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Dual-Time Topological Geometry and the Emergence of Temporal Asymmetry in Non-Equilibrium Dynamics

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Simple Summary

Many physical systems follow time-reversible laws at the microscopic level, yet exhibit irreversible behavior such as dissipation, memory, and aging at macroscopic scales. This work introduces a dual-time framework in which reversible and irreversible processes are described using two independent but coupled time parameters. By formulating dynamics on a bi-temporal geometric structure, the framework provides a unified way to represent both types of temporal evolution. Temporal asymmetry and the arrow of time emerge naturally from the geometry and topology of coupled time evolution, rather than being imposed phenomenologically. The approach recovers standard non-equilibrium models in appropriate limits while offering a new mathematical perspective on memory effects and non-Markovian dynamics.

Abstract

We develop a dual-time topological framework for the mathematical description of non-equilibrium systems, aimed at reconciling time-reversible microscopic dynamics with irreversible macroscopic behavior. The formulation introduces two independent but coupled temporal parameters: a reversible time associated with microscopic or generative dynamics, and an irreversible time governing dissipation, entropy production, and macroscopic evolution. Physical states are defined on a bi-temporal manifold, allowing reversible and irreversible processes to be treated within a unified geometric setting. Temporal evolution is described using independent temporal connections and their associated curvature. We show that nonvanishing temporal curvature induces path dependence in temporal evolution, providing a geometric origin for memory effects, non-Markovian dynamics, and aging phenomena. Temporal asymmetry emerges dynamically through symmetry breaking between the temporal sectors and through projection from the bi-temporal domain onto a single observable time parameter. The relationship between the dual-time formalism and conventional single-time non-equilibrium models is analyzed. Standard evolution equations are recovered in integrable or decoupling limits, demonstrating that the proposed framework constitutes a genuine generalization compatible with established approaches. By encoding irreversibility in the geometry and topology of temporal evolution, this work provides a mathematically consistent framework for the emergence of the arrow of time in non-equilibrium theoretical physics.Unlike conventional approaches in which irreversibility and memory are encoded phenomenologically at the level of effective equations, the present framework derives non-Markovian dynamics and temporal asymmetry from the geometry and topology of coupled temporal evolution. In particular, a representation theorem is established showing that a broad class of convolution-type non-Markovian equations arise as projections of local dual-time dynamics.

Keywords: dual time; temporal topology; non-equilibrium systems; irreversibility; gauge theory

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1. Introduction

The emergence of temporal asymmetry in physical systems remains one of the most fundamental problems in mathematical physics. While the microscopic laws governing classical and quantum systems are typically invariant under time reversal—as exemplified by Hamiltonian mechanics and unitary quantum evolution—macroscopic phenomena ubiquitously exhibit irreversible behavior, manifested through dissipation, entropy production, relaxation, and aging. Reconciling microscopic reversibility with macroscopic irreversibility has been a central theme since the foundational works of Boltzmann and Gibbs and continues to motivate modern research in non-equilibrium statistical mechanics and dynamical systems [1–4].

Conventional approaches to non-equilibrium dynamics are generally formulated in terms of a single temporal parameter. Irreversibility is incorporated through coarse-graining and projection techniques [5,6], stochastic modeling, or the introduction of dissipative operators generating contraction semigroups. In the quantum setting, irreversible evolution is commonly described using quantum dynamical semigroups and Lindblad-type master equations [7,8]. While these frameworks are mathematically well established and widely successful, they encode temporal asymmetry at the level of effective evolution equations rather than at the level of the underlying temporal structure. As a result, memory effects and non-Markovian dynamics must typically be introduced phenomenologically, for instance through integro-differential equations or time-dependent generators [9,10].

In recent years, there has been renewed interest in the mathematical and physical foundations of memory and non-Markovianity in non-equilibrium systems. It has been shown that memory effects can emerge naturally under coarse-graining even in systems close to equilibrium [11], while detailed kinetic analyses have revealed nontrivial non-Markovian behavior in barrier-crossing processes and driven systems [12,13]. Non-Markovian dynamics have also been investigated in active and viscoelastic media [14], as well as in quantum systems, where genuinely non-Markovian effects such as the quantum Mpemba effect have been identified [15]. These developments highlight both the ubiquity of memory effects in non-equilibrium physics and the structural limitations of purely single-time descriptions.

Motivated by these observations, the present work introduces a *dual-time* framework in which reversible and irreversible temporal evolution are treated from the outset as independent but coupled temporal directions. Rather than attempting to derive irreversibility within a single temporal dimension, we formulate dynamics on a bi-temporal manifold on which physical states are defined. Reversible microscopic dynamics and irreversible macroscopic evolution are associated with distinct temporal parameters, coupled through geometric and gauge-theoretic structures. This approach is conceptually related to earlier multi-time and extended phase-space constructions [16,17], but differs in that both temporal parameters are treated as fundamental geometric coordinates rather than auxiliary variables or bookkeeping devices.

The central idea of the framework is to describe temporal evolution using independent temporal connections and to characterize their interaction through *temporal curvature*. Geometric formulations of dynamics based on connections and curvature are well established in classical and quantum mechanics [18,19]. Here, these methods are extended to the temporal domain. Nonvanishing temporal curvature induces path dependence in bi-temporal evolution, providing a mathematically precise geometric origin for memory effects, non-Markovian dynamics, and aging phenomena without introducing explicit memory kernels or stochastic assumptions.

An effective single observable time parameter arises through projection from the bi-temporal domain, yielding a macroscopic arrow of time as an emergent feature rather than a fundamental postulate. This perspective is complementary to relational and emergent-time proposals, including the thermal time hypothesis [20], while remaining compatible with standard dynamical evolution in appropriate integrable or decoupling limits.

A key result of this work is an explicit representation theorem showing that a broad and widely used class of non-Markovian evolution equations with convolution-type memory kernels can be realized as projections of local, first-order dynamics on the bi-temporal manifold. In this construction, the irreversible temporal coordinate naturally plays the role of a memory or age variable. This correspondence is established rigorously and illustrated through worked classical and quantum examples with exponential memory kernels, thereby demonstrating that the dual-time framework is not merely conceptual but provides a constructive geometric realization of non-Markovian dynamics.

The paper is organized as follows. Section 2 introduces the mathematical preliminaries and conceptual framework. Section 3 defines the bi-temporal manifold and the associated state space. Section 4 develops the temporal connections and coupled evolution equations. Section 5 analyzes the induced temporal topology and gauge structure. Section 6 explains the emergence of temporal asymmetry through symmetry breaking and projection. Section 7 derives memory effects, non-Markovian dynamics, and aging phenomena. Section 8 discusses the reduction to standard non-equilibrium models, and Section 9 situates the framework within the broader literature. Section 10 concludes with a summary and outlook.

The present work makes the following contributions. First, it formulates a dual-time topological geometry for non-equilibrium systems based on a bi-temporal manifold equipped with independent temporal connections, temporal curvature, and an associated gauge structure, providing a unified setting for coupled reversible and irreversible evolution (Sections 3–5). Second, it shows that non-vanishing temporal curvature induces path dependence in temporal evolution and, after projection onto a single observable time, gives rise to memory effects, non-Markovian dynamics, and aging, thereby offering a geometric mechanism for the macroscopic arrow of time (Section 7). Third, it establishes a representation theorem for Volterra-type evolution equations with convolution kernels of the form $K(t) = Ce^{tB}D$, demonstrating that such non-Markovian single-time equations arise as projections of local dual-time dynamics, and illustrates this correspondence through explicit classical and quantum examples (Sections 7.4–7.6). Finally, it analyzes integrable and decoupling regimes in which the dual-time structure reduces to standard Hamiltonian, dissipative semigroup, and Markovian master-equation frameworks, showing that the proposed formalism genuinely extends—rather than replaces—conventional single-time descriptions (Section 8).

2. Mathematical Preliminaries and Conceptual Framework

This section introduces the mathematical and conceptual structures underlying the proposed dual-time formulation. The goal is to clarify the role of temporal parameters in non-equilibrium systems, establish notation, and specify the geometric and functional setting used throughout the paper.

2.1. Time in Mathematical Physics

In many areas of mathematical physics, temporal evolution is described by a single real parameter $t \in \mathbb{R}$, representing physical time. At the microscopic level, the associated dynamics are often time-reversal invariant, as exemplified by Hamiltonian systems in classical mechanics and unitary evolution in quantum theory [1,2,18]. In contrast, macroscopic descriptions of non-equilibrium phenomena exhibit intrinsic temporal asymmetry, manifested through dissipation, entropy production, and relaxation toward attractors or steady states [3,4].

From a mathematical perspective, this tension reflects the coexistence of two distinct classes of dynamical behavior:

- *Reversible evolution*, typically generated by skew-adjoint or Hamiltonian operators and associated with group-valued time evolution, and
- *Irreversible evolution*, associated with dissipative semigroups, contraction operators, or gradient flows.

The mathematical theory of strongly continuous semigroups provides a rigorous framework for irreversible evolution in both classical and quantum settings [7,21], while variational and metric approaches to gradient flows have become central tools in the analysis of dissipative systems [22]. In conventional single-time formulations, both reversible and irreversible aspects are encoded within the same temporal parameter, often relying on coarse-graining, projection methods, or phenomenological assumptions to account for irreversibility [5,6].

While such approaches are successful in many applications, they obscure the structural distinction between reversible and irreversible temporal behavior. In particular, memory effects and non-Markovian dynamics typically arise only after additional assumptions are imposed at the level of effective equations [9,10].

The present work adopts a different viewpoint by introducing two distinct temporal parameters:

1. a *reversible time* parameter, associated with microscopic or generative dynamics, and
2. an *irreversible time* parameter, associated with macroscopic evolution, dissipation, and memory.

These parameters are treated as independent coordinates at the mathematical level, allowing reversible and irreversible processes to be represented simultaneously without reduction to a single effective time from the outset.

2.2. Geometric and Functional Structures

We now introduce the basic geometric and functional framework used in the remainder of the paper.

Let $\mathcal{X}$ denote the configuration space of the system, assumed to be a smooth manifold, possibly infinite-dimensional depending on the application. Temporal evolution is described on an extended domain obtained by adjoining temporal variables to $\mathcal{X}$.

In contrast to conventional formulations, we consider temporal domains equipped with multiple independent parameters. The resulting structure naturally leads to a product manifold

$$\mathcal{M} := \mathcal{X} \times \mathcal{T},$$

where $\mathcal{T}$ is a temporal manifold whose precise structure will be specified in Section 3.

Physical states are represented as functions or sections over $\mathcal{M}$. Depending on context, these may take values in:

- a vector space (classical observables),
- a Hilbert space (quantum states),
- or a more general fiber, such as density operators or probability measures.

We assume that admissible states belong to function spaces with sufficient regularity to admit differentiation with respect to each temporal parameter. Typical examples include spaces of smooth functions, Sobolev spaces, or Banach spaces supporting strongly continuous semigroups [21]. Such functional-analytic settings are standard in the rigorous treatment of both reversible and dissipative dynamics.

To describe temporal evolution geometrically, we employ the language of connections and covariant derivatives. Each temporal parameter is associated with its own directional derivative and, more generally, with its own connection on the appropriate bundle over $\mathcal{M}$. This geometric formulation allows reversible and irreversible dynamics to be encoded as parallel transport along distinct temporal directions, extending standard constructions from geometric mechanics and gauge theory [18,19].

Throughout the paper, we adopt the following conventions:

- Greek indices label temporal coordinates.
- Latin indices label spatial or configurational coordinates.
- Summation over repeated indices is implied when appropriate.

This framework provides the mathematical foundation for the dual-time construction developed in subsequent sections, where the interaction between distinct temporal structures gives rise to nontrivial topology, memory effects, and emergent temporal asymmetry.

3. Dual-Time Manifold and State Space

In this section we introduce the dual-time structure that underlies the proposed framework. The key idea is to represent reversible and irreversible temporal evolution by means of two independent temporal parameters, treated on equal mathematical footing. This leads naturally to a bi-temporal manifold and to state functions defined on an extended domain.

3.1. Bi-Temporal Domain

Let $\mathcal{T}_r$ and $\mathcal{T}_i$ be smooth one-dimensional manifolds representing, respectively, reversible and irreversible temporal parameters. For concreteness, and unless otherwise stated, we assume

$$\mathcal{T}_r \cong \mathbb{R}, \qquad \mathcal{T}_i \cong \mathbb{R}_{\geq 0},$$

where $\mathcal{T}_r$ is equipped with a time-reversal symmetry, while $\mathcal{T}_i$ is oriented and lacks such symmetry. This distinction reflects the mathematical separation between reversible group-valued dynamics and irreversible semigroup evolution commonly used in non-equilibrium theory [4,21].

Definition 1 (Dual-Time Manifold)**.** *The* dual-time manifold *is defined as the product*

$$\mathcal{T} := \mathcal{T}_r \times \mathcal{T}_i,$$

with local coordinates (t_r, t_i)*.*

The two temporal coordinates are interpreted as follows:

- t_r parametrizes reversible microscopic or generative dynamics, typically associated with Hamiltonian or unitary evolution [18];
- t_i parametrizes irreversible macroscopic evolution, dissipation, and aging, commonly modeled by dissipative semigroups or gradient flows [4,22].

At the mathematical level, no *a priori* relation between t_r and t_i is assumed. In particular, neither parameter is required to be a function of the other, and both are treated as independent coordinates on $\mathcal{T}$. This distinguishes the present construction from auxiliary-time or bookkeeping approaches, as well as from two-time correlation formalisms in statistical mechanics [5,6].

Let $\mathcal{X}$ denote the configuration space of the system, assumed to be a smooth manifold. We define the extended space-time manifold

$$\mathcal{M} := \mathcal{X} \times \mathcal{T} = \mathcal{X} \times \mathcal{T}_r \times \mathcal{T}_i.$$

This construction generalizes the standard space-time manifold $\mathcal{X} \times \mathbb{R}$ by replacing the single temporal dimension with a bi-temporal structure. The product form ensures that standard local differential-geometric tools remain applicable, while allowing distinct temporal directions to be treated independently [19].

3.2. Physical States on Bi-Temporal Domains

Physical states are represented as functions or sections defined on the extended manifold $\mathcal{M}$. The precise nature of these states depends on the physical context, but the following abstract definition suffices for the present analysis.

Definition 2 (Bi-Temporal State)**.** *A* bi-temporal state *is a mapping*

$$\Psi : \mathcal{X} \times \mathcal{T}_r \times \mathcal{T}_i \longrightarrow \mathcal{V},$$

where $\mathcal{V}$ *is a vector space or a Hilbert space encoding the internal degrees of freedom of the system.*

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Typical choices for $\mathcal{V}$ include:

- $\mathbb{R}$ or $\mathbb{C}$ for scalar fields,
- a Hilbert space for quantum states,
- a space of density operators or probability measures for statistical descriptions [9].

We assume that Ψ belongs to a function space admitting partial derivatives with respect to both temporal variables. Typical examples include

$$\Psi \in C^1(\mathcal{T};\mathcal{H}) \quad \text{or} \quad \Psi \in L^2(\mathcal{T};\mathcal{H}),$$

where $\mathcal{H}$ is a Hilbert space associated with the spatial degrees of freedom. Such functional settings are standard in the analysis of evolution equations generated by both unitary groups and dissipative semigroups [21].

Initial and boundary conditions in the dual-time setting are specified independently for each temporal parameter. In particular:

- reversible evolution may be defined for all $t_r \in \mathbb{R}$,
- irreversible evolution is typically defined for $t_i \geq 0$, reflecting the directed nature of macroscopic time.

The bi-temporal formulation allows reversible and irreversible processes to coexist within a single mathematical object. Observable physical quantities are obtained by suitable projections or restrictions of Ψ, a procedure that will be discussed in later sections. This dual-time state space provides the foundation for introducing independent temporal connections, coupled evolution equations, and nontrivial temporal topology, developed in the following sections.

4. Dual Temporal Connections and Evolution

In this section we introduce the dynamical structure associated with the dual-time manifold. Temporal evolution is described geometrically by means of connections and covariant derivatives along each temporal direction. The interaction between reversible and irreversible dynamics is encoded through coupling between these temporal connections, extending standard geometric formulations of evolution equations to a bi-temporal setting [18,19].

4.1. Independent Temporal Connections

Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle over the extended manifold

$$\mathcal{M} = \mathcal{X} \times \mathcal{T}_r \times \mathcal{T}_i,$$

whose sections represent bi-temporal states. We assume that $\mathcal{E}$ is equipped with a connection that decomposes naturally along the temporal directions. Such a decomposition is standard in geometric mechanics and gauge theory, where evolution is represented as parallel transport along distinguished directions in the base manifold [18,19].

Definition 3 (Temporal Covariant Derivatives)**.** *We define two temporal covariant derivatives*

$$\nabla_r := \partial_{t_r} + \mathcal{A}_r, \qquad \nabla_i := \partial_{t_i} + \mathcal{A}_i,$$

where $\mathcal{A}_r$ and $\mathcal{A}_i$ are connection one-forms associated with reversible and irreversible time, respectively.

The operators ∇_r and ∇_i act on sections $\Psi \in \Gamma(\mathcal{E})$ and generate parallel transport along the t_r and t_i directions. No assumption is made *a priori* about the commutativity of these operators, allowing for genuinely coupled temporal evolution.

The reversible connection $\mathcal{A}_r$ is assumed to generate time-reversal symmetric dynamics. In many applications, it is natural to take $\mathcal{A}_r$ to be skew-adjoint, corresponding to Hamiltonian or unitary evolution and to group-valued time evolution [18]. In contrast, the irreversible connection $\mathcal{A}_i$ generates dissipative dynamics and is not required to satisfy any symmetry under time reversal; mathematically, it is naturally associated with contraction semigroups or dissipative generators [4,21].

4.2. Coupled Evolution Equations

The temporal evolution of a bi-temporal state Ψ is governed by differential equations involving both temporal covariant derivatives. The most general local evolution law compatible with the dual-time structure may be written as

$$\begin{aligned} \nabla_r \Psi &= \mathcal{L}_r(\Psi), \\ \nabla_i \Psi &= \mathcal{L}_i(\Psi), \end{aligned} \tag{1}$$

where $\mathcal{L}_r$ and $\mathcal{L}_i$ are linear or nonlinear operators acting on the state space.

The operators $\mathcal{L}_r$ and $\mathcal{L}_i$ need not be independent. Coupling between reversible and irreversible dynamics may be encoded explicitly through mixed terms in the operators or implicitly through the geometry of the temporal connections. Similar coupled structures arise in projection-based approaches to non-equilibrium dynamics, though there they are typically formulated within a single temporal parameter [5,6].

Consistency of the evolution requires that mixed second derivatives satisfy an integrability condition. Formally, one obtains

$$[\nabla_r, \nabla_i]\Psi = (\partial_{t_r}\mathcal{A}_i - \partial_{t_i}\mathcal{A}_r + [\mathcal{A}_r, \mathcal{A}_i])\Psi. \tag{2}$$

The commutator on the right-hand side defines a curvature associated with the temporal connections. In geometric terms, this curvature measures the obstruction to integrability of the bi-temporal evolution and plays a role analogous to field strength in gauge theory [19]. Nonvanishing temporal curvature corresponds to a nontrivial coupling between reversible and irreversible temporal evolution and will play a central role in the emergence of temporal asymmetry.

In the special case where

$$[\nabla_r, \nabla_i] = 0,$$

the dual-time evolution is integrable and the two temporal evolutions can be consistently separated. This limit corresponds to systems in which reversible and irreversible dynamics effectively decouple, recovering the structure of conventional single-time evolution equations.

More generally, nonzero temporal curvature induces path dependence in the bi-temporal evolution. That is, the state obtained by evolving first in t_r and then in t_i differs from that obtained by evolving in the opposite order. This path dependence provides a geometric mechanism for memory effects and irreversibility, anticipating the emergence of non-Markovian behavior discussed in later sections [9,10].

The dual temporal evolution equations (1) thus generalize conventional single-time evolution equations by allowing reversible and irreversible processes to coexist and interact within a unified geometric framework.

5. Temporal Topology and Gauge Structure

In this section we analyze the global geometric and topological properties induced by the dual temporal connections. We show that the coupling between reversible and irreversible temporal evolution naturally leads to nontrivial temporal topology and a gauge-theoretic structure on the bi-temporal manifold. The analysis builds on standard results from connection theory and gauge geometry, extended here to a multi-temporal setting [18,19,23].

5.1. Temporal Fiber Bundles

Let $\mathcal{E} \to \mathcal{M}$ be the vector bundle introduced in Section 4, equipped with temporal connections ∇_r and ∇_i. The temporal directions define a rank-two subbundle of the tangent bundle $T\mathcal{M}$, spanned locally by the vector fields ∂_{t_r} and ∂_{t_i}.

The restriction of the connection to this subbundle defines a *temporal connection* on the temporal manifold $\mathcal{T}$. Accordingly, the bundle $\mathcal{E}$ may be regarded as a fiber bundle over $\mathcal{T}$ with typical fiber given by sections over the configuration space $\mathcal{X}$. Such reductions are standard in the study of connections on product manifolds and foliated spaces [24].

Nonvanishing temporal curvature,

$$\mathcal{F}_{ri} := [\nabla_r, \nabla_i],$$

induces a nontrivial holonomy in the temporal directions. In particular, parallel transport of a state Ψ around a closed loop in $\mathcal{T}$ need not be trivial.

Proposition 1. *If $\mathcal{F}_{ri} \neq 0$, the temporal fiber bundle over $\mathcal{T}$ admits nontrivial holonomy, and the resulting parallel transport depends on the homotopy class of the temporal path.*

Proof 1. *This follows directly from standard results in connection theory: nonvanishing curvature implies path-dependent parallel transport and nontrivial holonomy for loops enclosing regions where the curvature is nonzero [23,24].*

Temporal holonomy provides a geometric mechanism for memory and history dependence. The state at a given point in $\mathcal{T}$ retains information about the temporal path by which it was reached, anticipating the emergence of memory effects and non-Markovian dynamics discussed in later sections.

5.2. Gauge Symmetries and Temporal Transformations

The dual-time formalism admits a natural gauge freedom associated with local transformations in the state space. Let G be a Lie group acting smoothly on the fibers of $\mathcal{E}$.

Definition 4 (Temporal Gauge Transformation)**.** *A temporal gauge transformation is a smooth map*

$$U : \mathcal{T} \longrightarrow G$$

acting on states by

$$\Psi \mapsto U\Psi.$$

Under such transformations, the temporal connection components transform according to the usual gauge-covariant rule

$$\mathcal{A}_\alpha \mapsto U\mathcal{A}_\alpha U^{-1} - (\partial_{t_\alpha} U)U^{-1}, \qquad \alpha \in \{r, i\},$$

and the associated temporal curvature transforms covariantly,

$$\mathcal{F}_{ri} \mapsto U\mathcal{F}_{ri}U^{-1}.$$

This gauge structure reflects the freedom to choose different local representations of reversible and irreversible dynamics without altering observable quantities. As in standard gauge theory, physical observables are required to be gauge invariant with respect to temporal transformations [23].

The presence of temporal gauge symmetry allows for the classification of dual-time systems according to topological invariants associated with the temporal curvature. Such invariants characterize inequivalent classes of temporal evolution and provide a natural language for distinguishing systems with fundamentally different irreversibility properties.

The temporal gauge structure introduced here generalizes standard gauge-theoretic constructions by extending them to multiple temporal dimensions. In doing so, it enables a unified treatment of reversible and irreversible dynamics within a single geometric and topological framework, while remaining consistent with established principles of differential geometry and gauge theory.

6. Emergence of Temporal Asymmetry

In this section we show how an effective temporal asymmetry emerges dynamically from the dual-time framework. The key mechanisms are symmetry breaking between the reversible and irreversible temporal sectors and the projection of bi-temporal evolution onto a single observable time parameter. These mechanisms provide a structural explanation for the arrow of time that does not rely on phenomenological assumptions or explicit entropy postulates.

6.1. Symmetry Breaking Between Temporal Sectors

At the level of the dual-time manifold $\mathcal{T} = \mathcal{T}_r \times \mathcal{T}_i$, the reversible and irreversible temporal parameters are treated symmetrically as independent coordinates. However, the associated temporal connections generally possess distinct mathematical properties.

In particular, the reversible temporal connection ∇_r is typically associated with generators that are skew-adjoint or Hamiltonian, giving rise to group-valued time evolution, while the irreversible temporal connection ∇_i is associated with dissipative or contractive generators, leading to semigroup evolution [4,21]. This structural distinction implies that the full temporal symmetry group of the bi-temporal manifold is not preserved by the dynamics.

Definition 5 (Temporal Symmetry). *A dual-time system is said to be temporally symmetric if the evolution equations are invariant under the exchange*

$$(t_r, \nabla_r) \longleftrightarrow (t_i, \nabla_i).$$

In general, such symmetry is explicitly or dynamically broken by the choice of temporal connections and by the presence of nonvanishing temporal curvature $\mathcal{F}_{ri}$. The irreversible temporal direction thereby acquires a distinguished role, even though no preferred time parameter is introduced at the kinematic level.

Proposition 2. *If the temporal curvature $\mathcal{F}_{ri}$ is nonzero and the generators of ∇_r and ∇_i belong to inequivalent operator classes (e.g. skew-adjoint versus dissipative), then temporal symmetry is broken.*

Proof 2. *The temporal curvature couples the two temporal evolutions. If the generators are structurally inequivalent, no automorphism exists that maps one temporal sector onto the other while preserving the evolution equations. Consequently, invariance under exchange of the two temporal directions is lost, implying symmetry breaking [4,21].*

This symmetry breaking is not imposed externally but arises from the intrinsic mathematical properties of the temporal connections and their coupling. In this sense, temporal asymmetry is a dynamical and geometric phenomenon rather than a kinematic assumption.

6.2. Observation-Induced Time Projection

While the bi-temporal formulation provides a complete description of system evolution, physical observations typically depend on a single effective time parameter. We formalize this reduction as a projection from the bi-temporal domain to a one-dimensional temporal manifold.

Let $\mathcal{O}$ denote a class of admissible observables acting on bi-temporal states. An observation is represented mathematically by a projection operator

$$\mathcal{P} : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}_{\text{obs}}),$$

where $\mathcal{E}_{\text{obs}}$ is an effective bundle over a one-dimensional time manifold $\mathcal{T}_{\text{obs}}$. Such projection procedures are familiar from coarse-graining and reduced-dynamics approaches in non-equilibrium statistical mechanics [5,6].

Definition 6 (Observable Time)**.** *An* observable time $t \in \mathcal{T}_{\text{obs}}$ *is defined as a parameter labeling equivalence classes of points in* $\mathcal{T}$ *under the projection* $\mathcal{P}$*.*

Operationally, this projection may correspond to restricting states to a submanifold

$$\Sigma \subset \mathcal{T},$$

or to integrating out one temporal parameter with respect to a suitable measure. In either case, the resulting observable dynamics inherit contributions from both reversible and irreversible temporal evolution.

Proposition 3. *If* $\mathcal{F}_{ri} \neq 0$*, the projected evolution on* $\mathcal{T}_{\text{obs}}$ *is generically non-Markovian.*

Proof 3. *Nonvanishing temporal curvature implies path dependence in the bi-temporal manifold* $\mathcal{T}$*. The projection* $\mathcal{P}$ *identifies distinct bi-temporal paths with the same observable time parameter, resulting in reduced dynamics that depend on the prior temporal history. This violates the semigroup property and leads to effective non-Markovian behavior [9,10].*

The emergence of a single observable time parameter thus results from the combined effects of temporal symmetry breaking and projection. The arrow of time observed at the macroscopic level is therefore an emergent property of the dual-time structure rather than a fundamental assumption.

This mechanism provides a mathematically consistent explanation for the coexistence of reversible microscopic laws and irreversible macroscopic behavior within a unified temporal framework, and it prepares the ground for the analysis of memory effects and aging phenomena in the following section.

7. Memory Effects and Non-Markovian Dynamics

In this section we analyze the dynamical consequences of the dual-time framework for memory effects, non-Markovian behavior, and aging phenomena. We show that these features arise naturally from the geometry of the bi-temporal manifold and the associated temporal curvature, without the introduction of phenomenological memory kernels or ad hoc nonlocal terms. A concrete representation of convolution-type memory within the dual-time framework is given in Section 7.4.

7.1. Bi-Temporal Origin of Memory

In standard single-time formulations of non-equilibrium dynamics, memory effects are typically introduced through integro-differential equations, nonlocal operators, or explicit memory kernels [5,6,9]. In contrast, within the dual-time framework, memory arises as a geometric consequence of coupled temporal evolution.

Let $\Psi(t_r, t_i)$ be a bi-temporal state evolving according to the coupled evolution equations introduced in Section 4. Consider two temporal paths $\gamma_1, \gamma_2 \subset \mathcal{T}$ connecting the same initial and final points in the bi-temporal domain.

If the temporal curvature $\mathcal{F}_{ri}$ vanishes, the evolution along γ_1 and γ_2 yields identical states. However, if $\mathcal{F}_{ri} \neq 0$, the resulting states generally differ.

Proposition 4. *Nonvanishing temporal curvature implies path-dependent bi-temporal evolution and hence encodes memory of past evolution.*

Proof 4. *Path dependence of parallel transport is a direct consequence of nonzero curvature in connection theory [23,24]. Since different temporal paths correspond to different histories of reversible and irreversible evolution, the final state retains information about its past trajectory.*

When the bi-temporal state is projected onto an observable single-time description, this path dependence manifests as effective memory. Formally, the reduced dynamics at time t depend not only on the instantaneous state but also on the prior evolution in the irreversible temporal direction t_i, leading to temporal nonlocality. This provides a geometric underpinning for memory effects observed in a wide range of non-equilibrium systems [11,12].

7.2. Emergence of Non-Markovian Dynamics

Let Φ_t denote the effective evolution operator acting on observable states after projection onto the one-dimensional time manifold $\mathcal{T}_{\text{obs}}$. In Markovian dynamics, the semigroup property holds:

$$\Phi_{t+s} = \Phi_t \circ \Phi_s.$$

In the dual-time framework, this property generally fails due to temporal curvature and projection effects.

Theorem 1. *If $\mathcal{F}_{ri} \neq 0$, the effective evolution $\{\Phi_t\}$ does not form a semigroup and the observable dynamics are non-Markovian.*

Proof 5. *Nonzero temporal curvature induces path dependence in bi-temporal evolution. Upon projection, different decompositions of $t + s$ correspond to inequivalent bi-temporal paths, violating the semigroup property that characterizes Markovian dynamics [9,21].*

This result provides a geometric explanation for non-Markovian dynamics: memory effects emerge not from explicit nonlocal terms or stochastic modeling, but from the reduction of higher-dimensional temporal evolution to a single effective time parameter. Recent studies of non-Markovian processes in classical and quantum systems highlight the ubiquity of such behavior in non-equilibrium settings [10,13].

7.3. Aging and Long-Time Behavior

Aging phenomena in non-equilibrium systems are characterized by the breakdown of time-translation invariance and by the dependence of system behavior on the time elapsed since preparation [4]. Such effects are commonly observed in glasses, driven systems, and active matter, and are often modeled phenomenologically.

Within the dual-time framework, aging arises naturally from the distinguished role of the irreversible temporal parameter t_i. Since t_i is typically defined on a half-line and enters the evolution through a dissipative connection, the system retains an intrinsic reference to its "age" as measured by irreversible time.

Proposition 5. *If the irreversible temporal evolution is non-stationary in t_i, then the projected dynamics exhibit aging behavior.*

Proof 6. *Non-stationarity in t_i implies that the evolution depends explicitly on the absolute value of t_i, not only on temporal differences. After projection, this dependence translates into observables that depend on the time elapsed since initialization, thereby breaking time-translation invariance [4,11].*

Thus, aging is not an additional assumption but a direct consequence of the dual-time structure and the asymmetry between reversible and irreversible temporal sectors.

Taken together, these results show that memory, non-Markovianity, and aging are intrinsic features of the dual-time topological framework. They emerge from the geometry and topology of temporal evolution and provide a unified mathematical explanation for a wide range of non-equilibrium phenomena observed in contemporary theoretical and applied studies.

7.4. A Representation Theorem for Convolution-Type Memory

In this subsection we show that a broad and widely used class of non-Markovian evolution equations with convolution memory kernels admits an explicit dual-time realization on the bi-temporal manifold $T = T_r \times T_i$. In particular, we prove that Volterra-type equations with kernels of the form

$$K(t) = C\,e^{tB}D$$

arise as projections of local evolution equations on T in which the irreversible time t_i plays the role of an "age" or memory coordinate.

Throughout this subsection, let X be a Banach space and let $A, B, C, D : X \to X$ be bounded linear operators. We view A as the generator of reversible dynamics in t_r, B as the generator of irreversible relaxation in t_i, and C, D as bounded coupling operators between the observable and memory sectors.

Definition 7 (Convolution-Type Memory Equation)**.** *Let $K : [0, \infty) \to \mathcal{L}(X)$ be given by*

$$K(t) := C\,e^{tB}D, \qquad t \geq 0. \tag{3}$$

A function $u : [0, \infty) \to X$ is said to solve the convolution-type memory equation if

$$\frac{d}{dt}u(t) = Au(t) + \int_0^t K(t-s)\,u(s)\,ds, \qquad t > 0, \tag{4}$$

with initial condition $u(0) = u_0 \in X$, and the integral is understood in the Bochner sense.

Equations of the form (4) are classical in the theory of Volterra integral equations and provide standard models of non-Markovian memory in both classical and quantum settings.

We now construct an explicit dual-time realization of (4) on the bi-temporal domain $T = T_r \times T_i \cong \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, with coordinates (t_r, t_i), where t_r represents the observable (or reversible) time and t_i represents an irreversible time associated with the "age" of stored memory.

Definition 8 (Dual-Time Memory Field)**.** *Given a candidate observable trajectory $u : [0, \infty) \to X$, a dual-time memory field is a function*

$$\chi : [0, \infty) \times [0, \infty) \to X, \qquad (t_r, t_i) \mapsto \chi(t_r, t_i),$$

interpreted as the contribution at observable time t_r from memory of age t_i.

The dual-time dynamics we consider are given by the following first-order system on T:

$$\frac{\partial}{\partial t_r}u(t_r) = Au(t_r) + \int_0^\infty C\,\chi(t_r, t_i)\,dt_i, \qquad t_r > 0, \tag{5a}$$

$$\left(\frac{\partial}{\partial t_r} + \frac{\partial}{\partial t_i}\right)\chi(t_r, t_i) = B\,\chi(t_r, t_i), \qquad t_r > 0,\ t_i > 0, \tag{5b}$$

with boundary and initial conditions

$$\chi(t_r,0) = Du(t_r), \qquad t_r \geq 0, \tag{5c}$$

$$\chi(0,t_i) = 0, \qquad t_i \geq 0, \tag{5d}$$

$$u(0) = u_0 \in X. \tag{5e}$$

Equation (5b) is a linear transport equation on T, with characteristics $t_r - t_i = \text{const}$, driven by the dissipative operator B and fed at the boundary $t_i = 0$ by the current observable state $u(t_r)$ via (5c). The observable equation (5a) couples u to the entire memory field χ by integrating over all ages $t_i \geq 0$.

The following result shows that this dual-time system is equivalent to the convolution memory equation (4).

Theorem 2 (Dual-Time Representation of Convolution Memory)**.** *Let $A, B, C, D \in \mathcal{L}(X)$ be bounded operators and let $K(t) = Ce^{tB}D$ as in (3).*

(i) Suppose $u \in C^1([0,\infty);X)$ solves the convolution equation (4). Define $\chi : [0,\infty) \times [0,\infty) \to X$ by

$$\chi(t_r,t_i) := \begin{cases} e^{t_iB}D\,u(t_r - t_i), & 0 \leq t_i \leq t_r, \\ 0, & t_i > t_r. \end{cases} \tag{6}$$

Then the pair (u,χ) satisfies the dual-time system (5).

(ii) Conversely, suppose (u,χ) is a pair of continuous functions with $u \in C^1([0,\infty);X)$, χ continuously differentiable on $(0,\infty) \times (0,\infty)$, satisfying (5). Then u solves the convolution equation (4) with kernel $K(t) = Ce^{tB}D$.

In particular, there is a one-to-one correspondence between solutions of the non-Markovian equation (4) and solutions of the local dual-time evolution (5) on T.

Proof 7. *We prove each direction separately.*

(i) From convolution dynamics to dual-time dynamics. *Let u solve (4) and define χ by (6). We first verify that χ satisfies (5b)–(5d).*

Fix (t_r,t_i) with $0 < t_i < t_r$. Differentiating (6) with respect to t_r and t_i yields

$$\partial_{t_r}\chi(t_r,t_i) = e^{t_iB}D\,\frac{d}{dt}u(t_r - t_i), \qquad \partial_{t_i}\chi(t_r,t_i) = Be^{t_iB}D\,u(t_r - t_i) - e^{t_iB}D\,\frac{d}{dt}u(t_r - t_i).$$

Adding these expressions we obtain

$$(\partial_{t_r} + \partial_{t_i})\chi(t_r,t_i) = Be^{t_iB}D\,u(t_r - t_i) = B\,\chi(t_r,t_i),$$

which is precisely (5b) on the interior of the domain.

For the boundary condition at $t_i = 0$ we have

$$\chi(t_r,0) = e^{0\cdot B}D\,u(t_r - 0) = Du(t_r),$$

which is (5c). For $t_r = 0$ and $t_i > 0$ we are in the case $t_i > t_r$ in (6), so $\chi(0,t_i) = 0$, giving (5d). Thus χ satisfies (5b)–(5d).

We now verify (5a). For each fixed $t_r \geq 0$ we have

$$\int_0^\infty C\,\chi(t_r,t_i)\,dt_i = \int_0^{t_r} Ce^{t_iB}D\,u(t_r - t_i)\,dt_i,$$

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since $\chi(t_r, t_i) = 0$ *for* $t_i > t_r$*. Making the change of variables* $s = t_r - t_i$ *(so* $t_i = t_r - s$ *and* $dt_i = -ds$*) we obtain*

$$\int_0^\infty C\,\chi(t_r, t_i)\,dt_i = \int_0^{t_r} Ce^{(t_r - s)B} D\,u(s)\,ds = \int_0^{t_r} K(t_r - s)\,u(s)\,ds.$$

Using this identity in (4) *shows that* (5a) *holds for* $t_r > 0$*. The initial condition* $u(0) = u_0$ *is the same in both formulations. Hence* (u, χ) *satisfies the dual-time system* (5).

(ii) From dual-time dynamics to convolution dynamics. *Conversely, suppose* (u, χ) *satisfies* (5) *with the stated regularity. We first solve* (5b) *along characteristics.*

Fix $t_r > 0$ *and* $t_i > 0$ *and consider the characteristic curve* $\gamma(\tau) = (\tau, t_i + \tau - t_r)$ *passing through* (t_r, t_i) *at* $\tau = t_r$*. Along this curve we have*

$$\frac{d}{d\tau}\chi(\gamma(\tau)) = (\partial_{t_r} + \partial_{t_i})\chi(\tau, t_i + \tau - t_r) = B\,\chi(\gamma(\tau)),$$

where we used (5b)*. Thus*

$$\frac{d}{d\tau}\chi(\gamma(\tau)) = B\,\chi(\gamma(\tau))$$

has the unique solution

$$\chi(\gamma(\tau)) = e^{(\tau - \tau_0)B}\chi(\gamma(\tau_0))$$

for any reference point τ_0 *on the same characteristic.*

If $0 < t_i < t_r$*, the characteristic through* (t_r, t_i) *intersects the boundary* $t_i = 0$ *at* $(t_r - t_i, 0)$*. Taking* $\tau_0 = t_r - t_i$ *and using the boundary condition* (5c) *gives*

$$\chi(t_r, t_i) = e^{(t_r - (t_r - t_i))B}\chi(t_r - t_i, 0) = e^{t_i B} D\,u(t_r - t_i).$$

If instead $t_i > t_r$*, the characteristic through* (t_r, t_i) *intersects the initial line* $t_r = 0$ *at* $(0, t_i - t_r)$*. Taking* $\tau_0 = 0$ *and using* (5d) *yields* $\chi(t_r, t_i) = 0$*. In summary,*

$$\chi(t_r, t_i) = \begin{cases} e^{t_i B} D\,u(t_r - t_i), & 0 \le t_i \le t_r, \\ 0, & t_i > t_r, \end{cases}$$

so χ *is necessarily of the form* (6).

Substituting this expression into (5a) *we obtain, as in part (i),*

$$\frac{d}{dt_r}u(t_r) = Au(t_r) + \int_0^{t_r} Ce^{(t_r - s)B} D\,u(s)\,ds = Au(t_r) + \int_0^{t_r} K(t_r - s)u(s)\,ds,$$

which is exactly the convolution equation (4)*. Thus* u *solves* (4) *with kernel* $K(t) = Ce^{tB}D$*.*

Remark 1 (Dual-Time Geometry and Temporal Curvature)**.** *The construction above shows that non-Markovian evolution with a convolution kernel* $K(t) = Ce^{tB}D$ *can be realized as the projection of a local, first-order dual-time evolution on* $T_r \times T_i$*.*

Geometrically, one may regard the pair (u, χ) *as a section of a vector bundle* $E \to M = X \times T$ *and the operators appearing in* (5) *as defining temporal covariant derivatives along the reversible and irreversible directions, in the sense of Section 4. The irreversible coordinate* t_i *enters only on a half-line, encoding the intrinsic directionality of macroscopic time, while the coupling operators* C *and* D *inject and read out memory from the* t_i*-sector. Unless the corresponding temporal connections decouple (for instance when* $C = D = 0$*), the resulting temporal curvature is nonvanishing, and the effective dynamics in* t_r *are non-Markovian after projection, consistently with the general discussion in Sections 6, 7.*

7.5. *Worked Example: Exponential Memory Kernel*

To make the abstract discussion of Section 7 more concrete, we now consider a standard scalar model with an exponential memory kernel and exhibit its dual-time realization. Such kernels arise, for example, as simple models of viscoelastic relaxation and as effective descriptions of generalized Langevin dynamics with a single environmental time scale [5,9].

Single-Time Formulation

Let $u : [0, \infty) \to X$ be an observable scalar (or vector-valued) quantity, where for this example we may take $X = \mathbb{R}$ or a finite-dimensional Hilbert space. Consider the non-Markovian evolution equation

$$\frac{d}{dt}u(t) = -\omega\, u(t) + \kappa \int_0^t e^{-\gamma(t-s)}\, u(s)\, ds, \qquad t > 0, \tag{7}$$

with parameters $\omega, \gamma > 0$ and coupling strength $\kappa \in \mathbb{R}$. The first term describes instantaneous relaxation with rate ω, while the convolution term introduces a decaying memory of the past with characteristic time γ^{-1}. The kernel

$$K(t) = \kappa e^{-\gamma t}, \qquad t \geq 0,$$

is of the form $K(t) = Ce^{tB}D$ with $B = -\gamma$, $C = \kappa$, $D = 1$ in the notation of Definition 7. Equation (7) is a special case of the convolution-type memory equation (4) with

$$A = -\omega, \qquad B = -\gamma, \qquad C = \kappa, \qquad D = 1. \tag{8}$$

Dual-Time Realization

We now realize (7) as the projection of a local dual-time evolution on $T = T_r \times T_i \cong \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, following the general construction of Section 7.4. Let t_r denote the reversible time parameter and t_i the irreversible "age" parameter. Define the bi-temporal observable and memory field

$$u : [0, \infty) \to X, \qquad \chi : [0, \infty) \times [0, \infty) \to X,$$

and consider the dual-time system

$$\frac{\partial}{\partial t_r}u(t_r) = -\omega\, u(t_r) + \int_0^\infty \kappa\, \chi(t_r, t_i)\, dt_i, \qquad t_r > 0, \tag{9a}$$

$$\left(\frac{\partial}{\partial t_r} + \frac{\partial}{\partial t_i}\right)\chi(t_r, t_i) = -\gamma\, \chi(t_r, t_i), \qquad t_r > 0,\ t_i > 0, \tag{9b}$$

with boundary and initial conditions

$$\chi(t_r, 0) = u(t_r), \qquad t_r \geq 0, \tag{9c}$$

$$\chi(0, t_i) = 0, \qquad t_i \geq 0, \tag{9d}$$

$$u(0) = u_0 \in X. \tag{9e}$$

Equation (9b) describes the irreversible evolution of the memory field along characteristics of slope -1 in the (t_r, t_i)-plane, damped at rate γ, while the boundary condition (9c) injects freshly created memory at zero age. The observable evolution (9a) couples u to the entire age distribution $\chi(t_r, \cdot)$ via the integral over t_i.

The following specialization of Theorem 2 shows that the dual-time system (9) is equivalent to the single-time non-Markovian equation (7).

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Proposition 6 (Dual-time representation of exponential memory). *Let $\omega, \gamma > 0$ and $\kappa \in \mathbb{R}$, and let u be a C^1 solution of (7). Define $\chi : [0,\infty) \times [0,\infty) \to X$ by*

$$\chi(t_r, t_i) := \begin{cases} \kappa e^{-\gamma t_i} u(t_r - t_i), & 0 \le t_i \le t_r, \\ 0, & t_i > t_r. \end{cases} \tag{10}$$

Then (u, χ) satisfies the dual-time system (9). Conversely, if (u, χ) satisfies (9) with $u \in C^1([0,\infty); X)$ and χ continuously differentiable on $(0,\infty) \times (0,\infty)$, then u solves the single-time equation (7).

Proof 8. *The proof is a direct specialization of Theorem 2 to the choice of operators (8). For completeness, we sketch the main steps.*

Assume first that u solves (7) and define χ by (10). For $0 < t_i < t_r$, differentiating with respect to t_r and t_i yields

$$\partial_{t_r} \chi(t_r, t_i) = \kappa e^{-\gamma t_i} \frac{d}{dt} u(t_r - t_i), \quad \partial_{t_i} \chi(t_r, t_i) = -\gamma \chi(t_r, t_i) - \kappa e^{-\gamma t_i} \frac{d}{dt} u(t_r - t_i).$$

Adding these two equations gives

$$(\partial_{t_r} + \partial_{t_i}) \chi(t_r, t_i) = -\gamma \chi(t_r, t_i),$$

which is (9b). The boundary and initial conditions (9c)–(9d) are immediate from (10). For the observable evolution, we compute

$$\int_0^\infty \kappa\, \chi(t_r, t_i)\, dt_i = \int_0^{t_r} \kappa^2 e^{-\gamma t_i} u(t_r - t_i)\, dt_i.$$

By the change of variables $s = t_r - t_i$ this becomes

$$\int_0^\infty \kappa\, \chi(t_r, t_i)\, dt_i = \kappa \int_0^{t_r} e^{-\gamma (t_r - s)} u(s)\, ds,$$

and substituting into (9a) we recover (7). The converse direction proceeds exactly as in the proof of Theorem 2: solving (9b) along characteristics yields (10), and inserting this expression into (9a) reproduces the convolution equation (7).

Geometric Interpretation and Temporal Curvature

In the language of Section 4, the pair (u, χ) can be viewed as a section of a vector bundle $E \to M = X \times T$ whose fiber at each (t_r, t_i) consists of an observable component and an age-resolved memory profile. The dual-time evolution (9a)–(9b) is generated by temporal covariant derivatives of the form

$$\nabla_r = \partial_{t_r} + A_r, \qquad \nabla_i = \partial_{t_i} + A_i,$$

where the connection components A_r and A_i act on the extended state (u, χ) by combining the reversible generator $-\omega$ with the dissipative generator $-\gamma$ and the coupling κ that injects and reads out memory from the t_i-sector. When $\kappa \neq 0$ and $\gamma > 0$, the temporal connections do not commute: the associated temporal curvature

$$F_{ri} = [\nabla_r, \nabla_i] = \partial_{t_r} A_i - \partial_{t_i} A_r + [A_r, A_i]$$

is nonvanishing on the memory sector. In particular, evolving first along t_r and then along t_i yields a different bi-temporal state from evolving in the opposite order, reflecting the fact that the contribution of older memory $(t_i > 0)$ is progressively damped.

Under projection to a single observable time parameter t, obtained by identifying $t \equiv t_r$ and integrating over the irreversible coordinate t_i, this path dependence appears as the convolution term in (7). The instantaneous rate of change $\dot{u}(t)$ depends on the entire history $\{u(s) : 0 \le s \le t\}$, with weights determined by the temporal curvature through the damping and coupling parameters γ and κ. In this way, the simple model (7) can be seen as a concrete manifestation of the general mechanism discussed in Section 7: nonvanishing temporal curvature on the bi-temporal manifold produces memory and non-Markovianity after projection onto a single observable time.

7.6. Example: Non-Markovian Open Quantum Two-Level System

We now illustrate the dual-time construction in a genuinely quantum setting by considering a two-level system coupled to a structured reservoir. Such systems are standard testbeds for quantum non-Markovianity and memory effects [9,10]. In particular, non-exponential relaxation and information backflow can be captured effectively by generalized master equations with convolution memory kernels.

Single-Time Non-Markovian Master Equation

Let $\mathcal{H} \cong \mathbb{C}^2$ be the system Hilbert space and let $\rho(t)$ denote the reduced density operator at physical time $t \geq 0$. We consider a two-level system with Hamiltonian

$$H_S = \frac{\Omega}{2}\sigma_z, \tag{11}$$

where σ_z is the Pauli z-matrix and Ω the level splitting. In the Markovian regime, spontaneous emission into a memoryless bath is described by a Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generator [7,8]

$$\mathcal{L}_{\mathrm{M}}[\rho] = -i[H_S,\rho] + \gamma_0\Big(\sigma_-\rho\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-,\rho\}\Big), \tag{12}$$

where $\sigma_\pm$ are the raising and lowering operators and $\gamma_0 > 0$ is the Markovian decay rate.

In a structured environment with a finite correlation time, microscopic derivations (e.g. via the Nakajima–Zwanzig projection operator method) lead instead to a generalized master equation of the form [9,10]

$$\frac{d}{dt}\rho(t) = -i[H_S,\rho(t)] + \int_0^t k(t-s)\,\mathcal{D}[\rho(s)]\,ds, \qquad t > 0, \tag{13}$$

where $\mathcal{D}$ is a dissipator of GKSL type,

$$\mathcal{D}[\rho] := \sigma_-\rho\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-,\rho\}, \tag{14}$$

and $k : [0,\infty) \to \mathbb{R}$ is a scalar memory kernel encoding reservoir correlations. A widely used and analytically tractable choice is the exponential kernel

$$k(t) = g^2 e^{-\gamma t}, \qquad t \geq 0, \tag{15}$$

with coupling amplitude $g > 0$ and reservoir correlation rate $\gamma > 0$. Substituting (15) into (13) yields

$$\frac{d}{dt}\rho(t) = -i[H_S,\rho(t)] + g^2\int_0^t e^{-\gamma(t-s)}\,\mathcal{D}[\rho(s)]\,ds. \tag{16}$$

Equation (16) is a Volterra-type evolution equation on the Banach space X of trace-class operators on $\mathcal{H}$, with kernel superoperator

$$K(t)[\rho] := g^2 e^{-\gamma t}\,\mathcal{D}[\rho], \qquad t \geq 0. \tag{17}$$

Comparing with Definition 7, we identify

$$A\rho = -i[H_S,\rho], \quad B = -\gamma\,\mathrm{id}_X, \quad C = g^2\mathcal{D}, \quad D = \mathrm{id}_X. \tag{18}$$

The non-Markovian master equation (16) is therefore a special case of the convolution memory equation (4) with operator-valued kernel $K(t) = Ce^{tB}D$ on X.

Dual-Time Realization in Liouville Space

We now realize the dynamics (16) as the projection of a local dual-time evolution on the bi-temporal manifold $T = T_r \times T_i \cong \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Let t_r denote the reversible temporal coordinate and t_i the irreversible "age" coordinate as in Section 3. We introduce:

- an observable density operator $\rho : [0, \infty) \to X$, and
- an age-resolved quantum memory field $\chi : [0, \infty) \times [0, \infty) \to X$,

and consider the dual-time system

$$\frac{\partial}{\partial t_r}\rho(t_r) = -i[H_S, \rho(t_r)] + \int_0^\infty C\,\chi(t_r, t_i)\,dt_i, \qquad t_r > 0, \tag{19a}$$

$$\left(\frac{\partial}{\partial t_r} + \frac{\partial}{\partial t_i}\right)\chi(t_r, t_i) = B\,\chi(t_r, t_i), \qquad t_r > 0,\ t_i > 0, \tag{19b}$$

with boundary and initial conditions

$$\chi(t_r, 0) = D\,\rho(t_r) = \rho(t_r), \qquad t_r \geq 0, \tag{19c}$$

$$\chi(0, t_i) = 0, \qquad t_i \geq 0, \tag{19d}$$

$$\rho(0) = \rho_0, \tag{19e}$$

where A, B, C, D are given by (18). Equation (19b) is a linear transport equation in the (t_r, t_i)-plane on the space of density operators, damped at rate γ along characteristics of slope -1. The boundary condition (19c) injects freshly created memory at zero age proportional to the current system state $\rho(t_r)$. The observable evolution (19a) couples the system to the entire age-distribution $\chi(t_r, \cdot)$ via the integral over t_i.

As a direct corollary of Theorem 2 we obtain:

Proposition 7 (Dual-time representation of a non-Markovian qubit)**.** *Let H_S, $\mathcal{D}$, g and γ be as above. Then:*

(i) *If ρ is a C^1 solution of the generalized master equation (16), the memory field $\chi : [0, \infty) \times [0, \infty) \to X$ defined by*

$$\chi(t_r, t_i) := \begin{cases} e^{t_i B} D\,\rho(t_r - t_i) = e^{-\gamma t_i}\rho(t_r - t_i), & 0 \leq t_i \leq t_r, \\ 0, & t_i > t_r, \end{cases} \tag{20}$$

yields a pair (ρ, χ) that satisfies the dual-time system (19).

(ii) *Conversely, if (ρ, χ) satisfies (19) with $\rho \in C^1([0, \infty); X)$ and χ continuously differentiable on $(0, \infty) \times (0, \infty)$, then ρ solves the single-time non-Markovian master equation (16).*

In particular, there is a one-to-one correspondence between solutions of the non-Markovian open-system dynamics (16) and solutions of the local dual-time evolution (19) on $T_r \times T_i$.

Proof. This is an immediate specialization of Theorem 2 to the operator choices (18). For completeness we outline the argument.

Assume first that ρ solves (16) and define χ by (20). For $0 < t_i < t_r$ we obtain

$$\partial_{t_r}\chi(t_r, t_i) = e^{-\gamma t_i}\frac{d}{dt}\rho(t_r - t_i), \quad \partial_{t_i}\chi(t_r, t_i) = -\gamma\,\chi(t_r, t_i) - e^{-\gamma t_i}\frac{d}{dt}\rho(t_r - t_i),$$

so that

$$(\partial_{t_r} + \partial_{t_i})\chi(t_r, t_i) = -\gamma\,\chi(t_r, t_i) = B\,\chi(t_r, t_i),$$

which is (19b). The boundary and initial conditions (19c)–(19d) follow immediately from (20). Moreover,

$$\int_0^\infty C\,\chi(t_r,t_i)\,dt_i = g^2\int_0^{t_r} e^{-\gamma t_i}\mathcal{D}[\rho(t_r-t_i)]\,dt_i.$$

With the change of variables $s = t_r - t_i$ this becomes

$$\int_0^\infty C\,\chi(t_r,t_i)\,dt_i = g^2\int_0^{t_r} e^{-\gamma(t_r-s)}\mathcal{D}[\rho(s)]\,ds,$$

and substitution into (19a) reproduces the single-time equation (16). The converse direction is obtained by solving (19b) along characteristics, which forces χ to take the form (20), and inserting this expression into (19a), exactly as in the proof of Theorem 2. □

Geometric Interpretation and Temporal Curvature

From the dual-time perspective, the pair (ρ,χ) is a section of a vector bundle $E \to M = X \times T$ whose fiber at each bi-temporal point (t_r,t_i) consists of the instantaneous reduced state and an age-resolved memory profile of the open system. The reversible generator $A\rho = -i[H_S,\rho]$ acts along the t_r-direction, while the irreversible generator $B = -\gamma\,\mathrm{id}_X$ governs relaxation of the memory field along the (t_r,t_i) characteristics. The superoperator $C = g^2\mathcal{D}$ injects the effect of memory back into the observable sector through (19a), and the boundary condition (19c) implements continuous writing of the current system state into the t_i-sector.

In the notation of Section 4, one may regard the dual-time dynamics as generated by temporal covariant derivatives

$$\nabla_r = \partial_{t_r} + A_r, \qquad \nabla_i = \partial_{t_i} + A_i,$$

acting on (ρ,χ), where A_r and A_i are block operators on the extended Liouville space encoding A, B, and the couplings C, D. Whenever $g \neq 0$ and $\gamma > 0$, these temporal connections do not commute: the corresponding temporal curvature

$$F_{ri} = [\nabla_r, \nabla_i]$$

is nonzero on the memory sector. Geometrically, evolving first in the reversible direction t_r and then in the irreversible direction t_i produces a different joint state (ρ,χ) than evolving in the opposite order, because intermediate memory of different ages has been damped at different rates. This path dependence is precisely the non-Markovian memory encoded in the convolution term of the single-time master equation (16).

Upon projection onto a single observable time $t \equiv t_r$, with the irreversible coordinate t_i integrated out as in (19a), the temporal curvature manifests as a history-dependent effective generator. The instantaneous rate of change $\dot\rho(t)$ depends on the entire history $\{\rho(s) : 0 \le s \le t\}$, weighted by the exponentially decaying kernel $k(t-s)$. In this way, the non-Markovian open quantum dynamics of a two-level system in a structured reservoir arise as a concrete realization of the general mechanism discussed in Section 7: nonvanishing temporal curvature on the bi-temporal manifold leads to non-Markovian behavior after projection to a single observable time.

8. Reduction to Standard Non-Equilibrium Models

In this section we analyze the relationship between the dual-time framework and conventional single-time descriptions of non-equilibrium dynamics. We identify conditions under which the dual-time evolution reduces to standard Markovian or semigroup-based models and clarify the scope of the generalization introduced in this work.

8.1. Single-Time Limits

The dual-time formulation admits several limiting regimes in which one of the temporal parameters becomes dynamically irrelevant or effectively slaved to the other. Representative cases include:

- *Reversible-time dominance*: the irreversible temporal connection vanishes or becomes constant, yielding effectively reversible dynamics;
- *Irreversible-time dominance*: reversible evolution becomes trivial or averages out, leaving purely dissipative evolution;
- *Integrable temporal structure*: the temporal curvature $\mathcal{F}_{ri}$ vanishes.

We focus on the integrable case, which provides the clearest connection to standard single-time models.

Proposition 8. *If* $\mathcal{F}_{ri} = 0$*, the dual temporal connections commute and there exists a coordinate transformation on* $\mathcal{T}$ *such that the evolution depends on a single effective time parameter.*

Proof 9. *Vanishing curvature implies that the temporal covariant derivatives commute. By Frobenius' theorem, the corresponding temporal distribution is integrable, allowing the introduction of adapted coordinates in which evolution proceeds along a single temporal direction [19,24].*

In this regime, the bi-temporal state reduces to

$$\Psi(x, t_r, t_i) = \psi(x, t),$$

where t is an effective time parameter determined by the chosen foliation of the temporal manifold $\mathcal{T}$.

8.2. Recovery of Known Frameworks

Under the reduction described above, the dual-time evolution equations reduce to familiar and widely studied models.

Hamiltonian and Unitary Dynamics.

If the irreversible temporal connection is absent or trivial and the reversible connection ∇_r generates unitary or Hamiltonian evolution, the framework reduces to standard Hamiltonian mechanics or Schrödinger dynamics [18].

Markovian Dissipative Dynamics.

If reversible evolution decouples and the irreversible temporal connection generates a contraction semigroup, the resulting dynamics are governed by a single-time evolution equation of the form

$$\partial_t \psi = \mathcal{L}\psi,$$

where $\mathcal{L}$ is a dissipative generator. This includes classical master equations, kinetic equations, and Lindblad-type quantum dynamics [4,7,8,21].

Stochastic and Kinetic Models.

When irreversible evolution dominates and fluctuations are incorporated through stochastic terms, the reduced framework recovers stochastic differential equations and kinetic models commonly used in non-equilibrium statistical mechanics [5,6].

Remark 2. *In all cases, the reduced single-time dynamics correspond to special limits of the dual-time framework in which temporal curvature vanishes or becomes dynamically negligible. Outside these limits, memory effects and non-Markovian behavior persist and cannot be captured by conventional single-time models [9,10].*

This analysis demonstrates that the dual-time formalism provides a genuine generalization of existing non-equilibrium frameworks. It reproduces standard models in appropriate limits while extending them to regimes characterized by coupled reversible and irreversible temporal evolution, nonvanishing temporal curvature, and emergent memory effects.

9. Relation to Existing Approaches

In this section we situate the proposed dual-time topological framework within the broader landscape of non-equilibrium mathematical physics. Rather than providing an exhaustive review, we highlight representative approaches and clarify the conceptual and mathematical distinctions relative to the present work.

9.1. Single-Time Non-Equilibrium Frameworks

Traditional approaches to non-equilibrium systems are formulated in terms of a single temporal parameter and typically rely on one or more of the following strategies:

- coarse-graining or projection from microscopic reversible dynamics,
- introduction of dissipative terms or stochastic forces,
- semigroup formulations encoding irreversibility.

Canonical examples include classical master equations, kinetic theory, and projection-operator methods [4–6], as well as quantum dynamical semigroups generated by Lindblad-type operators [7,8]. While these frameworks are mathematically well established and have broad applicability, irreversibility is encoded at the level of effective evolution equations rather than at the level of the underlying temporal structure.

As a result, memory effects and non-Markovian behavior typically enter through phenomenological modifications, such as integro-differential equations or time-dependent generators [9,10]. In contrast, the dual-time framework introduced here separates reversible and irreversible evolution already at the kinematic level by assigning independent temporal parameters. Irreversibility then emerges from the geometric and topological properties of temporal evolution rather than from additional assumptions imposed on single-time dynamics.

9.2. Extended Phase-Space and Two-Time Formalisms

Various extensions of classical and quantum mechanics introduce additional variables or time-like parameters to account for dissipation, memory, or delayed interactions. These include extended phase-space constructions and auxiliary-variable methods in non-equilibrium thermodynamics [17], as well as two-time formalisms commonly used in the theory of correlation and response functions.

Such approaches often employ multiple time arguments for observables or correlation functions, while retaining a single underlying physical time governing system evolution. By contrast, the present framework introduces a genuinely bi-temporal manifold on which states themselves are defined. Both temporal parameters enter symmetrically at the level of geometry and dynamics, and neither is treated as an auxiliary or derived quantity.

9.3. Thermal Time and Emergent Time Proposals

The idea that time may be emergent rather than fundamental has been explored in several contexts, including relational approaches to dynamics and the thermal time hypothesis [20]. In these frameworks, an effective temporal flow is defined in terms of states, entropy, or observables rather than being imposed a priori.

The dual-time framework shares with these approaches the view that the observed arrow of time may be emergent. However, it differs in that emergence arises from the projection of a higher-dimensional temporal structure rather than from state-dependent definitions of time. Moreover, the present formulation remains compatible with standard dynamical evolution equations in appropriate limits, ensuring continuity with established non-equilibrium models.

9.4. Geometric and Gauge-Theoretic Approaches

Geometric formulations of dynamics based on connections, curvature, and gauge symmetries are well established in classical and quantum mechanics [18,19]. More recently, geometric methods have been extended to non-equilibrium systems, including geometric formulations of thermodynamics, dissipation, and gradient flows.

The present work contributes to this line of research by extending gauge-theoretic and geometric methods to multiple temporal dimensions. Temporal curvature plays a central role, providing a unified geometric origin for memory effects, non-Markovianity, and temporal asymmetry. This distinguishes the approach from geometric formulations that remain confined to single-time evolution.

9.5. Conceptual Distinctions

The defining features of the dual-time topological framework may be summarized as follows:

- reversible and irreversible dynamics are represented by independent temporal parameters;
- temporal asymmetry emerges from curvature, symmetry breaking, and projection;
- memory and aging arise geometrically rather than phenomenologically;
- standard non-equilibrium models are recovered as special limiting cases.

Taken together, these features position the framework as a unifying and generalizing approach to non-equilibrium mathematical physics. It complements existing methods while offering new structural insights into the emergence of the arrow of time from the geometry and topology of temporal evolution.

10. Conclusions and Outlook

In this work we have developed a dual-time topological framework for the mathematical description of non-equilibrium systems, aimed at reconciling reversible microscopic dynamics with irreversible macroscopic behavior. By introducing two independent but coupled temporal parameters, the framework provides a unified geometric setting in which reversible and irreversible processes coexist already at the kinematic level, rather than being combined only at the level of effective equations.

The central mathematical structures of the framework are a bi-temporal manifold, independent temporal connections, and the associated temporal curvature. We have shown that nonvanishing temporal curvature induces path dependence in temporal evolution, leading naturally to memory effects, non-Markovian dynamics, and aging phenomena. Temporal asymmetry emerges dynamically through symmetry breaking between the temporal sectors and through projection onto a single observable time parameter, without the need for phenomenological assumptions, explicit memory kernels, or entropy-based postulates [4,9].

A key feature of the framework is its compatibility with established non-equilibrium models. Standard single-time evolution equations, including Hamiltonian dynamics, dissipative semigroup evolution, and Markovian master equations, are recovered in integrable or decoupling limits. This demonstrates that the proposed formalism constitutes a genuine generalization rather than an alternative formulation, extending conventional approaches to regimes characterized by coupled reversible and irreversible temporal evolution and nontrivial temporal topology [7,8,21].

Several directions for future work naturally arise from the present analysis. From a mathematical perspective, the classification of temporal topologies and curvature-induced invariants on bi-temporal manifolds warrants further investigation, particularly in relation to holonomy, gauge structure, and equivalence classes of temporal evolution. Extensions to infinite-dimensional configuration spaces, operator-algebraic formulations, and stochastic evolutions also present promising avenues for further study [19,24]. From a physical standpoint, applications to quantum non-equilibrium systems, open quantum dynamics, and systems exhibiting strong memory effects and aging are of particular interest.

More broadly, the dual-time framework provides a mathematically consistent approach to the emergence of the arrow of time in non-equilibrium theoretical physics. By encoding irreversibility in the geometry and topology of temporal evolution, it offers a unified perspective that bridges microscopic reversibility and macroscopic temporal asymmetry within a single coherent formalism, complementing existing non-equilibrium theories while revealing new structural mechanisms underlying temporal asymmetry.

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Appendix A Technical Proofs and Auxiliary Results

This appendix collects technical arguments and auxiliary results that support statements made in the main text. These results are standard consequences of differential geometry and operator theory, but are included here for completeness.

Appendix A.1 Integrability of the Dual-Time Distribution

We recall that the temporal covariant derivatives ∇_r and ∇_i define a rank-two distribution on the temporal manifold $\mathcal{T}$.

Lemma 1. *If the temporal curvature* $\mathcal{F}_{ri} = [\nabla_r, \nabla_i]$ *vanishes identically, then the temporal distribution spanned by* ∇_r *and* ∇_i *is integrable.*

Proof 10. *Vanishing curvature implies that the covariant derivatives commute. By Frobenius' theorem, any involutive distribution is integrable. Hence there exists a local foliation of* $\mathcal{T}$ *by one-dimensional submanifolds along which evolution may be parametrized by a single effective time variable.*

This result underlies the reduction to standard single-time dynamics discussed in Section 8.

Appendix A.2 Path Dependence and Holonomy

Let γ_1 and γ_2 be two smooth temporal paths in $\mathcal{T}$ connecting the same endpoints.

Lemma 2. *If* $\mathcal{F}_{ri} \neq 0$*, parallel transport of a bi-temporal state along* γ_1 *and* γ_2 *yields different final states in general.*

Proof 11. *Nonvanishing curvature implies nontrivial holonomy of the temporal connection. Standard results in connection theory show that parallel transport around closed loops enclosing regions of nonzero curvature depends on the homotopy class of the loop. The claim follows by composing* γ_1 *with the reverse of* γ_2*.*

This lemma provides the technical basis for the emergence of memory effects discussed in Section 7.

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Appendix B Illustrative Examples

In this appendix we present simple illustrative examples that demonstrate how the dual-time framework operates in concrete settings. These examples are not intended as exhaustive models, but as conceptual demonstrations.

Appendix B.1 Linear Finite-Dimensional Toy Model

Consider a finite-dimensional state vector $\Psi \in \mathbb{C}^n$ evolving according to

$$\nabla_r \Psi = A\Psi, \qquad \nabla_i \Psi = B\Psi,$$

where A is skew-adjoint and B is dissipative.

If $[A, B] \neq 0$, the temporal curvature

$$\mathcal{F}_{ri} = [A, B]$$

is nonvanishing. The resulting evolution depends on the order in which reversible and irreversible temporal evolution is applied, illustrating path dependence and memory even in a linear setting.

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Appendix B.2 Markovian Limit

If A and B commute, the temporal curvature vanishes and the evolution reduces to

$$\Psi(t_r, t_i) = e^{t_r A + t_i B}\Psi(0,0).$$

Choosing a foliation $t = t_i$ yields a standard Markovian semigroup evolution

$$\partial_t \psi = B\psi,$$

demonstrating explicitly how conventional single-time dynamics are recovered as a special case.

Appendix B.3 Emergent Memory under Projection

Let the observable state be defined by projection

$$\psi(t) = \Psi(t_r(t), t_i(t)).$$

If $\mathcal{F}_{ri} \neq 0$, different choices of the temporal path leading to the same observable time t yield different values of $\psi(t)$. This simple construction illustrates how non-Markovian behavior arises from projection of bi-temporal evolution.

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These appendices illustrate that the dual-time framework is mathematically consistent, technically well founded, and capable of reproducing both standard and nonstandard dynamical behavior within a unified geometric setting.

References

1. Boltzmann, L. *Lectures on Gas Theory*; Dover Publications, 1995. Translated from the original German edition.
2. Gibbs, J.W. *Elementary Principles in Statistical Mechanics*; Yale University Press, 1902.
3. Prigogine, I.; Lebowitz, J.L., Eds. *Nonequilibrium Statistical Mechanics*; Wiley, 1999.
4. Spohn, H. *Large Scale Dynamics of Interacting Particles*; Springer, 1991.
5. Zwanzig, R. *Nonequilibrium Statistical Mechanics*; Oxford University Press, 2001.
6. Mori, H. Transport, Collective Motion, and Brownian Motion. *Progress of Theoretical Physics* **1965**, *33*, 423–455.
7. Lindblad, G. On the Generators of Quantum Dynamical Semigroups. *Communications in Mathematical Physics* **1976**, *48*, 119–130.
8. Gorini, V.; Kossakowski, A.; Sudarshan, E.C.G. Completely Positive Dynamical Semigroups of N-Level Systems. *Journal of Mathematical Physics* **1976**, *17*, 821–825.
9. Breuer, H.P.; Petruccione, F. *The Theory of Open Quantum Systems*; Oxford University Press, 2002.
10. Rivas, Á.; Huelga, S.F.; Plenio, M.B. Quantum Non-Markovianity: Characterization, Quantification and Detection. *Reports on Progress in Physics* **2014**, *77*, 094001.
11. Meyer, H.; Pelagejcev, P.; Schilling, T. Emergence of Memory in Equilibrium versus Nonequilibrium Systems. *Physical Review Letters* **2024**, *132*, 147101.
12. Dalton, B.A.; Ayaz, C.; Kiefer, H.; Klimek, A.; Tepper, L.; Netz, R.R. Transition-path probability depends non-monotonically on memory time for non-Markovian systems. *Proceedings of the National Academy of Sciences* **2023**, *120*, e2220068120.
13. Lavacchi, L.; Dalton, B.A.; Netz, R.R. Non-Markovian equilibrium and non-equilibrium barrier-crossing kinetics in asymmetric double-well potentials. *European Physical Journal E* **2025**, *48*, 26.
14. Abbasi, A.; Netz, R.R.; Naji, A. Non-Markovian Modeling of Nonequilibrium Fluctuations and Dissipation in Active Viscoelastic Biomatter. *Physical Review Letters* **2023**, *131*, 228202.
15. Strachan, D.J.; Purkayastha, A.; Clark, S.R. Non-Markovian Quantum Mpemba Effect. *arXiv:2402.05756* **2024**. Preprint.
16. Dirac, P.A.M. Relativistic Quantum Mechanics. *Proceedings of the Royal Society of London A* **1932**, *136*, 453–464.
17. Grmela, M.; Öttinger, H.C. Dynamics and Thermodynamics of Complex Fluids. I. Development of a General Formalism. *Physical Review E* **1997**, *56*, 6620–6632.
18. Abraham, R.; Marsden, J.E. *Foundations of Mechanics*; Benjamin/Cummings, 1978.
19. Marsden, J.E.; Ratiu, T.S. *Introduction to Mechanics and Symmetry*; Springer, 1999.

doi:10.20944/preprints202601.2341.v1

20. Connes, A.; Rovelli, C. Von Neumann Algebra Automorphisms and Time–Thermodynamics Relation in Generally Covariant Quantum Theories. *Classical and Quantum Gravity* **1994**, *11*, 2899–2918.
21. Engel, K.J.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Springer, 2000.
22. Ambrosio, L.; Gigli, N.; Savaré, G. *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*; Birkhäuser, 2005.
23. Nakahara, M. *Geometry, Topology and Physics*; CRC Press, 2003.
24. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry, Vol. I*; Wiley, 1963.