattice.

5. *The previous example works if we change vector spaces by other algebraic structures: for instance, algebras, modules, rings, abelian groups etc.*
6. *Let $\mathcal{C}$ be a small category with binary products $\prod$ and coproducts $\coprod$ (see, for instance, [10]) and for $a, b \in Obj(\mathcal{C})$ (the set of objects of $\mathcal{C}$), define*

$$a \preceq b \text{ iff there exist a morphism } f : a \to b \text{ in } \mathcal{C}.$$

It is straightforward to check that $\langle Obj(\mathcal{C}), \preceq \rangle$ is a m-hyperlattice. In this case, $a \prod b \in a \curlywedge b$ and $a \coprod b \in a \curlyvee b$.

Definition 7. *The category MHL of hyperlattices is the one where the objects are hyperlattices and the morphism are just the usual morphisms of hyperalgebras. In other words, given $L_1, L_2 \in MHL$, a function $f : L_1 \to L_2$ is a morphism if for all $x, y, z \in L_1$ we have the following:*

1. *$z \in x \curlywedge y$ implies $f(z) \in f(x) \curlywedge f(y)$;*
2. *$z \in x \curlyvee y$ implies $f(z) \in f(x) \curlyvee f(y)$.*

Proposition 1. *Let L_1, L_2 be MHLs and $f : L_1 \to L_2$ be a function. Then f is a MHL-morphism iff for all $x, y \in L_1$, $x \preceq y$ implies $f(x) \preceq f(y)$.*

Proof. This is an immediate consequence of the characterization of $\preceq$ in terms of $\curlywedge$ (or $\curlyvee$): $x \preceq y$ iff $x \in x \curlywedge y$ (iff $y \in x \curlyvee y$). □

Remark 3. *It is worth noting that a version of the celebrated Principle of Duality for ordered sets (see, for instance, [11]) can be easily obtained for prosets, and so for m-hyperlattices. Indeed, if $\mathsf{P} = \langle P, \preceq \rangle$ is a proset, so is its* dual *is $\mathsf{P}^* = \langle P, \succeq \rangle$, where $a \succeq b$ iff $b \preceq a$, for every $a, b \in P$. Clearly, any statement Φ (just concerning $\preceq$) has its* dual statement *Φ^* (obtained from Φ by replacing $\preceq$ by $\succeq$). From this, P satisfies Φ^* iff P^* satisfies Φ^*. Since $\Phi^{**} = \Phi$, it follows that Φ holds in any P iff Φ^* holds in any P (Principle of Duality for prosets). This can be extended to m-hyperlattices, and so if $\mathsf{L} = \langle L, \curlywedge, \curlyvee \rangle$ is a hyperlattice (with underlying preorder $\preceq$), its dual $\mathsf{L}^* = \langle L, \curlywedge^*, \curlyvee^* \rangle$ (with underlying preorder $\succeq$) is also a m-hyperlattice, where the hyperoperations are defined by $a \curlywedge^* b := a \curlyvee b$ and $a \curlyvee^* b := a \curlywedge b$, for every $a, b \in L$. By the principle of duality for prosets, and by definition of m-hyperlattices, if a statement Φ (containing $\preceq$, $\curlywedge$, and $\curlyvee$) holds in any m-hyperlattice then its dual statement Φ^* (obtained from Φ by replacing $\preceq$ by $\succeq$, $\curlywedge$ by $\curlyvee$, and $\curlyvee$ by $\curlywedge$, respectively) also holds in any m-hyperlattice.*

Proposition 2. *Let $\mathsf{P} = \langle P, \preceq, \curlywedge, \curlyvee \rangle$ be a m-hyperlattice. If $x, y \in P$ such that $x \equiv y$, then for every $z \in P$ and $B \subseteq P$:*

1. *$x \preceq z$ iff $y \preceq z$.*
2. *$x \preceq B$ iff $y \preceq B$.*
3. *$x \curlyvee z = y \curlyvee z$.*
4. *$z \preceq x$ iff $z \preceq y$.*
5. *$B \preceq x$ iff $B \preceq y$.*
6. *$x \curlywedge z = y \curlywedge z$.*

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Proof. Items (1)-(2) and (4)-(5) follow from the transitivity of $\preceq$. For (3), observe that $Ub(x,z) = Ub(y,z)$ for every $z \in P$, by (1). Hence

$$x \curlyvee z = \mathsf{Min}(Ub(x,z)) = \mathsf{Min}(Ub(y,z)) = y \curlyvee z.$$

Item (6) is proved analogously. □

Remark 4. *Given non-empty sets $A, B \subseteq L$, the notation $A \preceq B$ established in Definition 4 can be extended to the hyperoperations $\curlywedge$ and $\curlyvee$ as follows: $A \curlywedge B := \bigcup\{a \curlywedge b \ : \ a \in A \ \text{ and } \ b \in B\}$, and $A \curlyvee B := \bigcup\{a \curlyvee b \ : \ a \in A \ \text{ and } \ b \in B\}$.*

Proposition 3. *Let L be an m-hyperlattice, and $x, y, z \in L$. Then:*

1. *If $\emptyset \neq A, B \subseteq L$ then $A\#B = B\#A$ for $\# \in \{\curlywedge, \curlyvee\}$.*
2. *If $\emptyset \neq A \subseteq B \subseteq L$ then $\mathsf{Max}(A) \preceq \mathsf{Max}(B)$.*
3. *If $\emptyset \neq A \subseteq B \subseteq L$ then $\mathsf{Min}(B) \preceq \mathsf{Min}(A)$.*
4. *If $x \preceq y$ then $x \curlywedge z \preceq y \curlywedge z$.*
5. *If $x \preceq y$ then $x \curlyvee z \preceq y \curlyvee z$.*
6. *If $\emptyset \neq A, B \subseteq L$ and $A \preceq B$ then $A\#c \preceq B\#c$ for $\# \in \{\curlywedge, \curlyvee\}$ and any $c \in L$.*
7. *Let $\emptyset \neq B \subseteq L$. If $a \preceq x$ and $a \preceq B$ then $a \preceq x \curlywedge B$.*
8. *Let $\emptyset \neq B \subseteq L$. If $x \preceq a$ and $B \preceq a$ then $x \curlyvee B \preceq a$.*
9. *Let $\emptyset \neq A, B \subseteq L$. If $A \preceq z$ and $B \preceq z$ then $A \curlyvee B \preceq z$.*
10. *Let $\emptyset \neq A, B \subseteq L$. If $x \curlywedge A \preceq z$ then $x \curlywedge (A \curlywedge B) \preceq z$.*
11. *Let $\emptyset \neq A, B \subseteq L$. If $B \curlywedge y \preceq z$ then $(A \curlywedge B) \curlywedge y \preceq z$.*
12. *$(x \curlywedge y) \curlywedge z \equiv x \curlywedge (y \curlywedge z)$.*
13. *$(x \curlyvee y) \curlyvee z \equiv x \curlyvee (y \curlyvee z)$.*

Proof.

1. It follows from the fact that $A\#B = \bigcup\{a\#b \ : \ a \in A, \ b \in B\}$ and by (R1) and (R4) from Remark 2.
2. Assume that $\emptyset \neq A \subseteq B \subseteq L$, and let $x \in \mathsf{Max}(A)$ and $y \in \mathsf{Max}(B)$. Since $x \in A$ then $x \in B$, hence $x \preceq y$. This shows that $\mathsf{Max}(A) \preceq \mathsf{Max}(B)$.
3. It follows from (2) by duality.
4. Assume that $x \preceq y$. By definition, $x \curlywedge z = \mathsf{Max}(A)$ and $y \curlywedge z = \mathsf{Max}(B)$, where $A = \{w \in L \ : \ w \preceq x \text{ and } w \preceq z\}$ and $B = \{w \in L \ : \ w \preceq y \text{ and } w \preceq z\}$. By definition of m-hyperlattices, $\emptyset \neq A \subseteq B \subseteq L$. By item (2), $x \curlywedge z \preceq y \curlywedge z$.
5. It follows from (4) by duality.
6. Fix $\# \in \{\curlywedge, \curlyvee\}$ and $c \in L$, and let $x \in A\#c$, $y \in B\#c$. Then $x \in a\#c$ and $y \in b\#c$ for some $a \in A$ and $b \in B$. By hypothesis, $a \preceq b$ and so $a\#c \preceq b\#c$, by items (4) and (5). From this, $x \preceq y$ and so $A\#c \preceq B\#c$.
7. Assume that $a \preceq x$ and $a \preceq B$. If $b \in B$ then $a \preceq b$, hence $a \preceq x \curlywedge b$. Let $c \in x \curlywedge B$. Hence, $c \in x \curlywedge b$ for some $b \in B$. Since $a \preceq x \curlywedge b$, it follows that $a \preceq c$. That is, $a \preceq x \curlywedge B$.
8. It follows from (7) by duality.
9. Suppose that $A \preceq z$ and $B \preceq z$, and let $a \in A$. Then, $a \preceq z$ and so, by (8), $a \curlyvee B \preceq z$, for every $a \in A$. Then $A \curlyvee B = \bigcup\{a \curlyvee b \ : \ a \in A, \ b \in B\} = \bigcup\{a \curlyvee B \ : \ a \in A\} \preceq z$.
10. Assume that $x \curlywedge A \preceq z$. Let $w \in x \curlywedge (A \curlywedge B)$. Then $w \in x \curlywedge u$ such that $u \in A \curlywedge B$. Hence, there exists $a \in A$ and $b \in B$ such that $u \in a \curlywedge b$. Since $a \curlywedge b \preceq a$ it follows that $u \preceq a$. By (4), $x \curlywedge u \preceq x \curlywedge a$, and then $w \preceq x \curlywedge a$ (given that $w \in x \curlywedge u$). Since $x \curlywedge a \subseteq x \curlywedge A \preceq z$ it follows that $x \curlywedge a \preceq z$ and so $w \preceq z$. That is, $x \curlywedge (A \curlywedge B) \preceq z$.
11. It follows from (10) and by commutativity of $\curlywedge$, namely: $A \curlywedge B = B \curlywedge A$ and $A \curlywedge x = x \curlywedge A$ for every A and x.
12. Let $a \in (x \curlywedge y) \curlywedge z$. Then, $a \in b \curlywedge z$ for some $b \in x \curlywedge y$. From $a \in b \curlywedge z \preceq b$ we infer that $a \preceq b$. From $b \in x \curlywedge y \preceq x$ it follows that $b \preceq x$. By transitivity, $a \preceq x$ (*). Now, $b \in x \curlywedge y \preceq y$ and so

$b \preceq y$. By (4), $b \curlywedge z \preceq y \curlywedge z$. But $a \in b \curlywedge z$, hence $a \preceq y \curlywedge z$ (**). From (*) and (**) we prove that $a \preceq x \curlywedge (y \curlywedge z)$, by (7). This shows that $(x \curlywedge y) \curlywedge z \preceq x \curlywedge (y \curlywedge z)$.
Conversely, let $a \in x \curlywedge (y \curlywedge z)$. Then, $a \in x \curlywedge b$, for some $b \in y \curlywedge z$. Since $b \in y \curlywedge z \preceq y$ it follows that $b \preceq y$ and so $x \curlywedge b \preceq x \curlywedge y$, by (4). But then $a \in x \curlywedge b \preceq x \curlywedge y$, which implies that $a \preceq x \curlywedge y$ (@). In turn, $a \in x \curlywedge b \preceq b$, hence $a \preceq b$. Since $b \in y \curlywedge z \preceq z$ it follows that $b \preceq z$, and then $a \preceq z$ (@@). By (@), (@@) and (7) (and by the fact that $B \curlywedge w = w \curlywedge B$) it follows that $a \preceq (x \curlywedge y) \curlywedge z$. That is, $x \curlywedge (y \curlywedge z) \preceq (x \curlywedge y) \curlywedge z$.
13. It follows from (12) by duality.

□

Remark 5.

1. *Let $\emptyset \neq A, B \subseteq L$. It is worth noting that $A \preceq B$* does not imply, *in general, that $A\#C \preceq B\#C$ for $\# \in \{\curlywedge, \curlyvee\}$ and $\emptyset \neq C \subseteq L$. Moreover: $a \preceq b$* does not imply *(in general) that $a\#C \preceq b\#C$ for $\# \in \{\curlywedge, \curlyvee\}$ and $\emptyset \neq C \subseteq L$. This shows that item (6) of Proposition 3 cannot be generalized to arbitrary non-empty sets C. To see an example, recall the m-hyperlattice L considered in Example 1 (2). Let p, q, r, s be 4 different propositional variables, and let $a = p \wedge q$, $b = p$, and $C = \{r, s\}$. Clearly, $a \preceq b$. However, $a \curlywedge C \not\preceq b \curlywedge C$. Indeed, $(p \wedge q) \wedge r \in a \curlywedge C$ and $p \wedge s \in b \curlywedge C$ but $(p \wedge q) \wedge r \not\preceq p \wedge s$. Analogously, $a \preceq b$ but $a \curlyvee C \not\preceq b \curlyvee C$: indeed, $(p \wedge q) \vee r \in a \curlyvee C$ and $p \vee s \in b \curlyvee C$ but $(p \wedge q) \vee r \not\preceq p \vee s$. In order to guarantee monotonicity for $\curlywedge$ and $\curlyvee$ w.r.t. sets, stability (see Definition 8) is required, as we shall see below in Proposition 5.*
2. *It can be proven that, in general, $A \curlywedge B \not\preceq A$, for $\emptyset \neq A, B \subseteq L$. Moreover, $A \curlywedge x \not\preceq A$ in general, for $A \neq \emptyset$. Analogously, $A \not\preceq A \curlyvee x$ and so $A \not\preceq A \curlyvee B$ in general. Examples can be found, once again, in the m-hyperlattice of Example 1 (2). Indeed, take $A = \{p, q\}$ and $x = r$ for p, q, r different propositional variables. Since $p \wedge r \not\preceq q$ then $\{p, q\} \curlywedge r \not\preceq \{p, q\}$. Analogously, since $q \not\preceq p \vee r$ then $\{p, q\} \not\preceq \{p, q\} \curlyvee r$. In order to guarantee the validity of these desirable properties, it is required stability once again, see Proposition 4 below.*

Definition 8. *Let L be a m-hyperlattice, and let $\emptyset \neq A, B \subseteq L$. We say that A and B are* ***similar****, and write $A \equiv B$, if $a \equiv b$ for every $a \in A$ and $b \in B$. That is: $a \preceq b$ and $b \preceq a$ for every $a \in A$ and $b \in B$. A non-empty subset $A \subseteq L$ is* ***stable*** *if $A \equiv A$.*

Proposition 4. *Let $A, B \subseteq L$ be stable subsets. Then $A \curlywedge B$ and $A \curlyvee B$ are stable subsets. In particular, for all $a \in A$ and $b \in B$ we have*

$$A \curlywedge B = a \curlywedge b \text{ and } A \curlyvee B = a \curlyvee b.$$

Proof. Let $x, y \in A \curlywedge B$. Then $x \in a \curlywedge b$ and $y \in a' \curlywedge b'$ for some $a, a' \in A$ and $b, b' \in B$. Since $a \equiv a'$ and $b \equiv b'$, (6) of Proposition 2 provides $a \curlywedge b = a' \curlywedge b'$ and we have

$$x, y \in a \curlywedge b = \mathrm{Max}(\mathrm{Lb}(\{a, b\})).$$

Since $x \in \mathrm{Lb}(\{a, b\})$ it follows that $x \preceq y$. Similarly we get $y \preceq x$, providing $x \equiv y$. Hence $A \curlywedge B$ is stable. For the final part, we already know that if $a, a' \in A$ and $b, b' \in B$ then $a \curlywedge a' \equiv b \curlywedge b'$ (Proposition 2 item (6) again). Then for all $x \in A$ and $y \in B$ we get

$$A \curlywedge B = \bigcup_{a \in A, b \in B} a \curlywedge b = x \curlywedge y.$$

The proof for $A \curlyvee B$ follows by duality. □

Proposition 5. *Let $A, B, C, D \subseteq L$ be stable sets such that $A \preceq B$ and $C \preceq D$. Then $A \curlywedge C \preceq B \curlywedge D$ and $A \curlyvee C \preceq B \curlyvee D$.*

Proof. Since A, B, C, D are stable, we can suppose by Proposition 4 that $A \curlywedge C = a \curlywedge c$ and $B \curlywedge D = b \curlywedge d$ for some $a \in A$, $b \in B$, $c \in C$ and $d \in D$ such that, by hypothesis, $a \preceq b$ and $c \preceq d$. Since $a \curlywedge c \preceq a \preceq b$ and $a \curlywedge c \preceq c \preceq d$, we get $a \curlywedge c \preceq b \curlywedge d$, providing that $A \curlywedge C = a \curlywedge c \preceq b \curlywedge d = B \curlywedge D$. Monotonicity of $\curlyvee$ follows from monotonicity of $\curlywedge$ by duality. □

Supremoinds and infimoids between stable sets can be characterized in a natural way:

Proposition 6. *Let $A, B \subseteq L$ be stable sets. Then $A \curlywedge B = \mathsf{Max}(Lb(A \cup B))$ and $A \curlyvee B = \mathsf{Min}(Ub(A \cup B))$.*

Proof. Let $a \in A$ and $b \in B$. By stability of both A and B, it is immediate to see that $Lb(A \cup B) = Lb(\{a, b\})$. From this and by Proposition 4, $A \curlywedge B = a \curlywedge b = \mathsf{Max}(Lb(\{a, b\})) = \mathsf{Max}(Lb(A \cup B))$. The case for $A \curlyvee B$ follows from the case for $\curlywedge$ by duality. □

As it would be expected, the *absorption laws* of lattices hold for m-hyperlattices in a suitable weaker form:

Proposition 7 (Absorption laws for m-hyperlattices)**.** *Let L be a m-hyperlattice, and let $x, y \in L$. Then:*

$$x \curlywedge (x \curlyvee y) \equiv x \equiv x \curlyvee (x \curlywedge y).$$

Proof. Let $A = x \curlywedge (x \curlyvee y)$. Since $\{x\}$ and $x \curlyvee y$ are stable, $A = \mathsf{Max}(Lb(\{x\} \cup (x \curlyvee y)))$, by Proposition 6. Since $x \preceq x$ and $x \preceq x \curlyvee y$, it follows that $x \preceq A$. Conversely, let $a \in A$. Then $a \preceq x$, since $a \in Lb(\{x\} \cup (x \curlyvee y))$. This shows that $A \preceq x$ and so $A \equiv x$. The second claim follows from the first by duality. □

Proposition 8. *Let $A, B, C \subseteq L$ be stable subsets. Then, for all $a \in A$, $b \in B$ and $c \in C$ it holds:*

$$\begin{aligned}
A \curlywedge (B \curlywedge C) &= a \curlywedge (b \curlywedge c), \\
(A \curlywedge B) \curlywedge C &= (a \curlywedge b) \curlywedge c, \\
A \curlyvee (B \curlyvee C) &= a \curlyvee (b \curlyvee c), \\
(A \curlyvee B) \curlyvee C &= (a \curlyvee b) \curlyvee c, \\
(A \curlywedge B) \curlyvee C &= (a \curlywedge b) \curlyvee c, \\
(A \curlyvee B) \curlywedge C &= (a \curlyvee b) \curlywedge c.
\end{aligned}$$

In particular,

$$A \curlywedge (B \curlywedge C) \equiv (A \curlywedge B) \curlywedge C \text{ and } A \curlyvee (B \curlyvee C) \equiv (A \curlyvee B) \curlyvee C.$$

Proof. Given $d \in b \curlywedge c$, by Proposition 4 we conclude that

$$A \curlywedge (B \curlywedge C) = A \curlywedge (b \curlywedge c) = a \curlywedge d.$$

Since $b \curlywedge c$ is stable and for $d, d' \in b \curlywedge c$ we have $a \curlywedge d = a \curlywedge d'$, we get $A \curlywedge (B \curlywedge C) = a \curlywedge (b \curlywedge c)$. The proof for the other cases is similar. □

Definition 9. *Let $\mathsf{P} = \langle P, \preceq, \curlywedge, \curlyvee \rangle$ be a m-hyperlattice. The sets $\mathsf{Min}(P)$ and $\mathsf{Max}(P)$ of minima and maxima elements of P will be denoted by $\bot$ (or 0) and $\top$ (or 1), respectively. We say that a m-hyperlattice P is* bounded *if $\top \neq \emptyset \neq \bot$.*

Remark 6. *Let $\emptyset \neq A \subseteq P$. If $\top \neq \emptyset$, then $A \equiv \top$ iff $A \subseteq \top$. Analogously, if $\bot \neq \emptyset$, then $A \equiv \bot$ iff $A \subseteq \bot$.*

To end this section, it will be shown that the denotation, in any concrete hyperlattice L, of any term in the signature of m-hyperlattices, produces a stable subset of L.

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Definition 10 (M-term)**.** *Let L be a m-hyperlattice. We define* ***M-terms*** *recursively as follows:*

1. *Every stable subset $A \subseteq L$ is a M-term.*
2. *If A and B are M-terms, then $A \curlywedge B$ and $A \curlyvee B$ are M-terms.*

Proposition 9. *Let $A \subseteq L$ be a M-term. Then A is a stable subset.*

Proof. Just use induction and Propositions 4 and 8. □

3. Sette Implicative Hyperlattices

In this section we recall the class of hyperalgebras introduced by Sette in [8] (that we called *Sette implicative hyperlattices*) as a basis of the hyperalgebraic semantics for da Costa logic C_ω he also proposed. The intuition behind this is to generalize the notion of implicative lattices to the context of hyperalgebras. As we shall see in Proposition 10, Sette's intuitions were right.

Definition 11 (Sette implicative hyperlattices, [8, Definition 2.3])**.** *A* Sette implicative hyperlattice *(or a* SIHL*)) is a hyperalgebra $\mathsf{L} = \langle L, \curlywedge, \curlyvee, \multimap \rangle$ such that the reduct $\langle L, \curlywedge, \curlyvee \rangle$ is a m-hyperlattice and the hyperoperator $\multimap$ satisfies the following properties, for every $x, y, z, z' \in L$:*

(I1) *$z \in x \multimap y$ implies that $x \curlywedge z \preceq y$;*

(I2) *$x \curlywedge z \preceq y$ implies that $z \preceq x \multimap y$;*

(I3) *$z \equiv z'$ and $z \in x \multimap y$ implies that $z' \in x \multimap y$.*

It is possible to give a more direct characterization of SIHLs.

Definition 12. *Let* P *be a m-hyperlattice, and let $x, y \in P$. The set $\mathsf{R}(x, y)$ is given by $\mathsf{R}(x, y) = \{z \in P : x \curlywedge z \preceq y\}$.*

Proposition 10. *Let $\mathsf{L} = \langle L, \curlywedge, \curlyvee, \multimap \rangle$ be a hyperalgebra such that $\langle L, \curlywedge, \curlyvee \rangle$ is a m-hyperlattice. Then,* L *is an SIHL iff $x \multimap y = \mathsf{Max}(\mathsf{R}(x, y))$, for every $x, y \in L$.*

Proof.
'Only if' part: Suppose that L is an SIHL, and let $z \in x \multimap y$. By (I1), $x \curlywedge z \preceq y$ and so $z \in \mathsf{R}(x, y)$. Now, let $z' \in \mathsf{R}(x, y)$. Then, $x \curlywedge z' \preceq y$ which implies that $z' \preceq x \multimap y$, by (I2). Since $z \in x \multimap y$, by hypothesis, then $z' \preceq z$. This proves that $\mathsf{R}(x, y) \preceq z$ and so $z \in \mathsf{Max}(\mathsf{R}(x, y))$. That is, $x \multimap y \subseteq \mathsf{Max}(\mathsf{R}(x, y))$. In order to prove the converse inclusion, let $z \in \mathsf{Max}(\mathsf{R}(x, y))$. Then, $z \in \mathsf{R}(x, y)$ such that $\mathsf{R}(x, y) \preceq z$. It holds that $x \multimap y \subseteq \mathsf{R}(x, y)$, by (I1), then

$$(*) \qquad x \multimap y \preceq z.$$

In turn, from $z \in \mathsf{R}(x, y)$ it follows that $x \curlywedge z \preceq y$ and so

$$(**) \qquad z \preceq x \multimap y,$$

by (I2). Now, let $z' \in x \multimap y$ (observe that $x \multimap y \neq \emptyset$, given that L is a hyperalgebra). By $(*)$ and $(**)$, $z' \preceq z$ and $z \preceq z'$, that is, $z \equiv z'$. By (I3), $z \in x \multimap y$. That is, $\mathsf{Max}(\mathsf{R}(x, y)) \subseteq x \multimap y$.

'If' part: Assume that $x \multimap y = \mathsf{Max}(\mathsf{R}(x, y))$, for every $x, y \in L$. Since L is an hyperalgebra, by hypothesis, then $x \multimap y \neq \emptyset$, for every $x, y \in L$. If $z \in x \multimap y$ then, by definition of $\multimap$, $z \in \mathsf{R}(x, y)$, which means that $x \curlywedge z \preceq y$. Hence, $\multimap$ satisfies (I1). Suppose now that $x \curlywedge z \preceq y$. Then $z \in \mathsf{R}(x, y)$, therefore $z \preceq \mathsf{Max}(\mathsf{R}(x, y)) = x \multimap y$. This shows that $\multimap$ satisfies (I2). Finally, suppose that $z \equiv z'$ and $z \in x \multimap y$. By Proposition 2 (6), and given that $z \in \mathsf{R}(x, y)$, it follows that $x \curlywedge z' = x \curlywedge z \preceq y$. That

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is, $z' \in \mathsf{R}(x,y)$. Since $\mathsf{R}(x,y) \preceq z$, it follows from Proposition 2 (5) that $\mathsf{R}(x,y) \preceq z'$. This implies that $z' \in \mathsf{Max}(\mathsf{R}(x,y)) = x \multimap y$. That is, $\multimap$ also satisfies (I3). □

Remark 7. *The latter proposition shows that Sette's intuitions when defining implicative hyperlattices based on Morgado hyperlattices were right: indeed, $x \multimap y = \mathsf{Max}(\mathsf{R}(x,y))$ seems to be the more natural generalization of the notion of implicative lattices to the realm of Morgado hyperlattices.*

Corollary 1. *Let $\mathsf{L} = \langle L, \curlywedge, \curlyvee \rangle$ be a m-hyperlattice. If $\mathsf{Max}(\mathsf{R}(x,y)) \neq \emptyset$ for every $x,y \in L$ then L is a SIHL such that $x \multimap y = \mathsf{Max}(\mathsf{R}(x,y))$, for every $x,y \in L$.*

Lemma 2. *Let L be a m-hyperlattice. Then:*

1. $y \in \mathsf{R}(x,y)$.
2. *For all $x,y \in L$ there exist $z \in x \multimap y$ such that $y \preceq z$.*
3. *If $y \equiv y'$ then $R(x,y) = R(x,y')$. In particular, $x \multimap y = x \multimap y'$.*
4. *If $x \equiv x'$ then $R(x,y) = R(x',y)$. In particular, $x \multimap y = x' \multimap y$.*
5. *If $x \equiv x'$ and $y \equiv y'$ then $R(x,y) = R(x',y')$. In particular, $x \multimap y = x' \multimap y'$.*
6. *If $z,z' \in R(x,y)$ then $z \curlywedge z' \subseteq R(x,y)$.*
7. *$R(x,y)$ is stable.*
8. *If $z \in R(x,y)$ then $R(x,z) \subseteq R(x,y)$. In particular, $x \multimap y \subseteq x \multimap z$.*
9. *$x \multimap y$ is stable.*
10. *For all $x,y,z \in L$, $z \preceq x \multimap y$ iff $z \curlywedge x \preceq y$.*

Proof.

1. It follows from $x \curlywedge y \preceq y$.
2. It follows from $y \in R(x,y)$ and $x \multimap y = \mathsf{Max}(R(x,y))$.
3. If $z \in R(x,y)$, then $x \curlywedge z \preceq y \preceq y'$, which means $z \in R(x,y')$. Similarly $z \in R(x,y)$.
4. Proposition 4 provides that $x \curlywedge z \equiv x' \curlywedge z$ for all $z \in L$. If $z \in R(x,y)$, then $x \curlywedge z \preceq y$ and since $x \equiv x'$, we get $x' \curlywedge z \preceq y$, providing $z \in R(x',y)$. Similarly $z \in R(x',y)$ imply $z \in R(x,y)$. Therefore $R(x,y) = R(x',y)$ and in particular, $x \multimap y = x' \multimap y$.
5. Just note that combining the previous items we get
$$R(x,y) = R(x',y) = R(x',y').$$
6. It follows from $x \curlywedge (z \curlywedge z') \preceq x \curlywedge z$ and $x \curlywedge (z \curlywedge z') \preceq x \curlywedge z'$.
7. It follows from the fact that $z \preceq z'$ implies $x \curlywedge z \preceq x \curlywedge z'$.
8. Let $z \in R(x,y)$ and $z' \in R(x,z)$. Then $x \curlywedge z \preceq y$ and $x \curlywedge z' \preceq z$. From $x \curlywedge z' \preceq z$ and $x \curlywedge z' \preceq x$ we get $x \curlywedge z' \preceq x \curlywedge z \preceq y$, providing $z' \in R(x,y)$.
9. Let $z,z' \in x \multimap y$. From $z \in \mathsf{Ub}(R(x,y))$ and $z' \in x \multimap y = \mathsf{Max}(\mathsf{Ub}(R(x,y)))$ we get $z' \preceq z$. Similarly $z \preceq z'$.
10. Let $z \preceq x \multimap y$. Then $z \preceq z'$ for all $z' \in x \multimap y$. By item (4) of Proposition 3 we get $x \curlywedge z \preceq x \curlywedge z'$ and by (I1), $x \curlywedge z' \preceq y$ establishing $x \curlywedge z \preceq y$. The converse is an immediate consequence of (I2).

□

In the rest of this Section, some basic but useful properties of SIHLs will be obtained. They will be used in Section 6.

Proposition 11. *Let L be a SIHL, and let $x,y,z \in L$. Then:*

1. $x \curlywedge (x \multimap y) \preceq y$.
2. $x \preceq y$ *iff* $x \multimap y = \top$.
3. $x \multimap x = \top$.
4. $x \multimap (y \multimap x) = \top$.

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5. $x \multimap (y \multimap (x \curlywedge y)) = \top$.
6. $(x \curlywedge y) \multimap x = \top$ *and* $(x \curlywedge y) \multimap y = \top$.
7. $x \multimap (x \curlyvee y) = \top$ *and* $y \multimap (x \curlyvee y) = \top$.

Proof.

1. Let $w \in x \curlywedge (x \multimap y) = \bigcup\{x \curlywedge z \ : \ z \in (x \multimap y)\}$. Then, $w \in x \curlywedge z$ for some $z \in (x \multimap y)$. But $z \in \mathsf{R}(x, y)$ and so $x \curlywedge z \preceq y$. From this, $w \preceq y$.
2. Suppose that $x \preceq y$, and let $z \in L$. Given that $x \curlywedge z \preceq x$ then $x \curlywedge z \preceq y$. That is, $\mathsf{R}(x, y) = L$. From this, $x \multimap y = \mathsf{Max}(\mathsf{R}(x, y)) = \mathsf{Max}(L) = \top$. Conversely, suppose that $x \multimap y = \mathsf{Max}(\mathsf{R}(x, y)) = \top$, and let $z \in x \multimap y$. Then, $x \curlywedge z \preceq y$. But $x \preceq z$ (since $z \in \top$), then $x \in x \curlywedge z$ (recall Remark 2). This implies that $x \preceq y$.
3. It follows from (2) and the fact that $x \preceq x$.
4. There exist $z \in y \multimap x$ such that $x \preceq z$. Using (2) we get
$$\top = x \multimap z \subseteq x \multimap (y \multimap x).$$
5. It follows from (2) and the fact that $x, y \in R(y, x \curlywedge y)$ (so there exist $z \in y \multimap (x \curlywedge y)$ with $x \preceq z$ and $y \preceq z$).
6. It follows from (2) and the fact that $x, y \in R(x, x \curlywedge y)$ (and also $x, y \in R(y, x \curlywedge y)$).
7. It follows from (2) and the fact that $x, y \in R(x, x \curlyvee y)$ (and also $x, y \in R(y, x \curlyvee y)$).

□

Proposition 12. *Let $A, B \subseteq L$ be stable non-empty subsets. Then for all $a \in A$ and all $b \in B$,*
$$A \multimap B = a \multimap b.$$
In particular, $A \multimap B$ is stable.

Proof. Follows from Lemma 2 items (5) and (9). □

Proposition 13. *Let L be a SIHL. Let $A, B, C \subseteq L$ such that $A \neq \emptyset$, $B \neq \emptyset$ and $C \neq \emptyset$. Let $x, y, z \in L$. Then:*

1. *$A \preceq B$ iff $a \multimap b = \top$ for every $a \in A$ and $b \in B$, iff $A \multimap B = \top$. In particular, $A \preceq B$ iff $a \multimap b = \top$ for some $a \in A$ and some $b \in B$.*
2. *$x \preceq y \multimap z$ iff $x \curlywedge y \preceq z$.*
3. *$A \preceq B \multimap C$ iff $a \preceq b \multimap c$ for every $a \in A$, $b \in B$ and $c \in C$ iff $a \curlywedge b \preceq c$ for every $a \in A$, $b \in B$ and $c \in C$ iff $A \curlywedge B \preceq C$.*
4. *$A \curlywedge (A \multimap B) \preceq B$.*

Proof.

1. Assume that $A \preceq B$, and let $a \in A$ and $b \in B$. Since $a \preceq b$, by hypothesis, we infer that $a \multimap b = \top$, by Proposition 11 (2). Now, suppose that $a \multimap b = \top$ for every $a \in A$ and $b \in B$. Then, $A \multimap B = \bigcup\{x \multimap y \ : x \in A, y \in B\} = \top$. Finally, suppose that $A \multimap B = \top$, and let $a \in A$ and $b \in B$. Then $a \multimap b \subseteq \top$. Let $z \in a \multimap b$. Then, $a \curlywedge z \preceq b$, and $z \in \top$, hence $a \preceq z$. From this, $a \in a \curlywedge z$ (by Remark 2) and so $a \preceq b$. This shows that $A \preceq B$.
2. Suppose that $x \preceq y \multimap z$, and let $w \in y \multimap z = \mathsf{Max}(\mathsf{R}(y, z))$. By hypothesis, $x \preceq w$ and so, by Proposition 3 (4), $x \curlywedge y \preceq w \curlywedge y \preceq z$ (since $w \in \mathsf{R}(y, z)$). The converse follows from (I2) of definition of SIHLs.
3. Assume that $A \preceq B \multimap C$ and let $a \in A$, $b \in B$ and $c \in C$. Since $b \multimap c \subseteq B \multimap C$ then $a \preceq b \multimap c$ and so, by (2), $a \curlywedge b \preceq c$, for every a, b, c, Hence $A \curlywedge B \preceq c$ for every $c \in C$ and so $A \curlywedge B \preceq C$. Conversely, assume that $A \curlywedge B \preceq C$ and let $a \in A$, $b \in B$ and $c \in C$. Then $a \curlywedge b \preceq c$ (since

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$a \curlywedge b \subseteq A \curlywedge B$) and so $a \preceq b \multimap c$, by (2), for every a, b, c. Hence $a \preceq B \multimap C$ for every $a \in A$ and so $A \preceq B \multimap C$.

4. Clearly $A \multimap B \preceq A \multimap B$. By (3), $(A \multimap B) \curlywedge A \preceq B$, and so $A \curlywedge (A \multimap B) \preceq B$, by Proposition 3 (1).

□

It is well-known that every implicative lattice is distributive; in particular, any Heyting algebra (which is nothing else than an implicative lattice with a bottom element) is distributive. By considering a suitable notion of distributive m-hyperlattices, it will be proven that the same results hold for implicative hyperlattices (see Corollary 2 below).

Definition 13. *A m-hyperlattice L is said to be* **distributive** *if, for all $x, y, z \in L$,*

$$x \curlywedge (y \curlyvee z) \equiv (x \curlywedge y) \curlyvee (x \curlywedge z) \text{ and } x \curlyvee (y \curlywedge z) \equiv (x \curlyvee y) \curlywedge (x \curlyvee z).$$

This relatively weak notion of distributive hyperlattices will be enough for proving, in Section 6, the soundness of some important axioms of positive intuitionistic logic w.r.t. implicative hyperlattices (see proof of Theorem 1, (1) $\Rightarrow$ (2)).

Proposition 14. *In any SIHL the following holds, for any $x, y, z \in L$:*

1. $x \curlywedge (y \curlyvee z) \equiv (x \curlywedge y) \curlyvee (x \curlywedge z)$.
2. $(x \multimap (y \multimap z)) \multimap ((x \multimap y) \multimap (x \multimap z)) = \top$.
3. $(x \multimap y) \multimap ((y \multimap z) \multimap (x \curlyvee y) \multimap z)) = \top$.
4. $(x \multimap y) \curlywedge (y \multimap z) \preceq (x \curlyvee y) \multimap z)$.
5. *If A is stable then: $A \curlywedge x \preceq z$ and $A \curlywedge y \preceq z$ implies that $A \curlywedge (x \curlyvee y) \preceq z$.*
6. $x \curlyvee (y \curlywedge z) \equiv (x \curlyvee y) \curlywedge (x \curlyvee z)$.

Proof.

1. Recall Proposition 6, and let

$$w \in (x \curlywedge y) \curlyvee (x \curlywedge z) = \text{Min}(\text{Ub}((x \curlywedge y) \cup (x \curlywedge z))).$$

Since $x \curlywedge y \preceq x \curlywedge (y \curlyvee z)$ and $x \curlywedge z \preceq x \curlywedge (y \curlyvee z)$, we have that

$$x \curlywedge (y \curlyvee z) \subseteq \text{Ub}((x \curlywedge y) \cup (x \curlywedge z)).$$

This implies that $w \preceq x \curlywedge (y \curlyvee z)$ and so $(x \curlywedge y) \curlyvee (x \curlywedge z) \preceq x \curlywedge (y \curlyvee z)$.
In order to prove the other inequality, let

$$w \in (x \curlywedge y) \curlyvee (x \curlywedge z) = \text{Min}(\text{Ub}((x \curlywedge y) \cup (x \curlywedge z))).$$

Then, $x \curlywedge y \preceq w$ and $x \curlywedge z \preceq w$. From this $y \preceq x \multimap w$ and $z \preceq x \multimap w$, by (I2), which implies that $y \curlyvee z \preceq x \multimap w$. Now, let $u \in y \curlyvee z$. Then $u \preceq x \multimap w$, which implies by item (10) of Lemma 2 that $x \curlywedge u \preceq w$. By Proposition 4, the latter implies that $x \curlywedge (y \curlyvee z) = x \curlywedge u \preceq w$. Therefore, $x \curlywedge (y \curlyvee z) \preceq (x \curlywedge y) \curlyvee (x \curlywedge z)$.

2. By Proposition 13 items (1) and (3), the following holds:

$$\begin{aligned}
&(x \multimap (y \multimap z)) \multimap ((x \multimap y) \multimap (x \multimap z)) = \top \text{ iff} \\
&x \multimap (y \multimap z) \preceq (x \multimap y) \multimap (x \multimap z) \text{ iff} \\
&(x \multimap (y \multimap z)) \curlywedge (x \multimap y) \preceq x \multimap z \text{ iff} \\
&[(x \multimap (y \multimap z)) \curlywedge (x \multimap y)] \curlywedge x \preceq z
\end{aligned}$$

Let $A = x \multimap (y \multimap z)$ and $B = x \multimap y$. Observe that A and B are stable, and we have that $B \curlywedge x \preceq y$, by Proposition 11 (1), and $y \curlywedge (x \curlywedge A) \preceq z$. Indeed, $x \curlywedge A \preceq y \multimap z$, by Proposition 13 (4), hence $y \curlywedge (x \curlywedge A) \preceq z$, by Proposition 13 (3).
Since $B \curlywedge x \preceq y$ and $y \curlywedge (x \curlywedge A) \preceq z$, we get, by Proposition 5, that $(B \curlywedge x) \curlywedge (x \curlywedge A) \preceq y \curlywedge (x \curlywedge A) \preceq z$. Now, note that $x \equiv x \curlywedge x$ and $(B \curlywedge x) \curlywedge (x \curlywedge A) \equiv (A \curlywedge B) \curlywedge x \curlywedge x$, providing that $(A \curlywedge B) \curlywedge x \preceq z$.

3. By Proposition 13 items (1) and (3), the following holds:

$$(x \multimap y) \multimap ((y \multimap z) \multimap (x \curlyvee y) \multimap z)) = \top \text{ iff } A \curlywedge (x \curlyvee y) \preceq z,$$

where $A = (x \multimap y) \curlywedge (y \multimap z)$. Let $w \in A$. By Proposition 8, $A \curlywedge (x \curlyvee y) = w \curlywedge (x \curlyvee y)$, since A is stable. By Proposition 4, $A \curlywedge x = w \curlywedge x$ and $A \curlywedge y = w \curlywedge y$, hence $(A \curlywedge x) \curlyvee (A \curlywedge y) = (w \curlywedge x) \curlyvee (w \curlywedge y)$. By item (1),

$$A \curlywedge (x \curlyvee y) = w \curlywedge (x \curlyvee y) \equiv (w \curlywedge x) \curlyvee (w \curlywedge y) = (A \curlywedge x) \curlyvee (A \curlywedge y).$$

From this, it is required to prove that $(A \curlywedge x) \curlyvee (A \curlywedge y) \preceq z$. By Proposition 3 (9), it is enough to prove that $A \curlywedge x \preceq z$ and $A \curlywedge y \preceq z$.
Since $A \curlywedge y \equiv (x \multimap y) \curlywedge [y \curlywedge (y \multimap z)]$ and $y \curlywedge (y \multimap z) \preceq z$, we get $A \curlywedge y \preceq z$. For the other part, note that $A \curlywedge x \equiv [x \curlywedge (x \multimap y)] \curlywedge (y \multimap z)$ and then

$$A \curlywedge x \equiv [x \curlywedge (x \multimap y)] \curlywedge (y \multimap z) \preceq y \curlywedge (y \multimap z) \preceq z.$$

4. It is an immediate consequence of the previous item (3), and items (1) and (3) of Proposition 13.
5. Assume that $A \curlywedge x \preceq z$ and $A \curlywedge y \preceq z$, and let $w \in A$. Then, $w \curlywedge x \preceq z$ and $w \curlywedge y \preceq z$, which implies that $w \preceq x \multimap z$ and $w \preceq y \multimap z$, by (I2). From this, $w \preceq (x \multimap z) \curlywedge (y \multimap z)$. By item (4) and transitivity, $w \preceq (x \curlyvee y) \multimap z$. Since A is stable, this shows that $A \preceq (x \curlyvee y) \multimap z$. By Proposition 13 (3), $A \curlywedge (x \curlyvee y) \preceq z$.
6. For the proof of $x \curlyvee (y \curlywedge z) \equiv (x \curlyvee y) \curlywedge (x \curlyvee z)$, note that $x \preceq x \curlyvee y$ and $y \curlywedge z \preceq x \curlyvee z$ provide that $x \curlyvee (y \curlywedge z) \preceq (x \curlyvee y) \curlywedge (x \curlyvee z)$. For the other inequality, observe that $x \curlywedge z \preceq x \curlyvee (y \curlywedge z)$, and $y \curlywedge z \preceq x \curlyvee (y \curlywedge z)$. By item (5) (taking $A = \{z\}$) it follows that $(x \curlyvee y) \curlywedge z \preceq x \curlyvee (y \curlywedge z)$. But clearly $(x \curlyvee y) \curlywedge x \preceq x \curlyvee (y \curlywedge z)$, and so $(x \curlyvee y) \curlywedge (x \curlyvee z) \preceq x \curlyvee (y \curlywedge z)$, by item (5) once again.

□

Corollary 2. *Every SIHL is a distributive m-hyperlattice.*

Proof. It follows from items (1) and (6) of Proposition 14. □

Proposition 15. *Let $A, B, C, D \subseteq L$ be stable subsets. Then*

$$A \curlywedge (B \curlyvee C) \equiv (A \curlywedge B) \curlyvee (A \curlywedge C) \text{ and } A \curlyvee (B \curlywedge C) \equiv (A \curlyvee B) \curlywedge (A \curlyvee C).$$

Proof. By an argument similar to the used in Proposition 8 we get

$$A \curlywedge (B \curlyvee C) = a \curlywedge (b \curlyvee c)$$
$$(A \curlywedge B) \curlyvee (A \curlywedge C) = (a \curlywedge b) \curlyvee (a \curlywedge c)$$

for all $a \in A$, $b \in B$ and $c \in C$. Since by Corollary 2 we have $a \curlywedge (b \curlyvee c) \equiv (a \curlywedge b) \curlyvee (a \curlywedge c)$, we get $A \curlywedge (B \curlyvee C) \equiv (A \curlywedge B) \curlyvee (A \curlywedge C)$. The proof for the other equation is similar. □

Now, Proposition 9 will be extended to SIHLs, showing that the denotation, in any concrete SIHL L, of any term in the signature of SIHLs, produces a stable subset of L.

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Definition 14 (S-term)**.** *Let L be a SIHL. We define **S-terms** recursively as follows:*

1. *Every stable subset $A \subseteq L$ is an S-term.*
2. *If A and B are S-terms, then $A \curlywedge B$, $A \curlyvee B$ and $A \multimap B$ are S-terms.*

Proposition 16. *Let $A \subseteq L$ be an S-term. Then A is a stable subset.*

Proof. Just use induction and Propositions 4, 8 and 12. □

Proposition 17. *Let $A, B, C, D \subseteq L$ be stable subsets such that $A \preceq B$ and $C \preceq D$. Then $B \multimap C \preceq A \multimap D$.*

Proof. By stability, we only need to prove that $b \multimap c \preceq a \multimap d$ for some $a \in A$, $b \in B$, $c \in C$ and $d \in D$. By Proposition 13 (3), the latter is equivalent to prove that $(b \multimap c) \curlywedge a \preceq d$. By hypothesis, $a \preceq b$ and $c \preceq d$. By stability and by Propositions 5 and 11 (1), $(b \multimap c) \curlywedge a \preceq (b \multimap c) \curlywedge b \preceq c \preceq d$. This concludes the proof. □

4. The Logic C_ω

Among the most influential contributions of the pioneering work of the Brazilian mathematician Newton da Costa (1929–2024) is the development of the logic C_ω, a system that belongs to his well-known hierarchy of paraconsistent logics C_n (for $1 \leq n \leq \omega$) introduced in [12]. C_ω logic, as well as the other systems C_n, is defined over the signature $\Sigma_\omega = \{\wedge, \vee, \to, \neg\}$. The original idea of da Costa was to consider C_ω as the *syntactic* limit of the hierarchy C_n, for $1 \leq n < \omega$. In fact, the Hilbert calculus for C_ω contains exactly all the axioms that belong to any C_n, for $1 \leq n < \omega$. However, as shown in [13], C_ω it is not the *deductive* limit of these calculi. Despite this, C_ω has several interesting features: it is based on positive intuitionistic logic instead of positive classical logic, as happens with C_n for $n < \omega$. Indeed, while the latter satisfy Peirce's law $((\alpha \to \beta) \to \alpha) \to \alpha$, the former do not (see [14, Theorem 15]). On the other hand, each C_n (for $n < \omega$) is finitely trivializable, while C_ω is not (see [14, Theorem 8]).

Definition 15 (Hilbert calculus for C_ω)**.** *The Hilbert calculus for C_ω over Σ_ω is defined as follows:*

Axiom schemas:

(AX1) $\alpha \to (\beta \to \alpha)$

(AX2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

(AX3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(AX4) $(\alpha \wedge \beta) \to \alpha$

(AX5) $(\alpha \wedge \beta) \to \beta$

(AX6) $\alpha \to (\alpha \vee \beta)$

(AX7) $\beta \to (\alpha \vee \beta)$

(AX8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(EM) $\alpha \vee \neg\alpha$

(cf) $\neg\neg\alpha \to \alpha$

Inference rule:

(MP) $$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

Remark 8. *It is worth noting that (Ax1)-(Ax8) plus MP constitutes a sound and complete Hilbert calculus for positive intuitionistic logic IPL^+, which is semantically characterized by the class of implicative lattices with 1 as the only designated value.*

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5. Sette Hyperalgebras for C_ω

In 1969, Sette and da Costa proposed in [7] the first semantics for C_ω by means of C_ω*-algebras*, based on the notion of C_n-algebras (afterwards called *da Costa algebras*) introduced 3 years before by da Costa in [6]. C_ω-algebras are implicative lattices expanded with an equivalence relation which is congruential with respect to the implicative lattice operations, and with an operator $'$ satisfying suitable properties in order to interpret the paraconsistent negation.

Also in 1969 (but only published in 1977), M. Fidel introduced a novel algebraic-relational non-deterministic semantics for all the calculi C_n (including C_ω), nowadays known as *Fidel structures*, proving for the first time the decidability of da Costa's calculi (see [15]). In 1986, A. Loparić proposed another semantical characterization for C_ω by means of valuation semantics over $\{0,1\}$, also known as *bivaluation semantics*, proving soundness and completeness (see [16]). In the same year, M. Baaz introduced in [17] a sound and complete Kripke-style Semantics for C_ω.

In Chapter 2 of his MSc dissertation from 1971 under supervision of da Costa ([8]), Sette introduced a class of hyperalgebras for C_ω called C_ω*-hyperlattices*. He proved that they correspond with C_ω algebras, inducing therefore a suitable semantics for C_ω.

In what follows, a slightly more general definition of Sette's hyperalgebras will be considered, giving in Theorem 1 a direct proof of soundness and completeness of $\mathcal{C}_\omega$ w.r.t. these hyperalgebras. Recall that $\top = \mathsf{Max}(L)$, thus $w \in \top$ if and only if $z \preceq w$ for every $z \in L$.

Definition 16 (Sette hyperalgebras for C_ω)**.** *A* Sette hyperalgebra for C_ω *(or a SHC$_\omega$) is a hyperalgebra $\mathsf{H} = \langle H, \curlywedge, \curlyvee, \multimap, \div \rangle$ over Σ_ω such that the reduct $\langle H, \curlywedge, \curlyvee, \multimap \rangle$ is a SIHL and the hyperoperator $\div$ satisfies the following properties, for every $x, y, w \in H$:*

(H1) *$y \in \div x$ and $w \in x \curlyvee y$ implies that $w \in \top$;*

(H2) *$y \in \div x$ and $w \in \div y$ implies that $w \preceq x$.*

By Remark 6 it is immediate to see that conditions (H1) and (H2) can be written in a concise way as follows:

(H1') $x \curlyvee \div x \equiv \top$;

(H2') $\div \div x \preceq x$,

for every $x \in H$.

Definition 17 (SHC$_\omega$ semantics)**.** *Let H be a SHC$_\omega$, and let $\Gamma \cup \{\varphi\}$ be a set of formulas over Σ_ω.*

1. *The Nmatrix associated to H is $\mathcal{M}_\mathsf{H} = \langle \mathsf{H}, \top \rangle$.*
2. *We say that φ is a semantical consequence of Γ w.r.t. H, denoted by $\Gamma \models_\mathsf{H} \varphi$, if $\Gamma \models_{\mathcal{M}_\mathsf{H}} \varphi$.*
3. *Let $\mathbb{SHC}_\omega$ be the class of SHC$_\omega$s. Then, φ is a semantical consequence of Γ w.r.t. SHC$_\omega$s, denoted by $\Gamma \models_{\mathbb{SHC}_\omega} \varphi$, if $\Gamma \models_\mathsf{H} \varphi$ for every $\mathsf{H} \in \mathbb{SHC}_\omega$.*

By using the notion of swap structures, in Section 6 it will we obtained, for the first time, a direct proof of soundness and completeness of C_ω w.r.t. Sette hyperalgebras semantics.

Remark 9. *It is worth noting that the original formulation of hyperalgebras for C_ω given in [8, Definition 2.3] considered, besides* **(H2')***, condition* **(H1'')** *$x \curlyvee \div x = \top$. The latter condition is clearly stronger than* **(H1')***. By virtue of Definition 17, in order to validate (EM) w.r.t. $\mathbb{SHC}_\omega$ it suffices to require the weaker condition* **(H1')***, as we did.*

6. Swap Structures for C_ω

With the aim of obtaining a more elucidative (non-deterministic) semantics for the paraconsistent logics known as *logics of formal inconsistency* (LFIs), in [18, Chapter 6] it was introduced a particular

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way to define non-deterministic matrices (or Nmatrices) called *swap structures*. This particular class of Nmatrices can be seen as non-deterministic twist structures (which, in turn, constitute a particular class of logical matrices), see [9]. Since the LFIs studied in [18] are based on classical logic, which is characterized by the two-elements Boolean algebra $\mathbf{2} := \{0,1\}$, the original swap structures were defined over **2**. In this section, swap structures for C_ω will be introduced, showing that they form a particular class of SHC$_\omega$s which characterize C_ω. As one could expect, they are defined over implicative lattices, the algebraic models for IPL^+.

Recall that, given an implicative lattice $\mathcal{A} = \langle A, \wedge, \vee, \to \rangle$ and $a \in A$, then $a \to a$ is the top element of A, which will be denoted by 1. From now on, given $z \in A \times A$, the first and second components of z will be denoted, respectively, by z_1 and z_2. That is, $z = (z_1, z_2)$.

Definition 18 (Swap structures for C$_\omega$)**.** *Let $\mathcal{A} = \langle A, \wedge, \vee, \to \rangle$ be an implicative lattice. Let $S_\mathcal{A} = \{z \in A \times A \ : \ z_1 \vee z_2 = 1\}$. The swap structure for C_ω over $\mathcal{A}$ is the hyperalgebra $\mathsf{S}(\mathcal{A}) = \langle S_\mathcal{A}, \check{\wedge}, \check{\vee}, \check{\to}, \check{\neg} \rangle$ over the signature Σ_ω such that the hyperoperators are defined as follows:*

$$z \check{\wedge} w = \{u \in S_\mathcal{A} \ : \ u_1 = z_1 \wedge w_1\} \qquad z \check{\to} w = \{u \in S_\mathcal{A} \ : \ u_1 = z_1 \to w_1\}$$
$$z \check{\vee} w = \{u \in S_\mathcal{A} \ : \ u_1 = z_1 \vee w_1\} \qquad \check{\neg} z = \{u \in S_\mathcal{A} \ : \ u_1 = z_2 \text{ and } u_2 \leq z_1\}$$

The hyperoperations in $S_\mathcal{A}$ can be described in a more succinct way as follows:

$$z \check{\wedge} w = (z_1 \wedge w_1, \_) \qquad z \check{\to} w = (z_1 \to w_1, \_)$$
$$z \check{\vee} w = (z_1 \vee w_1, \_) \qquad \check{\neg} z = (z_2, \_ \leq z_1)$$

Following the usual definitions for swap structures (see for instance [18, Chapter 6] and [19]), it is possible to associate an Nmatrix to each swap structure in a natural way:

Definition 19. *Let $\mathcal{A}$ be an implicative lattice. The Nmatrix associated to $\mathsf{S}(\mathcal{A})$ is $\mathcal{M}(\mathcal{A}) = \langle S_\mathcal{A}, D_\mathcal{A} \rangle$ where the set of designated truth-values is $D_\mathcal{A} = \{z \in S_\mathcal{A} \ : \ z_1 = 1\}$.*

Proposition 18. *Let $\mathsf{S}(\mathcal{A})$ be the swap structure for C_ω over an implicative lattice $\mathcal{A}$. Then:*

1. *$\mathsf{S}(\mathcal{A})$ is a SHC$_\omega$ which satisfies, for every $z \in S_\mathcal{A}$, condition **(H1″)**: $z \curlyvee \div z = \top$.*
2. *The preorder in $\mathsf{S}(\mathcal{A})$ is given as follows: $z \preceq w$ iff $z_1 \leq w_1$ in $\mathcal{A}$. Hence, $z \equiv w$ iff $z_1 = w_1$. Moreover, $D_\mathcal{A} = \top$.*
3. *$\mathcal{M}(\mathcal{A}) = \mathcal{M}_{\mathsf{S}(\mathcal{A})}$.*

Proof. It is immediate from the definitions and the properties of implicative lattices. Item (2) uses Remark 2, specifically: $z \preceq w$ iff $z \in z \curlywedge w$. $\square$

Definition 20 (Swap structures semantics for C_ω)**.** *Let $\Gamma \cup \{\varphi\}$ be a set of formulas over Σ_ω. Then, φ is a semantical consequence of Γ w.r.t. swap structures, denoted by $\Gamma \models^{SW}_{C_\omega} \varphi$, whenever $\Gamma \models_{\mathcal{M}(\mathcal{A})} \varphi$ for every implicative lattice $\mathcal{A}$.*

In order to prove our main result (Theorem 1 below), we recall here some well-known notions and results concerning (Tarskian) logics.

Given a Tarskian and finitary logic **L** and a set of formulas $\Delta \cup \{\varphi\}$ of **L**, the set Δ is said to be *φ-saturated in* **L** if the following holds: (i) $\Delta \nvdash_\mathbf{L} \varphi$; and (ii) if $\psi \notin \Delta$ then $\Delta, \psi \vdash_\mathbf{L} \varphi$.

It is immediate to see that any φ-saturated set in a Tarskian logic is deductively closed, i.e.: $\psi \in \Delta$ iff $\Delta \vdash_\mathbf{L} \psi$.

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By a classical result proven by Lindenbaum and Łoś, if $\Gamma \cup \{\varphi\}$ is a set of formulas of a Tarskian and finitary logic $\mathbf{L}$ such that $\Gamma \nvdash_{\mathbf{L}} \varphi$, then there exists a φ-saturated set Δ such that $\Gamma \subseteq \Delta$.[2] Since C_ω is a Tarskian and finitary logic, Lindenbaum-Łoś theorem holds for it. We arrive to our main result:

Theorem 1 (Soundness and completeness of C_ω w.r.t. hypertructures)**.**
Let $\Gamma \cup \{\varphi\}$ be a set of formulas over Σ_ω. The following assertions are equivalent:

1. $\Gamma \vdash_{C_\omega} \varphi$;
2. $\Gamma \models_{\mathbb{SHC}_\omega} \varphi$;
3. $\Gamma \models^{SW}_{C_\omega} \varphi$.

Proof.
(1) $\Rightarrow$ (2) (Soundness of C_ω w.r.t. SHC$_\omega$s). Assume that $\Gamma \vdash_{C_\omega} \varphi$. In order to prove that $\Gamma \models_{\mathbb{SHC}_\omega} \varphi$, it is enough to prove the following, for every $\mathsf{H} \in \mathbb{SHC}_\omega$ and every valuation $v : For(\Sigma_\omega) \to H$ over $\mathcal{M}_{\mathsf{H}}$: (i) if φ is an instance of an axiom of C_ω then $v(\varphi) \in \top$; and (ii) if $v(\varphi) \in \top$ and $v(\varphi \to \psi) \in \top$ then $v(\psi) \in \top$. So, let H and v.

Axiom (Ax1): Let $\alpha = \varphi \to (\psi \to \varphi)$ be an instance of (Ax1), and let $x = v(\varphi)$ and $y = v(\psi)$. Then, $v(\alpha) \in x \multimap (y \multimap x) = \top$, by Proposition 11 (4). Using a similar argument combined with Propositions 11, 13 and 14 we prove that if φ is an instance of the other axioms AX2-AX8 then $v(\varphi) \in \top$.

Axiom (EM): Let $\alpha = \varphi \vee \neg\varphi$ be an instance of (EM), and let $x = v(\varphi)$, $y = v(\neg\varphi)$. Then, $y \in \div x$ and so $v(\alpha) \in x \curlyvee y \subseteq x \curlyvee \div x = \top$, by (H1'). This means that $v(\alpha) \in \top$.

Axiom (cf): Let $\alpha = \neg\neg\varphi \to \varphi$ be an instance of (cf), and let $x = v(\varphi)$, $y = v(\neg\varphi)$, $z = v(\neg\neg\varphi)$. Then, $y \in \div x$, $z \in \div y$ and so $z \preceq x$, by (H2). By Proposition 11 (2), $z \multimap x = \top$. But then $v(\alpha) \in z \multimap x = \top$, that is, $v(\alpha) \in \top$.

Finally, in order to prove that trueness is preserved by MP, let $x = v(\alpha)$, $y = v(\beta)$, $z = v(\alpha \to \beta)$, and suppose that $x \in \top$ and $z \in \top$. Since $z \in x \multimap y$ then $x \curlywedge z \preceq y$, by (I1). Now, if $w \in H$ then $w \preceq x$ and $w \preceq z$ (since $x, z \in \top$) and so $w \preceq x \curlywedge z$, by definition of $\curlywedge$. From this, and the fact that $x \curlywedge z \neq \emptyset$, it follows that $w \preceq y$. Therefore, $y \in \top$.

(2) $\Rightarrow$ (3). It is immediate, by Proposition 18, items (1) and (3).

(3) $\Rightarrow$ (1) (Completeness of C_ω w.r.t. swap structures semantics). Suppose that $\Gamma \nvdash_{C_\omega} \varphi$. Then, by Lindenbaum-Łoś result mentioned above, there exists a φ-saturated set Δ in C_ω such that $\Gamma \subseteq \Delta$. Now, define a relation $\simeq_\Delta$ over $For(\Sigma_\omega)$ as follows: $\alpha \simeq_\Delta \beta$ iff $\Delta \vdash_{C_\omega} \alpha \to \beta$ and $\Delta \vdash_{C_\omega} \beta \to \alpha$. Since C_ω contains positive intuitionistic logic (recall Remark 8), it follows that $\simeq_\Delta$ is an equivalence relation. Moreover, it is a congruence w.r.t. the signature $\Sigma_\to = \{\wedge, \vee, \to\}$. Thus, if $A_\Delta = For(\Sigma_\omega)/_{\simeq_\Delta}$ then the following operations over A_Δ are well-defined:

$$[\alpha]_\Delta \wedge [\beta]_\Delta := [\alpha \wedge \beta]_\Delta, \qquad [\alpha]_\Delta \vee [\beta]_\Delta := [\alpha \vee \beta]_\Delta, \qquad \text{and} \qquad [\alpha]_\Delta \to [\beta]_\Delta := [\alpha \to \beta]_\Delta$$

where $[\alpha]_\Delta$ denotes the equivalence class of α w.r.t. $\simeq_\Delta$. Moreover, $\mathcal{A}_\Delta = \langle A_\Delta, \wedge, \vee, \to \rangle$ is an implicative lattice, therefore $1 = [\alpha \to \alpha]_\Delta$ is the top element, for every α. Let $\mathsf{S}(\mathcal{A}_\Delta)$ be the swap structure for C_ω over $\mathcal{A}_\Delta$ with domain $S_{\mathcal{A}_\Delta}$ as given in Definition 18. Let $v_\Delta : For(\Sigma_\omega) \to S_{\mathcal{A}_\Delta}$ be the canonical map given by $v_\Delta(\alpha) = ([\alpha]_\Delta, [\neg\alpha]_\Delta)$ for every α. Observe that $[\alpha]_\Delta \vee [\neg\alpha]_\Delta = [\alpha \vee \neg\alpha]_\Delta = 1$ then v_Δ is a well-defined map. Clearly it is a valuation over $\mathcal{M}(\mathcal{A}_\Delta)$ such that $v_\Delta(\alpha) \in \top$ iff $[\alpha]_\Delta = v_\Delta(\alpha)_1 = 1$ iff $\Delta \vdash_{C_\omega} \alpha$. From this, $v_\Delta(\alpha) \in \top$ for every $\alpha \in \Gamma$, while $v_\Delta(\varphi) \notin \top$, given that $\Delta \nvdash_{C_\omega} \varphi$. This shows that $\Gamma \not\models_{\mathcal{M}(\mathcal{A}_\Delta)} \varphi$ and so $\Gamma \not\models^{SW}_{C_\omega} \varphi$.

This completes the proof. □

7. Conclusion and Final Remarks

In this paper we describe the almost unknown concepts of Morgado hyperlattices and Sette implicative hyperlattices and hyperalgebras for C_ω, proving new properties about them. In particular,

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[2] For a proof of this result see, for instance, ([20], Theorem 22.2) or ([18], Theorem 2.2.6).

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using the notion of swap structures, we obtain a new and direct proof of soundness and completeness of da Costa logic C_ω w.r.t. hyperalgebraic semantics based on Sette hyperalgebras for C_ω.

In [21], R. Cignoli improved a construction of Kalman from 1958, obtaining an adjunction between the category and bounded distributive lattices and Kleene algebras by means of that he called a *Kalman functor*. This technique, based on the notion of twist structures, has been amply studied in the literature, and Kalman functor was adapted to other kinds of algebras (see, for instance, [22] and the references therein). As a future research, we aim to define a Kalman functor from the category of implicative m-hyperlattices to the category of Sette hyperalgebras for C_ω, based on the notion of swap structures. Some first steps to adapt the Kalman functor to the hyperalgebraic setting by means of swap structures have been taken in [9], in the context of LFIs.

Morgado hyperlattices, as well as Sette implicative hyperlattices and hyperalgebras for C_ω, can be an interesting ways to define, with appropriate adaptations, semantics for several non-classical logics. In addition to the applications to logic, the study and further development of hyperlattices of this kind may be of interest to the subject of hyperalgebras in general.

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