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*Article*

# Anomalies in the Heisenberg Uncertainty Relation for Special Deformed Boson Algebra

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**Abstract**

The Heisenberg uncertainty relation, concerning the product of the standard deviations of the position and momentum observables, is studied for a quantum oscillator which is characterized by deformed boson algebra. Special forms of the deformed number operator are found to induce the following anomalies: the ground state or any deformed coherent state of the oscillator under study exhibits minimum uncertainty, as the canonical boson algebra provides, but the product of the corresponding standard deviations is less than the canonical lower bound, $\hbar/2$. Above all, each standard deviation and, consequently, the uncertainty become arbitrarily small for the ground state, other states of the Fock basis, and deformed coherent states with arbitrarily small magnitude of the label, if the parameters defining the special deformed number operator are properly chosen.

**Keywords:** quantum oscillator; Heisenberg uncertainty relation; minimum uncertainty; deformed boson algebra; deformed coherent states; Fock basis

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## 1. Introduction

The Heisenberg uncertainty relation quantifies the minimum precision which characterizes the simultaneous measurements of two non-commuting observables [1–5]. Particularly, the product of the standard deviations of the two observables exhibits a lower bound which is determined by the commutation relation of the observables and the state of the quantum system. For example, the product of the standard deviations of the position and momentum observables of the canonical quantum oscillator exhibits the lower bound $(\hbar/2)$, which is realized by the ground state and any coherent state, where $\hbar$ is the reduced Planck constant. Thus, the value $(\hbar/2)$ quantifies the minimum uncertainty which characterizes the simultaneous measurement of the position and momentum observables.

The generalization of the Heisenberg uncertainty principle has been a leading topic of investigation for decades. See Refs. [6–11], and references therein. In Ref. [7], the generalization is performed by considering the commutator operator of position and momentum to be a quadratic polynomial of the momentum operator; this approach suggests the existence of both a minimum measureble lenght and maximum measureble momentum. In Ref. [6], the effects of the generalized commutator operator are studied on the energy eigenvalues and eigenvectors, coherent and squeezed states of a quantum oscillator, by expanding the position and momentum operators around the corresponding low-energy quantities.

The deformation of the canonical commutation relations of quantum operators is a basal approaches in quantum theory. In Refs. [12–17], the boson algebra of a quantum oscillator is generalized by defining the commutation or the $q$-commutation relations via the deformed number operator. The deformed boson algebra alters the canonical properties of a quantum oscillator and increases or decreases the value of the minimum uncertainty according to the properties of the deformed number operator [12,13].

Recently, a novel class of coherent states has been introduced in Ref. [17] via deformed boson algebra and a special deformation law of the number operator which is obtained from a generalization of the factorial; in the same reference, the corresponding Heisenberg uncertainty relation and the minimum uncertainty are also studied. Many forms of generalized coherent states have been introduced in the last decades [18–27]. Recently, coherent states of a quantum harmonic oscillator has been generalized in Ref. [28], by adopting positive Fox $H$ functions as auxiliary functions [29].

As a continuation of the above-described scenario, here, we consider a quantum oscillator which is characterized by special deformed boson algebra such that the generalized coherent states introduced in Ref. [28] are the eigenstates of the corresponding deformed annihilation operator. We aim to study if and how the Heisenberg uncertainty relation, the canonical value of the minimum uncertainty and the states realizing the minimum uncertainty are changed by the special deformed boson algebra and the special deformation law of the number operator.

The paper is organized as follows. Section 2 is devoted to the description of deformed boson algebra, deformed coherent states and the properties of generalized coherent states. The Heisenberg uncertainty relation is studied in Section 3 for the simultaneous measurement of the position and momentum observables by considering the deformed boson algebra and special deformation law of the number operator. In the same Section the uncertainty relation is evaluated for the states of the Fock basis and the corresponding deformed coherent states. Summary of the results and conclusions are reported in Section 4. Details of the calculations are provided in appendix.

## 2. Deformed Boson Algebra

For the sake of clarity, we report below the main properties of a deformed quantum oscillator which is described by deformed boson algebra, following Ref. [17], mainly. The deformed boson algebra is defined via the set $\{I, [\hat{N}], \hat{A}, \hat{A}^{\dagger}\}$ of operators acting over the Hilbert space of the quantum oscillator, and the corresponding commutation relations. The above-mentioned set is composed by the identity operator $\hat{I}$, the deformed number operator $[\hat{N}]$, the deformed annihilation operator $\hat{A}$, and the deformed creation operator $\hat{A}^{\dagger}$. The deformed number operator is defined via the deformed creation and annihilation operators as follows:

$$[\hat{N}] = \hat{A}^{\dagger}\hat{A}, \tag{1}$$

over the Hilbert space of the quantum oscillator. The Hamiltonian $\hat{H}$ of the deformed quantum oscillator is defined in terms of the deformed annihilation and creation operators as below,

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{A}^{\dagger}\hat{A} + \hat{A}\hat{A}^{\dagger}\right), \tag{2}$$

where $\omega$ is the frequency of the oscillator, or, equivalently, the mode. The above-reported deformed operators fulfill the following commutation relations over the Hilbert space of the quantum oscillator:

$$\left[\hat{A}, \hat{A}^{\dagger}\right] = [\hat{N}+1] - [\hat{N}], \tag{3}$$

$$\left[\hat{H}, \hat{A}^{\dagger}\right] = \frac{\hbar\omega}{2}\left([\hat{N}+1] - [\hat{N}] + 1\right)\hat{A}^{\dagger}, \tag{4}$$

$$[\hat{H}, \hat{A}] = -\frac{\hbar\omega}{2}\left([\hat{N}+1] - [\hat{N}] + 1\right)\hat{A}. \tag{5}$$

Briefly, the ground state $|0\rangle$ of the quantum oscillator is the eigenstate of the deformed annihilation operator $\hat{A}$ with vanishing eigenvalue,

$$\hat{A}|0\rangle = 0. \tag{6}$$

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The other elements of the Fock basis, $\{|n\rangle, \ \forall\, n \in \mathbb{N}_0\}$, are defined from the ground state, $|0\rangle$, via the deformed creation operator $\hat{A}^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{[n]!}}\left(\hat{A}^\dagger\right)^n |0\rangle, \tag{7}$$

for every $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of natural numbers, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ and $\mathbb{N} \equiv \{1, 2, \ldots\}$. The deformed annihilation and creation operators act over the Fock basis as below:

$$\hat{A}|n\rangle = \sqrt{[n]}|n-1\rangle, \tag{8}$$

for every $n \in \mathbb{N}$; and

$$\hat{A}^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle, \tag{9}$$

for every $n \in \mathbb{N}_0$. The elements of the Fock basis are eigenstates of the Hamiltonian operator, $\hat{H}|n\rangle = E_n|n\rangle$, and the eigenvalues are given by the form below,

$$E_n = \frac{\hbar\omega}{2}([n+1] + [n]), \tag{10}$$

for every $n \in \mathbb{N}_0$. The box function $[n]$ is a non-negative arithmetic function of the natural variable $n$ which describes the deformed number operator $[\hat{N}]$ over the Fock basis,

$$[\hat{N}]|n\rangle = [n]|n\rangle, \tag{11}$$

for every $n \in \mathbb{N}_0$, with $[0] = 0$. The factorial of the box function, $[n]!$, appearing in Eq. (7), exhibits the properties of the canonical factorial arithmetic function,

$$[n+1]! = [n+1][n]!, \tag{12}$$

for every $n \in \mathbb{N}_0$, with $[0]! = 1$.

The position operator $\hat{q}$ and the momentum operator $\hat{p}$ of the deformed quantum oscillator are defined in terms of the deformed annihilation and creation operators as below,

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{A} + \hat{A}^\dagger\right), \tag{13}$$

$$\hat{p} = \imath\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{A}^\dagger - \hat{A}\right), \tag{14}$$

and fulfill the following commutation relation:

$$[\hat{q}, \hat{p}] = \imath\hbar\left([\hat{N}+1] - [\hat{N}]\right), \tag{15}$$

where $\imath$ is the imaginary unit.

### 2.1. Deformed Coherent States

The deformed coherent state $|z\rangle$ is the eigenstate of the deformed annihilation operator $\hat{A}$ with eigenvalue $z$, i.e., $\hat{A}|z\rangle = z|z\rangle$, for every $z \in \mathbb{C}$, where $\mathbb{C}$ is the set of the complex numbers. The deformed coherent state is expressed in the Fock basis as below [17],

$$|z\rangle = \frac{1}{\sqrt{\mathcal{N}\left(|z|^2\right)}} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle, \tag{16}$$

for every $z \in \mathbb{C} \setminus \{0\}$, while the ground state $|0\rangle$ is provided by the vanishing eigenvalue, $z = 0$. The normalization factor $\mathcal{N}\left(|z|^2\right)$ is given by the following deformed exponential function:

$$\mathcal{N}(u) = \sum_{n=0}^{+\infty} \frac{u^n}{[n]!}, \tag{17}$$

for every $u > 0$. Naturally, the normalization factor $\mathcal{N}\left(|z|^2\right)$ is required to be positive and finite for every $z \in \mathbb{C} \setminus \{0\}$. This condition is realized for every $z \in \mathbb{C} \setminus \{0\}$ iff the deformed factorial $[n]!$ fulfills the following asymptotic relation [28]:

$$\lim_{n \to +\infty} \sup_{n' \geq n} \left([n']!\right)^{-1/n'} = 0. \tag{18}$$

Generalized coherent states are defined, among other ways, by requiring the conditions of normalizability, continuity in the label and resolution of the identity operator with a positive weight function [17,18,20–26]. Deformed coherent states are generalized coherent states if the above-reported properties hold. The first and second properties hold if the deformed factorial $[n']!$ fulfill relation (18). As far as the third property is concerned, the set of deformed coherent states, $\{|z\rangle, \ \forall z \in \mathbb{C}\}$, resolves the identity operator $I$ over the Hilbert space of the quantum oscillator with a positive weight function $U(u)$,

$$\int_{\mathbb{R}^2} U\left(|z|^2\right) |z\rangle\langle z| d^2 z = I, \tag{19}$$

where $\mathbb{R}$ is the set of real numbers, and $d^2 z = d\Re(z) d\Im(z)$, if the following relation [17,18,20–24]:

$$\pi \int_0^{\infty} \frac{U(u)}{\mathcal{N}(u)} u^n du = [n]!, \tag{20}$$

holds for every $n \in \mathbb{N}_0$. For example, relation (19) allows to represent the elements of the Fock basis as below,

$$|n\rangle = \pi^{-1} \int_{\mathbb{R}^2} \frac{U\left(|z|^2\right)}{\sqrt{\mathcal{N}\left(|z|^2\right)}} \frac{(z^*)^n}{\sqrt{[n]!}} |z\rangle d^2 z, \tag{21}$$

for every $n \in \mathbb{N}_0$.

Consider the following expression of the deformed factorial,

$$[n]! = \frac{\Gamma(b + a(n+1))}{\Gamma(b+a)} \frac{\Gamma(c + a'(n+1))}{\Gamma(c + a')} \frac{\Gamma(d + a')}{\Gamma(d + a'(n+1))}, \tag{22}$$

for every $n \in \mathbb{N}_0$, with $a, a' > 0$, $b, c \geq 0$, $d \geq c + 1$. The set $\{|z\rangle, \ \forall z \in \mathbb{C}\}$ of deformed coherent states, obtained from Eqs. (16) and (22), represents a class of generalized coherent states [28]. The corresponding normalization factor is given by a Wright generalized hypergeometric function,

$$\mathcal{N}(u) = \frac{\Gamma(b+a)\Gamma(c+a')}{\Gamma(d+a')} \, {}_2\Psi_2 \left[ u \middle| \begin{array}{c} (1,1), (d+a', a') \\ (b+a, a), (c+a', a') \end{array} \right], \tag{23}$$

for every $u > 0$. The Wright generalized hypergeometric function is a complex-valued function that is defined by the following power series [34,35]:

$$ {}_p\Psi_q \left[ z \middle| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_k, B_k)_1^q \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{k=1}^{q} \Gamma(\beta_k + B_k n)} \frac{z^n}{n!}, \tag{24}$$

for every $z \in \mathbb{C} \setminus \{0\}$, $\alpha_j, \beta_k \in \mathbb{C}$, $A_j, B_k > 0$. The Wright generalized hypergeometric function is an entire function of the complex variable $z$ if

$$\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > -1. \tag{25}$$

Note that relation (25) is fulfilled by the Wright generalized hypergeometric function appearing in Eq. (23).

The set $\{|z\rangle, \forall z \in \mathbb{C}\}$ of deformed coherent states resolves the identity operator according to Eq. (19) and the corresponding weight function is a product of a Wright generalized hypergeometric function and a Fox $H$ function [28],

$$U(u) = \pi^{-1} {}_2\Psi_2 \left[ u \middle| \begin{array}{c} (1,1), (d+a', a') \\ (b+a, a), (c+a', a') \end{array} \right] H^{2,0}_{1,2} \left[ u \middle| \begin{array}{c} (d, a') \\ (b, a), (c, a') \end{array} \right], \tag{26}$$

for every $u > 0$. The Fox $H$ function is defined via the following integral [30–35]:

$$H^{m,n}_{p,q} \left[ z \middle| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{array} \right] = \frac{1}{2\pi\imath} \int_{\mathcal{C}} \Xi^{m,n}_{p,q} \left[ s \middle| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{array} \right] z^{-s} ds, \tag{27}$$

where

$$\Xi^{m,n}_{p,q} \left[ s \middle| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{array} \right] = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j - B_j s) \prod_{j=n+1}^{p} \Gamma(\alpha_j + A_j s)}. \tag{28}$$

The poles of the Gamma functions $\Gamma(\beta_1 + B_1 s), \ldots, \Gamma(\beta_m + B_m s)$, are required to differ from the poles of the Gamma functions $\Gamma(1 - \alpha_1 - A_1 s), \ldots, \Gamma(1 - \alpha_n - A_n s)$. The above-requested condition is realized by the following constraint:

$$A_j \left( l + \beta_{j'} \right) \neq B_{j'} \left( \alpha_j - l' - 1 \right), \tag{29}$$

which is required to hold for every $j = 1, \ldots, n$, $j' = 1, \ldots, m$, and $l, l' \in \mathbb{N}_0$. The empty products coincide with unity. The allowed values of the indexes $n$ and $m$ are $0 \leq n \leq p$, $0 \leq m \leq q$, and $A_i, B_j > 0$, $\alpha_i, \beta_j \in \mathbb{C}$, for every $i = 1, \cdots, p$, and $j = 1, \cdots, q$. Refer to [34,35] for the existence condition, the domain of analiticity and the contour path $\mathcal{C}$ which is adopted in the definition (27) of the Fox $H$ function.

## 3. The Uncertainty Relation for Special Deformed Number Operator

Briefly, the Heisenberg uncertainty relation provides the lower bound for the product of the variances of the position and momentum observables that are simultaneously measured on a general state $|\psi\rangle$ of a quantum system [1–5],

$$\langle\psi|(\hat{q} - \langle\psi|\hat{q}|\psi\rangle)^2|\psi\rangle\langle\psi|(\hat{p} - \langle\psi|\hat{p}|\psi\rangle)^2|\psi\rangle \geq \frac{1}{4}|\langle\psi|[\hat{q}, \hat{p}]|\psi\rangle|^2. \tag{30}$$

Equivalently, the lower bound for the product of the standard deviations is given by the following form:

$$\sqrt{\langle\psi|(\hat{q} - \langle\psi|\hat{q}|\psi\rangle)^2|\psi\rangle\langle\psi|(\hat{p} - \langle\psi|\hat{p}|\psi\rangle)^2|\psi\rangle} \geq \frac{1}{2}|\langle\psi|[\hat{q}, \hat{p}]|\psi\rangle|. \tag{31}$$

The canonical commutation relation of the position and momentum observables,

$$[\hat{q}, \hat{p}] = \imath\hbar I, \tag{32}$$

provides the lower bound $(\hbar/2)^2$ for the product of the variances of the position and momentum observables, relation (30), or, equivalently, the lower bound $(\hbar/2)$ for the product of the standard deviations, relation (31). The lower bound is realized if the state $|\psi\rangle$ is the ground state or a canonical coherent state of a quantum oscillator. The Heisenberg uncertainty relation (30), or (31), holds, in general, if the position and momentum observables, $\hat{q}$ and $\hat{p}$, respectively, are substituted by two general non-commuting observables which are represented by the Hermitian operators $\hat{O}_1$ and $\hat{O}_2$, such that $\left[\hat{O}_1, \hat{O}_2\right] = \imath\hat{O}_3$ [1–4].

### 3.1. Special Deformation Law of the Canonical Number Operator

At this stage we start our analysis. The Heisenberg uncertainty relation (30), or, equivalently, (31), is studied below in the case where the commutation relation of the position and momentum operators is deformed according to Eq. (15). We focus on a special deformed number operator which is defined below over the Fock basis.

Consider the deformed factorial $[n]_{b,c,d}!$, given by Eq. (22), for every $n \in \mathbb{N}_0$, with $a = a' = 1$, $b, c \geq 0$, $d \geq c+1$ [28]. The corresponding box function $[n]_{b,c,d}$ is obtained from Eq. (12),

$$[n]_{b,c,d} = \frac{(n+b)(n+c)}{(n+d)}, \tag{33}$$

for every $n \in \mathbb{N}$, and $[0]_{b,c,d} = 0$, with $b, c \geq 0$, $d \geq c+1$. Similarly to the canonical case, the deformation law $[n]_{b,c,d}$ is an increasing arithmetic function of the natural variable $n$,

$$[n+1]_{b,c,d} > [n]_{b,c,d}, \tag{34}$$

for every $n \in \mathbb{N}_0$, as $d \geq c+1 \geq 1$. The resulting deformed number operator $\left[\hat{N}\right]_{b,c,d}$ is defined over the Fock basis by Eqs. (11) and (33). The box function $[n]_{b,c,d}$ provides the canonical natural variable $n$ as particular case,

$$[n]_{d,0,d} = n, \tag{35}$$

for every $n \in \mathbb{N}_0$, and $d \geq 1$. Thus, the special deformed number operator $\left[\hat{N}\right]_{b,c,d}$ provides the canonical number operator $\hat{N}$ for $b = d \geq 1$, and $c = 0$,

$$\left[\hat{N}\right]_{d,0,d} = \hat{N}, \tag{36}$$

over the Fock basis and, consequently, over the Hilbert space of the quantum oscillator.

### 3.2. Arbitrarily Small, Minimum Uncertainty for the Ground State

Let the quantum state $|\psi\rangle$ be a state of the Fock basis. The variance of the position and momentum observables are obtained from Eqs. (13) and (14), respectively,

$$\langle n|(\hat{q} - \langle n|\hat{q}|n\rangle)^2|n\rangle = \frac{\hbar}{2m\omega}([n+1] + [n]), \tag{37}$$

$$\langle n|(\hat{p} - \langle n|\hat{p}|n\rangle)^2|n\rangle = \frac{\hbar m\omega}{2}([n+1] + [n]), \tag{38}$$

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for every $n \in \mathbb{N}_0$. Consequently, the uncertainty relation for the product of the variances of the position and momentum observables results to be

$$\begin{aligned} \langle n|(\hat{q} - \langle n|\hat{q}|n\rangle)^2|n\rangle\langle n|(\hat{p} - \langle n|\hat{p}|n\rangle)^2|n\rangle &= \left(\frac{\hbar}{2}\right)^2([n+1]+[n])^2 \\ &\geq \left(\frac{\hbar}{2}\right)^2([n+1]-[n])^2, \end{aligned} \tag{39}$$

for every $n \in \mathbb{N}_0$. Equivalently, the uncertainty relation for the product of the corresponding standard deviations is

$$\sqrt{\langle n|(\hat{q} - \langle n|\hat{q}|n\rangle)^2|n\rangle\langle n|(\hat{p} - \langle n|\hat{p}|n\rangle)^2|n\rangle} = \frac{\hbar}{2}([n+1]+[n]) \geq \frac{\hbar}{2}|[n+1]-[n]|, \tag{40}$$

for every $n \in \mathbb{N}_0$. Note that $\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$, for every $n \in \mathbb{N}_0$. Relation (39), or, equivalently, relation (40), becomes an equality if the box function vanishes, $[n] = 0$; this property is realized uniquely by the vanishing value of the natural variable $n$, i.e., $n = 0$, if the general box function $[n]$ is assumed to be positive, $[n] > 0$, for every $n \in \mathbb{N}$, and an increasing arithmetic function of the natural variable $n$. The variances of the position and momentum observables are given by the forms below for the ground state,

$$\langle 0|\hat{q}^2|0\rangle = \frac{\hbar}{2m\omega}[1], \tag{41}$$

$$\langle 0|\hat{p}^2|0\rangle = \frac{\hbar m\omega}{2}[1]. \tag{42}$$

Thus, the minimum uncertainty in the simultaneous measurement of the position and momentum observables is realized among the states of the Fock basis uniquely by the ground state,

$$\langle 0|\hat{q}^2|0\rangle\langle 0|\hat{p}^2|0\rangle = \left(\frac{\hbar}{2}[1]\right)^2, \tag{43}$$

or, equivalently,

$$\sqrt{\langle 0|\hat{q}^2|0\rangle\langle 0|\hat{p}^2|0\rangle} = \frac{\hbar}{2}[1]. \tag{44}$$

The ground state $|0\rangle$ exhibits minimum uncertainty also for a quantum oscillator that is described by the canonical boson algebra.

At this stage, we consider the special deformed number operator $[\hat{N}]_{b,c,d}$, or, equivalently, the special deformation law $[n]_{b,c,d}$ that is defined in Section 3.1. According to the uncertainty relation (39), or, equivalently, relation (40), the minimum uncertainty realized by the ground states is less than, equal to, or greater than the canonical value, $\hbar/2$, if the deformed unity, $[1]_{b,c,d}$, is less than, equal to, or greater than unity,

$$[1]_{b,c,d} \lesseqqgtr 1. \tag{45}$$

Condition (45) holds if

$$d \gtreqqless b + c + bc, \tag{46}$$

for every allowed value of the involved parameters. We are interested in the realization of arbitrarily small values of the minimum uncertainty for the simultaneous measurement of the position and

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momentum observables. In this regard, consider the positive parameter $l$, that is not greater than unity, $0 < l \leq 1$. The deformed unity, $[1]_{b,c,d}$, is less than the parameter $l$,

$$[1]_{b,c,d} < l, \tag{47}$$

if the following condition holds:

$$l(d+1) > (b+1)(c+1). \tag{48}$$

According to the uncertainty relation (43), or, equivalently, relation (44), the minimum uncertainty, $[1]_{b,c,d}\hbar/2$, becomes arbitrarily small if the parameters $b, c, d$, defining the deformed number operator $[\hat{N}]_{b,c,d}$, fulfill condition (48) for arbitrarily small positive values of the positive parameter $l$, i.e., $0 < l \ll 1$. Note that condition (48) holds for sufficiently large values of the parameter $d$, if arbitrarily small values of the parameter $l$ are chosen.

At this stage, we search for states of the Fock basis, different from the ground state, such that the corresponding uncertainty in the simultaneous measurement of the position and momentum observables is not minimum, but can be arbitrarily reduced by properly choosing the values of the involved parameters $b, c, d$. Consider the following condition:

$$[n'+1]_{b,c,d} \leq r, \tag{49}$$

with $0 < r \leq 1/2$. Condition (49) holds for every natural number $n'$ such that

$$0 \leq n' \leq \frac{1}{2}\left(\sqrt{(2+b+c-r)^2 + 4(rd + r - bc - b - c - 1)} - 2 - b - c + r\right). \tag{50}$$

The argument of the square root, equivalent to the expression below, is positive for the allowed values of the involved parameters,

$$4r(d-c) + (b-c-r)^2 > 0, \tag{51}$$

as $d > c$. Condition (50) provides various values of the natural number $n'$ fulfilling relation (49) for $0 < r \leq 1/2$, by choosing sufficiently large values of the product $(rd)$,

$$r(d+2) \gg bc + 2(b+c+2). \tag{52}$$

The above-defined values $n'$ of the natural variable $n$ and the involved parameters fulfill the following constraint:

$$[n'+1]_{b,c,d} + [n']_{b,c,d} < 2r \leq 1. \tag{53}$$

Consequently, the variances of the position and momentum observables for the state $|n'\rangle$ are bounded as below,

$$\langle n'|\hat{q}^2|n'\rangle < \frac{\hbar r}{m\omega}, \tag{54}$$

$$\langle n'|\hat{p}^2|n'\rangle < \hbar m\omega r, \tag{55}$$

for every $n' \in \mathbb{N}_0$ such that constraint (50) is fulfilled. Thus, the corresponding uncertainty relation for the product of the variances of the position and momentum observables is

$$\langle n'|\hat{q}^2|n'\rangle\langle n'|\hat{p}^2|n'\rangle < (\hbar r)^2 \leq \left(\frac{\hbar}{2}\right)^2, \tag{56}$$

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for every state $|n'\rangle$. Equivalently, the uncertainty relation for the product of the corresponding standard deviations is

$$\sqrt{\langle n'|\hat{q}^2|n'\rangle\langle n'|\hat{p}^2|n'\rangle} < \hbar r \leq \frac{\hbar}{2}, \tag{57}$$

for every state $|n'\rangle$. Thus, the uncertainty, estimated by the product of the standard deviations, is arbitrarily less than the value $(\hbar r)$, for every state $|n'\rangle$ of the Fock basis which is determined by relations (50) and (52), and every positive value of the parameter $r$ which is arbitrarily smaller than the value $1/2$. The required condition is provided by sufficiently large values of the parameter $d$.

### 3.3. Arbitrarily Small, Minimum Uncertainty for Deformed Coherent States

Let the quantum states $|\psi\rangle$ be a deformed coherent state, $|z\rangle$, with $z \in \mathbb{C} \setminus \{0\}$. The variances of the position and momentum observables are evaluated via Eqs. (13) and (14), respectively,

$$\langle z|\big(\hat{q} - \langle z|\hat{q}|z\rangle\big)^2|z\rangle = \frac{\hbar}{2m\omega}\langle z|\big([\hat{N}+1] - [\hat{N}]\big)|z\rangle, \tag{58}$$

$$\langle z|\big(\hat{p} - \langle z|\hat{p}|z\rangle\big)^2|z\rangle = \frac{\hbar m\omega}{2}\langle z|\big([\hat{N}+1] - [\hat{N}]\big)|z\rangle, \tag{59}$$

for every $z \in \mathbb{C} \setminus \{0\}$. Thus, the uncertainty relation for the product of the variances of the position and momentum observables results to be

$$\langle z|\big(\hat{q} - \langle z|\hat{q}|z\rangle\big)^2|z\rangle\langle z|\big(\hat{p} - \langle z|\hat{p}|z\rangle\big)^2|z\rangle = \left(\frac{\hbar}{2}\right)^2\Big(\langle z|\big([\hat{N}+1] - [\hat{N}]\big)|z\rangle\Big)^2, \tag{60}$$

for every $z \in \mathbb{C} \setminus \{0\}$. Equivalently, the uncertainty relation for the product of the corresponding standard deviations is

$$\sqrt{\langle z|\big(\hat{q} - \langle z|\hat{q}|z\rangle\big)^2|z\rangle\langle z|\big(\hat{p} - \langle z|\hat{p}|z\rangle\big)^2|z\rangle} = \frac{\hbar}{2}\Big|\langle z|\big([\hat{N}+1] - [\hat{N}]\big)|z\rangle\Big|, \tag{61}$$

for every $z \in \mathbb{C} \setminus \{0\}$. The uncertainty relation (60), or (61), shows that any deformed coherent state $|z\rangle$ exhibits minimum uncertainty, for every $z \in \mathbb{C} \setminus \{0\}$. Note that the minimum uncertainty characterizing the ground state $|0\rangle$, is obtained from Eq. (60), or Eq. (61), by choosing vanishing value of the label, $z = 0$.

Consider the special deformation law of the number operator described by Eq. (33). The minimum uncertainty relation (60), or (61), is determined by the terms $\langle z|[\hat{N}+1]_{b,c,d}|z\rangle$ and $\langle z|[\hat{N}]_{b,c,d}|z\rangle$. The former term results to be a ratio of Wright generalized hypergeometric functions,

$$\langle z|[\hat{N}+1]_{b,c,d}|z\rangle = {}_5\Psi_5\left[|z|^2\middle|\begin{array}{c}(\alpha_j, A_j)_1^5\\(\beta_k, B_k)_1^5\end{array}\right]\Big/\,{}_2\Psi_2\left[|z|^2\middle|\begin{array}{c}(\alpha'_j, A'_j)_1^2\\(\beta'_k, B'_k)_1^2\end{array}\right], \tag{62}$$

for every $z \in \mathbb{C} \setminus \{0\}$, and $b, c \geq 0$, $d \geq c+1$, where $\alpha_1 = A_1 = A_2 = A_3 = A_4 = A_5 = 1$, $\alpha_2 = 2+b$, $\alpha_3 = 2+c$, $\alpha_4 = \alpha_5 = 1+d$, $\beta_1 = \beta_2 = 1+b$, $\beta_3 = \beta_4 = 1+c$, $\beta_5 = 2+d$, $B_1 = B_2 = B_3 = B_4 = B_5 = 1$, $\alpha'_1 = A'_1 = A'_2 = B'_1 = B'_2 = 1$, $\alpha'_2 = 1+d$, $\beta'_1 = 1+b$, $\beta'_2 = 1+c$. The latter term is given by the form below,

$$\langle z|[\hat{N}]_{b,c,d}|z\rangle = |z|^2, \tag{63}$$

for every $z \in \mathbb{C} \setminus \{0\}$, and $b, c \geq 0$, $d \geq c+1$.

The special deformation law $[n]_{b,c,d}$ of the number operator, given by Eq. (33), fulfills the following constraint:

$$\Big([n+1]_{b,c,d} - [n]_{b,c,d}\Big) \lesseqqgtr 1, \tag{64}$$

for every $n \in \mathbb{N}_0$, and every allowed value of the involved parameters such that $b \lesseqqgtr d$. Thus, the special deformed number operator $[\hat{N}]_{b,c,d}$ exhibits the following property:

$$\langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle \lesseqqgtr 1, \tag{65}$$

holding for every $z \in \mathbb{C} \setminus \{0\}$, and every allowed value of the involved parameters such that $b \lesseqqgtr d$. According to property (65), the minimum value of the product of the variances of the position and momentum observables is less than, equal to, or greater than the canonical value, $(\hbar/2)^2$,

$$\begin{aligned} \langle z|(\hat{q} - \langle z|\hat{q}|z\rangle)^2|z\rangle\langle z|(\hat{p} - \langle z|\hat{p}|z\rangle)^2|z\rangle &= \left(\frac{\hbar}{2}\right)^2 \left( \langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle \right)^2 \\ &\lesseqqgtr \left(\frac{\hbar}{2}\right)^2, \end{aligned} \tag{66}$$

for every $n \in \mathbb{N}_0$, and every allowed value of the involved parameters such that $b \lesseqqgtr d$. Equivalently, for the product of the standard deviations, the minimum uncertainty is less than, equal to, or greater than the canonical value, $\hbar/2$,

$$\sqrt{\langle z|(\hat{q} - \langle z|\hat{q}|z\rangle)^2|z\rangle\langle z|(\hat{p} - \langle z|\hat{p}|z\rangle)^2|z\rangle} = \frac{\hbar}{2} \langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle \lesseqqgtr \frac{\hbar}{2}, \tag{67}$$

for every $z \in \mathbb{C} \setminus \{0\}$, and every allowed value of the involved parameters such that $b \lesseqqgtr d$.

For small nonvanishing values of the magnitude of the label, $0 < |z|^2 \ll 1$, the minimum uncertainty, given by relation (67), is properly described by the following asymptotic behavior:

$$\langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle \sim [1]_{b,c,d} + \xi_{b,c,d}|z|^2, \tag{68}$$

as $|z| \to 0^+$. The parameter $\xi_{b,c,d}$ is defined as below,

$$\xi_{b,c,d} = \frac{(2+b)(2+c)(1+d)}{(1+b)(1+c)(2+d)} - 1. \tag{69}$$

for every allowed value of the involved parameters. The parameter $\xi_{b,c,d}$ is positive, $\xi_{b,c,d} > 0$, for every allowed values of the involved parameters which realize relation (47), or, equivalently, relation (48), with $0 < l \leq 1$. According to relation (68), the minimum value of the product of the standard deviations of the position and momentum observables is greater than the value $\left(\hbar[1]_{b,c,d}/2\right)$, for small nonvanishing values of the magnitude of the label,

$$[1]_{b,c,d} < \langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle < [1]_{b,c,d} + \eta_{b,c,d}|z|^2, \tag{70}$$

as $|z| \to 0^+$, with $z \neq 0$. The positive parameter $\eta_{b,c,d}$ is required to be greater than the parameter $\xi_{b,c,d}$, i.e., $\xi_{b,c,d} < \eta_{b,c,d}$. Naturally, the minimum uncertainty $\left(\hbar[1]_{b,c,d}/2\right)$, realized by the ground state $|0\rangle$, is obtained for vanishing value of the label, $|z| \to 0^+$.

Arbitrarily small values of the product of the standard deviations of the position and momentum observables,

$$\langle z| \left( [\hat{N}+1]_{b,c,d} - [\hat{N}]_{b,c,d} \right) |z\rangle < l + \epsilon \ll 1, \tag{71}$$

is obtained from the asymptotic form (68) if the following conditions hold: arbitrarily small, nonvanishing values of the magnitude of the label are required: $0 < |z|^2 \ll 1$, and $0 < \eta_{b,c,d}|z|^2 = \epsilon \ll 1/2$, and the involved parameter are requested to fulfill condition (48), for arbitrarily small values of the

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parameter $l$, i.e., $0 < l \ll 1/2$. The latter condition is realized by arbitrarily large values of the parameter $d$. Thus, according to relation (71), the product of the standard deviations of the position and momentum observables becomes arbitrarily smaller than the canonical value $\hbar/2$, by choosing arbitrarily small values of the magnitude of the label and arbitrarily large values of the involved parameter $d$.

## 4. Summary and Conclusions

In the present scenario, we have considered a quantum oscillator which is characterized by deformed boson algebra [12,17]. We have introduced the deformed number operator $\left[\hat{N}\right]_{b,c,d}$, defined via the nen-negative parameters $b, c, d$, which produces the canonical number operator $\hat{N}$, for $b = d \geq 1$, and $c = 0$. We have studied the Heisenberg uncertainty relation, concerning the simultaneous measurement of the position and momentum observables. The uncertainty is quantified by the product of the standard deviations of the two above-mentioned observables. The ground state and the deformed coherent states of the quantum oscillator exhibit minimum uncertainty; it is well known that this property holds for the canonical boson algebra. Instead, for the special deformed boson algebra under study, the minimum uncertainty realized by the ground state is less than, equal to, or greater than the canonical value, $\hbar/2$, if the deformed unity, $[1]_{b,c,d}$, is less than, equal to, or greater than unity. Additionally, the minimum uncertainty realized by the ground state of the quantum oscillator is arbitrarily less than the canonical value, $\hbar/2$, if the deformed unity is arbitrarily less than unity, $0 < [1]_{b,c,d} \ll 1$. The latter condition is realized by choosing arbitrarily large values of the parameter $d$.

Similarly to the properties of the ground state, the minimum uncertainty realized by the deformed coherent states is less than, equal to, or greater than the canonical value, $\hbar/2$, if the deformed number operator $\left[\hat{N}\right]_{b,c,d}$ is characterized by the following values of the involved parameters: $d > b \geq 0$, and $d \geq c + 1 \geq 1$; $d = b \geq 1 + c \geq 1$; or $b > d \geq c + 1 \geq 1$; respectively. Especially, for the deformed coherent states under study, the product of the standard deviations is arbitrarily less than canonical value, $\hbar/2$, if both the magnitude of the label and the deformed unity are arbitrarily less than unity. Again, the latter property is realized by choosing arbitrarily large values of the parameter $d$. The product of the standard deviations of the position and momentum observables becomes arbitrarily small for other states of the Fock basis, beside the ground state, if arbitrarily large values of the parameter $d$ are chosen.

In conclusion, a special deformation of the number operator is determined such that a quantum oscillator characterized by deformed boson algebra exhibits anomalies in the simultaneous measurement of the position and momentum observables: the uncertainty is arbitrarily small for the ground state, other states of the Fock basis and deformed coherent states with arbitrarily small magnitude of the label. The deformation of the boson algebra of a quantum oscillator allows to produce theoretical models such that the uncertainty in the simultanous measurement of the position and momentum observables becomes arbitrarily small for deformed coherent states. The proposal for physical systems which might show the present anomalies is beyond the purposes of the present study. However, low-energy mechanical oscillators or optomechanical resonators are suitable for the realization of deformed commutation rules [36–39]; thus, these systems might be the proper ground for the realization of the special deformation laws under study and perform further investigation of the present anomalies concerning the Heisenberg uncertainty relation.

## Appendix A. Details

Property (34) of the deformation law $[n]_{b,c,d}$ is provided by the following relation:

$$[n+1]_{b,c,d} - [n]_{b,c,d} = \frac{d(c+1) + b(d-c) + (2d+1)n + n^2}{(n+d)(n+d+1)} > 0, \tag{A1}$$

holding for every $n \in \mathbb{N}_0$, as $d \geq c+1$, and $b, c \geq 0$. Condition (49) is equivalent to the following relation:

$$r - \left[n' + 1\right]_{b,c,d} = -\frac{(b+1)(c+1) - r(d+1) + n'(b+c-r+2) + n'^2}{n' + d + 1} \geq 0, \tag{A2}$$

holding for every natural number $n'$ such that

$$(b+1)(c+1) - r(d+1) + n'(b+c-r+2) + n'^2 \leq 0. \tag{A3}$$

Relation (A3) provides in straightforward manner relations (50) and (52). Relation (53) is realized by every natural number $n'$ fulfilling relation (49), as relation (34) holds for every $n \in \mathbb{N}_0$, and every allowed value of the involved parameters.

Condition (47) is realized by the allowed values of parameters fulfilling inequality (48), which is obtained in straightforward manner from Eq. (33) for $n = 1$. Equality (62) is obtained in straightforward manner by expressing the deformed coherent state $|z\rangle$ in the Fock basis, Eq. (16), and considering the definition of the Wright generalized hypergeometric function, Eq. (24). Relation (63) is found in straightforward manner by expressing the deformed number operator $\left[\hat{N}\right]_{b,c,d}$ with the deformed creation and annihilation operators. The following relation:

$$[n]_{b,c,d} - [n+1]_{b,c,d} + 1 = \frac{(d-c)(d-b)}{(n+d)(n+d+1)} \gtreqless 0, \tag{A4}$$

holds for every $n \in \mathbb{N}_0$, if $b \lesseqgtr d$, as $d > c$. Relation (65) is obtained in straightforward manner by expressing the deformed coherent state $|z\rangle$ in the Fock basis, Eq. (16), and considering that relation (A4) holds for every $n \in \mathbb{N}_0$, if $b \lesseqgtr d$. The uncertainty relation (66), or, equivalently, relation (67), is obtained from the general form (30), or (31), respectively, by considering that relation (65) holds for every $z \in \mathbb{C} \setminus \{0\}$, if $b \lesseqgtr d$.

The asymptotic form (68) and the parameter $\xi_{b,c,d}$, given by Eq. (69), are obtained by considering the term $\langle z| \left( \left[\hat{N}+1\right]_{b,c,d} - \left[\hat{N}\right]_{b,c,d} \right) |z\rangle$, expressing the deformed coherent state $|z\rangle$ in the Fock basis, Eq. (16), and performing the power series expansion of the corresponding fraction of power series of the independent variable $|z|^2$ around $|z|^2 = 0$. Relation (70) is obtained from the asymptotic form (68) and the following inequality involving the small $o$ function:

$$o\left(|z|^2\right) < \left(\eta_{b,c,d} - \xi_{b,c,d}\right)|z|^2, \tag{A5}$$

which holds for $|z| \to 0^+$, if $\xi_{b,c,d} < \eta_{b,c,d}$. Relation (71) is obtained from relation (70), by considering relations (47) and (48). This concludes the demonstration of the present analytical results.

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