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Article

Novel Series Expansion of sin(x) by Synthesizing the Weierstrass Product with Explicit Zeta Function Components

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Abstract

In this paper, we derive explicit exponential series representations for the sine function involving even values of the Riemann zeta function. The result is obtained via logarithmic differentiation and integration of the Weierstrass product. We further demonstrate that the method extends naturally to the cosine function, yielding an analogous representation. These results highlight a structural connection between trigonometric entire functions and the distribution of their zeros.

Keywords: trigonometry; zeta function; sine function; logarithmic derivative; Weierstrass Product

Introduction

If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π [1]. The Weierstrass Product of $\sin x$ is given by

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

[3]. The sine product formula is important in mathematics because it has many applications, including the proofs of other problems. One such application is the calculation of the values of $\zeta(2)$ and $\zeta(4)$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Riemann zeta function [2].

The method is used because the properties of logarithms provide avenues to quickly simplify complicated functions to be differentiated. These properties can be manipulated after the taking of natural logarithms on both sides and before the preliminary differentiation [4].

Applications of the sine product formula include the calculation of certain values of the Riemann zeta function. The proof that $\zeta(2) = 2\pi/6$ is often called Euler's Theorem [2].

The value of $\zeta(4)$ is $4\pi/90$ [2].

Methods

Weierstrass Product:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Taking the logarithmic derivative,

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x - n\pi} + \frac{1}{x + n\pi} \right)$$

Expand the cotangent's partial fraction for $x \neq 0$,

$$\cot x = \frac{1}{x} + 2 \sum_{n=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} x^{2k-1}$$

Integrating term by term,

$$\ln(\sin x) = \ln x - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}} x^{2k}$$

Exponentiate to construct new series:

$$\sin x = x \exp \left(- \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}} x^{2k} \right)$$

Results

By all of these derivations we can conclude that:

$$\sin x = x \exp \left(- \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}} x^{2k} \right)$$

Expand the exponential,

$$\sin x = x \left(1 - \frac{1}{6}x^2 + \left(\frac{1}{72} - \frac{1}{90} \right) x^4 - \dots \right)$$

This given series is converging for $\sin x$ for $x \in \mathbb{R}$.

Discussion

The key features of the new derived series are:

- 1.Zeta Function Dependence
- 2.Convergence Properties
- 3.Hybrid Form

Extension to *cosx*

In this section, we demonstrate that the methodology developed for the sine function extends naturally to the cosine function. This confirms that the appearance of Riemann zeta values in the derived exponential series is not incidental, but rather a consequence of the underlying Weierstrass factorization framework.

It is well known that the cosine function admits the following Weierstrass product representation:

$$\cos x = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2} \right),$$

which converges uniformly on compact subsets of the complex plane.

Taking the natural logarithm, we obtain

$$\log (\cos x) = \sum_{n=1}^{\infty} \log \left(1 - \frac{4x^2}{(2n-1)^2\pi^2} \right).$$

Differentiating term by term, justified by uniform convergence away from the zeros of the cosine function, yields

$$\frac{d}{dx} \log (\cos x) = -\tan x.$$

On the other hand, differentiating the logarithmic product gives

$$\frac{d}{dx} \log (\cos x) = - \sum_{n=1}^{\infty} \frac{8x}{(2n-1)^2\pi^2 - 4x^2}.$$

This expression is consistent with the classical partial fraction expansion of the tangent function. Hence, we obtain the identity

$$\tan x = \sum_{n=1}^{\infty} \frac{8x}{(2n-1)^2\pi^2-4x^2}.$$

For $|x|<\frac{\pi}{2}$, the above series converges uniformly, allowing termwise integration. Integrating both sides with respect to x , we obtain

$$\log (\cos x) = - \sum_{n=1}^{\infty} \int \frac{8x}{(2n-1)^2\pi^2-4x^2} dx.$$

Evaluating the integral yields

$$\log (\cos x) = - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{2k}}{k(2n-1)^{2k}\pi^{2k}} x^{2k}.$$

Interchanging the order of summation (justified by absolute convergence), we obtain

$$\log (\cos x) = - \sum_{k=1}^{\infty} \frac{2^{2k}}{k\pi^{2k}} \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} \right) x^{2k}.$$

The inner sum over odd integers can be expressed in terms of the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = (1-2^{-2k})\zeta(2k).$$

Substituting this into the previous expression gives

$$\log (\cos x) = - \sum_{k=1}^{\infty} \frac{2^{2k}(1-2^{-2k})}{k\pi^{2k}} \zeta(2k) x^{2k}.$$

Exponentiating both sides, we arrive at the main result of this section.

For $|x|<\frac{\pi}{2}$, the cosine function admits the following exponential series representation involving even values of the Riemann zeta function:

$$\cos x = \exp \left(- \sum_{k=1}^{\infty} \frac{2^{2k}(1-2^{-2k})}{k\pi^{2k}} \zeta(2k) x^{2k} \right).$$

1. This representation parallels the corresponding expansion obtained for the sine function, confirming that the underlying method applies systematically to trigonometric entire functions.
2. The coefficients reflect the restriction of the zero set of the cosine function to odd multiples of $\frac{\pi}{2}$, encoded via the factor $(1-2^{-2k})$.
3. The method suggests further applicability to entire functions with explicitly known zero distributions.

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