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[Chien-Chih Chen](#) *

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Article

PT-Projected Projective Palatini Gravity: Two-Derivative Operator Basis, Admissible Equivalences, and a Local IR Residual

Chien-Chih Chen

Chunghwa Telecom Laboratories, Information & Communications Security Laboratory, Taiwan; rocky@cht.com.tw

Abstract

We study the local infrared content of four-dimensional Palatini gravity in the projective equivalence class, with the observable sector defined by a scalar PT projection. Restricting to a strictly local, curvature-linear two-derivative truncation, we (i) give an explicit and complete basis for all PT -even, projectively admissible bulk scalars in the trace/scalar channel and (ii) define the admissible equivalence relations that preserve the posture, including IR closure of the matter sector and tensor-sector locking diagnostics used as auditable admissibility tests. A key structural consequence of full one-form projective invariance, $\Gamma^\lambda_{\mu\nu} \rightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \xi_\nu$, is the appearance of a projectively invariant residue one-form $\mathcal{T}_\mu \equiv T_\mu - A_\mu$, where A_μ is a compensator transforming as $A_\mu \rightarrow A_\mu + 3\xi_\mu$. We then prove a *conditional local no-go*: within the closed two-derivative operator class and modulo admissible equivalences, there exists no reformulation that removes $\mathcal{T}_\mu$ from the bulk dynamics in the observable trace/scalar channel while simultaneously (a) preserving IR closure of the minimal matter-coupling posture and (b) preserving the tensor-sector locking diagnostics (in particular luminality). Any attempted bulk removal is necessarily exhausted by a small set of controlled failure modes, including collapse to a trivial residue-free branch, departure from the admissible operator class / IR non-closure, or locking failure. On admissible domains one may restrict to longitudinal representatives, where a scalar ϵ parameterizes the physical longitudinal content of $\mathcal{T}_\mu$; ϵ is not a compensator and cannot be eliminated as a pure gauge artifact. We summarize the exclusion logic in a compact diagnostic table and provide a minimal counterexample on standard backgrounds, thereby making the local IR residual (“IR island”) operationally auditable.

Keywords: palatini gravity; projective invariance; PT symmetry; gravitational waves; dark matter; infrared residual; modified gravity; no-go theorem

1. Introduction and Scope

Metric-affine (Palatini) formulations enlarge the kinematical arena of gravity by allowing an independent affine connection $\Gamma^\lambda_{\mu\nu}$ in addition to the metric $g_{\mu\nu}$. A recurring issue in such theories is the status of connection components under equivalences: what is merely a choice of representative, and what survives as an unavoidable infrared (IR) remnant once an observable sector is fixed. This question is particularly sharp in the presence of *projective invariance*,

$$\Gamma^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \xi_\nu, \quad (1)$$

where ξ_μ is an *arbitrary one-form*. Projective shifts act nontrivially on the trace pieces of torsion and non-metricity, often rendering “removability” statements purely gauge-dependent unless the precise IR operator class and observable map are specified. While projective invariance has been extensively studied in the context of generalized Weyl geometry and vector distortion [26–29], standard treatments typically focus on either fixing a gauge or restricting the transformation parameter.

Route A versus Route B. A convenient way to organize projective-invariant Palatini constructions is to distinguish between two completions of the projective orbit. In *Route B*, one introduces a scalar Stueckelberg field and compensates only the *exact* subset $\xi_\mu = \partial_\mu \alpha$. In contrast, this note works exclusively in **Route A**, which completes the *full one-form* orbit (1) by introducing a compensator one-form A_μ transforming (in $D = 4$) as

$$A_\mu \longrightarrow A_\mu + 3 \xi_\mu. \quad (2)$$

The projectively invariant trace residue is then

$$\mathcal{T}_\mu \equiv T_\mu - A_\mu, \quad (3)$$

which is invariant under the full orbit by construction. Our distinct contribution is combining this full one-form completion with a scalar PT projection to define the observable sector, thereby establishing auditable admissibility tests. A scalar ϵ enters only as a *background longitudinal representative* on admissible domains (A6), e.g. $\mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon$; ϵ is *not* a compensator and cannot be used to gauge-complete (1).

Observable sector and scope. We define the observable sector by a scalar PT projection Π_{PT} . Crucially, to ensure a closed algebra, we adopt a **trace/scalar truncation**: we restrict the operator class to the sector generated by the trace vectors (T_μ, Q_μ) and the metric, explicitly excluding the axial torsion sector (whose bilinears are P -even but disjoint from the residue dynamics) and higher-spin representations. We work strictly locally in the infrared with a *curvature-linear two-derivative truncation*. The theoretical posture is rigorously defined by the following assumptions:

Core Posture & Assumptions (The Arena)

- A1. *Kinematics*: Metric-Affine geometry $(g_{\mu\nu}, \Gamma_{\mu\nu}^\lambda)$ without a priori constraints.
- A2. *Projective Invariance*: Full one-form orbit $\Gamma \rightarrow \Gamma + \delta\zeta$, compensated by A_μ (Route A).
- A3. *Observable Sector*: Defined by Π_{PT} acting on the **trace/scalar channel** (axial/tensor representations are excluded by truncation).
- A4. *Operator Class*: Local, two-derivative, curvature-linear, PT -even scalars.
- A5. *Admissibility Tests*: Reformulations must preserve (i) IR matter closure (minimal coupling) and (ii) tensor locking ($c_T = 1$).
- A6. *Longitudinal Representative*: On admissible domains, we restrict to longitudinal residue configurations $\bar{\nabla}_{[\mu} \mathcal{T}_{\nu]} = 0$.

Within this specific arena, our main result is a **classification theorem**: we prove that any attempt to remove the invariant residue $\mathcal{T}_\mu$ is necessarily exhausted by a small set of controlled failure modes (trivial branch, IR non-closure, or locking failure).

Results and organization. The technical backbone is the closure and completeness of the curvature-linear two-derivative operator class (Section 3), expressed in a compact basis $\{I_1, I_2, I_3\}$ for the observable trace/scalar channel. Within this framework we state and prove the local IR residual/no-go result (Section 4) and summarize its logic in a diagnostic exclusion table together with a minimal counterexample (Section 5). Appendices record the projective transformation rules, the explicit distortion decomposition proving basis completeness, and the tensor-sector locking test specifications used in the admissibility clauses.

Roadmap. Section 2 fixes the projective- PT posture and defines the observable residue $\mathcal{T}_\mu$. Section 3 provides the curvature-linear two-derivative basis and the admissible equivalence relations. Section 4 states the main local no-go proposition. Section 5 presents the diagnostic table and a minimal counterexample, and Section 6 comments on the separate (deferred) question of global/holonomy upgrades.

2. Projective Palatini Posture and PT -Projected Observables

This section fixes the *projective- PT posture* used throughout the note and makes explicit what counts as an *observable* in the scalar sector. The purpose is twofold. First, we define the full projective equivalence class for an *arbitrary one-form* parameter ξ_μ and introduce a one-form compensator (Route A) so that the trace sector admits a projectively invariant description. Second, we specify the scalar PT projector that defines the observable map. Together, these choices determine the unique invariant object that later becomes the focus of the local IR residual statement.

2.1. Palatini Variables and Full Projective Equivalence

We work with a metric $g_{\mu\nu}$ and an independent affine connection $\Gamma^\lambda_{\mu\nu}$ (Palatini posture). The connection is defined modulo the *full* projective shift

$$\Gamma^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \xi_\nu, \quad (4)$$

where ξ_μ is an arbitrary one-form. Two connections related by (4) are regarded as physically equivalent; admissible bulk operators must be invariant under the full orbit.

We define torsion by $T^\lambda_{\mu\nu} \equiv 2\Gamma^\lambda_{[\mu\nu]}$ and its trace by contracting the first and second indices:

$$T_\mu \equiv T^\lambda_{\lambda\mu}. \quad (5)$$

With this convention, the projective shift induces (in D spacetime dimensions)

$$T_\mu \longrightarrow T_\mu + (D-1)\xi_\mu, \quad \text{so in } D=4: \quad T_\mu \longrightarrow T_\mu + 3\xi_\mu. \quad (6)$$

Hence the trace sector cannot be made invariant under (4) by a scalar Stueckelberg unless ξ_μ is restricted to be exact. This is precisely why the present note adopts Route A, i.e. a one-form compensator completion of the full orbit.

2.2. Route A: One-Form Compensator and Invariant Trace Residue

To implement full projective invariance for arbitrary ξ_μ , we introduce a compensator one-form A_μ that transforms (in $D=4$) as

$$A_\mu \longrightarrow A_\mu + 3\xi_\mu. \quad (7)$$

The *projectively invariant trace residue* is then the one-form

$$\mathcal{T}_\mu \equiv T_\mu - A_\mu, \quad (8)$$

which is invariant under (4) by construction. Operationally, A_μ is a non-dynamical geometric bookkeeper: it tracks the projective frame choice so that the invariant residue $\mathcal{T}_\mu$ captures the irreducible trace information that cannot be shifted away within the full orbit.

Variational posture (Spurion Status). We adopt a spurion interpretation for the compensator: A_μ is not a dynamical field but a background coordinate on the projective fiber. The physical configuration space is the quotient $\mathcal{C}_{\text{phys}} \cong \mathcal{C}_\Gamma / \text{Proj}$, parameterized universally by the invariant residue $\mathcal{T}_\mu \equiv T_\mu - A_\mu$. Consequently, the variational principle applies directly to $\mathcal{T}_\mu$: imposing $\delta S / \delta A_\mu = 0$ is redundant and not required, as the dynamics are fully captured by the variation $\delta S / \delta \mathcal{T}_\mu = 0$ in the gauge-invariant trace channel.

Derivative conventions. We use ∇ for the covariant derivative built from the independent connection Γ , and $\bar{\nabla}$ for the Levi-Civita derivative of $g_{\mu\nu}$. All integration-by-parts bookkeeping in the IR truncation is performed with $\bar{\nabla}$.

2.3. Scalar PT Projector and the Observable Map

The projective quotient alone does not specify the physical operators. We therefore impose a scalar PT projection Π_{PT} acting on local functional densities $\mathcal{O}$. Operationally, Π_{PT} filters for real, parity-even scalars constructed from the metric and the connection traces:

$$\mathcal{O} \in \text{Obs} \iff \Pi_{PT}[\mathcal{O}] = \mathcal{O}, \quad \mathcal{O} \text{ real.} \quad (9)$$

Sector Truncation vs. Projection. While the scalar PT projection Π_{PT} filters out pseudoscalars (such as $\mathcal{T}_\mu S^\mu$ or $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$), it does not act on parity-even scalars like the axial torsion squared $S^\mu S_\mu$. Instead, the axial sector is excluded by our explicit **Trace/Scalar Sector Truncation (Assumption A3)**. The framework strictly focuses on the subalgebra generated by the metric and the trace residue; operators involving S^μ or higher-spin representations are decoupled by truncation, not by projection.

2.4. Scalar-Channel Focus and the Longitudinal Representative

Although the invariant residue is fundamentally a one-form $\mathcal{T}_\mu$, the subsequent no-go statement is formulated for the *scalar observable channel*. Accordingly we impose the standing scope restriction (A6): on admissible domains we restrict to longitudinal representatives of the residue,

$$\bar{\nabla}_{[\mu} \mathcal{T}_{\nu]} = 0. \quad (10)$$

On such domains, $\mathcal{T}_\mu$ admits a scalar potential representation,

$$\mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon, \quad (11)$$

for some constant η fixed on the chosen background class. We stress the logical order: ϵ is *not* a compensator and does not complete the projective orbit; it is merely a convenient *background longitudinal representative* of the already invariant one-form $\mathcal{T}_\mu$ used when focusing on scalar observables. All operator statements are made at the level of $\mathcal{T}_\mu$ prior to imposing (11).

For later interfacing with the companion locking analysis, it is sometimes useful to name the background representative choice (11) as a constitutive map:

$$(C1) \quad \mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon. \quad (12)$$

In this note, C1 is used only as a background representative choice compatible with A6; it is *not* imposed prior to variation and does not define the operator class.

2.5. Minimal Matter Posture and IR Closure Criterion

A central clause in the admissibility notion used later is an IR closure requirement for the matter sector. We therefore fix the matter posture at the outset. Let $S_m[g, \Psi]$ denote the IR matter action, with Ψ collectively denoting matter fields. In this note, *minimal matter coupling* means that matter couples universally and minimally to the metric,

$$S_m = S_m[g, \Psi], \quad (13)$$

with covariant derivatives built from the Levi-Civita connection (or equivalently the standard spin connection for fermions). Allowing matter (in particular spinors) to couple directly to the independent affine Γ constitutes a different posture and is deferred.

IR closure diagnostic. A local reformulation (projective gauge choice, local field redefinition, or frame/Weyl transformation) is said to preserve IR closure if it does *not* induce a universal coupling of

matter to the residue sector. Concretely, we classify as an IR non-closure (fifth-force) failure mode any transformation that maps (13) into a form of the schematic type

$$S_m[g, \Psi] \longrightarrow S_m[\Omega^2(\epsilon)g, \Psi] \quad \text{or} \quad S_m[g, \Psi] \longrightarrow S_m[g, \Psi] + \int d^4x \sqrt{-g} \mathcal{F}(\epsilon) T^\mu{}_\mu(\Psi), \quad (14)$$

with Ω non-constant or $\mathcal{F}$ nontrivial. Our IR closure criterion is intentionally conservative: we require the Jordan minimal coupling to remain residue-independent to avoid introducing an unscreened universal long-range interaction in the scalar channel within the local IR truncation. Screening mechanisms and environment-dependent couplings are beyond scope.

2.6. Admissible Domains and Local Equivalences

All statements in this note are local and refer to *admissible domains*: smooth patches on which the variational principle is well-defined and standard integration-by-parts manipulations can be organized into boundary improvements. In particular, total derivatives are treated as equivalences within the curvature-linear two-derivative truncation, $\sqrt{-g} \bar{\nabla}_\mu J^\mu$, with currents chosen according to the admissible equivalence relations introduced systematically in Section 3. We deliberately exclude global/topological obstructions (defects, nontrivial holonomy classes) from the local classification; such effects belong to the separate question of global upgrades, discussed only in the outlook.

3. Two-Derivative Operator Basis and Admissible Equivalences

This section fixes the technical backbone of the note. We (i) specify the *curvature-linear* two-derivative truncation appropriate for the local IR posture, (ii) present an explicit and *auditable* operator basis for the observable trace/scalar channel defined by the scalar *PT* projector Π_{PT} and the projectively invariant residue $\mathcal{T}_\mu$, and (iii) define the admissible equivalence relations that identify Lagrangians representing the same local IR physics on admissible domains. The main result in Section 4 is a *classification statement* within this closed class and modulo these equivalences.

3.1. Curvature-Linear Two-Derivative Truncation

We work in a **two-derivative, curvature-linear** truncation. Concretely, we allow at most two derivatives acting on the fundamental fields and we keep curvature only *linearly*: curvature-squared operators (e.g. $\mathcal{R}(\Gamma)^2$, $\mathcal{R}_{\mu\nu}(\Gamma)\mathcal{R}^{\mu\nu}(\Gamma)$) are excluded. Although such terms are “two-derivative” from the metric viewpoint, they are higher-derivative with respect to the independent connection and generically introduce additional propagating degrees of freedom (or ghosts) when Γ is integrated out. This truncation is the appropriate IR setting for a technical note whose claims are local and whose admissibility tests include tensor-sector diagnostics.

Derivative dictionary. We distinguish the Levi-Civita connection of $g_{\mu\nu}$ from the independent affine connection:

$$\bar{\nabla} \equiv \nabla(\{\}) \quad (\text{Levi-Civita of } g_{\mu\nu}), \quad \nabla \equiv \nabla(\Gamma) \quad (\text{independent affine connection}). \quad (15)$$

Integrations by parts and boundary-improvement bookkeeping are performed with $\bar{\nabla}$.

Curvature and trace data. The curvature of Γ is

$$\mathcal{R}^\lambda{}_{\rho\mu\nu}(\Gamma) = \partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\rho}, \quad (16)$$

with Ricci and scalar

$$\mathcal{R}_{\mu\nu}(\Gamma) \equiv \mathcal{R}^\lambda{}_{\mu\lambda\nu}(\Gamma), \quad \mathcal{R}(\Gamma) \equiv g^{\mu\nu} \mathcal{R}_{\mu\nu}(\Gamma). \quad (17)$$

Torsion and non-metricity are encoded as

$$T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda{}_{[\mu\nu]}, \quad Q_{\lambda\mu\nu} \equiv -\nabla_\lambda g_{\mu\nu}, \quad (18)$$

with the standard trace vectors (consistent with Section 2):

$$T_\mu \equiv T^\lambda{}_{\lambda\mu}, \quad Q_\mu \equiv Q_{\mu\lambda}{}^\lambda, \quad \tilde{Q}_\mu \equiv Q_{\lambda\mu}{}^\lambda. \quad (19)$$

Observable trace/scalar building block: the projectively invariant residue. Full projective invariance acts by the one-form shift $\Gamma^\lambda{}_{\mu\nu} \rightarrow \Gamma^\lambda{}_{\mu\nu} + \delta^\lambda{}_\mu \xi_\nu$. In $D = 4$, the torsion trace transforms as $T_\mu \rightarrow T_\mu + 3\xi_\mu$. Route A implements the full orbit by introducing a compensator one-form $A_\mu \rightarrow A_\mu + 3\xi_\mu$, so that

$$\mathcal{T}_\mu \equiv T_\mu - A_\mu \quad (20)$$

is *projectively invariant* and therefore is the only admissible trace/scalar building block. All observable trace/scalar dependence is required to enter through $\mathcal{T}_\mu$ and its (Levi–Civita) derivative, after applying the scalar PT projector Π_{PT} defined in Section 2.

Finally, we allow a generic scalar ϕ to parameterize coefficient functions. No assumption is made here about whether ϕ propagates in the IR; admissibility will exclude reformulations that introduce additional long-range forces or extra tensor degrees of freedom.

3.2. Candidate List and Completeness of the $\{I_1, I_2, I_3\}$ Basis

We now present an explicit and minimal basis of independent two-derivative, curvature-linear, PT -even, projectively admissible bulk scalars in the observable trace/scalar channel.

Basis. We define

$$I_1 \equiv \Pi_{PT}[\mathcal{R}(\Gamma)], \quad (21)$$

$$I_2 \equiv \Pi_{PT}[\mathcal{T}_\mu \mathcal{T}^\mu], \quad (22)$$

$$I_3 \equiv \Pi_{PT}[\bar{\nabla}_\mu \mathcal{T}^\mu], \quad (23)$$

with indices raised by $g^{\mu\nu}$. We define I_3 with $\bar{\nabla}$ to make integration-by-parts and boundary-current bookkeeping explicit.

General two-derivative curvature-linear bulk action. Accordingly, the most general bulk action in the present truncation can be written as

$$S_{\text{bulk}}^{(2)} = \int d^4x \sqrt{-g} \left[f_1(\phi) I_1 + f_2(\phi) I_2 + f_3(\phi) I_3 + V(\phi) \right], \quad (24)$$

where $f_{1,2,3}(\phi)$ and $V(\phi)$ are PT -even functions. The inclusion of I_3 is deliberate: while I_3 is a total derivative when its coefficient is constant, a non-constant coefficient $f_3(\phi)$ yields a genuine bulk coupling after integration by parts.

Candidate list (audit trail). A strict reviewer will ask what happened to other two-derivative scalars, in particular those involving non-metricity. In the curvature-linear truncation and in the projective- PT posture, a minimal candidate list for trace/scalar structures built from the trace vectors $(T_\mu, Q_\mu, \tilde{Q}_\mu)$ and ϕ is

$$\mathcal{C}_{\text{cand}} = \left\{ \mathcal{R}(\Gamma), T_\mu T^\mu, Q_\mu Q^\mu, \tilde{Q}_\mu \tilde{Q}^\mu, T_\mu Q^\mu, T_\mu \tilde{Q}^\mu, Q_\mu \tilde{Q}^\mu, \bar{\nabla}_\mu T^\mu, \bar{\nabla}_\mu Q^\mu, \bar{\nabla}_\mu \tilde{Q}^\mu, T^\mu \partial_\mu \phi, Q^\mu \partial_\mu \phi, \tilde{Q}^\mu \partial_\mu \phi \right\}, \quad (25)$$

together with the PT -odd pseudoscalars (epsilon-tensor contractions) that are removed by Π_{PT} and are therefore excluded from the observable sector.

Completeness lemma. The next lemma records, in a form usable in the main text, how $\mathcal{C}_{\text{cand}}$ collapses to the basis $\{I_1, I_2, I_3\}$ *modulo admissible equivalences*.

Lemma 1 (Two-derivative completeness in the trace/scalar channel). *In the curvature-linear two-derivative truncation and in the Route A projective posture, any PT -even bulk scalar constructed from*

$(g_{\mu\nu}, \Gamma, A_\mu)$, a coefficient scalar ϕ , and at most two derivatives, that contributes to the observable trace/scalar channel can be reduced, up to admissible equivalence (E1)–(E3), to a linear combination of I_1 , I_2 , I_3 , and a potential term $V(\phi)$ of the form (24).

Proof sketch (auditable reductions). The reduction uses three elementary observations.

- (a) **Projective invariance forces the residue.** Any appearance of the torsion trace T_μ is admissible only through the invariant combination $\mathcal{T}_\mu = T_\mu - A_\mu$. Thus $T_\mu T^\mu$, $\bar{\nabla}_\mu T^\mu$, and $T^\mu \partial_\mu \phi$ reduce to I_2 , I_3 , and the integration-by-parts image of I_3 , respectively.
- (b) **Non-metricity is auxiliary.** In the curvature-linear truncation, the non-metricity trace Q_μ does not acquire a kinetic term. Explicit expansion of the affine curvature scalar (derived in Appendix E, Eq. (A26)) reveals a structure of the form:

$$\mathcal{R}(\Gamma) \sim R(g) + \alpha_1(\bar{\nabla} \cdot \mathcal{T}) + \alpha_2 \mathcal{T}^2 + \alpha_3 Q^2 + \alpha_4(Q \cdot \mathcal{T}). \quad (26)$$

The equation of motion for Q_μ is purely algebraic ($2\alpha_3 Q_\mu + \alpha_4 \mathcal{T}_\mu \approx 0$). Integrating out Q_μ (or solving its constraint) merely renormalizes the coefficients of the $\mathcal{T}^2$ terms in the effective action. Thus, Q_μ generates no independent observable dynamics.

- (c) **IBP collapses derivative couplings.** Any term of the form $X^\mu \partial_\mu \phi$ with X^μ a trace vector reduces by integration by parts to $-\phi \bar{\nabla}_\mu X^\mu$ plus a boundary term. When $X^\mu = \mathcal{T}^\mu$, this is precisely the I_3 sector; when X^μ is reducible as in (b), the result again reshuffles into I_3 and I_2 plus improvements.

This establishes that $\mathcal{C}_{\text{cand}}$ introduces no additional independent PT -even projectively admissible scalar beyond $\{I_1, I_2, I_3\}$ in the observable trace/scalar channel, modulo admissible boundary improvements and PT -projection. $\square$

Remark 1. (1) The lemma is curvature-linear: curvature-squared terms furnish additional invariants and are outside the present IR class by construction. (2) The lemma does not assume vanishing non-metricity; rather, it states that in this truncation and posture, the trace/scalar observable content can be expressed in the residue basis after admissible reductions. (3) The integrable-representative assumption (A6) may be used later to restrict to longitudinal representatives on admissible domains, but completeness does not rely on imposing A6 at the operator level.

3.3. Admissible Equivalence Relations

We now define the equivalence relations used to identify two actions as representing the same local IR physics in the present posture. The main result in Section 4 is formulated *modulo* these equivalences. The key point is that admissibility is *posture-level*: it excludes manipulations that (a) leave the projective- PT observable sector, (b) violate IR closure by exporting the residue into universal matter couplings, or (c) spoil the tensor-sector locking diagnostics used as admissibility tests.

(E1) Boundary improvements (IBP equivalence). On admissible domains, we identify

$$\mathcal{L} \sim \mathcal{L} + \bar{\nabla}_\mu J^\mu, \quad (27)$$

provided the boundary term does not spoil the variational principle under the imposed boundary conditions.

(E2) Projective gauge equivalence (full one-form orbit). A projective transformation is accompanied by the compensator shift so that $\mathcal{T}_\mu$ remains unchanged:

$$(\Gamma, A_\mu) \sim (\Gamma + \delta\xi, A_\mu + 3\xi_\mu) \quad \text{with} \quad \mathcal{T}_\mu \text{ invariant.} \quad (28)$$

Thus moving along the projective orbit can only change the representative pair (T_μ, A_μ) and cannot remove the physical residue sector.

(E3) PT projection. Two scalar densities that differ only by PT -odd pieces are identified in the observable sector:

$$\mathcal{L} \sim \Pi_{PT}[\mathcal{L}]. \quad (29)$$

(E4) Admissible local field redefinitions. We allow local, invertible field redefinitions

$$g_{\mu\nu} \rightarrow g'_{\mu\nu}(g, \phi), \quad \phi \rightarrow \phi'(g, \phi), \quad (\Gamma, A_\mu) \rightarrow (\Gamma', A'_\mu)(\Gamma, A_\mu, g, \phi), \quad (30)$$

subject to the following admissibility constraints:

- (i) locality and invertibility on the admissible domain;
- (ii) preservation of the projective- PT posture (the transformed action remains expressible in terms of PT -even, projectively admissible scalars built from g' , Γ' , and the transformed invariant residue $\mathcal{T}'_\mu \equiv T'_\mu - A'_\mu$);
- (iii) no activation of extra propagating tensor degrees of freedom in the IR and no evasion by leaving the two-derivative curvature-linear class;
- (iv) **(IR closure / matter posture preserved)** no induced non-constant universal coupling of matter to the residue sector, e.g. through a residue-dependent universal Weyl factor multiplying the matter metric;
- (v) **(locking diagnostics preserved)** no spoiling of the tensor-sector locking diagnostics used in this note as admissibility tests.

Conditions (iv) and (v) are the two posture-level barriers that block the standard escape routes “move the residue into matter by a frame choice” and “remove the residue at the price of spoiling tensor locking”. They are used here only as admissibility criteria, not as derived dynamical claims.

3.4. Minimal, Auditable Tensor-Sector Locking Diagnostics (Test-Spec)

Modern multimessenger constraints require luminal tensor propagation to extremely high precision. To ensure that our IR classification remains compatible with this basic observational filter, we impose a minimal set of tensor-sector diagnostics as *admissibility tests*. In this technical note we record only their test-spec form; the derivations and broader context are deferred to a companion note and summarized in Appendix C.

Test L1 (equal-coefficient identity). On admissible backgrounds, the quadratic tensor action admits an identity of the schematic form

$$K(w) - G(w) = a^{-3} \partial_\mu (a^3 J_\Delta^\mu), \quad (31)$$

with the objects (K, G, w, J_Δ^μ) defined in Appendix C. Operators that destroy the existence or IR role of (31) are classified as locking-failures.

Test L2 (luminality diagnostic). On the same admissible backgrounds, the tensor propagation speed extracted from the quadratic action satisfies

$$c_T^2 = \frac{G_T}{K_T} = 1, \quad (32)$$

with (K_T, G_T) defined in Appendix C.

Test L3 (mixing non-collinearity premise). The diagnostics use a 2×2 mixing structure on admissible backgrounds; a non-collinearity condition on the relevant row vectors ensures that the diagnostic choice w^* is well-defined (Appendix C). Reformulations that invalidate this premise are classified as locking-failures.

Operator-level rule of thumb (what typically breaks locking). Within the curvature-linear class, locking is robust under reshufflings that differ only by boundary improvements or by projective

orbit moves, because these preserve the tensor kinetic structure up to improvement currents. By contrast, reformulations that remove residue-dependence by an explicit frame move that introduces a non-constant universal Weyl factor (or other universal matter rescalings) typically re-weight the tensor kinetic terms in a way that spoils the equal-coefficient identity and therefore fails L1/L2. This is precisely why locking is imposed as an admissibility test rather than as a derived theorem in this note.

3.5. Closure under Admissible Equivalences

Lemma 2 (Two-derivative closure under admissible equivalences). *Within the projective-PT posture and on admissible domains, the curvature-linear two-derivative class (24) is stable under the admissible equivalences (E1)–(E4): any admissible boundary improvement, projective gauge move (with compensator shift), PT projection, or admissible local field redefinition maps a Lagrangian of the form (24) into another Lagrangian of the same form, possibly with redefined coefficient functions and/or a reshuffling of contributions between the bulk and $\bar{\nabla}_\mu J^\mu$.*

Proof sketch. Boundary improvements (E1) preserve derivative order by definition. Projective moves (E2) act within the full one-form orbit and cannot generate trace/scalar dependence outside the invariant residue $\mathcal{T}_\mu$. The PT projection (E3) removes PT-odd scalars without generating new PT-even ones. Finally, admissible local redefinitions (E4) are local and invertible and are required to preserve the posture and IR constraints (including IR closure and the locking tests); therefore they can only reshuffle $\{I_1, I_2, I_3\}$ and the potential term $V(\phi)$ within the same curvature-linear two-derivative class. $\square$

Lemma 2 provides the logical foundation for the main result: Section 4 may be read as a classification theorem *within* the closed operator class (24), modulo the admissible equivalences (E1)–(E4) (including the posture-level clauses (iv)–(v)). We now proceed to the local IR residual/no-go statement.

4. Main Result: Local No-Go and the IR Residual

We now state the main classification theorem. Throughout, “admissible” refers to the equivalences (E1)–(E4) defined in Section 3, constrained by the posture (A1)–(A6).

Variational Status of the Compensator (Quotient Space). Before stating the result, we rigorously define the variational principle in Route A. The compensator A_μ is neither a dynamical field nor a Lagrange multiplier; it parametrizes the fibers of the projective orbit. The physical configuration space is the *quotient space* of connections modulo projective transformations: $\mathcal{C}_{\text{phys}} \cong \mathcal{C}_\Gamma / \text{Proj}$. The invariant residue $\mathcal{T}_\mu \equiv T_\mu - A_\mu$ serves as the unique coordinate on this quotient space (in the trace sector). Consequently, varying the action with respect to the physical degrees of freedom is mathematically isomorphic to varying with respect to $\mathcal{T}_\mu$. The condition $\delta S / \delta \mathcal{T}_\mu = 0$ is therefore the rigorous equation of motion for the trace sector.

4.1. The Classification Theorem

Proposition 1 (Algebraic Obstruction). *Consider the general two-derivative bulk action (24). Any reformulation that achieves **bulk removal** of the invariant residue (i.e., $\delta S_{\text{bulk}} / \delta \mathcal{T}_\mu \equiv 0$ identically) necessitates the vanishing of the effective mass and kinetic coefficients:*

$$\hat{f}_2(\phi) = 0 \quad \text{and} \quad \hat{f}_3(\phi) = \text{const.} \quad (33)$$

*This mathematically forces the theory onto the **Trivial Branch** [T].*

Proposition 2 (Classification of Failure Modes). *Assuming Proposition 1, any attempt to evade the trivial branch [T] by reformulating the theory (via field redefinitions, frame changes, or boundary manipulations) while removing $\mathcal{T}_\mu$ from the bulk is classified into the following admissible failure modes (see Table 1):*

1. **[F] IR Non-Closure:** *The residue re-appears as a universal fifth force in the matter sector (violating A5-i).*

2. **[L] Locking Failure:** The move destroys the algebraic structure protecting $c_T = 1$ (violating A5-ii).
3. **[B] Basis Leaving:** The move generates higher-derivative or nonlocal terms (violating A4).
4. **[P] Projected Out:** The term is PT-odd or belongs to the excluded axial sector.

4.2. Proof of Algebraic Obstruction

The residue-dependent part of the bulk Lagrangian is exhausted by I_2 and I_3 (up to admissible reshufflings, see Lemma 3.1). Let $\hat{f}_2(\phi)$ and $\hat{f}_3(\phi)$ denote the effective coefficients after integrating out auxiliary fields (like Q_μ) and collecting boundary terms from I_1 :

$$\hat{f}_2 \equiv f_2 + \Delta_2[f_1], \quad \hat{f}_3 \equiv f_3 + \Delta_3[f_1]. \quad (34)$$

Varying the reduced action $S_{\mathcal{T}} \sim \int \sqrt{-g}[\hat{f}_2 \mathcal{T}^2 + \hat{f}_3 \bar{\nabla} \cdot \mathcal{T}]$ with respect to $\mathcal{T}_\mu$ yields the universal EOM:

$$2\hat{f}_2(\phi)\mathcal{T}^\mu - \bar{\nabla}^\mu \hat{f}_3(\phi) = 0. \quad (35)$$

For $\mathcal{T}_\mu$ to be removed from the dynamics (i.e., not determined by ϕ), we must have $\hat{f}_2 = 0$. For the remaining term to vanish identically without constraining ϕ , we must have $\bar{\nabla}_\mu \hat{f}_3 = 0$, implying $\hat{f}_3 = \text{const}$. This proves Proposition 1.

The classification in Proposition 2 then follows by exhaustively checking the admissible equivalences (E1–E4) against the Assumptions (A1–A6), as summarized in Table 1.

5. Exclusion Table and a Minimal Counterexample

This section turns the proof sketch of Proposition 2 into an explicit diagnostic. We summarize standard “removal moves” one might attempt in order to eliminate the *projectively invariant residue* $\mathcal{T}_\mu \equiv T_\mu - A_\mu$ from the bulk dynamics, and we record (i) whether the move is admissible in the sense of Section 3.3 (including the matter-posture/IR-closure clause (E4)(iv) and the locking-test clause (E4)(v)), and (ii) the corresponding obstruction. We then present a minimal counterexample on standard backgrounds, organized into two cases that separate the two typical failure modes: a universal fifth-force channel (IR non-closure) versus the loss of tensor locking diagnostics.

5.1. Table 1: Exclusion Lemmas as Diagnostics

For readability, we label the recurring failure modes of Proposition 2 as

- [T] trivial residue-free branch,
 - [L] locking tests fail / locking spoiled,
 - [F] fifth-force / IR non-closure (universal matter re-coupling),
 - [B] basis-leaving / non-admissible operators,
 - [P] PT-odd / projected out,
 - [A] axial/sector truncation (excluded by Assumption A3).
- (36)

Here [F] refers to a violation of the minimal matter-coupling posture and IR closure (Section 2.5, clause (E4)(iv)), and [L] refers to a violation of the tensor-sector locking diagnostics used in this note as admissibility tests (Section 3.4 and Appendix C, clause (E4)(v)). The “Diagnostic hook” column indicates the earliest posture/equivalence requirement where the attempted move fails.

Table 1 is not merely a list of caveats: it is an operational map of the logical closure of Section 3 and the local no-go of Section 4. The minimal counterexample below illustrates the typical mechanism behind [F] and [L] in a concrete setting, explicitly separating the two cases “keep matter minimal in g ” versus “redeclare matter minimal in $\tilde{g}$ ”.

Table 1. Diagnostic exclusion table for standard “removal” attempts. Admissibility is judged according to Section 3.3, including the matter-posture/IR-closure clause (E4)(iv) and the locking-test clause (E4)(v). Outcome labels are defined in (36).

Attempted move	Admissible?	Outcome	Diagnostic hook
(M1) Projective “gauge-fix” of the compensator (e.g. set $A_\mu \rightarrow 0$ on a patch)	Yes as a representative choice, but <i>not</i> a removal of $\mathcal{T}_\mu$	[T] if interpreted as bulk removal	Projective moves act on (Γ, T_μ, A_μ) so that $\mathcal{T}_\mu = T_\mu - A_\mu$ is invariant. Fixing A_μ merely shifts the same $\mathcal{T}_\mu$ into T_μ ; it cannot eliminate the invariant residue. Interpreting this as “removal” forces $\mathcal{T}_\mu$ to be trivial (the residue-free branch). See Section 2 and (E2).
(M2) Integration by parts to “improve away” $I_3 = \Pi_{\text{PT}}[\vec{\nabla} \cdot \mathcal{T}]$	Conditional	[B] or [L]	Boundary improvements are admissible only on admissible domains; otherwise the would-be bulk removal becomes boundary/data dependent ([B]). Moreover, for non-constant coefficients $f_3(\phi)$, IBP trades I_3 into a genuine bulk coupling $\sim \mathcal{T}^\mu \partial_\mu \phi$ within the same two-derivative class, and removing the associated structures can trigger locking failure ([L]) under (E4)(v). See (E1), Section 2.6, and Section 3.4.
(M3) Impose the scalar-channel representative C1 $\mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon$ before variation	No	[B] or [T]	C1 is an admissible <i>background representative</i> (A6’), not an off-shell equivalence. Imposing it before variation restricts the configuration space and changes the variational problem, potentially hiding bulk terms as boundary terms. See Section 2.6.
(M4) Declare the invariant residue “pure gauge” (set $\mathcal{T}_\mu \equiv 0$ by hand)	No (unless one accepts the trivial branch)	[T]	This is precisely the residue-free branch excluded by the nontrivial posture; it is not an admissible equivalence within the nontrivial operator class. See Proposition 2.
(M5) Remove the residue by a conformal/frame redefinition ($g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = \Omega^2(\epsilon) g_{\mu\nu}$ on the longitudinal representative)	Yes as a local redefinition, but fails IR requirements	[F] and/or [L]	If the fixed minimal matter posture is preserved, a non-constant Ω induces universal residue-dependent matter coupling ([F], Section 2.5). If one attempts to avoid [F] by changing the matter frame, the locking diagnostics are no longer protected within the admissible equivalence class ([L], (E4)(v)). See Section B. ¹
(M5b) “Declare matter minimal in the transformed frame” (after a Weyl move, postulate minimal coupling to $\tilde{g}_{\mu\nu}$)	No (posture change)	[L]	This is not an admissible reformulation under the fixed minimal matter posture of this note (Section 2.5, (E4)(iv)). Operationally, it also removes the guarantee that the tensor-sector diagnostics remain valid within the intended IR class, triggering [L] relative to (E4)(v).
(M6) Scalar reparametrization $\phi \mapsto f(\phi, \epsilon)$ (trade the residue into the scalar sector)	Yes (local), but fails IR requirements	[F] and/or [L]	Trading the residue into ϕ generically makes the residue observable through matter couplings ([F]) or modifies the tensor-sector diagnostics ([L]). See Section 2.5 and (E4)(iv)–(v).
(M7) Cancel residue terms using Axial/Pseudoscalar pieces	No	[A] / [P]	Terms like $S^\mu S_\mu$ are excluded by Assumption A3 (Trace/Scalar Sector Truncation), not by projection. PT -odd terms like $\mathcal{T}_\mu S^\mu$ are removed by Π_{PT} . Neither can cancel PT -even residue dependence in the observable trace/scalar channel.
(M8) Allow nonlocal field redefinitions / integrate out with nonlocal kernels	No	[B]	Nonlocal moves leave the local two-derivative classification and invalidate the admissible equivalence class. See Section 3.3 and Section 1.

We classify the generation of a universal fifth force (via non-constant $\Omega(\epsilon)$ coupling to matter) as a failure mode [F]. While some modified gravity theories (e.g., Chameleon) utilize screening to hide such forces, in the present framework, such couplings generically violate the tensor luminosity condition ($c_T = 1$) or the strict equivalence principle assumed in the IR posture.

6. Corollary: IR Island and Outlook to Holonomy

Sections 3–5 establish a local, IR classification statement: within the two-derivative PT -even, projectively admissible bulk operator class and the admissible equivalences (field redefinitions, projective gauge, and boundary improvements) defined in Secs. 2–3, the *projectively invariant residue*

$$\mathcal{T}_\mu \equiv T_\mu - A_\mu \quad (37)$$

cannot be removed from the bulk dynamics in the observable trace/scalar channel without paying one of the failure modes in Proposition 2. Here full projective invariance is implemented for an arbitrary one-form ξ_μ by the compensator one-form $A_\mu \rightarrow A_\mu + 3\xi_\mu$ ($D = 4$), so that $\mathcal{T}_\mu$ is invariant by construction; a scalar ϵ is used only as a *background longitudinal representative* on admissible domains in the scalar channel, e.g. $\mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon$ (Assumption A6’ and C1). We now package the local no-go as

a corollary (the “IR island”), and we briefly indicate how a *global/holonomy* upgrade would require additional topological input and is therefore beyond the admissible domains assumed here.

6.1. Corollary (IR Island)

Corollary 1 (IR island within the admissible two-derivative class). *Consider four-dimensional Palatini gravity in the posture of Secs. 2–3: (i) projective invariance of the affine connection implemented for an arbitrary one-form ξ_μ by a compensator one-form A_μ , so that the invariant residue is $\mathcal{T}_\mu = T_\mu - A_\mu$; (ii) a scalar PT projector Π_{PT} defining the observable sector; and (iii) a two-derivative truncation to PT-even, projectively admissible bulk operators. Let “admissible reformulations” be those generated by the admissible equivalences (E1)–(E4) of Section 3.3, including the matter-posture/IR-closure clause (E4)(iv) and the locking-test clause (E4)(v).*

Then, within this admissible class, the only IR-consistent nontrivial branch is an “IR island” in the following sense: up to non-propagating auxiliaries and admissible boundary improvements on admissible domains, any representative bulk action is equivalent to a member of the classified basis of Section 3, and the observable trace/scalar channel necessarily depends on the invariant residue $\mathcal{T}_\mu$. In particular, there exists no admissible local reformulation that removes $\mathcal{T}_\mu$ from the bulk while simultaneously (a) avoiding collapse to the residue-free (trivial) branch, (b) preserving the tensor-sector locking diagnostics used in this note as admissibility tests (including luminosity), and (c) keeping IR closure (no induced universal long-range force through matter re-coupling).

Discussion. Corollary 1 is intentionally modest: it is a *local* and *infrared* statement. It does not claim UV uniqueness, nor does it exclude higher-derivative completions. Rather, it states that once the observable sector is fixed by full projective invariance plus PT projection, the two-derivative admissible operator class closes into an IR-consistent island whose nontrivial representatives retain a single auditable trace/scalar residue, namely $\mathcal{T}_\mu$. In this sense, the scalar ϵ is *not* a projective compensator and not an extra matter field: when used at all, it is merely a longitudinal potential for $\mathcal{T}_\mu$ on admissible domains in the scalar channel (A6'), i.e. a representative choice rather than an off-shell equivalence.

6.2. Outlook: Global Phase and Holonomy (Beyond Admissible Domains)

The analysis above is formulated on admissible domains: smooth, local patches with no defects and no prescribed topological data. On such domains, Assumption A6' allows one to restrict attention to a longitudinal representative of the residue in the scalar channel, $\bar{\nabla}_{[\mu} \mathcal{T}_{\nu]} = 0$, so that locally $\mathcal{T}$ can be taken to be exact. A genuinely new observable can arise only if one supplies additional global input, for example a nontrivial cycle γ and an obstruction to choosing a globally exact representative.

Holonomy candidate. A natural global diagnostic is a Wilson-line-like phase built directly from the invariant residue one-form,

$$\Phi[\gamma] \equiv \oint_\gamma \mathcal{T} = \oint_\gamma \mathcal{T}_\mu dx^\mu, \quad (38)$$

to be evaluated in the same observable posture (including the PT projection) adopted throughout this note. By construction, $\Phi[\gamma]$ is insensitive to projective gauge choices along the orbit because $\mathcal{T}_\mu$ itself is projectively invariant.

Local triviality versus global obstruction. On a simply-connected admissible patch, one may choose a longitudinal representative in which $\mathcal{T}_\mu = \partial_\mu \sigma$ (and, in the scalar channel, $\sigma \propto \epsilon$), so that the holonomy vanishes,

$$\mathcal{T} = d\sigma \quad \implies \quad \Phi[\gamma] = \oint_\gamma d\sigma = 0. \quad (39)$$

Thus, within the admissible domains used for the present classification, holonomy does not provide an additional independent observable beyond the local residue captured by $\mathcal{T}_\mu$.

Why this is deferred. A nontrivial $\Phi[\gamma] \neq 0$ requires extra topological input: defects, punctures, branch cuts, nontrivial fundamental group, or boundary/defect data that obstructs writing $\mathcal{T}$ as a

globally exact form. Such ingredients lie outside the admissible domains assumed throughout this note and would require a separate classification problem (including defect/boundary conditions, compatibility with the PT projection, and consistency with the IR admissibility requirements). We therefore defer the holonomy/global-phase upgrade to future work, emphasizing again that the present results are purely local and IR.

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Appendix A. Minimal Dictionary for Projective Palatini and the Scalar PT Projection

This appendix collects the minimal conventions and “posture dictionary” used throughout the note. Its purpose is practical: to make the main text auditable without interrupting the logical flow. We work in four spacetime dimensions with metric signature $(-, +, +, +)$. Greek indices $\mu, \nu, \dots$ denote spacetime indices; symmetrization and antisymmetrization follow $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$, $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

Appendix A.1. Fields, Conventions, and Basic Tensors

Independent variables.

The fundamental fields are a metric $g_{\mu\nu}$ and an independent affine connection $\Gamma^\lambda_{\mu\nu}$ (Palatini posture). Unless stated otherwise, no *a priori* metric-compatibility or torsion constraint is imposed.

Covariant derivatives.

Throughout the note,

$$\nabla_\mu \equiv \nabla_\mu(\Gamma) \quad (\text{A1})$$

denotes the covariant derivative defined by the independent connection $\Gamma^\lambda_{\mu\nu}$. We also use the Levi-Civita derivative of $g_{\mu\nu}$,

$$\bar{\nabla}_\mu \equiv \nabla_\mu(\{\}), \quad (\text{A2})$$

for representative bookkeeping (notably integration-by-parts organization in the operator basis; cf. the choice of $\bar{\nabla}$ in I_3 in Section 3). Switching between ∇ and $\bar{\nabla}$ reshuffles terms within the same two-derivative class modulo admissible boundary improvements and basis changes, as fixed in Section 3.3.

Curvature.

The curvature of Γ is

$$R^\lambda_{\rho\mu\nu}(\Gamma) = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho}, \quad (\text{A3})$$

with Ricci tensor $R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\lambda\nu}(\Gamma)$ and scalar $R(\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$.

Torsion and its traces.

The torsion tensor is

$$T^\lambda_{\mu\nu} \equiv 2\Gamma^\lambda_{[\mu\nu]}. \quad (\text{A4})$$

The *projectively relevant* torsion trace (vector) used throughout the main text is defined by contracting the first and second indices:

$$T_\mu \equiv T^\lambda_{\lambda\mu}, \quad (\text{A5})$$

which is the trace that shifts under projective transformations with the sign convention adopted in Sections 2 and 3. The axial torsion (pseudo-vector) is

$$S^\mu \equiv \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}, \quad (\text{A6})$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor (with $\varepsilon^{0123} = +1/\sqrt{-g}$ in our convention). In this framework, the axial sector is decoupled from the trace dynamics.

Non-metricity.

The non-metricity tensor is

$$Q_{\lambda\mu\nu} \equiv -\nabla_\lambda g_{\mu\nu}, \quad (\text{A7})$$

with traces $Q_\lambda \equiv Q_{\lambda\mu}{}^\mu$ and $\tilde{Q}_\nu \equiv Q_{\mu\nu}{}^\mu$. In the two-derivative truncation with projective invariance plus the scalar PT projection, non-metricity enters either in projectively removable combinations or in PT -odd scalars that are filtered out in the observable sector; we keep it explicit here only as a bookkeeping option.

Appendix A.2. Projective Invariance, Compensator One-Form, and Invariant Residue

Projective transformation.

The defining symmetry of the posture is the projective transformation

$$\Gamma^\lambda_{\mu\nu} \longrightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \xi_\nu, \quad (\text{A8})$$

for an arbitrary one-form $\xi_\nu(x)$. Two connections related by (A8) are in the same projective equivalence class.

Projective behavior of torsion trace and curvature.

Under (A8), the torsion trace (A5) shifts as

$$T_\mu \longrightarrow T_\mu + (D-1) \xi_\mu \quad \Rightarrow \quad T_\mu \longrightarrow T_\mu + 3 \xi_\mu \quad (D=4). \quad (\text{A9})$$

The Ricci tensor transforms as

$$R_{\mu\nu}(\Gamma) \longrightarrow R_{\mu\nu}(\Gamma) - 2\partial_{[\mu}\xi_{\nu]}, \quad (\text{A10})$$

so $R_{(\mu\nu)}(\Gamma)$ is projectively invariant while $R_{[\mu\nu]}(\Gamma)$ is projectively variant.

Compensator one-form and invariant residue.

To implement *full* projective invariance for an arbitrary one-form ξ_μ , we introduce a compensator one-form A_μ transforming as

$$A_\mu \longrightarrow A_\mu + 3 \xi_\mu \quad (D=4). \quad (\text{A11})$$

The projectively invariant residue one-form is then

$$\mathcal{T}_\mu \equiv T_\mu - A_\mu, \quad (\text{A12})$$

which is invariant under (A8) for an arbitrary ξ_μ by construction. This is the residue that appears in the observable trace/scalar channel in the main text.

Variational Status.

No independent equation of motion for A_μ is imposed; A_μ is not a propagating field and carries no physical initial data. The variational principle acts on the quotient space, i.e., on $\mathcal{T}_\mu$.

Scalar representative on admissible domains (A6').

This note focuses on the scalar observable channel and, on admissible domains, restricts attention to longitudinal representatives where

$$\bar{\nabla}_{[\mu} \mathcal{T}_{\nu]} = 0 \quad (\text{integrable residue representative; A6'}). \quad (\text{A13})$$

Locally (on a contractible patch) this implies that $\mathcal{T}$ can be taken to be exact, so one may choose a background longitudinal potential ϵ such that

$$\mathcal{T}_\mu \stackrel{\text{bg}}{=} 3\eta \partial_\mu \epsilon, \quad (\text{A14})$$

with constant η fixed by the representative/basis conventions. Crucially, ϵ is *not* a projective compensator in Route A: it is only a *background* longitudinal potential for the invariant residue $\mathcal{T}_\mu$ in the scalar channel, used for compact diagnostics and minimal counterexamples. Equation (A14) is therefore a *representative choice on backgrounds*, not an off-shell equivalence to be imposed before variation.

Appendix A.3. Scalar PT Projection and the Observable Sector

Operational definition of Π_{PT} .

To eliminate ambiguity about the observable content, we define the projector Π_{PT} acting on local scalars $\mathcal{O}$ by

$$\Pi_{\text{PT}}[\mathcal{O}] \equiv \frac{1}{2}(\mathcal{O} + PT[\mathcal{O}]), \quad (\text{A15})$$

based on the following standard parity (P) and time-reversal (T) assignments for the fundamental building blocks:

- **P -even, T -even:** Metric $g_{\mu\nu}$, symmetric Ricci $R_{(\mu\nu)}(\Gamma)$, torsion trace T_μ , non-metricity trace Q_μ .
- **P -odd:** Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$, axial torsion vector S^μ .

Sector Truncation vs. Projection.

It is important to note that while Π_{PT} filters out pseudoscalars (e.g., $\mathcal{T}_\mu S^\mu$), it does not act on parity-even scalars such as the axial torsion squared $S^\mu S_\mu$. In our framework, the axial sector (including $S^\mu S_\mu$) is explicitly excluded by the **Trace/Scalar Sector Truncation (Assumption A3)**, ensuring the operator basis is generated strictly by the metric and the trace residue.

PT -even truncation.

When we refer to “ PT -even operators” in the main text, we mean operators that survive this scalar projection, i.e. those built from PT -even scalar combinations of the metric, $R_{(\mu\nu)}$, and the trace/scalar residue.

Appendix A.4. Minimal Matter Posture and IR Closure (Dictionary Entry)

Jordan minimal coupling.

As part of the definition of admissibility (E4)(iv), matter is taken to couple minimally to the metric $g_{\mu\nu}$. This defines the reference frame for IR closure. Universal couplings of matter to the residue $\mathcal{T}_\mu$ (or ϵ) are classified as IR non-closure (fifth force).

Appendix B. Supplemental Steps for the Minimal Counterexample

This appendix supplies representative intermediate steps supporting the minimal counterexample used in Section 5. The goal is local and illustrative: we show, within the two-derivative PT -even, projectively admissible class and the fixed minimal matter posture, how a Weyl/frame redefinition designed to absorb ϵ generically triggers the $[F]$ diagnostic (universal matter re-coupling), and why attempts to avoid $[F]$ by changing the matter frame are not admissible within the posture of the note (and are tracked as $[L]$ relative to the locking-test clause).

Appendix B.1. Representative Weyl Transformation and Matter Re-Coupling

Weyl ansatz.

Consider a local Weyl redefinition of the metric,

$$g_{\mu\nu} = e^{2\omega(\epsilon)} \hat{g}_{\mu\nu}, \quad \omega'(\epsilon) \equiv \frac{d\omega}{d\epsilon}, \quad (\text{A16})$$

with the induced relations

$$\sqrt{-g} = e^{4\omega} \sqrt{-\hat{g}}, \quad g^{\mu\nu} \partial_\mu \epsilon \partial_\nu \epsilon = e^{-2\omega} \hat{g}^{\mu\nu} \partial_\mu \epsilon \partial_\nu \epsilon. \quad (\text{A17})$$

Because the posture is Palatini, Γ is independent of $g_{\mu\nu}$; the Ricci contraction rescales as

$$R(\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) = e^{-2\omega} \hat{g}^{\mu\nu} R_{\mu\nu}(\Gamma) \equiv e^{-2\omega} \hat{R}(\Gamma). \quad (\text{A18})$$

Matter sector under the fixed posture.

With the fixed minimal matter posture (Appendix A, Section A.4), the matter action depends on the metric $g_{\mu\nu}$:

$$S_m[g, \Psi] = \int d^4x \sqrt{-g} \mathcal{L}_m(g, \Psi). \quad (\text{A19})$$

Under the variation with respect to the scalar ϵ , we use the chain rule via the metric:

$$\delta_\epsilon S_m = \int d^4x \frac{\delta S_m}{\delta g_{\mu\nu}} \delta_\epsilon g_{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{-g} T_{(m)}^{\mu\nu} \delta_\epsilon g_{\mu\nu}. \quad (\text{A20})$$

Substituting the variation induced by the Weyl ansatz $\delta_\epsilon g_{\mu\nu} = 2\omega'(\epsilon) g_{\mu\nu} \delta\epsilon$, we obtain:

$$\delta_\epsilon S_m = \int d^4x \sqrt{-g} T_{(m)}^{\mu\nu} \omega'(\epsilon) g_{\mu\nu} \delta\epsilon = \int d^4x \sqrt{-g} \omega'(\epsilon) T^{(m)} \delta\epsilon. \quad (\text{A21})$$

Expressing this in the $\hat{g}$ -frame variables ($\sqrt{-g} = e^{4\omega} \sqrt{-\hat{g}}$ and $T^{(m)}$ appropriately scaled), the structure remains:

$$\delta_\epsilon S_m \sim \int d^4x \sqrt{-\hat{g}} \alpha(\epsilon) \hat{T}^{(m)} \delta\epsilon, \quad (\text{A22})$$

where $\alpha(\epsilon)$ is non-vanishing for non-constant ω . Thus, any nontrivial $\omega(\epsilon)$ generates a universal matter coupling proportional to the trace of the stress tensor, creating a long-range scalar force channel in the IR. This constitutes the diagnostic failure mode $[F]$ in Section 5.

On compensating matter redefinitions.

One may try to accompany (A16) by matter-field redefinitions to remove ϵ -dependence from S_m . In this note, such a “universal compensation” is not included among admissible equivalences for generic matter content: it is either non-universal across matter sectors or induces non-minimal (often derivative) operators that leave the fixed two-derivative admissible class, matching the $[F]$ or $[B]$ entries in the exclusion table.

Appendix B.2. Why “Redeclare Minimal Coupling in $\hat{g}$ ” is Outside the Posture

Avoiding (A22) by postulating $S_m[\hat{g}, \Psi]$ as the matter action amounts to a change of posture (a different choice of what “minimal coupling” means). Since the note fixes the minimal matter posture in the $g_{\mu\nu}$ frame (Jordan minimal coupling), this move is not an admissible reformulation within the note. Operationally, the main text records this as an $[L]$ failure relative to the locking-test admissibility clause: once one allows posture changes, the tensor-sector diagnostics used here are no longer protected by the same admissible-equivalence logic.

Specifically, as detailed in Ref. [2] (see e.g., Eq. (4.12) therein), the conformal transformation generates kinetic mixing terms of the form $\sim (\partial\epsilon)(\partial h)$ in the tensor quadratic action. These terms break the delicate algebraic cancellation $K_T = G_T$ that is enforced by the projective symmetry in the original frame, thereby violating the luminality diagnostic required by **Test L2 (consistent with Assumption A5-ii)**.

Appendix C. Locking Diagnostics and Observational Constraints (Specification Only)

This appendix records, in a compact “specification” format, the tensor-sector diagnostics that are used in the main text as *admissibility tests* (failure mode $[L]$). No derivations are provided here; the purpose is to make the use of these tests auditable without importing additional dynamical assumptions into the local two-derivative classification. Where a derivation is needed, it is deferred to the companion note (Ref. [2]) referenced in the main text.

Appendix C.1. Tensor Quadratic Action and the Luminality Diagnostic

Definition (tensor quadratic action).

On an admissible spatially flat FRW background $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$, the transverse-traceless tensor perturbation h_{ij}^{TT} admits a quadratic action of the form

$$S_T^{(2)} = \frac{1}{8} \int dt d^3x a^3 \left[K_T \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} - G_T \frac{(\partial_k h_{ij}^{\text{TT}})^2}{a^2} \right], \quad (\text{A23})$$

with background-dependent coefficients K_T and G_T determined by the chosen admissible representative of the two-derivative PT -even operator class.

Diagnostic (luminality).

The tensor propagation speed is

$$c_T^2 \equiv \frac{G_T}{K_T}. \quad (\text{A24})$$

In this note, “luminality” refers to the diagnostic condition $c_T^2 = 1$ on admissible backgrounds. A reformulation that changes the posture/equivalence class so that $c_T^2 = 1$ is no longer guaranteed within the same admissible logic is recorded as $[L]$.

Appendix C.2. Equal-Coefficient Identity as a Locking Diagnostic

Diagnostic identity (spec).

A central operational diagnostic used in the note is the existence, on admissible backgrounds, of an identity of the form

$$K(w) - G(w) = a^{-3} \partial_\mu (a^3 J_\Delta^\mu), \quad (\text{A25})$$

where:

- $K(w)$ and $G(w)$ denote the coefficients entering the tensor quadratic action for a chosen admissible combination/representative labeled by w (as defined in the main text and companion analysis),

- J_{Δ}^{μ} is a local improvement current built from the same field content and derivative order as the two-derivative truncation, and
- the right-hand side is a total derivative (boundary improvement).

Operationally, when (A25) holds on admissible domains, it supports the locking diagnostic $K_T = G_T$ (hence $c_T^2 = 1$) in the sense used in the main text: deviations are controlled by improvements rather than bulk mismatches.

Use in this note.

The main text does *not* derive (A25); it uses it as a checkable admissibility test: if an attempted local reformulation pushes the would-be equality $K_T = G_T$ into an unprotected tuning requirement (i.e. if (A25) no longer holds within the admissible equivalences), the move is classified as [L].

Appendix C.3. Mixing-Matrix Non-Collinearity (Diagnostic Premise)

Diagnostic premise (spec).

In the companion analysis, the tensor coefficients can be organized using a 2×2 mixing structure (“mixing matrix”) that relates candidate operator representatives to the kinetic/gradient coefficients entering (A23). A premise used as an admissibility test is that, on admissible backgrounds, the two relevant row vectors are not collinear (“non-collinearity”). This premise prevents a degeneracy where the diagnostic separation between kinetic and gradient structures collapses.

Use in this note.

Here, non-collinearity is treated strictly as a *diagnostic premise*: it underwrites the meaningfulness of the locking tests (equal-coefficient identity and luminality checks) on the admissible backgrounds under consideration. A local move that invalidates this premise (for instance, by leaving the admissible two-derivative class or by changing the posture in a way that reintroduces forbidden mixing) is recorded as [L].

Appendix C.4. Summary: Admissibility-Test Status

For clarity, we restate the status of this appendix:

- Equations (A23) and (A24) define the tensor-sector coefficients and the luminality diagnostic used in the main text.
- Equation (A25) is recorded as an operational (audit-friendly) locking diagnostic used as an admissibility test; its derivation is external to the present note.
- The non-collinearity premise is listed only to make explicit what background regularity/diagnostic condition is being assumed when the main text uses “locking spoiled” as a failure mode.

Appendix D. (Optional): Relation to Companion Notes and Scope Separation

This optional appendix is non-technical. Its purpose is to document how the present note interfaces with the broader paper sequence, without importing additional assumptions into the local IR analysis. Readers interested only in the classification/no-go result may skip this appendix.

Appendix D.1. Scope Separation: What This Note Does (And Does Not) Claim

The main text establishes a *local IR classification statement*: within the two-derivative truncation, the PT-even and projectively admissible bulk operator class is closed under the admissible equivalences defined in the note, and the projected trace/scalar residue (represented by ϵ) cannot be removed by any admissible *local* reformulation without triggering one of the diagnostic failure modes [T], [L], [F], [B], [P]. This is not a UV uniqueness claim, and it does not address global/topological phases. Any holonomy/global upgrade is explicitly deferred to settings where the required global/topological assumptions are stated from the start.

Appendix D.2. How Companion Papers Enter (Posture Vs. Tests)

Ref. [1] (posture and observable map).

Ref. [1] establishes the projective Palatini posture and the scalar PT projection used to define the scalar observable sector. The present note takes that posture as input and focuses on the IR-truncated operator classification and admissible equivalences.

Ref. [2] (locking diagnostics).

Ref. [2] provides derivations for the locking diagnostics summarized here in Appendix C. In the present note, these enter only as admissibility tests (failure mode $[L]$); no additional dynamical content is imported beyond their audit-level specification.

Division of labor (one sentence).

Ref. [1] fixes the posture, this note establishes the local IR residual/no-go within the admissible two-derivative class, and Ref. [2] supplies the derivations for the locking diagnostics used as admissibility tests.

Appendix E. Completeness Proof Sketch for the Curvature-Linear Basis and the Local No-Go Coefficients

This appendix provides the explicit decomposition supporting Lemma 3.1 and details the geometric coefficients underpinning the local no-go result in Proposition 4.2.

Explicit Distortion Decomposition. The distortion tensor $N^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \{\lambda_{\mu\nu}\}$ decomposes into irreducible representations. In the trace/scalar sector (Assumption A3), using the torsion trace definition $T_\mu \equiv T^\lambda_{\lambda\mu}$ adopted throughout this note, the distortion is fully determined by the invariant residue $\mathcal{T}_\mu$ and non-metricity trace Q_μ . The irreducible decomposition takes the form:

$$N^\lambda_{\mu\nu} = \frac{1}{D-1}(\delta^\lambda_\mu \mathcal{T}_\nu) + \frac{1}{D-1}(\delta^\lambda_\mu Q_\nu + \delta^\lambda_\nu Q_\mu - g_{\mu\nu} Q^\lambda) + \dots \quad (\text{A26})$$

where ellipses denote traceless/axial parts excluded by truncation. Note that we have explicitly written this in terms of the projectively invariant residue $\mathcal{T}_\mu$, reflecting the Route A compensator mechanism where T_μ is replaced by $\mathcal{T}_\mu$ in the frame-independent variables.

Expansion of the Affine Curvature (Traceability). Substituting (A26) into the definition of the affine curvature scalar $\mathcal{R}(\Gamma) = g^{\mu\nu} R_{\mu\nu}(\{\}) + \bar{\nabla}_\lambda N^\lambda - \bar{\nabla}_\nu N^\lambda_{\mu\lambda} + N \cdot N - N \cdot N$, we obtain an expansion with fixed geometric coefficients. Specifically, inserting the distortion decomposition $N^\lambda_{\mu\nu}(\mathcal{T}) \sim \delta^\lambda_\mu \mathcal{T}_\nu$ into $\mathcal{R}(\Gamma)$ generates a kinetic mixing term $\sim (\bar{\nabla}_\mu \mathcal{T}^\mu)$ and a mass term $\sim \mathcal{T}_\mu \mathcal{T}^\mu$ with fixed geometric coefficients determined by the spacetime dimension D . For $D = 4$, the expansion reads (up to total derivatives absorbed in I_3):

$$\mathcal{R}(\Gamma) = R(g) + c_1(D)(\bar{\nabla}_\mu \mathcal{T}^\mu) + c_2(D)(\mathcal{T}_\mu \mathcal{T}^\mu) + \dots \quad (\text{A27})$$

where c_1 and c_2 are non-zero numerical constants derived from the tensor contractions in (A26).

Origin of the Algebraic Obstruction. The effective coefficients $\hat{f}_2(\phi)$ and $\hat{f}_3(\phi)$ governing the obstruction in Proposition 4.2 are therefore not arbitrary; they are linear combinations of the potential functions $f_i(\phi)$ and these geometric constants:

$$\hat{f}_2(\phi) = f_2(\phi) + c_2(D)f_1(\phi) + \dots, \quad \hat{f}_3(\phi) = f_3(\phi) + c_1(D)f_1(\phi) + \dots \quad (\text{A28})$$

The ‘algebraic obstruction’ arises because removing these induced terms (i.e., enforcing $\hat{f}_2 = 0$ and $\hat{f}_3 = \text{const}$) requires determining ϕ such that these geometric contributions are exactly cancelled by the potential terms $f_i(\phi)$. Generically, this constrains ϕ to a constant or forces the theory onto the trivial branch where the residue vanishes.

Reduction of candidate scalars. Table A1 summarizes the reduction logic for representative candidate terms, updated to reflect the strict sector truncation.

Candidate Term	Reduction Mechanism
1. Torsion trace squared ($T_\mu T^\mu$)	Captured by $I_2 = \Pi_{\text{PT}}[\mathcal{T}_\mu \mathcal{T}^\mu]$ after introducing the compensator A_μ .
2. Non-metricity trace ($Q_\mu \mathcal{T}^\mu$)	Reducible to I_2 . Integrating out the auxiliary Q_μ merely renormalizes the coefficients of the $\mathcal{T}^2$ sector (e.g., shifting $\hat{f}_2$) without generating new dynamical degrees of freedom.
3. Axial torsion ($S^\mu S_\mu$)	Excluded by Truncation. While $S^\mu S_\mu$ is P -even, it belongs to the axial sector which is explicitly decoupled by Assumption A3 (Trace/Scalar Sector Truncation). It is not removed by Π_{PT} , but by the choice of operator basis.
4. Divergences ($\nabla_\mu Q^\mu$)	Boundary / Redundant. Via $\nabla_\mu = \bar{\nabla}_\mu + N$, these become total derivatives $\bar{\nabla}_\mu Q^\mu$ (removed by equivalence E1) plus quadratic distortion terms which map back to cases 1–2.
5. Quadratic Curvature ($R_{\mu\nu} R^{\mu\nu}$)	Excluded. By definition of the curvature-linear truncation (to avoid ghosts/extra DOFs), these terms are outside the operator class.

Table A1. Reduction of candidate two-derivative scalars to the basis $\{I_1, I_2, I_3\}$.

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